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DOUBLE ERGODICITY OF THE POISSON BOUNDARY AND APPLICATIONS TO BOUNDED COHOMOLOGY

V.A. Kaimanovich

Abstract

We prove that the Poisson boundary of any spread out non-degenerate symmetric random walk on an arbitrary locally compact second countable group G is doubly $\mathcal{M}^{\mathrm{sep}}$-ergodic with respect to the class $\mathcal{M}^{\mathrm{sep}}$ of separable coefficient Banach G-modules. The proof is direct and based on an analogous property of the bilateral Bernoulli shift in the space of increments of the random walk. As a corollary we obtain that any locally compact σ-compact group G admits a measure class preserving action which is both amenable and doubly $\mathcal{M}^{\mathrm{sep}}$-ergodic. This generalizes an earlier result of Burger and Monod obtained under the assumption that G is compactly generated and allows one to dispose of this assumption in numerous applications to the theory of bounded cohomology.

1 Introduction and Statement of the Results

Given a group G, a *Banach G-module* (E, π) is a Banach space E endowed with an isometric linear G-representation π. For a topological group G the module (E, π) is called *continuous* if the action of G on E is norm continuous. A *coefficient G-module* (E, π) is the contragredient module of a separable continuous Banach G-module, i.e., E is the dual of some separable Banach space $E^\flat$, and π consists of operators adjoint to a continuous action of G on $E^\flat$, see [BuM, Definition 1.1.2], [M, Definition 1.2.1]. Denote by $\mathcal{M}^{\mathrm{sep}}$ the class of all separable coefficient modules. Note that if G is locally compact, then any separable coefficient module is automatically continuous [BuM, Proposition 1.1.4], [M, Proposition 3.3.2].

Definition 1 ([BuM, Definition 5], [M, Definition 11.1.1]). Let G be a locally compact group, and (X, λ) be a Lebesgue space endowed with a measure class preserving action of G. Given a coefficient G-module (E, π), the space (X, λ) is called *(E, π)-ergodic* if any G-equivariant weak* measurable function $f : X \to E$ is a.e. constant. The space (X, λ) is *doubly (E, π)-ergodic* if its square is (E, π)-ergodic with respect to the diagonal action. If $\mathcal{M}$ is a class of coefficient Banach modules, then the space (X, λ) is called *$\mathcal{M}$-ergodic* (resp. *doubly $\mathcal{M}$-ergodic*) if it is (E, π)-ergodic (resp. doubly (E, π)-ergodic) for any coefficient module $(E, \pi) \in \mathcal{M}$.

If E is a coefficient module with trivial G-action π, then (E, π)-ergodicity obviously coincides with the usual ergodicity (absence of non-trivial G-invariant subsets in X), but in general (E, π)- and $\mathcal{M}$-ergodicity are stronger, see [M, Example 11.4.3]

Definition 2 ([MS1, Definition 2.3]). A Lebesgue space (X, λ) endowed with a measure class preserving action of a locally compact group G is called a *strong G-boundary* if this action is amenable and doubly $\mathcal{M}^{\mathrm{sep}}$-ergodic.

Burger and Monod [BuM, Theorem 6] (also see [M, Theorem 11.1.3]) proved that for any compactly generated locally compact group G there exists a finite index open subgroup G^* which has a strong boundary; if G is either connected or totally disconnected, then $G^* = G$. This result plays a fundamental role in the theory of continuous bounded cohomology and its applications to rigidity problems [BuM], [M], [MS1,2].

The proof of Burger and Monod is rather involved. Namely, they first establish this property for a semi-simple Lie group (acting on the associated Furstenberg boundary) and for the group of automorphisms of a homogeneous tree (acting on the geometric boundary of the tree) by using the classical Mautner lemma and its discrete counterpart [LM], and then apply the solution of Hilbert's 5th problem in order to pass to the general case. As follows from [K5, Theorem 2], the strong boundary of the group G^* in the theorem of Burger and Monod can actually be chosen to be the Poisson boundary of a *certain* absolutely continuous probability measure on G^* [BuM, Remark 3.5.1].

The aim of the present paper is to give a direct proof of the following general result:

Theorem 3. *The Poisson boundary of any spread out non-degenerate symmetric probability measure on a locally compact second countable group G is a strong G-boundary.*

Amenability of the action on the Poisson boundary being well known [Z], the main point in Theorem 3 is in establishing the double ergodicity. The proof relies on the same idea as the proof of the usual double ergodicity of the Poisson boundary for symmetric random walks [G] (also see [BL], [K2,4] which consists of reducing the problem to the ergodicity of the bilateral Bernoulli shift on the space of increments of the random walk.

