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PROPERTY (T) AND KAZHDAN CONSTANTS FOR DISCRETE GROUPS

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Abstract

Let Γ be a group generated by a finite set S. We give a sufficient condition for Γ to have Kazhdan's property (T). This condition is easy to check and gives Kazhdan constants. We give examples of groups to which this method applies. We prove that in some setting generic presentations define groups which satisfy this condition and thus have property (T). Moreover we prove that small changes in the presentation of a group satisfying this condition do not change the fact that the group has property (T).

1 Introduction

Let Γ be a group generated by a finite set S. Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ be a unitary representation of Γ. We say that π almost has invariant vectors if for every $\varepsilon > 0$ there exists a non-zero vector $u_\varepsilon \in \mathcal{H}_\pi$ such that $\|\pi(s)u_\varepsilon - u_\varepsilon\| < \varepsilon\|u_\varepsilon\|$ for every $s \in S$. In [K], Kazhdan defined property (T); namely, we say that the group Γ has *Kazhdan's property (T)* if every unitary representation of Γ which almost has invariant vectors has a non-zero invariant vector. The fact that a given group has property (T) does not depend on a set of generators.

One knows (see for example [HV]) that the group Γ generated by the set S has property (T) if and only if there exists $\varepsilon(S) > 0$ (which sometimes cannot be taken independently of S – see [GeZ]) such that for every unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ without non-zero invariant vectors one has

$$\max_{s\in S} \big\|\pi(s)\xi - \xi\big\| \geq \varepsilon(S)\|\xi\| \tag{1}$$

for every $\xi \in \mathcal{H}_\pi$.

A positive $\varepsilon(S)$ for which the inequality (1) is satisfied is called a *Kazhdan constant* for Γ with respect to S.

One can also define property (T) without supposing that a given group is finitely generated. Namely, one says that such a group has property (T) if every unitary representation which almost has invariant vectors for every finite subset S, has invariant vectors. But it can be shown that groups with property (T) are necessarily finitely generated. This was an idea of Kazhdan, who introduced property (T) in order to prove Siegel's conjecture which states that lattices in semi-simple Lie groups are finitely generated [K]. For lattices, property (T) is inherited from Lie groups and Kazhdan proved that lattices in simple Lie groups of rank at least 2 have property (T) (see [K], [DK], [V]). Since then, property (T) was used to solve several other problems. In the context of this paper, an important application of property (T) is due to Margulis who in [M1] used residually finite groups with property (T) to give the first explicit examples of expanding graphs. Here we present a result which enables one to show that a given group has property (T) if a certain graph is an expander.

The question concerning Kazhdan constants was asked by Serre (for examples of the computation see [BH], [BeCJ], [BeM], [Bu], [CMS], [Co], [O], [Sh1]). Explicit Kazhdan constants are useful for several applications. For instance, they can be used to estimate isoperimetric constants of expanding graphs (see [HV], [L], [S]).

For more information about property (T) see for instance [HV], [L], [M3], [Z].

Till recently the only known infinite groups with property (T) were related to lattices in semi-simple Lie groups.

In this paper we obtain a result, which enables one to prove property (T) and to estimate Kazhdan constants for many discrete groups given by presentations.

Let Γ be a group generated by a finite set S such that S is symmetric, i.e. $S = S^{-1}$, and the identity element e does not belong to S.

Definition 1. *We define a finite graph $L(S)$, in the following way:*

1. *vertices of $L(S) = \{s; s \in S\}$;*
2. *edges of $L(S) = \{(s, s'); s, s', s^{-1}s' \in S\}$.*

Let us suppose that the graph $L(S)$ is connected. This condition is not restrictive, because for a finitely generated group Γ one can always find a finite, symmetric generating set S, not containing e, such that $L(S)$ is connected (for instance $S \cup S^2 \setminus e$ will do). For a vertex $s \in L(S)$ let $\deg(s)$ denote its degree, i.e. the number of edges adjacent to s. Let Δ be a discrete Laplace operator acting on functions defined on vertices of $L(S)$, i.e. for $f \in l^2(L(S), \deg)$

$$\Delta f(s) = f(s) - \frac{1}{\deg(s)} \sum_{s' \sim s} f(s'),$$

where $s' \sim s$ means that the vertex s' is adjacent to the vertex s.

The operator Δ is a non-negative, self-adjoint operator on $l^2(L(S), \deg)$. If $L(S)$ is connected then zero is a simple eigenvalue of Δ. Let $\lambda_1(L(S))$ be the smallest non-zero eigenvalue of Δ acting on $l^2(L(S), \deg)$.

Theorem 1. *Let Γ be a group generated by a finite subset S, such that S is symmetric and $e \notin S$. If the graph $L(S)$ is connected and*

$$\lambda_1(L(S)) > \tfrac{1}{2} \tag{2}$$

then Γ has Kazhdan's property (T). *Moreover*

$$\frac{2}{\sqrt{3}}\left(2 - \frac{1}{\lambda_1(L(S))}\right)$$

is a Kazhdan constant with respect to the set S.

Remark. The condition stated in Theorem 1 is optimal. In order to see this, let us consider the group $\Gamma = \mathbb{Z}^2$ with the set of generators

$$S = \big\{(1,0), (-1,0), (0,1), (0,-1), (1,1), (-1,-1)\big\}\,.$$

Then the graph $L(S)$ consists of vertices and edges of the hexagon. One can compute that for such a graph the spectrum of Δ consists of $0, \frac{1}{2}, 1\frac{1}{2}$ and 2. Thus $\lambda_1(L(S)) = 1/2$ and the group $\mathbb{Z}^2$ does not have property (T).

