GAFA, Geom. funct. anal.
Vol. 13 (2003) 935 – 974
1016-443X/03/050935-40
DOI 10.1007/s00039-003-0436-5

GAFA Geometric And Functional Analysis

GROWTH AND PERCOLATION ON THE UNIFORM INFINITE PLANAR TRIANGULATION

O. Angel

Abstract

A construction as a growth process for sampling of the uniform infinite planar triangulation (UIPT), defined in [AnS], is given. The construction is algorithmic in nature, and is an efficient method of sampling a portion of the UIPT.

By analyzing the progress rate of the growth process we show that a.s. the UIPT has growth rate r^4 up to polylogarithmic factors, in accordance with heuristic results from the physics literature. Additionally, the boundary component of the ball of radius r separating it from infinity a.s. has growth rate r^2 up to polylogarithmic factors. It is also shown that the properly scaled size of a variant of the free triangulation of an m-gon (also defined in [AnS]) converges in distribution to an asymmetric stable random variable of type 1/2.

By combining Bernoulli site percolation with the growth process for the UIPT, it is shown that a.s. the critical probability $p_c = 1/2$ and that at p_c percolation does not occur.

1 Introduction

Since the 1960's there has been a combinatorial study of the properties of random finite planar maps chosen uniformly among members of one of a number of classes of planar maps. Many of their statistical properties have been studied. For example, Tutte [T2] has shown that almost all members of a certain class of planar maps have no non-trivial symmetries. Later by other means this result was extended to many other classes of planar structures [RW2].

Such planar classes include the class of general planar maps as well as the classes of triangulations (where all faces are triangles), quadrangulations and maps with other possible faces, and others. Members of a class may also have restrictions on degrees of vertices or on their connectivity. While understanding each particular class involves different techniques for handling the particular difficulties the class presents, many results appear to hold universally for any class of planar structures examined. It generally appears that there is a single universality class describing all (sufficiently unconstrained) classes of planar structures.

Another aspect common to many of the statistical results regarding uniformly chosen planar structures is dealing with asymptotics of the structure's properties as the size tends to infinity. Thus there are results about the distribution of degrees in a uniformly chosen triangulation [GR], the size of 3-connected and 4-connected components (or more general cores) [BeRW], [BFSS], [GW], and probabilistic 0-1 laws [BeCR]. Many of these results may be viewed as a finite version of corresponding results regarding an infinite planar structure of the same class. Considering infinite versions of such results may make the results more concise, though some of the precision of the finite case may be lost. For example, asymmetry of all but an exponentially small portion of planar maps translates to the simple (but no longer quantitative) fact that the infinite object a.s. has no symmetries.

The uniform infinite planar triangulation (UIPT) is one case of such an infinite object. It is defined by considering the uniform measure on rooted planar triangulations with n vertices and taking a weak limit as n tends to infinity. This gives rise to a probability measure supported on infinite planar triangulations, with the UIPT denoting a sample. The UIPT was suggested in [BenS] and shown to exist in [AnS] where some of its properties are studied. Of course a similar limit may be taken for any class of planar structures.

In this paper we consider triangulations only because the techniques presented in this paper apply to them more easily. However, there is no deep reason to prefer triangulations to other classes of planar maps, and the techniques may be extended to other classes as well. While local properties such as degree distribution are dependent on the class, large scale properties appear to be independent of local definitions. This is demonstrated in the infinite setting in [AnS] where two types of triangulations are studied – with or without double edges – and a simple relation between them is given. This relation implies that on a large scale the two objects have the same properties.

Schaeffer [Sc] found a bijection between certain types planar maps and labeled trees. Chassaing and Schaeffer [CS] recently used this bijection to show a connection between the asymptotic distribution of the radius of a random map and the integrated super-Brownian excursion. They deduce from this connection that the diameter of such a map of size n scales as $n^{1/4}$. The results presented here on the volume growth of the UIPT are a kind of infinite version of their result. While they work with planar quadrangulations and we with triangulations, it appears that such local differences are insignificant when large scale observations such as diameter, growth, separation, etc. are concerned. These are consequences of having a single universality class.

Physicists study similar random planar structures under the title of *2-dimensional quantum gravity*. The essential idea here is to develop a quantum theory of gravity by extending to higher dimensions the concept of Feynman integrals for one-dimensional paths. Triangulations (or other classes of maps) are viewed as discretized versions of 2-dimensional manifolds. In two dimensions this gives rise to a rich theory, much of which appears to be missing for higher dimensions. More often physicists are interested not in the discretized planar triangulation but in a continuous scaling limit of it which is believed to exist.

Once the basic structure is defined, models of statistical physics may be introduced on it and in many cases solved (e.g. [BoK] and others). Physicists applied here the methods of random matrix models [DiGZ]. Through these methods and other heuristics many conjectures were made on the structure of such triangulations. In particular, it is believed that the Hausdorff dimension of the scaling limit of 2-dimensional quantum gravity is 4 [AW]. This is a continuous form of the volume growth results of sections 5 and 6. For a good general exposition of quantum gravity see [ADJ], as well as [A], [D].

A further important motivation for understanding random triangulations (and random planar structures in general) stems from the KPZ relation [KPZ]. On a random surface many models of statistical physics become easier to analyze than in the Euclidean plane, as some of the geometric aspects of the problem disappear or can be disregarded. The KPZ relation, while not rigorously understood, is a relation between critical exponents of models on a random planar surface and the corresponding exponents in the plane.

For example non-intersection exponents for Brownian motion in the Euclidean plane or half plane correspond to asymptotic non-intersection exponents for random walks on a random surface. Similar relations hold for exponents governing behavior of self avoiding or loop erased random walks, boundary geometry of clusters in percolation, Ising or Potts model, and more. Using this relation the values of Brownian motion intersection exponents were calculated [Du1,2]. Later a rigorous derivation of the same values was found using the *SLE* process [LSW1,2,3].

The general structure of this paper is as follows: In section 2 a process for sampling the UIPT is presented – the peeling process. The UIPT is produced as the output of a growth process with relatively simple steps. This makes possible the analysis of the following sections. In section 3 a key aspect of the sampling process is considered – the boundary size of the finite triangulation generated after a number of steps. This boundary size, apart from being of independent interest in connection with the question of separation, proves to be essential for understanding the growth of the UIPT and (to a lesser extent) Bernoulli percolation. In sections 4, 5, 6 respectively asymptotic results on the boundary size of the ball, the hull's volume and the ball's volume are proved. Finally, in section 7, percolation of the UIPT is studied. The analysis here is based on a significant simplification of Bernoulli percolation derived from the construction of section 2, and depends only weakly on results in other parts of the paper.

We now proceed to state the main results of the paper, followed by some needed background and results (both general and specific).

We use the following notation: $X_n \sim Y_n$ will mean that $X_n/Y_n \to 1$, while $X_n \approx Y_n$ will mean that $\log X_n / \log Y_n \to 1$.

1.1 Main results. We consider in this paper only type II triangulations, i.e. planar triangulations with possibly double edges but no loops. The results on growth and percolation may be translated to type III triangulations through the relation between the two UIPT laws [AnS], and to type I triangulations through a similar decomposition. Precise definitions of these classes appear in [AnS].

The UIPT is the law of a measure on infinite rooted planar maps. Let B_r be the ball of radius r (w.r.t. the graph metric) around the root in the UIPT, thus B_r is a finite sub-triangulation of the UIPT. The complement of the ball is generally not connected. Denote by $\overline{B}_r$ the hull of the ball consisting of B_r together with all finite components of the complement. If $|T|$ is the number of vertices in a triangulation T, then we prove

Theorem 1.1. *For any $\varepsilon > 0$, a.s.*

$$\limsup_{r\to\infty} \frac{|\overline{B}_r|}{r^4 \log^{6+\varepsilon} r} < \infty ,$$

and a.s.

$$\lim_{r\to\infty} \frac{|\overline{B}_r| \log^{32/3+\varepsilon} r}{r^4} = \infty .$$

We also prove for the ball itself:

Theorem 1.2. *A.s.*

$$\limsup_{r\to\infty} \frac{|B_r|}{r^4 \log^{6+\varepsilon} r} < \infty\,,$$

and for any $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{|B_r| \log^{33/2+\varepsilon} r}{r^4} = \infty\,.$$

Analyzing $|\overline{B}_R|$ is a step in the analysis of $|B_r|$. Since $|B_r| \leq |\overline{B}_R|$, the upper bounds are the same. The lower bound on $|B_r|$ does not follow immediately, and further estimates are needed. This causes the lower bound on $|B_r|$ to be slightly weaker, however, this is of little significance as in the above (and following) results the powers of the logarithms are probably not the best possible. In proving these theorems the quantities in question are expressed as cumulative sums of random variables which are neither independent nor identically distributed, but are independent enough for some methods. Thus the above results are a sort of law of iterated logarithm (LIL) for these sums. The proof as well as other recent results [CS] suggest

CONJECTURE 1.3. *The random variables* $r^{-4}|B_r|$ *and* $r^{-4}|\overline{B}_r|$ *converge in distribution.*

This is roughly a converse of the result of [CS], stating that the radius of a uniform planar quadrangulation of size N scales as $N^{1/4}$.

While it is the ball growth rate of graphs which historically was the focus of more extensive research, the hull is also of independent interest (even apart from its submission to analysis). For one thing, there are questions regarding separation properties and the isoperimetric inequality in random planar maps. Hulls of balls (not necessarily around the root vertex) are candidates for having small boundary sizes compared to their volume.

Let $\partial\overline{B}_r$ denote the outer boundary of B_r, i.e. the vertices with neighbors in the infinite component of the complement. For $\partial\overline{B}_r$ we prove the following:

Theorem 1.4. *A.s. the size of the outer boundary of the ball of radius r, satisfies*

$$\limsup_{r\to\infty} \frac{|\partial\overline{B}_r|}{r^2 \log^3 r} < \infty\,,$$

and for any $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{|\partial\overline{B}_r| \log^{6+\varepsilon} r}{r^2} = \infty\,.$$

In particular this demonstrates that the hull's boundary is roughly the square root of their volume. While generally a large set with a small boundary size will not be a ball centered at the root vertex, it is plausible that it is not very different from a ball around some vertex. This suggests an anchored isoperimetric inequality, saying that the minimal boundary size for a connected set S in the UIPT of size n containing the root $\min_{0\in S,|S|=n}|\partial S|$ scales as $\sqrt{n}$.

This is in contrast to the finite case where a heuristic argument suggests a possible scaling of $n^{1/4}$. The argument is that in a uniform map of size n typical distances are on the order of $n^{1/4}$ (see [CS] and the argument of section 6). Hence it is likely that to separate such a map into two roughly equal parts one needs a cycle of length of order $n^{1/4}$. Of course, there is no need for the two exponents to coincide, as the finite version of the anchored isoperimetric inequality for the UIPT is to find the minimal boundary of a set of fixed size containing the root, and not of a set of roughly half the vertices.

When the UIPT is sampled using the growth process it is easy to add random colors to the vertices. This results in a sample of Bernoulli site percolation on the UIPT. Hence, percolation on the UIPT is effectively reduced to a simple Markov chain. Using this approach we see that as far as percolation goes the UIPT is similar to the triangular lattice in $\mathbb{R}^2$:

Theorem 1.5. *On the UIPT, the critical probability for site percolation is a.s. 1/2. Moreover, at $p = 1/2$ a.s. there are no infinite clusters.*

The transition bertween the annealed and quenched versions of this theorem is made with a 0-1 result for the UIPT and for percolation on it that is a consequence of the peeling construction. Percolation on the UIPT will be studied further in a future paper [An].

1.2 Further background. The results in this section either appear in [AnS] and are repeated here for completeness or are general facts needed.

1.2.1 Counting triangulations. Many of the results derived about the UIPT are in essence consequences of the asymptotics of the formula for the number of triangulations of a given size. These asymptotics are common to many other classes of planar structures. The following enumerative result due to Mullin [M], is derived using the techniques introduced by Tutte [T1], [GoJ].