If G is σ-compact but not second countable, then the Poisson boundary is, generally speaking, not well-defined. However, in this situation G has a second countable quotient G/K by a compact normal subgroup $K \subset G$. Due to this fact Theorem 3 implies

Theorem 4. *Any locally compact σ-compact group has a strong boundary.*

Theorem 4 generalizes the aforementioned result of Burger and Monod, and, by superseding it in the corresponding arguments, allows one to eliminate the assumption of being compactly generated in numerous applications (see [BuM], [M], [MS1,2]). For instance, the higher degree Lyndon–Hochschild–Serre exact sequence from [BuM, Theorem 13] (also see [M, Theorem 12.0.3]) now holds without this assumption, which leads to the following product formula (of Künneth type) for continuous bounded cohomology in degree 2 with separable coefficient modules in full generality (cf. [BuM, Theorem 14] and [M, Corollary 12.0.4]):

Theorem 5. *Let $G = G_1 \times \cdots \times G_n$ be a product of locally compact second countable groups G_i, and E be a separable coefficient G-module. Write $G_i' = \prod_{j \neq i} G_j$, and let $E^{G_i'}$ denote the submodule of E consisting of G_i'-invariant elements. Then there is a natural isomorphism of topological vector spaces*

$$H^2_{\mathrm{cb}}(G, E) \cong H^2_{\mathrm{cb}}\left(G, \sum_{i=1}^{n} E^{G_i'}\right) \cong \bigoplus_{i=1}^{n} H^2_{\mathrm{cb}}(G_i, E^{G_i'})\,.$$

In section 2 we establish an ergodicity result for the Bernoulli shift (Theorem 6), which is then used in section 3 for proving $\mathcal{M}^{\mathrm{sep}}$-ergodicity of the product of the Poisson boundaries of a given measure μ and the reflected measure $\check{\mu}$ (Theorem 17). Finally, the proofs of Theorem 3 and Theorem 4 are completed in section 4 at the end of the paper.

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2 Ergodicity of the Bernoulli Shift

Let (X, λ) be a probability measure space. Denote by $\boldsymbol{X} = X^{\mathbb{Z}}$ the space of sequences $\boldsymbol{x} = (x_i)$, $x_i \in X$ indexed by integers $i \in \mathbb{Z}$, and let $\boldsymbol{\lambda} = \lambda^{\mathbb{Z}}$ be the associated product measure on $\boldsymbol{X}$. The shift T acting on $\boldsymbol{X}$ by the formula $(T\boldsymbol{x})_i = x_{i+1}$ preserves the measure $\boldsymbol{\lambda}$ and is called the (bilateral) *Bernoulli shift* over the space (X, λ). The Bernoulli shift is ergodic in the usual sense, i.e., $\boldsymbol{X}$ contains no non-trivial T-invariant sets. The following result is a generalization of this property.

Theorem 6. *Let $T : (\boldsymbol{X}, \boldsymbol{\lambda}) \hookleftarrow$ be the bilateral Bernoulli shift over a probability space (X, λ). If E is a separable Banach space, and $f : \boldsymbol{X} \to E$, $\pi : X \to \mathrm{Iso}(E)$ are measurable maps such that a.e.*

$$f(T\boldsymbol{x}) = \pi(x_1) f(\boldsymbol{x}) \,, \tag{7}$$

then f is a.e. constant.

In the course of the proof we shall need the space $L^1(\boldsymbol{X}, \boldsymbol{\lambda}, E)$ of classes (mod 0) of measurable functions $\varphi : \boldsymbol{X} \to E$ endowed with the norm

$$|||\varphi||| = \int \|\varphi(\boldsymbol{x})\| \, d\boldsymbol{\lambda}(\boldsymbol{x}) \,,$$

where $\|\cdot\|$ is the norm in E (note that the Borel structures on a separable dual Banach space induced by the norm topology, the weak topology, and the weak* topology all coincide, see [M, Lemma 3.3.3]). A function $\varphi \in L^1(\boldsymbol{X}, \boldsymbol{\lambda}, E)$ is called a *cylinder function* if it can be factored through the finite product space $\prod_{i=-n}^{n} (X, \lambda)$ for a certain $n > 0$.