The condition stated in Theorem 1 involves only finitely many relations in the presentation of the group Γ. On the other hand groups which are homomorphic images of groups with property (T) have this property as well. Thus it is natural to ask: Is every discrete group with property (T) a homomorphic image of a group with property (T) which is finitely presented? (This question was answered positively by Shalom [Sh2].)

The condition (2) in Theorem 1 is elementary and easy to check. In order to prove property (T) for a group Γ, using Theorem 1, we do not need to know anything about unitary representations of Γ. This is the reason why Theorem 1 enables one to find infinitely many new groups with property (T), to prove that generic presentations define groups have property (T) (see section 7) and to show that if a given group satisfies condition (2) and therefore has property (T) then groups which do not differ too much from Γ also have property (T) (see section 8).

A condition similar to the one given in Theorem 1 was established in [Żu1] and enabled one to find infinitely many new groups with property (T) (see [BaS], [Bar], [Bou], [GP]).

The result presented in [Żu1] used a cohomological definition of property (T) (see [De], [Gu]) and was related to the work of Garland [Ga] (see also [Pa]). Results presented here do not rely on the cohomological definition of property (T).

One can also prove some geometric versions of Theorem 1, concerning configurations of vectors in finite dimensional Euclidean spaces $\mathbb{R}^n$. Let us be more precise.

Definition 2. *Let $P(S)$ be the set of vectors $v_{s_1},\dots,v_{s_n}\in\mathbb{R}^n$ where $n=|S|$ and $s_1,\dots,s_n$ are different elements of S such that*

1. $\|v_s-v_{s'}\|=\|v_{s^{-1}s'}\|$ *if* $s,s',s^{-1}s'\in S$,
2. $\|v_s\|=\|v_{s^{-1}}\|$ *for* $s\in S$.

We define the constant $K(S)$:

$$K(S)=\frac{4}{\sqrt{3}|T|}\min_{v_{s_1},\dots,v_{s_n}\in P(S)}\frac{\|\sum_{s\in S}v_s\deg(s)\|^2}{\sum_{s\in S}\|v_s\|^2\deg(s)}\,.$$

In section 4 we prove

Theorem 2. *Let S be a finite, symmetric set of generators for a group Γ, which does not contain the identity. If $K(S)>0$ then Γ has property* (T) *and $K(S)$ is a Kazhdan constant with respect to S.*

Theorem 2 is stronger than Theorem 1. Other geometric versions of Theorem 1 are presented in section 4 (Theorems 7 and 8).

The condition (2) in Theorem 1 is easy to be satisfied and can be applied to groups given by generic presentations. Let us be more precise.

We consider the following model $\mathcal{M}$ for random groups, which is related to Gromov's model ([Gr2]). Let us consider presentations with relations of length 3. Let d (called density as before) be between 0 and 1. Let $P_{\mathcal{M}}(m,d)$ be a set of presentations with m generators, relations of length 3 and density d, i.e. the number of relations is between $c^{-1}(2m-1)^{3d}$ and $c(2m-1)^{3d}$, where $c>1$ is any fixed constant. For simplicity, we will suppose in the proofs that the number of relations is equal to $(2m-1)^{3d}$.

In [Gr2, p. 273] Gromov proves that in his model a generic presentation with density less than 1/2 defines an infinite hyperbolic group and that a generic presentation with density greater than 1/2 defines a trivial group. In our model $\mathcal{M}$ we have the analogue:

Theorem 3. *For $d<1/2$ one has*

$$\lim_{m\to\infty}\frac{\#\{P\in P_{\mathcal{M}}(m,d);\Gamma(P)\text{ is infinite, hyperbolic group}\}}{\#P_{\mathcal{M}}(m,d)}=1\,.$$

We prove that a generic presentation in the model $\mathcal{M}$ with density greater than 1/3 defines a group with property (T), i.e.

Theorem 4. *For $d > 1/3$ one has*

$$\lim_{m\to+\infty} \frac{\#\{P \in P_{\mathcal{M}}(m,d); \Gamma(P) \text{ has property (T)}\}}{\#P_{\mathcal{M}}(m,d)} = 1\,.$$

Theorems 3 and 4 imply that a generic presentation in the model $\mathcal{M}$ with density between 1/3 and 1/2 defines an infinite hyperbolic group with property (T), i.e.

Corollary 1. *For $1/3 < d < 1/2$ one has*

$$\lim_{m\to\infty} \frac{\#\{P\in P_{\mathcal{M}}(m,d); \Gamma(P) \text{ is infinite, hyperbolic with property (T)}\}}{\#P_{\mathcal{M}}(m,d)} = 1\,.$$

Results about genericity of property (T) are related to the results concerning genericity of expanding graphs [Bo2], [Żu2]. The proof of hyperbolicity of random groups follows [Gr2].

In order to analyze the situation when $d = 1/3$ we introduce another model $\mathcal{F}$ which more precisely describes the number of relations we put in the presentation. It corresponds to the density 1/3 but it has an additional parameter v. We show (Theorem 10) that for v sufficiently large, a random group in this model has property (T).