Proposition 1.6. *For $n, m \geq 0$, not both 0, the number of rooted type II triangulations of a disc with $m+2$ boundary vertices and n internal vertices is*

$$\varphi_{n,m} = \frac{2^{n+1}(2m+1)!(2m+3n)!}{m!^2 n!(2m+2n+2)!} .$$

The case $n = m = 0$ for type II triangulations warrants special attention. A triangulation of a 2-gon must have at least one internal vertex so there are no triangulations with $n = m = 0$, yet the above formula gives $\varphi_{0,0} = 1$. It will be convenient to use this value rather than 0 for the following reason: Typically a triangulation of an m-gon is used not in itself but is used to close an external face of size m of some other triangulation by "gluing" it in. When the external face is a 2-gon, there is a further possibility of closing the hole by gluing the two edges to each other with no additional vertices. Setting $\varphi_{0,0} = 1$ takes this possibility into account. The formula will therefore be used also for $n = m = 0$.

Using Stirling's formula, the asymptotics of this are found to be

$$\varphi_{n,m} \sim C_m \alpha^n n^{-5/2},$$

where $\alpha = 27/2$ and

$$C_m = \frac{\sqrt{3}(2m+1)!}{4\sqrt{\pi}m!^2}(9/4)^m \sim C9^m m^{1/2}.$$

The power terms $n^{-5/2}$ and $m^{1/2}$ are common to many classes of planar structures, and are the first symptoms of universality. They arise from the common observation that a cycle partitions the plane into two parts (Jordan's curve theorem) and that the two parts may generally be triangulated (or for other classes, filled) independently of each other. This leads to a similar recursion formula for the number of maps which are translated to a quadratic equation for the number of maps. The similar form of the equations leads to algebraic singularities of the same order, and hence the asymptotics (see [GoJ], [BFSS]).

We also define and use the partition function for triangulations of an $(m+2)$-gon:

$$Z_m(t) = \sum \varphi_{n,m} t^n.$$

The following appears (up to a reparametrization) in [GoJ, §2.9] as an intermediate step in the derivation of Proposition 1.6.

Proposition 1.7. *If $t = \theta(1-2\theta)^2$ with $\theta \in [0, 1/6]$, then*

$$Z_m(t) = \frac{(2m)!((1-6\theta)m + 2 - 6\theta)}{m!(m+2)!}(1-2\theta)^{-(2m+2)}.$$

As a corollary, by setting $\theta = 1/6$ and $t = 2/27 = \alpha^{-1}$ we get

$$Z_m = Z_m(\alpha^{-1}) = \frac{(2m)!}{m!(m+2)!}\left(\frac{9}{4}\right)^{m+1}.$$

1.2.2 Locality. The free (Boltzmann) triangulation of a disc is defined in [AnS]:

Definition 1.8. *The* free distribution *on rooted triangulations of an* $(m+2)$*-gon, denoted* μ_m*, is the probability measure that assigns weight*

$$\alpha^{-n}/Z_m(\alpha^{-1})$$

to each rooted triangulation of the $(m+2)$*-gon having* n *internal vertices.*

One more distribution on triangulations of a disc is given by the UIPT of a disc. The limit of uniform distributions on triangulations of the sphere with N vertices as $N \to \infty$ exists. By conditioning on the root having m distinct neighbors with a single edge to each, and removing the triangles incident on the root (and choosing a new root) an immediate corollary is that there also exists a uniform distribution on infinite triangulations of an m-gon. The UIPT of a disc may be embedded in the disc having a single accumulation point. Since any point of the Riemann sphere may be mapped to infinity, by applying an inversion of the plane, a locally finite embedding on the outside of a disc is found.

Definition 1.9. *A rooted triangulation* A *is* rigid *if no triangulation includes two distinct sub-triangulations with coinciding roots, both isomorphic to* A*.*

Note that for a sub-triangulation T to be isomorphic to A, it is not enough that the graph of T is the same as A's, but it is needed that the two embeddings are equivalent and that the triangles of T are exactly the triangles of A. A simple case is when the triangles of a triangulation form a connected set in the dual graph of the triangulation, and then the triangulation is rigid. This is not a necessary condition, but it suffices for current needs.

For a finite rooted triangulation A with k external faces, define the event $R_i(A)$ as the set of all infinite rooted triangulations T of the plane that include A as a sub-triangulation with the roots coinciding, and such that the component of T in the i'th face of A is infinite. As for an infinite triangulation of a disc, the infinite part of the triangulation in such an embedding is generally located in some finite region of the plane with some accumulation point, and the infinite face is just another triangle.

The basic tools for the construction of the next section are the following results (see Proposition 4.10 and Theorem 5.1 of [AnS]).

Theorem 1.10. *Let* A *be a rigid rooted triangulation, with* n *vertices, some of which are on* k *external boundary components of sizes*

$m_1+2,\dots,m_k+2$. *The events* $R_i(A)$ *are almost disjoint (i.e.* $\tau(R_i(A)\cap R_j(A))=0$ *for* $i\neq j$*), and the probability that the i'th face is the infinite one is*

$$\frac{\alpha^{3-n}}{C_1}C_{m_i}\prod_{j\neq i}Z_{m_j}\,.$$

Theorem 1.11. *Let A be a finite rigid triangulation, and assume A has k external faces of sizes* $m_1+2,\dots,m_k+2$*. Conditioned on the event* $R_i(A)$*, let* T_j *denote the component of the UIPT in the j'th face. Then,*

1. *The triangulations* T_j *are independent.*
2. T_i *has the same law as the UIPT of an* (m_i+2)*-gon.*
3. *For* $j\neq i$*,* T_j *has the same law as the free triangulation of an* (m_j+2)*-gon.*

1.2.3 Stable random variables. For any $\alpha\in(0,2)$ a completely asymmetric stable random variable of type α will be denoted by S_α. These are real random variables with the property that the sum of n i.i.d. copies of S_α is distributed like $n^{1/\alpha}S_\alpha$. The completely asymmetric stable random variables are characterized by having density functions with super-polynomial decay on the left (as in the last three properties below). We will need the following facts about stable random variables (see [ST], [Z]):

Fact 1.12. 1. *As* $t\to\infty$*,* $\mathbb{P}(S_\alpha>t)\sim ct^{-\alpha}$*.*

2. *For* $0<\alpha<1$*, a.s.* $S_\alpha>0$*.*

3. *For* $0<\alpha<1$ *there exists* $c>0$ *such that for small positive t,* $\mathbb{P}(S_\alpha<t)<\exp\left(-ct^{\alpha/(\alpha-1)}\right)$*.*

4. *For* $1<\alpha<2$ *there exists* $c>0$ *such that for* $t>0$*,* $\mathbb{P}(S_\alpha<-t)<\exp\left(-ct^{\alpha/(\alpha-1)}\right)$*.*

Of particular interest is $S_{3/2}$ having the Airy distribution, with noted connections to random planar structures [BFSS], and $S_{1/2}$ having the Levi distribution.

2 A Growth Process

A possible method of sampling the UIPT is by adding one triangle at a time to a finite sub-triangulation, each time adding a triangle with the appropriate distribution conditioned on the sub-triangulations sampled so far. This has some advantages over adding all the vertices a given distance from the root at once. Primarily, this method has simple steps with a (relatively) simple distribution which can be written explicitly. However, the number of steps it takes to reach a given distance from the root is not fixed and has to be estimated.

The idea behind this construction and the heuristics for the Hausdorff dimension of the scaling limit of 2-dimensional quantum gravity may be found in section 4.7 of [ADJ]. Following them, we call the process "peeling", as it is similar to peeling an apple by cutting a thin strip going around the apple in circles. The name is especially appropriate when the peeling is made in an ordered manner, as in the sections regarding growth estimates, however, we use the name also in the context of section 7 where the process advances in a more chaotic manner.

Consider first the case of a free triangulation T of an $(m+2)$-gon. Call the boundary vertices $x_0,\dots,x_{m+1}$. There is a single triangle $t\in T$ containing the edge (x_{m+1},x_0). (This edge is chosen for simplicity; by symmetry the following discussion holds for any other boundary edge as well.) Denote the third vertex of t by y. There are two possibilities: either y is an internal vertex of the triangulation, or else $y=x_i$ for some $1\in[1,m]$. In the former case, the rest of the triangulation is a triangulation of an $(m+3)$-gon, hence the sum of the weights of all such triangulations is $\alpha^{-1}Z_{m+1}$ (the factor α^{-1} accounts for y). Thus,

$$\mu_m\big(y\notin\{x_0,\dots,x_{m+1}\}\big)=\frac{Z_{m+1}}{\alpha Z_m}\,. \tag{2.1}$$

Similarly, if $y=x_i$, then we have the triangle (x_{m+1},x_0,x_i) and two triangulations of an $(i+1)$-gon and an $(m-i+2)$-gon. The weight of a triangulation, $\alpha^{-|T|}$, is multiplicative, and since any pair of triangulations for the two components is possible, the total weight is $Z_{i-1}Z_{m-i}$ (see Figure 2.1). Therefore

$$\mu_m(y=x_i)=\frac{Z_{i-1}Z_{m-i}}{Z_m}\,, \tag{2.2}$$

and conditioned on $y=x_i$ the two components are filled with independent free triangulations.

The case $y=x_1$ is especially interesting, for then we have the triangle (x_{m+1},x_0,x_1). If this is also the triangle supported on the boundary edge (x_0,x_1), then after adding it the vertex x_0 is no longer on the boundary of the remaining triangulation. Otherwise we still need to triangulate the 2-gon (x_0,x_1). This was accounted for in the above formula since we set $\varphi_{0,0}=1$ and not 0. The triangulation of a 2-gon with no internal vertices has no internal triangles and so the edges are glued together.

The same thing may happen in the case $y=x_m$ with the edge (x_m,x_{m+1}). In the case that $m=1$ and $y=x_1=x_m$ there is a possibility

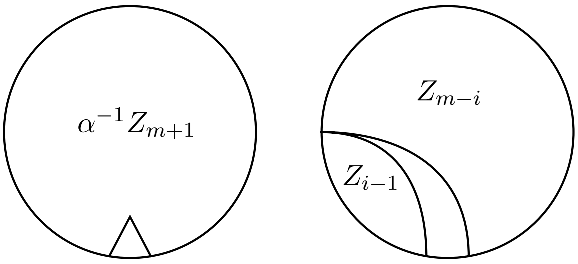

Figure 2.1: The two possibilities when a triangle is added in a free triangulation.

that both 2-gons are glued, in which case there are no discs left to triangulate and hence no additional triangles.

Equations (2.1) and (2.2) allow us to sample a free triangulation of any disc by adding triangles one at a time. If the disc is separated then the process is split into two independent processes for the resulting discs. Since we know that the free triangulation is finite, the process a.s. terminates at some time. Note that we can choose which boundary edge to build on at each iteration in any way we want to without effecting the resulting distribution.

A similar construction allows us to sample the UIPT. Suppose we wish to sample the UIPT of an $(m+2)$-gon, with external boundary vertices $x_0,\dots,x_{m+1}$. Heuristically, the same notion of the total weight for each possibility appears as before, with C_m being the total weight of infinite triangulations of an $(m+2)$-gon. This is so since the weight for triangulations of size N is roughly $C_m N^{-5/2}$, and the power term will cancel with an identical term in the denominator when $N \to \infty$.