LEMMA 8. *Cylinder functions are dense in $L^1(\boldsymbol{X}, \boldsymbol{\lambda}, E)$*

Proof. Fix $\varepsilon > 0$, cover the space E with a countable family of Borel sets E_i of diameters not exceeding ε, and take a point e_i in each of these sets. For a function $\varphi \in L^1(\boldsymbol{X}, \boldsymbol{\lambda}, E)$ let $Z_i = \varphi^{-1}(E_i)$, and put

$$\psi_1 = \sum_{i=1}^{\infty} \mathbf{1}_{Z_i} e_i \,,$$

so that

$$|||\varphi - \psi_1||| \le \varepsilon \,, \qquad |||\psi_1||| = \sum_{i=1}^{\infty} \boldsymbol{\lambda}(Z_i) \, \|e_i\| < \infty \,.$$

Further, take an integer N in such a way that

$$|||\psi_1 - \psi_2||| < \varepsilon \,, \quad \text{where} \quad \psi_2 = \sum_{i=1}^{N} \mathbf{1}_{Z_i} e_i \,,$$

and approximate each of the sets Z_i, $1 \le i \le N$, by a cylinder set Z_i' with

$$\boldsymbol{\lambda}(Z_i \triangle Z_i') < \frac{\varepsilon}{N \|e_i\|} \,.$$

Then

$$|||\psi_2 - \psi_3||| < \varepsilon \quad \text{for} \quad \psi_3 = \sum_{i=1}^{N} \mathbf{1}_{Z_i'} e_i \,,$$

whence $|||\varphi - \psi_3||| < 3\varepsilon$. □

Proof of Theorem 6. It follows the same idea as the famous Hopf's proof of the ergodicity of the geodesic flow on negatively curved manifolds (e.g., see [K2]). Namely, we show that f must be a.e. constant on the elements of both the "strongly stable" and the "strongly unstable" partitions of the space $\boldsymbol{X}$ (actually, one could just show that f must be constant on the elements of any coordinate partition of $\boldsymbol{X}$; our proof is, however, more ostensive).

Let us denote by $\boldsymbol{X}_-$ and $\boldsymbol{X}_+$ the spaces of sequences of elements of X indexed by integers $i \le 0$ and $i > 0$, respectively, and denote by $\boldsymbol{\lambda}_-$ (resp. $\boldsymbol{\lambda}_+$) the corresponding product measures obtained from the measure λ. Clearly, the concatenation map $(\boldsymbol{x}_-, \boldsymbol{x}_+) \mapsto \boldsymbol{x}_-\boldsymbol{x}_+$ establishes an isomorphism of the product $(\boldsymbol{X}_- \times \boldsymbol{X}_+, \boldsymbol{\lambda}_- \otimes \boldsymbol{\lambda}_+)$ and the space $(\boldsymbol{X}, \boldsymbol{\lambda})$. Denote by $\eta_\pm$ the preimage partitions of the space $\boldsymbol{X}$ determined by its coordinate projections onto the spaces $\boldsymbol{X}_\pm$, so that two sequences $\boldsymbol{x}, \boldsymbol{x}' \in \boldsymbol{X}$ are η_--*equivalent* (resp. η_+-*equivalent*), i.e., they belong to the same element of the partition η_- (resp. η_+), if and only if $x_i = x_i'$ for all $i \le 0$ (resp. for all $i > 0$).

The partitions $\eta_\pm$ have the following obvious property: if φ is a cylinder function, and $\boldsymbol{x}, \boldsymbol{x}' \in \boldsymbol{X}$ are η_--equivalent (resp. η_+-equivalent), then $\varphi(T^{-n}\boldsymbol{x}) = \varphi(T^{-n}\boldsymbol{x}')$ (resp. $\varphi(T^n\boldsymbol{x}) = \varphi(T^n\boldsymbol{x}')$) for all sufficiently large $n > 0$. Therefore, η_- (resp. η_+) can be considered as a counterpart of the strongly unstable (resp. strongly stable) foliation of the geodesic flow on a negatively curved manifold.

We shall prove that f is a.e. constant on the elements of both partitions η_- and η_+, which would then imply the claim of the theorem.