One has to suppose that $d \geq 1/3$ because for $d < 1/3$ generic groups do not have property (T). More precisely, if F_2 denotes a free group of rank two, then

Theorem 5. *For $d < 1/3$ one has*

$$\lim_{m\to+\infty} \frac{\#\{P \in P_{\mathcal{M}}(m,d); \Gamma(P) \text{ has a quotient } F_2\}}{\#P_{\mathcal{M}}(m,d)} = 1\,.$$

We will also show that small changes in the presentations of a group satisfying the condition (2) do not change the fact that the group has property (T).

We know that if we add any relation to a presentation of a group with property (T), we still obtain a group with property (T). This is because any homomorphic image of a group with property (T) still has this property.

Now, let us analyze the situation, when we remove a relation from the presentation of a Kazhdan group Γ.

Let us denote by $\deg(S)$ the minimal degree of vertices in $L(S)$, i.e.

$$\deg(S) = \min\left\{\deg(s); s \in L(S)\right\}.$$

Theorem 6. *Let Γ be a group generated by a finite, symmetric set S, such that $e \notin S$ and $L(S)$ is connected. Let us suppose that $\lambda_1 = \lambda_1(L(S)) > 1/2$, i.e. Γ has property* (T). *Then for any $t \in \mathbb{N}$ such that*

$$t \leq \tfrac{1}{11}\left(\lambda_1 - \tfrac{1}{2}\right)\deg(S)$$

after removing any t relations of length three from any presentation of Γ *with generators* S*, we obtain a group with Kazhdan's property* (T).

Theorem 6 shows that if $\lambda_1(L(S))$ is sufficiently greater than $1/2$ and if the number of neighbors of each vertex in $L(S)$ is sufficiently large we can remove several relations and still obtain groups with property (T). This is for instance the case of groups discussed in section 5.1, for which $\lambda_1(L(S))$ can be arbitrarily close to 1 and the degree of vertices in $L(S)$ can be arbitrarily large.

The paper is organized as follows. In section 2 we introduce the ingredients of the proof of Theorem 1, which is given in section 3. In section 4 we present other conditions implying property (T) (Theorems 2, 7 and 8). Examples of some groups to which these theorems apply are given in section 5. In section 6 and section 7 we give proofs concerning properties of random groups, in particular we show that generic presentations provide groups which have property (T) (Theorem 4). Finally, in section 8 we show that if a given group satisfies condition (2) in Theorem 1 the groups obtained by small changes in the presentation of this group, still have property (T).

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2 Conditions Implying Property (T)

In this section we present the ingredients of the proof of Theorem 1.

2.1 The operators d and d^*. Let T be a subset of $S \times S$ defined as follows:

$$T = \left\{(s,s'); s,s',s^{-1}s' \in S\right\}.$$

For $r = 0,1$ and 2 let C^r be the spaces defined as follows:

$$C^0 = \{u; u \in \mathcal{H}_\pi\},$$
$$C^1 = \left\{f : S \to \mathcal{H}_\pi; f(s^{-1}) = -\pi(s^{-1})f(s) \text{ for all } s \in S\right\},$$
$$C^2 = \{g : T \to \mathcal{H}_\pi\}.$$

Let us define linear operators $d : C^0 \to C^1$ and $d : C^1 \to C^2$ as follows:

$$\begin{array}{rcll} du(s) & = & \pi(s)u - u & \text{for all } u \in C^0, \\ df((s,s')) & = & f(s) - f(s') + \pi(s)f(s^{-1}s') & \text{for all } f \in C^1. \end{array}$$

For any $u \in C^0$ and $(s,s') \in T$ one has

$$\begin{aligned} ddu((s,s')) &= du(s) - du(s') + \pi(s)du(s^{-1}s') \\ &= \big(\pi(s)u - u\big) - \big(\pi(s')u - u\big) + \pi(s)\big(\pi(s^{-1}s')u - u\big) = 0\,. \end{aligned}$$

Thus

$$dd = 0\,.$$

For $s \in S$ we define a number $n(s)$ in the following way

$$n(s) = \#\big\{s' \in S; (s,s') \in T\big\}\,.$$

We suppose that the graph $L(S)$ is connected. This implies in particular that $n(s) > 0$ for every $s \in S$.

We remark that

$$\begin{aligned} n(s) &= n(s^{-1})\,, \\ \sum_{s\in S} n(s) &= |T|\,. \end{aligned}$$

Let $\langle\,\cdot\,,\cdot\,\rangle_{\mathcal{H}_\pi}$ be the scalar product in the Hilbert space $\mathcal{H}_\pi$. We define the scalar products on C^0, C^1 and C^2 in the following way:

$$\begin{array}{ll} \langle u,w\rangle_{C^0} = \langle u,w\rangle_{\mathcal{H}_\pi}|T| & \text{for } u,w \in C^0, \\ \langle f,g\rangle_{C^1} = \sum\limits_{s\in S} \big\langle f(s),g(s)\big\rangle_{\mathcal{H}_\pi} \cdot n(s) & \text{for } f,g \in C^1, \\ \langle h,g\rangle_{C^2} = \sum\limits_{t\in T} \big\langle h(t),g(t)\big\rangle_{\mathcal{H}_\pi} & \text{for } h,g \in C^2. \end{array}$$

When it is clear to which scalar product we are referring we will omit the subscript for simplicity.