To be precise, we consider as before the triangle (x_{m+1}, x_0, y) containing the edge (x_{m+1}, x_0). The UIPT is the limit of finite uniform measures, and for the measure with N additional vertices, the probability that y is a new vertex is a simple ratio. Taking the limit we find that

$$\begin{aligned}\mathbb{P}\big(y \notin \{x_0,\dots,x_{m+1}\}\big) &= \lim_{N\to\infty} \frac{\varphi_{m+1,N-1}}{\varphi_{m,N}} \\ &= \frac{C_{m+1}}{\alpha C_m}\,. \end{aligned} \tag{2.3}$$

The case $y = x_i$ is similar. Then we have two sub-triangulations: T_1 with boundary size $i+1$ and T_2 with boundary size $m-i+2$. One of them is infinite, and by Theorem 1.11 the infinite one is a UIPT and the finite is a free triangulation. We have two more possibilities, as in Figure 2.2, with probabilities (from Theorem 1.10):

$$\mathbb{P}\big((y = x_i) \cap T_1 \text{ is infinite}\big) = \frac{Z_{m-i}C_{i-1}}{C_m}$$
$$\mathbb{P}\big((y = x_i) \cap T_2 \text{ is infinite}\big) = \frac{C_{m-i}Z_{i-1}}{C_m}\,. \tag{2.4}$$

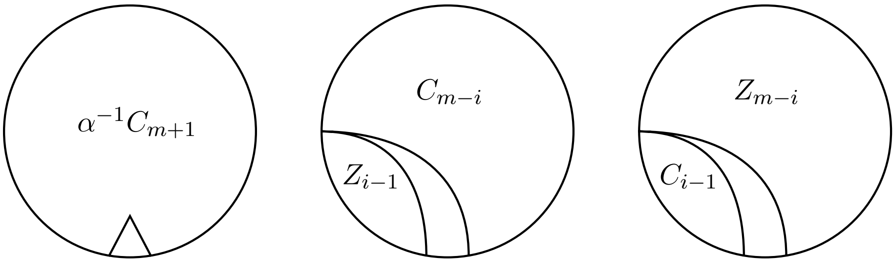

Figure 2.2: The possibilities when a triangle is added in the UIPT.

We can use (2.3),(2.4) to sample a neighborhood of the root in the UIPT, as follows: Start with the root triangle, and proceed to add new triangles to the triangulation. When the added triangle partitions the triangulation to a finite and infinite part, fill the finite part with a free triangulation. In either case the remaining triangulation is now a UIPT of some polygon.

To sample a ball of radius r, proceed as above as long as there are any vertices on the boundary that are at distance less than r from the root, and add triangles incident on such vertices. Since the ball is a.s. finite (from the tightness of its size, Corollary 4.5 of [AnS]), this process will a.s. terminate, giving a sample of the ball of radius r. Termination can also be deduced directly from the stated process by induction on r: Suppose that a.s. at some time all vertices at distance less than r from the root are in the interior, then the boundary contains only vertices at distance r or greater, so any new vertex we add is at distance at least $r + 1$. Hence the set of boundary vertices at distance r can only decrease. If we extend our triangulation at the boundary edge (u, v), then with probability bounded away from 0, v will not be in the new boundary. Thus a.s. after some finite number of iterations all vertices at distance r will also be in the interior. Since we have shown a process that a.s. terminates in a finite time and outputs a neighborhood of the root in the UIPT, this gives an alternative proof of the existence of the UIPT probability measure.

NOTE. In the construction above there is at each step complete freedom in the choice of which edge to build on. Typically, especially for analyzing the growth rate of the UIPT, we will go around the triangulation in a fixed direction. This means that we take a vertex v, on the boundary, and add the triangle incident on the edge to its right as long as v remains on the outer boundary. Whenever a hole is formed we fill it with a free triangulation. As soon as v leaves the boundary we look along the old boundary counterclockwise from v to the first vertex that is in the new boundary. This will guarantee that we find all vertices at distance r from the root before proceeding to $r+1$.

For other uses we may choose the edge to build on differently. Thus for analyzing percolation the edge we choose will depend on the colors of vertices previously visited. The results of the next section do not depend on the choice of the edge at each step.

The peeling method is somewhat flexible, and lends itself to sampling from a number of classes of planar objects. For example, in order to sample a type I triangulation, where looped edges are allowed, one only needs to find the appropriate values of Z_m, C_m and to include the possibility that the third vertex of the added triangle coincides with one of the other two. If the values of C_m, Z_m have the same asymptotics, then the peeling process will proceed in much the same way and the following results may apply to that class as well.

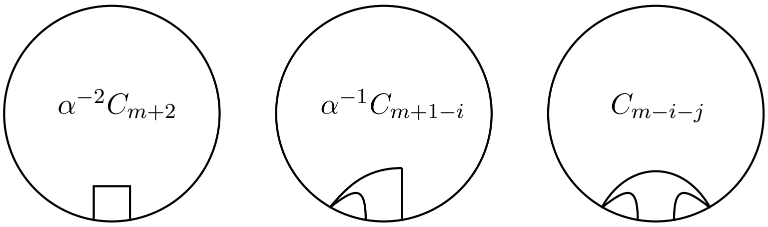

Figure 2.3: Some possibilities when sampling an infinite quadrangulation of the plane.

To sample a uniform infinite quadrangulation of the plane, a structure which may be defined and proved to exist in much the same way as the UIPT, the peeling process adds a quadrangle each time. The number of possibilities now grows as there may be two, one or no new vertices, as in Figure 2.3. In each case also the infinite face needs to be distinguished from the others. However, the process stays essentially the same, and using formulas for the asymptotic number of quadrangulations of a disc and for the partition function of the same, the process may be analyzed similarly.

3 Markov Chain

To study the growth rate of the UIPT we will analyze the peeling process for constructing the UIPT. At first we will focus only on the evolution of the outer boundary of the generated sub-triangulations as triangles are added.

Suppose we have a finite sub-triangulation with outer boundary of size $m+2$. We add to it the triangle on the outer face containing a given boundary edge e. Two of the vertices of this triangle are given by the ends of e. The probability that the third vertex is not in the sub-triangulation is given by (2.3), and on this event the size of the boundary increases by 1. The probability that the third vertex is some vertex on the outer boundary is given by (2.4). In this case the triangulation of the outer face is partitioned to a finite part with boundary of size $i+1$ for some $i \geq 1$, and an infinite part with boundary of size $m-i+2$. Thus the size of the outer boundary decreases by i. The probability of that is $2\frac{C_{m-i}Z_{i-1}}{C_m}$. (The factor of 2 comes from the fact that there are 2 vertices at distance i from e).

Let M_n be the size of the outer boundary after n such triangles have been added. We see that the evolution of M_n can be described as a Markov chain satisfying $M_{n+1} = M_n + X_n$ where the distribution of X_n is given by

$$\begin{aligned} \mathbb{P}(X_n = 1 | M_n = m) &= \frac{C_{m+1}}{\alpha C_m} = \frac{2m+3}{3m+3}\,, \\ \mathbb{P}(X_n = -k | M_n = m) &= \frac{2C_{m-k}Z_{k-1}}{C_m}\,, \\ &= \frac{2(2k-2)!}{(k-1)!(k+1)!} \cdot \frac{m!^2(2m-2k+1)!}{(m-k)!^2(2m+1)!}\,, \end{aligned} \tag{3.1}$$

for $k \in \mathbb{N}$.

Denote these probabilities by $p_{-1,m}$ and $p_{k,m}$ respectively. When m is large these probabilities converge to a limit distribution. Denote by p_{-1} (resp. p_k) the limit of the probability of having $X = 1$ (resp. $X = -k$), then

$$\begin{aligned} p_{-1} &= \lim_{m\to\infty} p_{-1,m} = \tfrac{2}{3}\,, \\ p_k &= \lim_{m\to\infty} p_{k,m} = \frac{2(2k-2)!}{(k-1)!(k+1)!4^k}\,. \end{aligned} \tag{3.2}$$

Note. It is worthwhile observing the relation,

$$p_{k,m} = p_k \frac{(m)_k}{(m+1/2)_k}\,,$$

where

$$(x)_k = x(x-1)\dots(x-k+1) = \frac{\Gamma(x+1)}{\Gamma(X-k+1)}$$

is the descending factorial notation. This implies that for $k > 0$, the probabilities $p_{k,m}$ are increasing in m and converge to p_k. From this it follows that if $m_1 < m_2$ then the step distribution from m_1 (i.e. the law of X_n conditioned on $M_n = m_1$) stochastically dominates the step distribution from m_2.

Another consequence is that for any $a < 1$ there is a $c = c(a) > 0$, such that $k < am$ implies $p_{k,m} > cp_k$. In other terms, the probability of steps of size up to a constant fraction of m is within a constant factor of the limit probability. Of course for steps larger than m the probability is 0. Hence the step distribution from m is similar to the limit distribution conditioned to be at most m.

In light of the convergence of the step distributions, the states of the Markov chain, M_n, can be viewed as a random walk on the integers, with step distribution depending slightly on the location. The Markov chain will be shown to be transient, and so the step distributions indeed converge. A central difficulty arises from the fact that the limit distribution has infinite variation (indeed $\mathrm{Var}(X|M = m) \sim c\sqrt{m}$).

Let X be a sample of the limit distribution. Because $p_k \sim ck^{-5/2}$, X has a finite α'th moment iff $\alpha < 3/2$. Since $\mathbb{E}X = 0$ and X is bounded by 1, the theory of stable random variables (see [ST], [Z]) tells us that if $X_0, X_1, \dots$ were i.i.d. copies of X, then $n^{-2/3}\sum_{i<n} X_i$ would have converged in distribution to a totally asymmetric stable random variable of type 3/2. Of course, in our case, the steps X_i are neither independent nor are they equally distributed. Instead their distribution depends on the sum of their predecessors. Still, this gives an indication that M_n should be studied at the scale of $n^{2/3}$. In fact, the sequence M_n with proper scaling appears to converge to a stable process conditioned to remain positive.

Since M_n is finite, the expectation of X_n is not 0, but rather the random walk has some drift. A straightforward calculation using generating functions leads to the following result:

Lemma 3.1.

$$\mathbb{E}(X_n|M_n = m) = \frac{4^m m!^2}{(2m+1)!} \sim \sqrt{\frac{\pi}{2m}}\,.$$

Sketch of proof. In view of the distribution of X_n, calculating the expectation involves substituting $x = 1$ and finding the coefficient of y^m in

$$\sum_{k,m} y^m C_{m-k} Z_{k-1} k x^k \,.$$

This may be re-written as

$$\sum_k k Z_{k-1} (xy)^k \sum_{m \geq k} C_{m-k} y^{m-k}.$$

However, using the binomial formula we have

$$\sum C_l y^l = \frac{\sqrt{3}}{4\sqrt{\pi}} (1-9y)^{-3/2},$$

and

$$\sum k Z_{k-1} (xy)^k = \frac{2-(2+9xy)\sqrt{1-9xy}}{54xy} \,.$$

Combining these identities, gives a closed form for the above double sum, from which coefficients may be extracted. □

This leads us to believe that the rate of growth of M_n is roughly equal to $M_n^{-1/2}$. This corresponds (again) to a growth rate of $n^{2/3}$. In fact, a lower bound on $\mathbb{E}M_n$ follows from the convexity of $x \to x^{-1/2}$:

$$EX_n \geq c\mathbb{E}(M_n^{-1/2}) \geq c(\mathbb{E}M_n)^{-1/2},$$

$$\mathbb{E}M_{n+1} = \mathbb{E}(M_n + X_n) \geq \mathbb{E}M_n + c(\mathbb{E}M_n)^{-1/2},$$

and therefore $\mathbb{E}M_n \geq c' n^{2/3}$.

This rate of growth is indeed correct and we prove the following variation on the LIL for M_n:

Theorem 3.2. *A.s.*

$$\limsup \frac{M_n}{n^{2/3} \log n} < \infty \,.$$

The proof is similar to the proof of the law of iterated logarithms, with some modifications to accommodate the positive expectation and unbounded variation of the steps. While a better upper bound with an iterated logarithm may hold, we will not attempt to prove such an upper bound here, since it is the power term we are primarily interested in. Additionally, in evaluating the growth rate of the UIPT, other steps add logarithmic factors, so the end result will not improve significantly by having a tighter bound here.