For a function $\varphi \in L^1(\boldsymbol{X}, \boldsymbol{\lambda}, E)$ denote by

$$I_+(\varphi) = \int \big\|\varphi(\boldsymbol{x}_-\boldsymbol{x}_+) - \varphi(\boldsymbol{x}'_-\boldsymbol{x}_+)\big\| d\boldsymbol{\lambda}_-(\boldsymbol{x}_-) d\boldsymbol{\lambda}_-(\boldsymbol{x}'_-) d\boldsymbol{\lambda}_+(\boldsymbol{x}_+)$$

its mean oscillation along the elements of the partition η_+, so that $I_+(\varphi) = 0$ if and only if the function φ is a.e. constant on the elements of η_+. If φ' is another function from $L^1(\boldsymbol{X}, \boldsymbol{\lambda}, E)$, then

$$\begin{aligned}\big\|\varphi(\boldsymbol{x}_-\boldsymbol{x}_+) - \varphi(\boldsymbol{x}'_-\boldsymbol{x}_+)\big\| &\le \big\|\varphi'(\boldsymbol{x}_-\boldsymbol{x}_+) - \varphi'(\boldsymbol{x}'_-\boldsymbol{x}_+)\big\| \\ &\quad + \big\|\varphi(\boldsymbol{x}_-\boldsymbol{x}_+) - \varphi'(\boldsymbol{x}_-\boldsymbol{x}_+)\big\| + \big\|\varphi(\boldsymbol{x}'_-\boldsymbol{x}_+) - \varphi'(\boldsymbol{x}'_-\boldsymbol{x}_+)\big\|,\end{aligned}$$

whence

$$\big|I_+(\varphi) - I_+(\varphi')\big| \le 2|||\varphi - \varphi'|||\,. \tag{9}$$

If φ is a cylinder function, then $I_+(\varphi \circ T^n) = 0$ for all sufficiently large n.

On the other hand, as follows from (7), for any $n > 0$,

$$\begin{aligned} I_+(f\circ T^n) &= \int \big\| f(T^n(\boldsymbol{x}_-\boldsymbol{x}_+)) - f(T^n(\boldsymbol{x}'_-\boldsymbol{x}_+))\big\| d\boldsymbol{\lambda}_-(\boldsymbol{x}_-)d\boldsymbol{\lambda}_-(\boldsymbol{x}'_-)d\boldsymbol{\lambda}_+(\boldsymbol{x}_+) \\ &= \int \big\| \pi(x_n)\dots\pi(x_2)\pi(x_1)(f(\boldsymbol{x}_-\boldsymbol{x}_+) \\ &\qquad - f(\boldsymbol{x}'_-\boldsymbol{x}_+))\big\| d\boldsymbol{\lambda}_-(\boldsymbol{x}_-)d\boldsymbol{\lambda}_-(\boldsymbol{x}'_-)d\boldsymbol{\lambda}_+(\boldsymbol{x}_+) \\ &= \int \big\| f(\boldsymbol{x}_-\boldsymbol{x}_+) - f(\boldsymbol{x}'_-\boldsymbol{x}_+)\big\| d\boldsymbol{\lambda}_-(\boldsymbol{x}_-)d\boldsymbol{\lambda}_-(\boldsymbol{x}'_-)d\boldsymbol{\lambda}_+(\boldsymbol{x}_+) = I_+(f), \end{aligned}$$

where $x_i \in X$ are the coordinates of $\boldsymbol{x}_+$. Therefore, by (9)

$$\begin{aligned} I_+(f) = I_+(f\circ T^n) &\le I_+(\varphi\circ T^n) + 2|||f\circ T^n - \varphi\circ T^n||| \\ &= I_+(\varphi\circ T^n) + 2|||f-\varphi|||\,, \end{aligned}$$

because T preserves the measure $\boldsymbol{\lambda}$. Then Lemma 8 implies that $I_+(f) = 0$. In the same way $I_-(f)$ also vanishes, therefore f is a.e. constant. □

Remark 10. For a general map $\pi : (\boldsymbol{X}, \boldsymbol{\lambda}) \to \mathrm{Iso}(E)$ one can only claim that $\|f(\boldsymbol{x})\|$ is a.e. constant (which follows from the usual ergodicity of the Bernoulli shift) as shown in the simplest example of the space $E \cong \mathbb{R}^2$. In this situation for an arbitrary function $f : \boldsymbol{X} \to E$ with a.e. constant $\|f(\boldsymbol{x})\|$ one can satisfy the formula $f(T\boldsymbol{x}) = \pi(\boldsymbol{x})f(\boldsymbol{x})$ by defining $\pi(\boldsymbol{x})$ as the rotation which moves the vector $f(\boldsymbol{x})$ to $f(T\boldsymbol{x})$.

3 The Poisson Boundary

Let us first briefly recall the basic definitions and facts connected with the Poisson boundary of random walks on groups, see [K3] and the references therein.