Let $d^* : C^1 \to C^0$ be the adjoint operator of $d : C^0 \to C^1$, i.e.

$$\langle du,f\rangle_{C^1} = \langle u,d^*f\rangle_{C^0} \text{ for } u \in C^0 \text{ and } f \in C^1.$$

One has the following explicit expression for d^*:

Lemma 1. *For $f \in C^1$ we have*

$$d^*f = -2\sum_{s\in S} f(s)\frac{n(s)}{|T|}.$$

Proof. The expression for $d^* : C^1 \to C^0$ is a consequence of the following equalities:

$$\langle du,f\rangle_{C^1} = \sum_{s\in S}\big\langle du(s),f(s)\big\rangle_{\mathcal{H}_\pi} n(s) = \sum_{s\in S}\big\langle \pi(s)u - u, f(s)\big\rangle_{\mathcal{H}_\pi} n(s)$$

$$
\begin{aligned}
&= \sum_{s\in S}\big(\langle u,\pi(s^{-1})f(s)\rangle_{\mathcal{H}_\pi} n(s) - \langle u, f(s)\rangle_{\mathcal{H}_\pi} n(s)\big)\\
&= \sum_{s\in S}\big(- \langle u, f(s^{-1})\rangle_{\mathcal{H}\pi} n(s^{-1}) - \langle u, f(s)\rangle_{\mathcal{H}_\pi} n(s)\big)\\
&= -2\sum_{s\in S}\langle u, f(s)\rangle_{\mathcal{H}_\pi} n(s) = \left\langle u, -2\sum_{s\in S}\frac{f(s)n(s)}{|T|}\right\rangle_{\mathcal{H}_\pi}|T|\\
&= \left\langle u, -2\sum_{s\in S}\frac{f(s)n(s)}{|T|}\right\rangle_{C^0}. \qquad \square
\end{aligned}
$$

We need an estimation on the norm of the operator $d^*: C^1 \to C^0$.

Lemma 2. *The norm $\|d^*\|_{C^1\to C^0}$ of the operator $d^*: C^1\to C^0$ is bounded by 2, i.e.*

$$\|d^*\|_{C^1\to C^0} \le 2\,.$$

Proof. Let $f\in C^1$ be such that $\|f\|_{C^1}=1$. Then we have

$$
\begin{aligned}
\|d^* f\|^2_{C^0} &= \left\| - 2\sum_{s\in S} f(s)\frac{n(s)}{|T|}\right\|^2_{\mathcal{H}_\pi}|T|\\
&\le 4\bigg(\sum_{s\in S}\|f(s)\|^2_{\mathcal{H}_\pi}\cdot n(s)\bigg)\bigg(\sum_{s\in S}\frac{n(s)}{|T|^2}\bigg)|T|\\
&= 4\|f\|^2_{C^1}\bigg(\sum_{s\in S}\frac{n(s)}{|T|}\bigg) = 4\,,
\end{aligned}
$$

which ends the proof of Lemma 2. $\square$

2.2 Property (T) The operators d and d^* are related to Kazhdan's property (T) and their spectral analysis enables one to estimate Kazhdan constants as shown in Theorem 1.

Let B^1 be the kernel of $d: C^1\to C^2$, i.e.

$$B^1 = \{f\in C^1;\ df = 0\}\,.$$

We will need an estimation on the norm of the operator $d^*: B^1 \to C^0$.

Lemma 3. *The norm $\|d^*\|_{B^1\to C^0}$ of the operator $d^*: B^1\to C^0$ is bounded by $\sqrt{3}$, i.e.*

$$\|d^*\|_{B^1\to C^0}\le\sqrt{3}\,.$$

Proof. First of all for $f\in B^1$ we have

$$
\begin{aligned}
\langle f,f\rangle &= \sum_{s\in S}\big\langle f(s),f(s)\big\rangle n(s) = \sum_{(s,s')\in T}\big\langle f(s^{-1}s'), f(s^{-1}s')\big\rangle\\
&= \sum_{(s,s')\in T}\big\langle \pi(s)f(s^{-1}s'), \pi(s)f(s^{-1}s')\big\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{(s,s')\in T} \big\langle f(s)-f(s'), f(s)-f(s')\big\rangle \\
&= \sum_{(s,s')\in T} \big(\langle f(s),f(s)\rangle + \langle f(s'),f(s')\rangle - 2\langle f(s),f(s')\rangle\big) \\
&= 2\langle f,f\rangle - 2\sum_{(s,s')\in T} \big\langle f(s),f(s')\big\rangle\,,
\end{aligned}
$$

which gives

$$2\sum_{(s,s')\in T} \big\langle f(s),f(s')\big\rangle = \langle f,f\rangle\,.$$

Thus we get

$$
\begin{aligned}
\|d^*f\|^2_{C^0} &= \Big\| -2\sum_{s\in S} f(s)\frac{n(s)}{|T|}\Big\|^2 |T| = \Big\| \sum_{(s,s')\in T} \big(f(s)+f(s')\big)\Big\|^2 \frac{1}{|T|} \\
&\le \sum_{(s,s')\in T} \big\|f(s)+f(s')\big\|^2 \\
&= \sum_{(s,s')\in T} \big(\|f(s)\|^2 + \|f(s')\|^2 + 2\langle f(s),f(s')\rangle\big) \\
&= 2\langle f,f\rangle + \langle f,f\rangle = 3\langle f,f\rangle\,,
\end{aligned}
$$

which ends the proof of Lemma 3. □

The following proposition expresses the relation between spectral properties of the operators d and d^* and Kazhdan's property (T).