Let $\mathcal{F}_n$ denote the σ-field generated by the random variables $M_0, \dots, M_n$. Thus $\mathcal{F}_0$ is the trivial σ-field. $\mathbb{E}_{\mathcal{F}_n}$ will denote expectation w.r.t. $\mathcal{F}_n$, i.e., expectation conditioned on the past. Lemma 3.1 may therefore be stated as $\mathbb{E}_{\mathcal{F}_n} X_n \sim c M_n^{-1/2}$.

Proof. We use the fact that for any λ_0 there is an absolute constant c_1, such that for $t < \lambda_0$

$$e^t < 1 + t + c_1|t|^{3/2}.$$

Denote by $\overline{X}_n$ the normalized steps: $\overline{X}_n = X_n - \mathbb{E}_{\mathcal{F}_n} X_n$. Using the bounds $0 < \mathbb{E}_{\mathcal{F}_n} X_n \le 1$, the 3/2th moment of $\overline{X}_n$ is estimated.

$$\begin{aligned}
\mathbb{E}_{\mathcal{F}_n} |\overline{X}_n|^{3/2} &= p_{-1,M_n}(1 - \mathbb{E}_{\mathcal{F}_n} X_n)^{3/2} + \sum_{k=1}^{M_n} p_{k,M_n}(k + \mathbb{E}_{\mathcal{F}_n} X_n)^{3/2} \\
&< 1 + \sum_{k=1}^{M_n} p_k (k+1)^{3/2} \\
&< c_2 \sum_{k=1}^{M_n} k^{-1} \\
&< c_3 \log M_n \\
&< c_3 \log n\,.
\end{aligned}$$

A lower bound with a different constant also holds hence this is best possible.

For any small $\lambda > 0$ we have

$$\begin{aligned}
\mathbb{E}_{\mathcal{F}_n} e^{\lambda \overline{X}_n} &\le \mathbb{E}_{\mathcal{F}_n}\big(1 + \lambda \overline{X}_n + c_1(\lambda |\overline{X}_n|)^{3/2}\big) \\
&\le 1 + c_4 \lambda^{3/2} \log n \\
&< \exp\big(c_4 \lambda^{3/2} \log n\big)\,.
\end{aligned}$$

While the $\overline{X}_n$ are not independent, the last bound is deterministic (independent of the history). Therefore we may multiply for $m \le i < n$ to get

$$\mathbb{E} \exp\left(\lambda \sum_{i=m}^{n-1} \overline{X}_i\right) < \exp\big(c_4 \lambda^{3/2} (n-m) \log n\big)\,,$$

and so

$$\mathbb{P}\left(\sum_{i=m}^{n-1} \overline{X}_i > t\right) < \exp\big(-t\lambda + c_4 \lambda^{3/2} (n-m) \log n\big)\,.$$

For small enough λ, this is a bound on the probability that the sum of the steps over an interval $[m, n)$ is much bigger than the sum of their expectations. λ will be chosen to minimize this bound w.r.t. t, m, n. The optimal λ is such that

$$\sqrt{\lambda} = \frac{2t}{3c_4 (n-m) \log n}\,,$$

and then the bound becomes

$$\mathbb{P}\Big(\sum_{i=m}^{n-1}\overline{X}_i > t\Big) < \exp\left(-c_5\frac{t^3}{(n-m)^2\log^2 n}\right).$$

If t is such that $t^3 = (3/c_5)(n-m)^2\log^3 n$, then $\lambda = O((n-m)^{-2/3})$ tends to 0 as the size of the interval is large. Substituting the given values for t and λ we get, for sufficiently large intervals,

$$\mathbb{P}\Big(\sum_{i=m}^{n-1}\overline{X}_i > c_6(n-m)^{2/3}\log n\Big) < n^{-3}.$$

For small intervals this can be made to hold as well by increasing c_6.

Since the sum over all intervals $[m,n)$ with $m<n$ of n^{-3} is finite, by Borel–Cantelli, all but finitely many intervals satisfy

$$\sum_{i=m}^{n-1}\overline{X}_i < c_6(n-m)^{2/3}\log n\,. \tag{3.3}$$

Take $c_7 = c_6+1+a$. To conclude the proof of the theorem it is sufficient to show that for each n where $M_n > c_7 n^{2/3}\log n$ there is an interval $[m,n)$ where the above bound is violated. For this we use an upper bound on $\mathbb{E}_{\mathcal{F}_n}X_n$: There is an $a>0$ such that $\mathbb{E}_{\mathcal{F}_n}X_n < aM_n^{-1/2}$.

Suppose n is such that $M_n > c_7 n^{2/3}\log n$. Take

$$m = 1+\max\{m<n \mid M_m \le n^{2/3}\}\,.$$

By the choice of m, for each $i\in[m,n)$ we have $M_i > n^{2/3}$ and therefore $\mathbb{E}_{\mathcal{F}_i}X_i < an^{-1/3}$. We then have

$$\sum_{m}^{n-1}\mathbb{E}_{\mathcal{F}_i}X_i < (n-m)an^{-1/3} < an^{2/3}.$$

On the other hand,

$$\begin{aligned}\sum_{m}^{n-1}\overline{X}_i &= M_n - M_m - \sum_{m}^{n-1}\mathbb{E}_{\mathcal{F}_i}X_i\\ &> c_7 n^{2/3}\log n - n^{2/3} - an^{2/3}\\ &> c_6 n^{2/3}\log n\,,\end{aligned}$$

contradicting (3.3), thus we found an interval $[m,n)$ as claimed. □

Next, we consider the hitting probabilities of the Markov chain. Those will later be used in establishing a lower bound. The proof is another straightforward use of generating functions, similar to the proof of Lemma 3.1.

Claim 3.3. *Starting the Markov chain at n, the probability of it ever hitting m is given by*

$$\mathbb{P}(n \to m) = 1 - \frac{(n)_{m+1}}{(n+1/2)_{m+1}}\,.$$

Sketch of proof. Fix m and assume the probability in question is a_n, with $a_m = 1$. For any $n \neq m$ we must have

$$\begin{aligned} a_{n,m} &= \sum_{k=-1}^{n} p_{n,k} a_{n-k,m} \\ &= \frac{C_{n+1}}{\alpha C_n} a_{n+1,m} + \sum_{k=1}^{n} \frac{2C_{n-k}Z_{k-1}}{C_n} a_{n-k,m}\,. \end{aligned}$$

If $g(t) = (t\alpha)^{-1} - 1 + 2\sum Z_{k-1}t^k$, and $f(t) = \sum C_n a_n t^n$, the above translates to

$$f(t)g(t) = \tfrac{C_0}{\alpha t} + \beta t^m,$$

where β is determined by boundary conditions. Since $g(t) = 2(1-9t)^{3/2}/27t$ is known, we can find $f(t)$, express it as a power series and divide the coefficients by C_n to get a_n. □

Note. In particular, the probability of returning to 0 from n is $\frac{1}{2n+1}$ and thus the Markov chain is transient. This formula also holds for $m \geq n$, since the numerator vanishes. For $n \gg m$ the hitting probability is $\frac{m+1}{2n} + O(n^{-2})$.

As a simple consequence, the probability of never returning to n when starting from n is found. The only way to avoid returning to n is by having $X = 1$ and afterward not hitting n from $n+1$. Since every number is visited at least once, we get

Claim 3.4. *The number of visits to n when the Markov chain is started from 0, is a geometric random variable with mean*

$$\frac{3n+3}{2n+3}\frac{(n+3/2)_{n+1}}{(n+1)!} \sim c\sqrt{n}\,.$$

Theorem 3.5. *For any $\varepsilon > 0$, a.s.*

$$\lim_{n\to\infty} \frac{M_n \log^{2+\varepsilon} n}{n^{2/3}} = \infty\,.$$

Proof. Consider the time T_n when the Markov chain first hits 2^n. Fix some $\varepsilon > 0$ and set $a_n = 2^n n^{-(1+\varepsilon)}$. We consider the behavior of the Markov chain from time T_n until it leaves the interval $I_n = [a_n, 2^{n+1}]$. For this we consider the probability of the Markov chain taking a step from I_n into the interval $J_n = [a_n/2, a_n]$. The probability of taking a step from

M to $M' < M$ is monotone increasing in M' and monotone decreasing in M, hence among all steps from $M \in I_n$ to $M' \in J_n$ the step from 2^{n+1} to $\lceil a_n/2 \rceil$ has the smallest probability. Using Stirling's formula, the probability of such a step is

$$\begin{aligned}\mathbb{P}(M' = a_n/2 \mid M = 2^{n+1}) &= \mathbb{P}(X = a_n/2 - 2^{n+1} \mid M = 2^{n+1}) \\ &> c_1 \frac{(2^n)^{-5/2}}{n^{(1+\varepsilon)/2}} .\end{aligned}$$

It follows that the probability of taking a step into J_n from any $M \in I_n$ is at least

$$c_1 \frac{(2^n)^{-5/2}}{n^{(1+\varepsilon)/2}} |J_n| = c_2 \frac{(2^n)^{-3/2}}{n^{3(1+\varepsilon)/2}} .$$

Therefore for any k,

$$\begin{aligned}\mathbb{P}(\text{visiting } J_n \text{ after } T_n) \geq \mathbb{P}(T_{n+1} - T_n > k)\cdot& \\ \mathbb{P}(\text{visiting } J_n \text{ after } T_n | T_{n+1} - T_n > k)& \\ \geq \left(1 - \left(1 - c_2 \frac{(2^n)^{-3/2}}{n^{3(1+\varepsilon)/2}}\right)^k\right) \mathbb{P}(T_{n+1} - T_n > k) .&\end{aligned}$$

Setting $k = n^{3(1+\varepsilon)/2} 2^{3n/2}$ so that the first factor is roughly constant, gives

$$\mathbb{P}(T_{n+1} - T_n > n^{3(1+\varepsilon)/2} 2^{3n/2}) \leq c_3 \mathbb{P}(\text{visiting } J_n \text{ after } T_n) .$$

However, by Claim 3.3 the probability that after time T_n the Markov chain visits $\lfloor a_n \rfloor$ is approximately $n^{-1-\varepsilon}/2$. Since the Markov chain can increase by at most 1 at each step, if J_n is visited after time T_n, then so is $\lfloor a_n \rfloor$. Therefore

$$\mathbb{P}(T_{n+1} - T_n > n^{3(1+\varepsilon)/2} 2^{3n/2}) \leq \frac{c_4}{n^{1+\varepsilon}} .$$

Thus a.s. $T_{n+1} - T_n < n^{3(1+\varepsilon)/2} 2^{3n/2}$ for all but finitely many n. Summing, we get that for some c_5, a.s. for all but finitely many n, $T_n < c_5 n^{3(1+\varepsilon)/2} (2^n)^{3/2}$.

This gives a sequence of points lying below the graph of M_n. By interpolating this translates to the lower bound on M_n for any n. Formally, let m be such that $T_m \leq n < T_{m+1}$. Since the Markov chain reached 2^m before time n, necessarily $n \geq 2^m$, and by the bound on T_{m+1}, for all but finitely many n,

$$\begin{aligned}n < T_{m+1} &< c_5 (m+1)^{3(1+\varepsilon)/2} (2^{m+1})^{3/2}, \\ n^{2/3} &< c_6 m^{1+\varepsilon} 2^m.\end{aligned}$$

Since J_m is not visited after T_m (for large m),

$$M_n > 2^m m^{-(1+\varepsilon)}$$

$$
\begin{aligned}
&> c_6^{-1} n^{2/3} m^{-2(1+\varepsilon)} \\
&> c_7 n^{2/3} \log^{-2(1+\varepsilon)} n \,. \qquad \square
\end{aligned}
$$

4 Boundary Growth

Based on Theorems 3.2 and 3.5 we now turn to study the growth rate of the UIPT. The UIPT is sampled starting with only a root triangle, and adding triangles in the outer face of the sub-triangulation we have. The behavior of M_n – the size of the outer boundary after n triangles have been added – is known. We wish now to relate this to the distance between the root and the boundary.