Let G be a locally compact group, and μ a Borel probability measure on G. The measure μ is called *spread out* (*étalée* in the French terminology) if there exists a convolution power μ^{*n} which is not singular with respect to the Haar measure class on G. The measure μ is *non-degenerate* if the minimal closed semigroup $S \subset G$ with $\mu(S) = 1$ is G.

The (right) *random walk* determined by the measure μ is the Markov chain on G with the transition probabilities $\pi_g = g\mu$. Denote by $G^{\mathbb{Z}_+}$ the space of *sample paths* $\boldsymbol{g} = (g_0, g_1, \dots)$ of the random walk, and by $\mathbf{P}$ the probability measure on $G^{\mathbb{Z}_+}$ corresponding to the initial distribution concentrated at the identity e of the group. The map

$$\Phi_+ : \boldsymbol{h}_+ \mapsto \boldsymbol{g} = (g_0, g_1, g_2, \dots)\,, \quad \begin{cases} g_0 = e\,, \\ g_n = g_{n-1}h_n = h_1h_2\cdots h_n\,,\ n > 0 \end{cases} \tag{11}$$

establishes an isomorphism of the space $(G^{\mathbb{Z}_+}, \mathbf{P})$ and the space $\boldsymbol{G}_+$ of sequences $\boldsymbol{h}_+ = (h_1, h_2, \dots)$ of *increments* h_n endowed with the product measure $\boldsymbol{\mu}_+$ (we use for product spaces the notation introduced in the proof of Theorem 6).

The path space $G^{\mathbb{Z}_+}$ is endowed with the coordinate-wise action of G, and for an arbitrary (σ-finite) initial distribution θ on G the associated measure on the path space is the convolution

$$\mathbf{P}_\theta = \theta\mathbf{P} = \int g\mathbf{P}\, d\theta(g)\,.$$

The right Haar measure m on G is preserved by the random walk, so that the measure $\mathbf{P}_m$ on $G^{\mathbb{Z}_+}$ is shift-invariant.

Any Borel measure on a second countable group G turns it into a Lebesgue space (see [R], [CFS] for a definition; any Polish space with a Borel probability measure on it is a Lebesgue space). Then the path space $(G^{\mathbb{Z}_+}, \mathbf{P}_m)$ is also a Lebesgue space, so that the following definition makes sense:

Definition 12. The *Poisson boundary* $\Gamma = \Gamma(G, \mu)$ of the random walk (G, μ) on a locally compact second countable group G is the space of ergodic components of the time shift in the space of sample paths $(G_+^{\mathbb{Z}}, \mathbf{P}_m)$. The Poisson boundary is endowed with the *harmonic measure class* $[\nu_m]$ which is the image of the class of the measure $\mathbf{P}_m$ under the canonical projection $\mathbf{bnd} : G^{\mathbb{Z}_+} \to \Gamma$ (NB: the projection $\mathbf{bnd}$ is defined in the measure category only!). For any probability measure θ on G absolutely continuous with respect to the Haar measure m (notation: $\theta \prec m$) the *harmonic measure* $\nu_\theta = \mathbf{bnd}(\mathbf{P}_\theta)$ is absolutely continuous with respect to the measure class $[\nu_m]$.

The time shift on the path space $G^{\mathbb{Z}_+}$ commutes with the coordinate-wise action of the group G, so that the latter action descends to an action of G on Γ which preserves the harmonic measure class, and $\nu_{g\theta} = g\nu_\theta$ for any $g \in G$ and $\theta \prec m$. Below we shall always consider the Poisson boundary Γ as a G-space endowed with the G-invariant measure class $[\nu_m]$.

Theorem 13 [Z]. *If μ is a spread out probability measure on a locally compact second countable group G, then the action of G on the Poisson boundary $\Gamma(G, \mu)$ is amenable.*

If the measure μ is spread out, then the harmonic measure ν_θ is well defined and absolutely continuous with respect to the harmonic measure class $[\nu_m]$ for an *arbitrary* initial probability distribution θ. If, in addition, μ is non-degenerate, then all the harmonic measures ν_θ actually belong to the class $[\nu_m]$. In particular, in this situation the harmonic measure class is represented by the canonical probability measure $\nu = \mathbf{bnd}(\mathbf{P})$ corresponding to the initial distribution concentrated at the group identity.