Proposition 1. *If there exists $c>0$ such that for every $f\in B^1$*

$$\langle dd^*f,f\rangle \ge c\langle f,f\rangle \tag{3}$$

then $c/\sqrt{3}$ is a Kazhdan constant.

Proof. First of all we prove that the inequality (3) implies that the operator $dd^*: B^1\to B^1$ has a bounded inverse. By (3) the image $dd^*(B^1)$ is closed in B^1. If $dd^*(B^1)$ were different from B^1, there would exist a non-zero vector $u\in B^1$ which would be orthogonal to the image of B^1 by dd^*. Then we would have, by (3),

$$0 = \big\langle u, dd^*(u)\big\rangle \ge c\langle u,u\rangle$$

which is a contradiction.

Thus $dd^*: B^1\to B^1$ has a bounded inverse $(dd^*)^{-1}: B^1\to B^1$ and $\|(dd^*)^{-1}\|_{B^1\to B^1}\le c^{-1}$.

Now, suppose that $c/\sqrt{3}$ is not a Kazhdan constant. Let us consider a unitary representation π without non-zero invariant vectors, such that for some $0 < \varepsilon < c/\sqrt{3}$ there exists $u \in \mathcal{H}_\pi$ such that $\|u\|_{\mathcal{H}\pi} = 1$ and $\|\pi(s)u - u\|_{\mathcal{H}_\pi} \leq \varepsilon$ for every $s \in S$. Then

$$\|du\|_{B^1}^2 = \sum_{s\in S} \big\|du(s)\big\|_{\mathcal{H}\pi}^2 n(s) = \sum_{s\in S} \big\|\pi(s)u - u\big\|_{\mathcal{H}_\pi}^2 n(s) \leq \sum_{s\in S} \varepsilon^2 n(s) = \varepsilon^2 |T|$$

which gives $\|du\|_{B^1} \leq \varepsilon\sqrt{|T|}$. Let us consider $d^*(dd^*)^{-1}du \in C^0$. We have

$$\begin{aligned}\big\|d^*(dd^*)^{-1}du\big\|_{C^0} &\leq \|d^*\|_{B^1\to C^0} \cdot \big\|(dd^*)^{-1}\big\|_{B^1\to B^1} \cdot \|du\|_{B^1} \\ &\leq \sqrt{3} \cdot c^{-1} \cdot \varepsilon\sqrt{|T|} < \sqrt{|T|}\,.\end{aligned}$$

By definition of the norm in C^0 one has then $d^*(dd^*)^{-1}du = u'$ where $\|u'\|_{\mathcal{H}_\pi} < 1$. So the vector $u - u'$ is non-zero. Finally

$$d(u - u') = du - d\big(d^*(dd^*)^{-1}\big)du = du - du = 0$$

which means that for every $s \in S$

$$\pi(s)(u - u') - (u - u') = 0\,.$$

Thus $u - u'$ is a non-zero invariant vector, which leads to a contradiction and ends the proof of Proposition 1. □

3 Proof of Theorem 1

Let us define the operator $D : C^1 \to C^2$ as follows:

$$Df\big((s_1, s_2)\big) = f(s_1) - f(s_2)\,,$$

where $f \in C^1$ and $(s_1, s_2) \in T$.

3.1 Relation between the operators d and D. The advantage of the operator D over the operator d is the fact that the definition of D does not involve the representation π. The following statement expresses the relation between the operators D and d, which is essential for the proof of Theorem 1.

PROPOSITION 2. *For every $f \in C^1$ one has*

$$\tfrac{1}{3}\langle df, df\rangle = \langle Df, Df\rangle - \langle f, f\rangle\,.$$

Proof. First of all we need the following:

LEMMA 4. *For every $f \in C^1$ one has*

$$\langle f, f\rangle = \sum_{(s,s')\in T} \big\langle f(s^{-1}s'), f(s^{-1}s')\big\rangle\,, \tag{4}$$

$$df\big((s, s')\big) = -df\big((s', s)\big)\,, \tag{5}$$

$$df\big((s, s')\big) = -\pi(s)df\big((s^{-1}, s^{-1}s')\big) = \pi(s')df\big(((s')^{-1}, (s')^{-1}s)\big)\,, \tag{6}$$

$$\tfrac{1}{3}\langle df, df\rangle = \sum_{(s,s')\in T} \big\langle df((s, s')), \pi(s)f(s^{-1}s')\big\rangle\,. \tag{7}$$