Throughout this and the next two sections triangles are added in an ordered manner, going around the boundary counterclockwise, and adding all the triangles incident on a vertex before moving to the next. In this manner we are sure to encounter all vertices at distance r from the root before moving on to $r + 1$.

Initially, we investigate the number of steps it takes the above process to sample the ball of radius r around the root (as well as its hull). This translates to estimates on the size of the ball's boundary. Let T_r be the time (number of triangles added) when we have found the outer boundary of the ball of radius r around the root. We prove

Theorem 4.1. *A.s.*

$$
\limsup_{r\to\infty} \frac{T_r}{r^3 \log^3 r} < \infty \,,
$$

and for any $\varepsilon > 0$

$$
\lim_{r\to\infty} \frac{T_r \log^{6+\varepsilon} r}{r^3} = \infty \,.
$$

Let C_n denote the set of vertices on the outer boundary at time n (thus $|C_n| = M_n + 2$). Clearly after some number of steps we have gone round a full circle, and added every triangle incident on a vertex in C_n. Denote the time it takes to accomplish this by D_n, i.e.,

$$
D_n = \inf \left\{ d | C_{n+d} \cap C_n = \emptyset \right\} .
$$

Clearly $T_{r+1} = T_r + D_{T_r}$. It is plausible that the process proceeds in an approximately fixed rate along the boundary, so that D_n is roughly linear in M_n. We therefore expect $T_{r+1} - T_r = D_{T_r}$ to be on the order of $M_{T_r} \approx T_r^{2/3}$, indicating that T_r grows like r^3. Together with Theorems 3.2 and 3.5 this implies that $|C_{T_r}|$ grows like r^2, i.e., the size of the outer boundary of the ball of radius r is quadratic in r. Note that T_r is not the volume of the ball but only the number of steps made. At each step when $X_t < 0$, a free triangulation is glued to the sub-triangulation. The bulk of the volume lies in those.

First we prove the following estimate:

LEMMA 4.2. *Starting with M boundary vertices, let D denote the number of steps of the peeling process until every vertex originally on the boundary is in the interior. For some $a, c > 0$, $\mathbb{P}(D > aM) < e^{-cM}$.*

NOTE. It is impossible to get an exponential bound for $\mathbb{P}(D \leq bM)$ for any b, due to the thick tail on the negative side of X_n. In fact, for every $b > 0$ there will a.s. be infinitely many n's for which $D_n < bM_n$. This is so, since the expected number of visits to m is approximately $\sqrt{m}$ (by Claim 3.4) and the probability of making a single step of size at least $(1-\varepsilon)m$ is about $m^{-3/2}$, thus such large jumps are made infinitely often.

Proof of Lemma 4.2. Let Z_t be the number of vertices in C_0 that are removed from the boundary at time t. D is just the smallest number such that $\sum^D Z_t = M$, since once a vertex is in the interior of the triangulation it cannot return to the boundary at a later stage. When building a triangle on an edge, with probability $1-p_{-1,M_t}$ the new triangle does not introduce a new vertex, and some vertices leave the outer boundary. Since an endpoint of the edge is in C_0, with half that probability the vertices leaving the boundary are from C_0. Hence at each step, $\mathbb{P}(Z_t > 0)$ is bounded away from 0.

Since vertices are removed from the boundary at a positive rate bounded from 0, for some a the probability of taking more than aM steps for all vertices in C_0 to be removed decays exponentially in M. □

Proof of Theorem 4.1. By Theorem 3.5, M_{T_r} grows like a power of T_r. By Lemma 4.2, the probability that $D_{T_r} > aM_{T_r}$ decays exponentially in M_{T_r} so this a.s. fails only for finitely many values of r. Using Theorem 3.2 this gives for large enough r

$$D_{T_r} < aM_{T_r} < c_1 T_r^{2/3} \log T_r \,,$$

$$T_{r+1} < T_r + c_1 T_r^{2/3} \log T_r \,,$$

and so

$$T_r < c_2 r^3 \log^3 r \,.$$

For the lower bound, note that the vertices are added to the boundary only one at a time (when $X_n = 1$). Since all the vertices in $C_{T_{r+1}}$ have been added to the boundary after time T_r, it follows that

$$T_{r+1} - T_r > M_{T_{r+1}} \,.$$

Together with Theorem 3.5 this implies that for any $\varepsilon > 0$, a.s. for all but finitely many r,

$$T_{r+1} - T_r > \frac{T_{r+1}^{2/3}}{\log^{2+\varepsilon} T_{r+1}} > \frac{T_r^{2/3}}{\log^{2+\varepsilon} T_r} .$$

And so for any $\varepsilon > 0$, for large enough r,

$$T_r > r^3 \log^{-(6+\varepsilon)} r . \qquad \square$$

Combining this theorem with Theorems 3.2 and 3.5 gives

COROLLARY 4.3. *The size of the boundary of the B_r's hull, given by M_{T_r}, a.s. satisfies*

$$\limsup_{r\to\infty} \frac{M_{T_r}}{r^2 \log^3 r} < \infty ,$$

and for any $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{M_{T_r} \log^{6+\varepsilon} r}{r^2} = \infty .$$

5 Hull Volume Growth

Knowing how long it takes to sample B_r in the UIPT, we turn our attention to the distribution of its hull's volume. So far, new vertices were added at times when $X_t = 1$. Since $\mathbb{P}(X_t = 1)$ remains away from 0, the number of vertices added that way is linear in T_r. This shows a volume growth of at least r^3 (up to logarithmic factors).

There are many additional vertices that are added whenever $X_t < 0$. At those times a portion of the boundary is closed off and filled with a free triangulation of an $(|X_t|+1)$-gon. Using Proposition 1.7 and the derivative of $Z_m(t)$ we find

PROPOSITION 5.1. *The expected number of internal vertices in a free triangulation of an $(m+2)$-gon is*

$$(Z_m)^{-1} \sum n\varphi_{n,m}\alpha^{-n} = \frac{(m+1)(2m+1)}{3} .$$

Roughly, the idea is to estimate Y_t, the number of vertices added at time t. By the above, $\mathbb{E}(Y_t|X_t) \sim cX_t^2$. Since $\mathbb{P}(X_t = -k) \approx ck^{-5/2}$ for $k < M_t$, and the probability is 0 for $k > M_t$, the expected number of vertices added at time t is (up to multiplicative constants)

$$\mathbb{E}(Y_t|M_t) \approx \sum \mathbb{P}(X_t = -k)\mathbb{E}(Y_t|K_t = -k) \approx \sum^{M_t} ck^{-1/2} \approx cM_t^{1/2} .$$

Now, $M_t \approx t^{2/3}$ and therefore the expected number of vertices added at time t is roughly $t^{1/3}$. Summing that up to time T_r, the number of vertices added is roughly $T_r^{4/3} \approx r^4$. This implies a growth of r^4 as was suggested in the physics literature ([AW], [ADJ] and others).

Note that in the above discussion it was convenient to consider instead of B_r, its hull. This enables us to add a free triangulation which may include vertices further away from the root. To estimate the ball volume growth the distance of vertices in such free triangulations needs to be considered as well.

For the steps X_t of the Markov chain with distribution given by (3.1), define

$$V_T(\gamma) = \sum^T |X_t|^\gamma .$$

Lemma 5.2. *Let $V_T(\gamma)$ be as above. For $\gamma > 3/2$, and for any $\varepsilon > 0$, a.s.*

$$\limsup \frac{V_T(\gamma)}{T^{2\gamma/3} \log^{2\gamma/3+\varepsilon} T} < \infty .$$

Proof. Since the distribution of $|X_t|$ is stochastically increasing w.r.t. M_t, and since the limit distribution (3.2) satisfies $\mathbb{P}(|X| > \lambda) = O(\lambda^{-3/2})$,

$$\mathbb{P}\big(|X_t|^\gamma > \lambda | M_t\big) = O(\lambda^{-3/(2\gamma)})$$

uniformly for all t. It follows that for some constants $a, b > 0$ (possibly depending on γ) we have stochastic domination by a stable random variable $|X_t|^\gamma \prec aS_{3/(2\gamma)} + b$. Since this domination holds even when conditioning on M_t, the sum is dominated by i.i.d. copies giving

$$V_T(\gamma) = \sum^T |X_t|^\gamma \prec aT^{2\gamma/3} S_{3/(2\gamma)} + bT .$$

Therefore as $T \to \infty$

$$\begin{aligned}\mathbb{P}\big(V_T(\gamma) > T^{2\gamma/3} \log^{2\gamma/3+\varepsilon} T\big) &< \mathbb{P}\big(aS_{3/(2\gamma)} > \log^{2\gamma/3+\varepsilon} T - o(1)\big) \\ &= O(\log^{-(1+3\varepsilon/(2\gamma))} T) .\end{aligned}$$

Restricting our attention to times $T_n = 2^n$, we see that a.s. for only finitely many such times $V_T(\gamma) > T^{(2\gamma)/3} \log^{2\gamma/3+\varepsilon} T$. By the monotonicity in T of $V_T(\gamma)$, the desired result holds for all T. □

While the $\log T$ term might be replaceable by an iterated logarithm, the exponent of T is the correct one, as is evident from the following lemma.

Lemma 5.3. *Let X_t be the steps of the Markov chain with distribution described by* (3.1)*. Then for any $\varepsilon > 0$, a.s. for all but finitely many n*

$$\big|\{t \in [2^n, 2^{n+1}) \text{ s.t. } |X_t| > (2^n)^{2/3} n^{-(2+\varepsilon)}\}\big| > n^2 .$$

Proof. First, estimate the probability that $|X_t|$ is very large. As long as $\gamma < M_t/3$

$$\begin{aligned}\mathbb{P}\big(|X_t| \geq \gamma | M_t\big) &= \sum_{k=\gamma}^{M_t} \mathbb{P}(X_t = -k | M_t) \\ &> \sum_{\gamma}^{M_t/2} c_1 p_k \\ &> \sum_{\gamma}^{M_t/2} c_1 k^{-5/2} \\ &> c_2 \gamma^{-3/2}.\end{aligned}$$

Thus the probability that $|X_t| < \gamma$ for all $t \in [2^n, 2^{n+1})$ satisfies

$$\begin{aligned}\mathbb{P}\Big(\max_{2^n \leq t < 2^{n+1}} \{|X_t|\} < \gamma\Big) &< (1 - c_2\gamma^{-3/2})^{2^n} \\ &< \exp\left(-c_2 2^n \gamma^{-3/2}\right).\end{aligned}$$

Set $\gamma = \gamma(n) = n^{-(2+\varepsilon)}(2^n)^{2/3}$. Theorem 3.5 implies that a.s. for all but finitely many n and for all $t \geq 2^n$, $M_t > 3\gamma$, and hence the last bound is valid. Thus

$$\mathbb{P}\Big(\max_{2^n \leq t < 2^{n+1}} \{|X_t|\} < n^{-(2+\varepsilon)}(2^n)^{2/3}\Big) < \exp(-c_2 n^{3+3\varepsilon/2}).$$

Since this is summable, a.s. for all but finitely many n there is a large $|X_t|$. Since ε is arbitrary this concludes the proof.

Similarly, the probability that there are at most k times when $|X_t| < \gamma$ is at most a binomial coefficient times the previous probability and so is at most

$$\tbinom{2^n}{k}(1 - c_2\gamma^{-3/2})^{2^n-k} < 2^{nk}\exp(-c_3 n^{3+3\varepsilon/2}).$$

For $k = n^2$, this bound too is summable, hence a.s. for all but finitely many n there are at least n^2 large $|X_t|$'s in the interval $[2^n, 2^{n+1})$. □

Next, we consider the volume of the hulls. At any step in the peeling construction where $X_t < 0$, a free triangulation of a $(|X_t|+1)$-gon is added to the triangulation. Denote by Y_t the number of vertices added to the triangulation at step t. If $X_t = 1$, then $Y_t = 1$, otherwise it is the size of the free triangulation added.