The *reflected measure* $\check{\mu}$ is the image of the measure μ under the map $g \mapsto g^{-1}$. We shall use for various objects associated with the measure $\check{\mu}$ the same notation as for the measure μ adding to them the "check" sign ˇ. It will be convenient for what follows to identify the path space $(G^{\mathbb{Z}_+}, \check{\mathbf{P}})$ with the product space $(\boldsymbol{G}_-, \boldsymbol{\mu}_-)$ via the map

$$\Phi_- : \boldsymbol{h}_- \mapsto \check{\boldsymbol{g}} = (\check{g}_0, \check{g}_1, \check{g}_2, \dots)\,, \quad \begin{cases} \check{g}_n = e\,, & n = 0\,; \\ \check{g}_n = h_0^{-1} h_{-1}^{-1} \cdots h_{-n+1}^{-1}\,, & n > 0\,. \end{cases} \tag{14}$$

Since the space of bilateral sequences $(\boldsymbol{G}, \boldsymbol{\mu})$ is the product $(\boldsymbol{G}_- \times \boldsymbol{G}_+, \boldsymbol{\mu}_- \otimes \boldsymbol{\mu}_+)$, we obtain the isomorphism

$$\boldsymbol{h} \mapsto (\boldsymbol{h}_-, \boldsymbol{h}_+) \mapsto (\check{\boldsymbol{g}}, \boldsymbol{g})$$

of the space $(\boldsymbol{G}, \boldsymbol{\mu})$ and the product $(G^{\mathbb{Z}_+}, \check{\mathbf{P}}) \times (G^{\mathbb{Z}_+}, \mathbf{P})$. Now put

$$\Psi(\boldsymbol{h}) = \big(\check{\mathbf{bnd}} \circ \Phi_-(\boldsymbol{h}_-), \mathbf{bnd} \circ \Phi_+(\boldsymbol{h}_+)\big) \in \check{\Gamma} \times \Gamma\,, \tag{15}$$

so that $\Psi(\boldsymbol{\mu}) = \check{\nu} \otimes \nu$. As follows from (11) and (14), a.e.

$$\Psi(T\boldsymbol{h}) = h_1^{-1} \Psi(\boldsymbol{h}) \tag{16}$$

(see [K4] for more details).

Now we are ready to prove

Theorem 17. *Let μ be a non-degenerate spread out measure on a locally compact second countable group G. Then the product of the Poisson boundaries $\check{\Gamma}$ and Γ of the measures $\check{\mu}$ and μ, respectively, is $\mathcal{M}^{\mathrm{sep}}$-ergodic with respect to the diagonal action of G.*

Proof. Let (E, π) be a separable coefficient Banach G-module, and $f : \check{\Gamma} \times \Gamma \to E$ be a G-equivariant weak* measurable function. By using the map Ψ (15) define a E-valued measurable function F on the space $(\boldsymbol{G}, \boldsymbol{\mu})$ as

$$F(\boldsymbol{h}) = f \circ \Psi(\boldsymbol{h})\,.$$

Then by (16), a.e.

$$F(T\boldsymbol{h}) = f\big(h_1^{-1}\Psi(\boldsymbol{h})\big) = \pi(h_1^{-1}) f(\Psi(\boldsymbol{h})) = \pi(h_1^{-1}) F(\boldsymbol{h})\,,$$

which in view of Theorem 6 implies that f is a.e. constant. □

Remark 18. The assumption that the measure μ is non-degenerate is essential in Theorem 17 as it guarantees that the measure $\check{\nu} \otimes \nu$ belongs to the product of harmonic measure classes on the Poisson boundaries $\check{\Gamma}$ and Γ. The simplest example when $\mu = \delta_e$ on a countable group G shows that Theorem 17 fails without this assumption.

4 Proofs of Main Theorems

Proof of Theorem 3. This is an immediate consequence of Theorem 13 and Theorem 17 in the particular case when the measure μ is symmetric, i.e., $\check{\mu} = \mu$. □

Proof of Theorem 4. For any locally compact σ-compact group G there exists a compact normal subgroup K such that the quotient group G/K is second countable, see [KaK] and [M, Scholium 5.3.11]. Now, if (X, λ) is a strong G/K-boundary (which exists by Theorem 3), then it is also a strong G-boundary for the G-action defined using the projection $G \to G/K$. Indeed, both amenability and double ergodicity are preserved under passing to compact extensions, see the discussion of the notion of an amenable action for groups which are not second countable at the end of [M, Section 5.3] and [M, Lemma 11.1.9], respectively. □

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Vadim A. Kaimanovich, CNRS UMR 6625, IRMAR, Université Rennes-1, Campus Beaulieu, 35042 Rennes, France kaimanov@univ-rennes1.fr

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