Proof. First we have

$$\sum_{(s,s')\in T} \big\langle f(s^{-1}s'), f(s^{-1}s')\big\rangle = \sum_{s''\in S} \sum_{(s,s')\in T;s^{-1}s'=s''} \big\langle f(s''), f(s'')\big\rangle$$
$$= \sum_{s''\in S} \big\langle f(s''), f(s'')\big\rangle n(s'') = \langle f, f\rangle \,,$$

which proves (4). Secondly

$$\begin{aligned} df\big((s,s')\big) &= f(s) - f(s') + \pi(s)f(s^{-1}s') \\ &= -\big(f(s') - f(s) + \pi(s)\pi(s^{-1}s')f((s')^{-1}s)\big) \\ &= -\big(f(s') - f(s) + \pi(s')f((s')^{-1}s)\big) = -df\big((s',s)\big) \,, \end{aligned}$$

which shows (5). Now the following equalities

$$\begin{aligned} df\big((s,s')\big) &= f(s) - f(s') + \pi(s)f(s^{-1}s') \\ &= -\pi(s)\big(-\pi(s^{-1})f(s) + \pi(s^{-1})f(s') - f(s^{-1}s')\big) \\ &= -\pi(s)\big(f(s^{-1}) - f(s^{-1}s') + \pi(s^{-1})f(s(s^{-1}s'))\big) \\ &= -\pi(s)df\big((s^{-1}, s^{-1}s')\big) \,, \end{aligned}$$

prove the first part of (6). The second part of (6) follows from

$$df\big((s,s')\big) = -df\big((s',s)\big) = -\big(-\pi(s')df(((s')^{-1},(s')^{-1}s))\big) \,.$$

Finally, the equality (7) will be a consequence of the following equalities. Because of (6) we have

$$\begin{aligned} &\sum_{(s,s')\in T} \big\langle df((s,s')), \pi(s)f(s^{-1}s')\big\rangle \\ &\quad = \tfrac{1}{3} \sum_{(s,s')\in T} \big(\langle df((s,s')), \pi(s)f(s^{-1}s')\rangle \\ &\qquad + \langle -\pi(s)df((s^{-1},s^{-1}s')), \pi(s)f(s^{-1}s')\rangle \\ &\qquad\quad + \langle \pi(s')df(((s')^{-1},(s')^{-1}s)), \pi(s)f(s^{-1}s')\rangle\big) \\ &\quad = \tfrac{1}{3} \sum_{(s,s')\in T} \big(\langle df((s,s')), \pi(s)f(s^{-1}s')\rangle + \langle df((s^{-1},s^{-1}s')), -f(s^{-1}s')\rangle \\ &\qquad + \langle df(((s')^{-1},(s')^{-1}s)), \pi((s')^{-1}s)f(s^{-1}s')\rangle\big) \\ &\quad = \tfrac{1}{3} \sum_{(s,s')\in T} \big(\langle df((s,s')), \pi(s)f(s^{-1}s')\rangle. \\ &\qquad + \langle df((s,s')), -f(s')\rangle + \langle df((s,s')), f(s)\rangle\big) \\ &\quad = \tfrac{1}{3} \sum_{(s,s')\in T} \big\langle df((s,s')), f(s) - f(s') + \pi(s)f(s^{-1}s')\big\rangle \end{aligned}$$

$$= \tfrac{1}{3} \sum_{(s,s')\in T} \big\langle df((s,s')), df((s,s'))\big\rangle \qquad = \tfrac{1}{3}\langle df, df\rangle\,,$$

which shows (7) and ends the proof of Lemma 4. □

Now, by definitions of d and D we have

$$df\big((s,s')\big) = Df\big((s,s')\big) + \pi(s)f(s^{-1}s')\,.$$

Thus by (7) we get the following equalities

$$\begin{aligned}\langle Df, Df\rangle &= \sum_{(s,s')\in T} \big\langle df((s,s')) - \pi(s)f(s^{-1}s'), df((s,s')) - \pi(s)f(s^{-1}s')\big\rangle\\ &= \sum_{(s,s')\in T} \big\langle df((s,s')), df((s,s'))\big\rangle + \sum_{(s,s')\in T} \big\langle f(s^{-1}s'), f(s^{-1}s')\big\rangle\\ &\qquad - 2\sum_{(s,s')\in T} \big\langle df((s,s')), \pi(s)f(s^{-1}s')\big\rangle\\ &= \langle df, df\rangle + \langle f, f\rangle - \tfrac{2}{3}\langle df, df\rangle = \tfrac{1}{3}\langle df, df\rangle + \langle f, f\rangle\,,\end{aligned}$$

which ends the proof of Proposition 2. □

3.2 Reduction to the graph $L(S)$. The operators d and d^* act on infinite dimensional spaces in general. But we will show that their spectral analysis can be reduced to the finite dimensional case, namely to the spectral analysis of the operator Δ acting on $l^2(L(S),\mathrm{deg})$. More precisely we have

PROPOSITION 3. *For every $f\in C^1$ one has*

$$\tfrac{1}{3}\langle df, df\rangle + \tfrac{1}{2}\lambda_1(L(S))\langle d^*f, d^*f\rangle \geq 2\left(\lambda_1(L(S)) - \tfrac{1}{2}\right)\langle f, f\rangle\,.$$

Proof. By definition every $f\in C^1$ is a function on the graph $L(S)$, which we will denote by f as well. We have then

$$\langle f, f\rangle_{C^1} = \langle f, f\rangle_{L(S)}\,,$$

$$\langle Df, Df\rangle_{C^2} = \sum_{(s,s')\in T} \big\langle f(s) - f(s'), f(s) - f(s')\big\rangle = 2\langle \Delta f, f\rangle_{L(S)}\,.$$

We need the following:

LEMMA 5. *For every $f\in C^1$ the function $F: L(S)\to \mathcal{H}_\pi$, defined as*

$$F(s) = f(s) + \frac{d^*f}{2} \quad \text{for } s\in S\,,$$

is orthogonal to the constant functions on $L(S)$.