Lemma 5.4. *For any* $\varepsilon > 0$, *a.s.*

$$\limsup_{T\to\infty} \frac{\sum^T Y_t}{T^{4/3}\log^{2+\varepsilon} T} < \infty.$$

The basic tool used in the proof is the following lemma:

LEMMA 5.5. *There are constants c_1, c_2 such that conditioned on $X_t = -k$, for any γ there is a coupling of $k^{-2}Y_t$ and $c_1 S_{3/2} + \gamma$ such that the latter is larger with probability at least $1 - e^{-c_2\gamma}$. This is written as*

$$k^{-2}Y_t \prec c_1 S_{3/2} + \gamma \qquad \textit{with prob. } 1 - e^{-c_2\gamma^3}\,.$$

Proof of Lemma 5.4. Lemma 5.5 may be rephrased as: For each t, conditioned on X_t,

$$Y_t \prec c_1 X_t^2 S_{3/2} + \gamma X_t^2 \qquad \text{with prob. } 1 - e^{-c_2\gamma^3}\,.$$

When conditioning on the whole sequence $\{X_t\}$, the Y_t's become independent. Summing up to time T, and using the formula for the sum of independent stable random variables we find that

$$\sum^T Y_t \prec c_1 V_T(3)^{2/3} S_{3/2} + \gamma V_T(2) \qquad \text{with prob. } 1 - Te^{-c_2\gamma^3},$$

and therefore

$$\mathbb{P}\Big(\sum^T Y_t > \gamma V_T(2) + c_1 V_T(3)^{2/3} \log^{2/3+\varepsilon} T\Big) < \mathbb{P}(S_{3/2} > \log^{2/3+\varepsilon} T) + Te^{-c_2\gamma^3}.$$

Set $\gamma = c_3 \log^{1/3} T$ with $c_3 = (2/c_2)^{1/3}$. The second term in the RHS is T^{-1}, and is dominated by the first. Thus

$$\mathbb{P}\Big(\sum^T Y_t > c_3 V_T(2) \log^{1/3} T + c_1 V_T(3)^{2/3} \log^{2/3+\varepsilon} T\Big) < c_4 \log^{-(1+3\varepsilon/2)} T\,.$$

Considering only the times $T = 2^n$, these failure probabilities are convergent and so a.s. failure occurs only a finite number of those times. Since $\sum^T Y_t$ is monotone as are $V_T(2), V_T(3)$, it follows that a.s. for all but finitely many T

$$\sum^T Y_t < c_3 V_{2T} \log^{1/3} T + c_1 V_{2T}(3)^{2/3} \log^{2/3+\varepsilon} T\,.$$

Using Lemma 5.2 to estimate $V_T(2), V_T(3)$ we get

$$\lim \frac{c_3 V_{2T} \log^{1/3} T}{T^{4/3} \log^{2+\varepsilon} T} < \infty\,,$$

as well as

$$\limsup \frac{c_1 V_{2T}(3)^{2/3} \log^{2/3+\varepsilon} T}{T^{4/3} \log^{2+\varepsilon} T} < \infty\,. \qquad \square$$

Proof of Lemma 5.5. By Theorem 1.11, conditioned on $X_t = -k$, Y_t is distributed as the size of a free triangulation of a $(k+1)$-gon. From the formula for $\varphi_{n,k-1}$ given by Proposition 1.6, it can be seen that

$$\mathbb{P}(k^{-2}Y_t > u \mid X_t = -k) < c_3 u^{-3/2}$$

for some universal c_3. For a real random variable R we use the notation $\Phi_R(t) = \mathbb{P}(R < t)$. The asymptotics of $\Phi_Y(t), \Phi_{S_{3/2}}(t)$ imply that for some universal t_0, c_1, for $t > t_0$

$$\Phi_{c_1 S_{3/2}}(t) < \Phi_Y(t)\,.$$

For $t > t_0$, monotonicity implies

$$\Phi_{c_1 S_{3/2}}(t-\gamma) < \Phi_Y(t)\,,$$

while for $t \le t_0$

$$\Phi_{c_1 S_{3/2}}(t-\gamma) < \Phi_{c_1 S_{3/2}}(t_0-\gamma) < e^{-c_2\gamma^3}\,.$$

Thus for any t

$$\Phi_{c_1 S_{3/2}}(t-\gamma) < \Phi_Y(t) + e^{-c_2\gamma^3}\,,$$

which is the claimed domination. □

The estimates resulting from comparing Y_t to a stable r.v. $S_{3/2}$ are tight. For one thing, Y_t also stochastically dominates a suitable normalized stable random variable. A more direct proof that the exponent 4/3 of Lemma 5.4 is correct comes from the following lemma.

Lemma 5.6. *Let Y_t be the number of vertices added to the triangulation at time t. For any $\varepsilon > 0$, a.s.*

$$\lim_{T\to\infty} \frac{n^{8/3+\varepsilon}}{(2^n)^{4/3}} \max_{2^n \le t < 2^{n+1}} \{Y_t\} = \infty\,.$$

Proof. Consider only the number of vertices added at times when $|X_t|$ is large. Fix $\varepsilon > 0$. By Lemma 5.3, a.s. for all but finitely many n there are at least n^2 times in the interval $[2^n, 2^{n+1})$ for which $|X_t| > (2^n)^{2/3} n^{-(2+\varepsilon)}$.

From the formulas for $\varphi_{n,m}$ and Z_m it is easily seen (see also Proposition 6.4) that for large γ, uniformly for all k

$$\mathbb{P}(Y_t > \gamma k^2 | X_t = -k) > c_1 \gamma^{-3/2},$$

and so the probability that this fails for the above times when $|X_t|$ is large is at most

$$(1 - c_1\gamma^{-3/2})^{n^2} < \exp(-c_1 n^2 \gamma^{-3/2})\,.$$

For $\gamma = n^{4/3-\varepsilon}$ this is finitely summable and so a.s. for all but finitely many n there is some $2^n \le t_n < 2^{n+1}$ with

$$|X_{t_n}| > (2^n)^{2/3} n^{-(2+\varepsilon)},$$

and

$$Y_{t_n} > n^{4/3-\varepsilon} X_{t_n}^2\,,$$

and so

$$\lim Y_{t_n} \frac{n^{8/3+3\varepsilon}}{(2^n)^{4/3}} = \infty\,.$$ □

Proof of Theorem 1.1. Since $|\overline{B}_r| = \sum_{t<T_r} Y_t$, the first part follows from Theorem 4.1 and Lemma 5.4, and the second part follows from Theorem 4.1 and Lemma 5.6. □

6 Ball Volume Growth

In order to get a lower bound on the $|B_r|$ we need to find (with good probability) a large number of vertices within a short distance of the root. When investigating the hull $\overline{B}_r$ we found a number of steps at which a free triangulation of an m-gon was added to the triangulation. We therefore wish to estimate not only the number of vertices in a free triangulation of a disc, but the number of such vertices that are close to the boundary.

Definition 6.1. *For a triangulation T of a disc and a vertex $v \in T$, the* height *of v, denoted h_v, is the distance from v to the boundary.*

Typically we expect a free triangulation of an m-gon to have size roughly m^2. We will see that typically, most vertices in such a triangulation have height at most roughly $\sqrt{m}$. This implies that typical distances in T are on the scale of $|T|^{1/4}$, conforming with the result of [CS]. The following methods may also be used to estimate the maximal height in a free triangulation.

Lemma 6.2. *Let T be a free triangulation of an m-gon. For any $\varepsilon > 0$ there are $c_1, c_2 > 0$, such that for sufficiently large m*

$$\mu_m(A) > c_2 ,$$

where A is the event consisting of all triangulations, T, of an $(m+2)$-gon with

$$\big|\{u \in T\,,\ h_u < \sqrt{m}\log^{3+\varepsilon} m\}\big| > c_1 m^2 .$$

Based on this lemma, the proof of Theorem 1.2 is straightforward – simply find a free triangulation of a large cycle near the root and it will follow that there are many vertices near the root.

Proof of Theorem 1.2. Fix $\varepsilon > 0$ and let

$$L_r = \big\{t < T_r \ s.t.\ |X_t| > r^2 \log^{-(6+5\varepsilon/3)} r\big\} .$$

Using the lower bound on T_r (Theorem 4.1) together with Lemma 5.3 to find times when $|X_t|$ is large, we get that for some c_1, a.s. for all but finitely many r

$$|L_r| > c_1 \log^2 r .$$

At each time $t \in L_r$ a free triangulation is glued to a cycle, with every vertex in the cycle at distance at most r from the root. For each such

t the probability that the event in Lemma 6.2 fails to occur for this free triangulation is at most c_2. Thus the probability of failure at all $t \in L_r$ is at most $\exp\left(-c_3 \log^2 r\right)$ for some $c_3 > 0$, which is finitely summable. Thus a.s. for all but finitely many r the event of Lemma 6.2 occurs at some time $t \in L_r$.

By the definition of L_r and from the lemma, this implies that there are at least $r^4 \log^{-(12+10\varepsilon/3)} r$ vertices at bounded distance from the cycle, and thus also at bounded distance from the root. To find this bound, note that $|X_t| < M_t$ and $t < T_r$. Theorem 3.2 bounds M_t in terms of t and Theorem 4.1 bounds T_r in terms of r. Combined they give

$$|X_t| < c_3 r^2 \log^3 r\,,$$

hence the bound on height in Lemma 6.2 is

$$\sqrt{|X_t|}\log^{3+\varepsilon}|X_t| < c_4 r \log^{9/2+\varepsilon} r\,.$$

Summarizing, for all but finitely many r we found $r^4 \log^{-(12+10\varepsilon/3)} r$ vertices in $B_{r'}$ with $r' = r + c_4 r \log^{9/2+\varepsilon} r$. By monotonicity of $|B_r|$ this suffices. □

In proving Lemma 6.2 we make use of the following distribution, closely related to the free triangulation of a disc.

Definition 6.3. *The free* marked *triangulation of an $(m+2)$-gon is a distribution on triangulations of the $(m+2)$-gon with a marked internal vertex, that assigns a rooted triangulation T marked at v probability*

$$\tilde\mu_m(T,v) = \tilde Z_m^{-1} \alpha^{-|T|},$$

where

$$\tilde Z_m = \sum n \varphi_{n,m} \alpha^{-n}.$$

This may also be defined as the annealed distribution for marking a random vertex in a free triangulation of the disc. Proposition 5.1 implies

$$\tilde Z_m = Z_m \mathbb{E}_{\mu_m}|T| = \frac{1}{6}\binom{2m+2}{m}\left(\frac{9}{4}\right)^{m+1},$$

hence the relation between $\tilde\mu_m$ and μ_m may be written as

$$\tilde\mu_m(T) = \frac{3|T|}{(m+1)(2m+1)}\mu_m(T)\,.$$

Clearly, conditioned on the size of T, the triangulation marginal of $\tilde\mu_m$ and μ_m are equal, and conditioned on the triangulation, each internal vertex has probability $|T|^{-1}$ of being the marked one.

The size distribution of the free marked triangulation is interesting, as is evident from the following proposition.

Proposition 6.4. *The distributions of $m^{-2}|T|$ with respect to $\tilde{\mu}_m$ converge to that of $2/3S_{1/2}$, where $S_{1/2}$ is an asymmetric stable random variable.*

$S_{1/2}$ with the Levi distribution, is one of very few cases of stable random variables for which there is a (more or less) closed form for the density function. $S_{1/2}$ has the same law as g^{-2} where g is a standard Gaussian random variable. In particular, for large m, the size of a free marked triangulation of an m-gon is distributed approximately as $2/3g^{-2}$. As a consequence, the size of a free (unmarked) triangulation of an m-gon also converge in distribution.