Proof. Let us consider any $u\in\mathcal{H}_\pi$. Then

$$\langle F, u\rangle_{L(S)} = \sum_{s\in S}\left\langle f(s) + \frac{d^*f}{2}, u\right\rangle_{\mathcal{H}_\pi} n(s)$$

$$= \Big\langle \sum_{s\in S} f(s)n(s), u \Big\rangle_{\mathcal{H}_\pi} + \langle d^* f, u\rangle_{\mathcal{H}_\pi} \sum_{s\in S} \frac{n(s)}{2}$$
$$= \left\langle -\frac{d^* f\cdot |T|}{2}, u \right\rangle_{\mathcal{H}_\pi} + \langle d^* f, u\rangle_{\mathcal{H}_\pi} \cdot \frac{|T|}{2} = 0\,,$$

which ends the proof of Lemma 5. □

Now by definition of $\lambda_1 = \lambda_1(L(S))$ we have

$$\begin{aligned}
\langle \Delta f, f\rangle_{L(S)} &= \left\langle \Delta\left(f + \frac{d^* f}{2}\right), f + \frac{d^* f}{2} \right\rangle_{L(S)} \\
&\geq \lambda_1 \left\langle f + \frac{d^* f}{2}, f + \frac{d^* f}{2} \right\rangle_{L(S)} \\
&= \lambda_1 \langle f, f\rangle_{L(S)} + \frac{\lambda_1}{2}\langle f, d^* f\rangle_{L(S)} \\
&= \lambda_1 \langle f, f\rangle_{L(S)} + \frac{\lambda_1}{2}\left\langle \sum_{s\in S} \frac{f(s)n(s)}{|T|}, d^* f \right\rangle |T| \\
&= \lambda_1 \langle f, f\rangle_{C^1} - \frac{\lambda_1}{4}\langle d^* f, d^* f\rangle_{C^0}\,.
\end{aligned}$$

So finally,

$$\begin{aligned}
2\lambda_1\langle f, f\rangle_{C^1} - \langle f, f\rangle_{C^1} &\leq 2\langle \Delta f, f\rangle_{L(S)} - \langle f, f\rangle_{C^1} + \tfrac{\lambda_1}{2}\langle d^* f, d^* f\rangle_{C^0} \\
&= \langle Df, Df\rangle_{C^2} - \langle f, f\rangle_{C^1} + \tfrac{\lambda_1}{2}\langle d^* f, d^* f\rangle_{C^0} \\
&= \tfrac{1}{3}\langle df, df\rangle_{C^2} + \tfrac{\lambda_1}{2}\langle d^* f, d^* f\rangle_{C^0}\,,
\end{aligned}$$

which ends the proof of Proposition 3 by applying Proposition 2. □

Now we are in a position to prove Theorem 1. If $f \in B^1$, one has $df = 0$ and Proposition 3 gives the inequality:

$$\langle dd^* f, f\rangle \geq 2\left(2 - \frac{1}{\lambda_1(L(S))}\right)\langle f, f\rangle\,.$$

By Proposition 1 this implies Theorem 1.

4 Geometric Conditions

Theorem 1 required the analysis of the spectrum of Δ acting on $l^2(L(S), \deg)$. This section is devoted to the study of geometric conditions concerning the configurations of vectors in finite dimensional Euclidean spaces which imply property (T).

Proof of Theorem 2. By Proposition 1,

$$C(S) = \frac{1}{\sqrt{3}} \min_{f\in B^1} \frac{\langle d^* f, d^* f\rangle}{\langle f, f\rangle}\,,$$

is a Kazhdan constant with respect to S. The function f can be represented by $n = |S|$ vectors $v_{s_1}, \dots, v_{s_n} \in \mathcal{H}_\pi$, where $v_{s_i} = f(s_i)$.

Let us check that conditions 1 and 2 in Definition 2 are satisfied. The condition that $f \in B^1$ means that $df = 0$, i.e.

$$0 = f(s) - f(s') + \pi(s) f(s^{-1}s') \quad \text{if} \quad s, s', s^{-1}s' \in S\,.$$

In particular

$$\|v_s - v_{s'}\| = \|v_{s^{-1}s'}\|\,.$$

Secondly

$$\|v_s\| = \|f(s)\| = \|\pi(s) f(s^{-1})\| = \|f(s^{-1})\| = \|v_{s^{-1}}\|\,.$$

Now

$$\begin{aligned}\langle d^* f, d^* f\rangle &= \Big\langle -2\sum_{s\in S} f(s)\frac{n(s)}{|T|}, -2\sum_{s\in S} f(s)\frac{n(s)}{|T|}\Big\rangle |T| \\ &= 4\Big\|\sum_{s\in S} v_s \deg(s)\Big\|^2 \frac{1}{|T|}\end{aligned}$$

and

$$\langle f, f\rangle = \sum_{s\in S} \|f(s)\|^2 n(s) = \sum_{s\in S} \|v_s\|^2 \deg(s)\,.$$

Finally, we can suppose that the vectors $v_{s_1}, \dots, v_{s_n}$ are in some $\mathbb{R}^n$. Thus

$$K(S) \le C(S)\,,$$

which by Proposition 1 ends the proof of Theorem 2. □

In cases of a large symmetry of the generating set, one can obtain better results concerning sufficient conditions for property (T) presented here and estimates of Kazhdan constants. Such symmetries were used in [PZ] to improve estimates of Kazhdan constants for $SL(n,\mathbb{Z})$.