Proof. The probability of a free marked triangulation having size n, namely

$$\tilde{\mu}_m(|T| = n) = \frac{n\varphi_{n,m}}{\tilde{Z}_m\alpha^{-n}}$$

may be rewritten as

$$\frac{8n(m+2)}{3(2n+2m+1)(2n+2m+2)} \cdot \left(\left(\frac{4}{27}\right)^n \binom{3n}{n}\right) \cdot \prod_{i=1}^{2m} \frac{n+i/3}{n+i/2} \,.$$

For large n, the second term is roughly $\sqrt{3n/4\pi}$. For $n \gg m \gg 1$, the first term is roughly $2m/3n$ while the last is roughly $e^{-m^2/3n}$. These approximations are uniform for all $n \gg m \gg 1$. It follows that

$$\tilde{\mu}_m\big(|T| = n\big) \sim \frac{mn^{-3/2}}{\sqrt{3\pi}} e^{-m^2/3n}.$$

And by a change of variable, for any t

$$\lim_{m\to\infty} \tilde{\mu}_m\big(|T| > tm^2\big) = \frac{1}{\sqrt{3\pi}} \int_t^\infty x^{-3/2} e^{-1/3x} dx \,,$$

Hence the distribution of $m^{-2}|T|$ with respect to $\tilde{\mu}$ converges as $m \to \infty$ to that of $2/3S_{1/2}$. □

The free marked triangulation has the advantage over the free triangulation that it may be sampled via a peeling process similar to the peeling process for the UIPT. As each triangle is added, there are three possibilities, shown in Figure 6.1. The triangle may partition the disc into two, in which case there is a marked and unmarked triangulation in both parts, corresponding to the infinite and free parts in the construction of the UIPT. The second option is that a new vertex is added. The third is that the added vertex is the marked vertex.

As before, in order to analyze the sampling process, assume that triangles are added to the boundary in order, adding all triangles incident to a vertex before moving to the next vertex counterclockwise. As before, this does not effect the resulting measure or the distribution of the number of required steps, but will make understanding the height distribution easier.

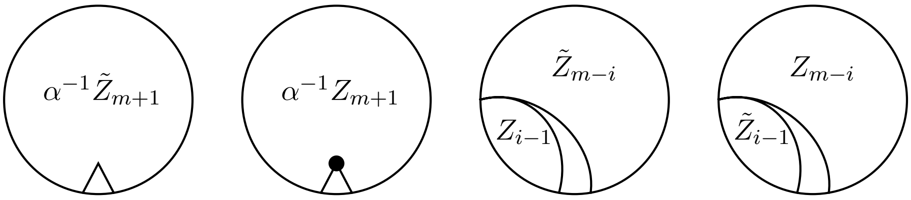

Figure 6.1: The possibilities when a triangle is added in the free marked triangulation.

As before, we consider the size of the boundary, M_n, after n steps were taken. $M_0 = m$ and at each step we have $M_{n+1} = M_n + X_n$ where the distribution of X_n is as in equations (2.3), (2.4) with $\tilde{Z}_m$ taking the role of C_m:

- $X_t = 1$ and the new vertex is unmarked with probability $\tilde{Z}_{m+1}/\alpha\tilde{Z}_m$.
- $X_t = 1$ and the new vertex is marked with probability $Z_{m+1}/\alpha\tilde{Z}_m$.
- $X_t = -k$ with probability $\tilde{Z}_{m-k}Z_{k-1}/\tilde{Z}_m$.

It is easy to see that as $M \to \infty$ these converge to the same probabilities as for the UIPT. However, here the drift is reversed: the small M_n's are likelier to give negative X. Here the expectation of the steps is deceiving, as it is positive. Still, the probabilities for large negative steps are sufficient to cause the Markov chain to terminate within roughly $M_0^{3/2}$ steps. Since each layer contains roughly M_0 vertices, this implies that the height of the marked vertex is at most in the order of $\sqrt{M_0}$.

LEMMA 6.5. *For M_n as above, for some $c > 0$*

$$\mathbb{P}(\max M_n > \lambda M_0) < c\lambda^{-1}.$$

Proof. By Proposition 6.4, for a free marked triangulation T of an $(m+2)$-gon,

$$\tilde{\mu}_m\big(|T| > tm^2\big) < c_1 t^{-1/2},$$

and, for some $c_2 > 0$,

$$\tilde{\mu}_m\big(|T| > m^2\big) > c_2\,.$$

Let M_n be the sequence observed in the process of sampling a triangulation T of an (M_0+2)-gon. Let A be the event that for some n, $M_n > \lambda M_0$. Conditioned on A, T includes a free marked triangulation of an (M_n+2)-gon, and so

$$\mathbb{P}\big(|T| > \lambda^2 M_0^2 \mid A\big) > c_2\,.$$

Thus

$$\mathbb{P}(A) < \frac{\mathbb{P}(|T| > \lambda^2 M_0^2)}{c_2} < \frac{c_1}{c_2\lambda}\,. \qquad \square$$

Lemma 6.6. *For M_n as above, let s be the smallest n such that either $M_n \le M_0/2$ or the process terminates at time n. For any $m \ge M_0$ and $\varepsilon > 0$ there is a $c = c(\varepsilon)$ such that*

$$\mathbb{P}(s > M_0^{3/2}\log^{2+3\varepsilon} m) < c\log^{-(1+\varepsilon)} m\,.$$

As before, let $\mathcal{F}_n$ denote the sigma field generated by the random variables $M_0, \dots, M_n$. Expectation with respect to $\mathcal{F}_n$ is denoted by $\mathbb{E}_{\mathcal{F}_n}$ (expectation conditioned on the past).

Proof. This is similar to the proof of Theorem 3.5. To avoid difficulties arising from the termination of the process, continue the Markov chain after termination with $X_t = -1$ after 0 is hit. With this continuation the following bounds hold universally. If we find that $M_n < M_0/2$, then clearly $s < n$ whether or not the process terminated.

We show that at each step there is a not too small probability of the Markov chain taking a step into $[0, M_0/2)$, and consequently this happens after a small number of steps. Direct inspection of the step probabilities reveals that, for some universal c_1,

$$\mathbb{P}(X_n = -k \mid M_n) > c_1(M_n - k)^{-1/2} M_n^{1/2} k^{-5/2}.$$

Since $k < M_n$, it follows that the probability of taking a step from M_n to the interval $[0, M_0/2)$ satisfies

$$\mathbb{P}(M_{n+1} < M_0/2 \mid M_n) > c_2 M_0^{1/2} M_n^{-2}.$$

Define the stopping time t as the first time the Markov chain leaves the interval $[M_0/2, M_0\log^{1+\varepsilon} m]$ (or terminates). As long as M_n is in the interval the probability of leaving the interval by having $M_{n+1} < M_0/2$ is at least $c_2 M_0^{-3/2}\log^{-(2+2\varepsilon)} m$. It follows that

$$\begin{aligned}\mathbb{P}(t > M_0^{3/2}\log^{2+3\varepsilon} m) &< (1 - c_2 M_0^{-3/2}\log^{-(2+2\varepsilon)} m)^{M_0^{3/2}\log^{2+3\varepsilon} m} \\ &< \exp\left(-c_2\log^{\varepsilon} m\right)\,.\end{aligned}$$

Clearly $s = t$ unless the process left the interval with $M_t > M_0\log^{1+\varepsilon} m$. By Lemma 6.5 the probability of that is bounded by $c_3\log^{-(1+\varepsilon)} m$, and therefore

$$\mathbb{P}(s > M_0^{3/2}\log^{2+3\varepsilon} m) < c_3\log^{-(1+\varepsilon)} m + \exp\left(-c_2\log^{\varepsilon} m\right)$$

as claimed. $\square$

Lemma 6.7. *Let (T, v) be a free marked triangulation of an $(m+2)$-gon. For any $\varepsilon > 0$, for all large enough m*

$$\tilde{\mu}_m\big(h_v > \sqrt{m}\log^{3+\varepsilon} m\big) < 1/2\,.$$

Proof. Consider for $k < \log_2 m$ the times S_k defined as the smallest s such that $M_s \leq 2^{-k}m$. We show that $S_{k+1} - S_k$ is unlikely to be large and give a bound on the distance between the boundary at time S_{k+1} and the boundary at time S_k.

Using the Lemma 6.6 applied to the process starting at time S_k with boundary size $M_{S_k} \leq 2^{-k}m$, and with m in the lemma's formulation being the boundary size of the original disc, we get the bound

$$\mathbb{P}\big(S_{k+1} - S_k > (2^{-k}M_0)^{3/2}\log^{2+\varepsilon} m\big) < c\log^{-(1+\varepsilon/3)} m\,.$$

Therefore with probability at least $1 - c\log^{\varepsilon/3} m$, for all relevant $k < \log_2 m$,

$$S_{k+1} - S_k < (2^{-k}m)^{3/2}\log^{2+\varepsilon} m\,. \tag{6.1}$$

For large m this probability is at least $3/4$.

As before, let T_r denote the time when the rth layer is complete, i.e. the smallest t such that the distance from the original boundary to the boundary at time t is r. At time T_{r+1} all vertices on the boundary have height $r + 1$ and so they were all added after time T_r. Since vertices are added at the boundary one at a time, we have $T_{r+1} - T_r \geq M_{T_{r+1}}$.

If (6.1) holds, then the number of rounds completed between S_k and S_{k+1} is at most

$$1 + \frac{S_{k+1} - S_k}{2^{-(k+1)}m} < 1 + 2\sqrt{m}\log^{2+\varepsilon} m\,,$$

and by summing up, with probability at least $3/4$, the total number of rounds completed before the process either terminates or reaches $M_n = 0$ is at most $c_1\sqrt{m}\log^{3+\varepsilon} m$.

If the process terminated before $M_n = 0$, then $h_v < c_1\sqrt{m}\log^{3+\varepsilon} m$ as required. In the case that the process did not terminate and reached $M_n = 0$, in the remaining 2-gon there is a free marked triangulation, and with probability $3/4$ the number of vertices inside is at most 30 and so $\mathbb{P}(h_v > 30 + c_1\sqrt{m}\log^{3+\varepsilon} m) < 1/2$. By changing ε the constants c_1 and 30 may be disposed of. □

Proof of Lemma 6.2. Let (T, v) be a free marked triangulation of an m-gon. Define $t = |T|$ and

$$s = \big|\{u \in T \mid h_u < \sqrt{m}\log^{3+\varepsilon} m\}\big|\,.$$

The probability that $h_v < \sqrt{m}\log^{3+\varepsilon} m$, bounded by Lemma 6.7, is clearly the expectation of s/t with respect to $\tilde{\mu}_m$, i.e.

$$\sum_T \tfrac{s}{t}\tilde{\mu}_m(T) > \tfrac{1}{2}\,.$$

Using the relation between $\tilde{\mu}_m$ and μ_m, this translates to

$$\sum_T \frac{s}{t} \frac{3t}{(m+1)(2m+1)} \mu_m(T) > 1/2\,,$$

or

$$\mathbb{E}_{\mu_m} s = \sum_T s\mu_m(T) > \tfrac{m^2}{3}\,.$$

To finish, note that since $s < t$ and $\mu_m(t > \gamma m^2) < c_1\gamma^{-3/2}$, the expectation of s restricted to any event of probability p is bounded by $c_2 m^2 p^{1/3}$ for some c_2. Therefore, if the probability that $s < c_3 m^2$ is p, the expectation of s is at most

$$pc_3 m^2 + c_2 m^2 (1-p)^{1/3}.$$

For any $c_3 < 1/3$ this gives a lower bound on p independent of m. □

7 Percolation on the UIPT

The peeling construction may also be used to understand percolation on the UIPT. Bernoulli site percolation with parameter p is the probability measure on colorings of the graph's vertices where each vertex is colored black with probability p and white otherwise, independently of all other vertices. Percolation is defined as the event that the root vertex is in an infinite connected black component. The resulting measure is denoted by $\mathbb{P}_p$.

Since the graph in question, namely the UIPT, is a random graph there is a distinction between annealed and quenched statements about percolation (though since there is no interaction between the percolation and the underling graph the underlying probability measure is the same). Annealed statements are averaged on all planar triangulations. Quenched statements on percolation are about the coloring conditioned on the triangulation. In particular, the critical probability p_c for annealed percolation is the infimum of all p such that $\mathbb{P}_p(\text{percolation}) > 0$. This means that in a positive measure of triangulations (w.r.t. τ) the probability of percolation is positive. The quenched problem is what is the least p such that on τ-almost all triangulations there is a positive probability of percolation.

It turns out the the annealed and quenched behaviours are the same, so Theorem 1.5 is a more general result than the following theorem.

Theorem 7.1. *The annealed critical probability for site percolation on the UIPT is 1/2.*

Proof. Suppose each vertex is colored independently at random, black with probability p and white otherwise. Assume with out loss of generality that the root vertex is black. We sample the connected black component containing the root vertex by adding triangles one by one to the triangulation, as in the peeling construction of the UIPT. As new vertices are added we color them randomly, independently of all previous events. This samples site percolation on (a part of) the UIPT.

If at any time all the vertices on the outer boundary of the sampled sub-triangulation we see are white, then we have found a cycle of white vertices enclosing the root, and so the black connected cluster containing the root is necessarily finite. On the other hand, if the process continues indefinitely and never reaches an all white outer boundary, then the root is in an infinite connected black component.

Recall that we are free to choose the edge in the outer boundary on which we attach a triangle at any time. At all times we will choose a boundary edge that has one black and one white endpoint (unless the outer boundary of the sub-triangulation is monochromatic). Choosing the edge in this manner will guarantee two things. First, the black (resp. white) vertices on the outer boundary lie on a continuous arc along the boundary as in Figure 7.1(a,b). Second, since each new vertex is connected to existing vertices of both colors (as long as the boundary is not monochromatic), all black (resp. white) vertices form a single connected cluster.

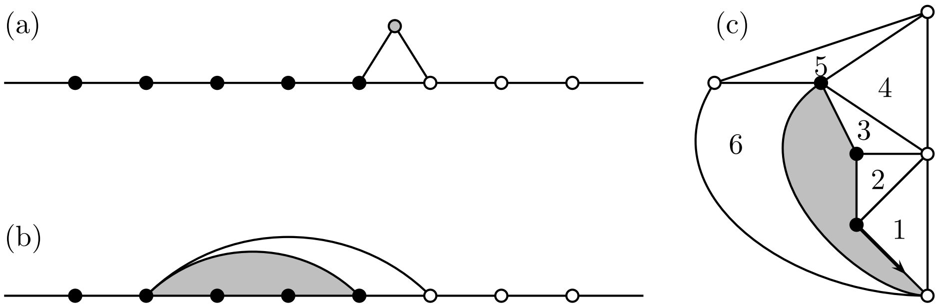

Figure 7.1: Sampling annealed percolation on the UIPT.

When a new vertex is added, we color it randomly (as in Figure 7.1(a), showing a segment of the boundary). When the new triangle includes only old vertices, as in Figure 7.1(b), the event of percolation does not depend on the finite part of the triangulation (in the shaded region). Thus, to determine whether percolation occurs we only need to keep track of the outer boundary and of the number of black and white vertices in it.

Figure 7.1(c) shows a possible outcome of the first few steps. The triangles are numbered in the order they are added. After the sixth triangle is added, we know that the black cluster of the root is finite. Since the part of the triangulation enclosed in the hole is finite, percolation cannot occur.

Let B_n (resp. W_n) denote the number of black (resp. white) vertices on the boundary at time n. As before, let $M_n + 2 = B_n + W_n$ denote the size of the boundary after n steps. Also let X_n denote the step size, with distribution given by (3.1). If $X_n = 1$, then the new vertex is colored black or white with the given bias. Therefore conditioned on (B_n, W_n):

$$(B_{n+1}, W_{n+1}) = \begin{cases} (B_n + 1, W_n \quad) & \text{with prob. } p \\ (B_n \quad, W_n + 1) & \text{with prob. } 1-p \end{cases} \qquad \text{if } X_n = 1\,.$$

On the other hand, if $X_n < 0$, then $|X_n|$ vertices from the outer boundary become internal vertices. In this case it is equally likely that the removed vertices are on the left or the right of the new triangle (the triangulation's geometry is independent of previously chosen colors). Note however, that if $B_n + X_n < 0$ and the removed vertices are on the black side, this does not result in a negative number of black vertices on the boundary. Instead, all the black vertices, and some "borrowed" white ones will be removed. Therefore conditioned on (B_n, W_n),

$$(B_{n+1}, W_{n+1}) = \begin{cases} f(B_n + X_n, W_n \qquad) & \text{with prob. } 1/2 \\ f(B_n \qquad, W_n + X_n) & \text{with prob. } 1/2 \end{cases} \qquad \text{if } X_n < 0\,,$$

where $f(B, W) = (B, W)$ unless $B < 0$ or $W < 0$,

$$f(B,W) = \begin{cases} (B,W) & \text{if } B, W \geq 0\,, \\ (0, B+W) & \text{if } B < 0\,, \\ (B+W, 0) & \text{if } W < 0\,. \end{cases}$$

We see that if $p = 1/2$, then X_n is added with equal probabilities to B or W, but if $p \neq 1/2$ there is a bias, and since $\mathbb{E}X_n \to 0$, the less probable color tends to die out, as we now prove.

Suppose $p < 1/2$. Consider the Markov chain (B'_n, W'_n) not limited to positive values, with evolution

$$(B'_{n+1}, W'_{n+1}) = \begin{cases} (B'_n + 1, W'_n \quad) & \text{with prob. } p \\ (B'_n \quad, W'_n + 1) & \text{with prob. } 1-p \end{cases} \qquad \text{if } X_n = 1\,.$$

$$(B'_{n+1}, W'_{n+1}) = \begin{cases} (B'_n + X_n, W'_n \qquad) & \text{with prob. } 1/2 \\ (B'_n \qquad, W'_n + X_n) & \text{with prob. } 1/2 \end{cases} \qquad \text{if } X_n < 0\,,$$

coupled in the natural way to (B_n, W_n). The difference $B'_{n+1} - B'_n$ is equal to the $B_{n+1} - B_n$ except in two cases:

- $W_n + X_n < 0$ and additional black vertices are removed. In this event $B'_{n+1} - B'_n > B_{n+1} - B_n$.
- $B_n + X_n < 0$ and additional white vertices are removed. In this event percolation does not occur.

Therefore, conditioned on the event of percolation, $B_n \le B'_n$. We will see that a.s. for some n, $B'_n \le 0$ and so $\mathbb{P}_p(\text{percolation}) = 0$. Denote by $\mathcal{F}_n$ the σ-field generated by the random variables $B_0, W_0, \dots, B_n, W_n$. Note that M_n and B'_n are determined by B, W up to time n, and hence are $\mathcal{F}_n$-measurable,

$$\mathbb{E}_{\mathcal{F}_n} B'_{n+1} - B'_n \le 1/2 \mathbb{E}_{\mathcal{F}_n} X_n - (1/2 - p)\mathbb{P}(X_n = 1 | \mathcal{F}_n) .$$

Recall that if $m > m'$, then the distribution of X_n conditioned on $M_n = m$ is stochastically dominated by the distribution conditioned on $M_n = m'$. Since a.s. $M_n \to \infty$, from some time on M_n is larger than any fixed M. Since $\mathbb{E}_{\mathcal{F}_n} X_n \sim c M_n^{-1/2} \to 0$ from some time on B'_n is dominated by a random walk with steps bounded from above ($X_n \le 1$) and negative expectation. Such a random walk will a.s. tend to $-\infty$, and since conditioned on percolation B'_n remains positive, $\mathbb{P}_p(\text{percolation}) = 0$.

Moreover, if whenever $B_n = 0$ we “reset” by setting $B'_n = 0$, then $B_n \le B'_n$ holds unconditionally, and when B'_n is large enough it still has negative expected change. With this modification B'_n does not tend to $-\infty$ but has a stationary distribution with exponential decay. By Borel–Cantelli it follows that $B_n \le B'_n = O(\log n)$.

When $p > 1/2$, the roles of B_n and W_n are interchanged, and since $B_n + W_n$ grows like $n^{2/3}$ it follows that a.s. $B_n \to \infty$. In particular with positive probability at all times $B_n > 0$, i.e., $\mathbb{P}_p(\text{percolation}) > 0$. □

Thus, we see that if $p < 1/2$, then a.s. the black clusters are finite, while if $p > 1/2$, then with positive probability there is an infinite black cluster (since white clusters are all finite, it is unique). To see that this is a.s. the case we prove the following 0-1 law. A general 0-1 law for triangulations is proved in [BeCR]. However, we also need a 0-1 law for annealed percolation on the UIPT.

Theorem 7.2. *The probability of any event invariant to finite changes in the triangulation is 0 or 1. Moreover, the same is true for the annealed percolation on the UIPT with any p.*

Proof of Theorem 1.5. $p_c = 1/2$ is an event independent of finite changes in the UIPT. Since the annealed $p_c = 1/2$, $\tau(p_c = 1/2) > 0$.

For $p = 1/2$ we see that the probability of having an infinite black (or, by symmetry, white) cluster is either 0 or 1. Thus, we see that either a.s. there are no monochromatic infinite clusters, or a.s. there are infinite clusters of both colors. To rule out the latter possibility note that there are a.s. infinitely many times when $|X_t| > M_t/2$. Conditioned on $|X_t| > M_t/2$, with probability at least 1/2 one of the colors is completely removed from the outer boundary. Thus, a.s. one of the colors dies out infinitely often and has only finite clusters. By symmetry, $\mathbb{P}_{1/2}(\text{percolation}) \leq 1/2$, and hence it is 0. □

Proof of Theorem 7.2. We represent the UIPT as a function of an infinite sequence of independent random variables, such that a.s. changing a finite number of them will only change a finite sub-triangulation. Since any event determined by the tail of such a sequence has probability 0 or 1, this is sufficient.

The basis will be the peeling construction of the UIPT. For each m, n, i define an independent random variable $Z_{m,n,i}$, with the appropriate distribution for a step when the boundary size is m. Such a random variable includes the third vertex of an added triangle as well as a sample of the free triangulation when appropriate. $Z_{m,n,i}$ is used at time t under 3 conditions:

- $M_t = m$
- $\max_{s<t} M_s = n$
- If s is the first time that $M_s = n$, then $t - s = i$.

For example, if the observed sequence of M's is $1, 2, 3, 1, 2, 3, 1, 2$, then the next step uses $Z_{2,3,5}$: $M_t = 2$, the maximal value seen so far is 3, first reached 5 steps ago.

Since a.s. $M_t \to \infty$, changing a finite number of the $Z_{m,n,i}$ will only change the evolution of the triangulation in a finite number of steps. Once M_t is sufficiently large the evolution will not be changed at all.

The exact same proof also works for percolation on the UIPT, adding the random colors of new vertices to the variables $Z_{m,n,i}$. □

Acknowledgements. The author thanks Oded Schramm, Itai Benjamini and Bálint Virág for useful discussions, as well as the anonymous referee for helpful comments. Part of this research was done during visits of the author to Microsoft Research. The author thanks his hosts for those visits.

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Omer Angel, Department of Mathematics, Weizmann Institute of Science, Rehovot, 76100, Israel `omer@wisdom.weizmann.ac.il`

Submitted: September 2002
Revision: March 2003