In certain situations we can impose further conditions on the set of vectors in $P(S)$ which was used to define the constant $K(S)$ in Theorem 2. Namely let us suppose that there exists a finite subgroup $H \subset \Gamma$ such that

$$hSh^{-1} = S \tag{8}$$

for every $h \in H$.

We define

$$\deg_H(s) = \#\{s' ; s^{-1}s' \in SH\}\,.$$

Definition 3. *Let $P_H(S)$ be the set of vectors $v_{s_1}, \dots, v_{s_n} \in \mathbb{R}^n$ where $n = |S|$ and $s_1, \dots, s_n$ are different elements of S such that*

1. $\|v_s - v_{s'}\| = \|v_{s^{-1}s'}\|$ *if* $s, s', s^{-1}s' \in S$,
2. $\|v_{s_1} - v_{s_2}\| = \|v_{s_3}\|$ *if for* $s_1, s_2, s_3 \in S$ *and for some* $h_0 \in H$ *we have* $s_1^{-1}s_2 = s_3 h_0$,

3. $\|v_s\| = \|v_{s^{-1}}\|$ *for* $s \in S$,
4. $v_s = v_{s'}$ *if* $s^{-1}s' \in H$,
5. $\sum_{s \in S} v_s \deg_H(s) = 0$.

Theorem 7. *If the set $P_H(S)$ can consist only of zero vectors then Γ has property* (T).

The proof of Theorem 7 is similar to the proof of Theorem 2.

In the case of a very symmetric set of generators S as above, we can consider a modified version of the graph $L(S)$ which will be denoted $L(S,H)$.

We consider the situation when Γ is a group generated by a finite, symmetric subset S, such that $e \notin S$. Let $H \subset \Gamma$ be a finite subgroup such that $hSh^{-1} = S$ for every $h \in H$ and $S \cap H = \emptyset$.

Definition 4. *We define a finite graph $L(S,H)$ as follows:*

1. *vertices of* $L(S,H) = \{s; s \in S\}$,
2. *edges of* $L(S,H) = \{(s,s'); s,s' \in S$ *and* $s^{-1}s' \in SH\}$.

As a corollary of Theorem 7 we get

Theorem 8. *Let Γ be a group generated by a finite symmetric subset S, such that $e \notin S$. Let $H \subset \Gamma$ be a finite subgroup such that $hSh^{-1} = S$ for every $h \in H$ and $S \cap H = \emptyset$. If the graph $L(S,H)$ is connected and*

$$\lambda_1\big(L(S,H)\big) > \tfrac{1}{2}$$

then Γ has property (T).

5 Examples

5.1 $\widetilde{A}_2$-groups. In [CMS], the family of groups acting co-compactly on buildings of type $\widetilde{A}_2$ was constructed. These groups are parameterized by an integer q which is a power of a prime number. They admit a presentation such that $L(S)$ is the incidence graph of the projective plane $\mathbb{P}^2(\mathbb{F}_q)$ over the finite field $\mathbb{F}_q$, i.e.

$$\text{vertices of } L(S) = \big\{\text{points } p \text{ and lines } l \text{ such that } p, l \in \mathbb{P}^2(\mathbb{F}_q)\big\},$$
$$\text{edges of } L(S) = \big\{(p,l); p \in l\big\}.$$

5.1.1 Kazhdan constants. In [CMS], the best possible Kazhdan constants for these groups were computed:

$$\text{Kazhdan constant } = \sqrt{2\varepsilon_q}\,,$$

where

$$\varepsilon_q = 1 - \frac{q}{q^2+q+1}\left(\left(\sqrt{q} + \sqrt{q^{-1}}\right) + 1\right).$$

5.1.2 Computations by Feit and Higman. In [FH], Feit and Higman computed the spectrum of the Laplace operator on graphs which are incidence graphs of finite projective planes.

PROPOSITION 4 (Feit, Higman). *Let L be the incidence graph of $\mathbb{P}^2(\mathbb{F}_q)$. Then*

$$\lambda_1(L) = 1 - \frac{\sqrt{q}}{q+1}\,.$$

Proof. Let f be an eigenfunction of Δ acting on $l^2(L,\deg)$ corresponding to an eigenvalue λ. Let v be a vertex of L such that $f(v) \neq 0$. Because of the duality between the points and the lines in $\mathbb{P}^2(\mathbb{F}_q)$ we can suppose that v corresponds to a point p in $\mathbb{P}^2(\mathbb{F}_q)$. We can also suppose that $f(v) = 1$. The group of automorphisms of $\mathbb{P}^2(\mathbb{F}_q)$ fixing the point p acts transitively on the lines containing p, the points different from p and finally on the lines which do not contain p. Thus there are a, b, $c \in \mathbb{R}$ such that in the space of the eigenfunctions of Δ on $l^2(L,\deg)$ with the eigenvalue λ there is a function $\overline{f}$ such that $\overline{f}(v) = 1$, and which is equal to a, b and c on the vertices corresponding respectively to the lines containing p, to the points different from p and to the lines which do not contain p. As $\overline{f}$ is an eigenfunction of Δ with the eigenvalue λ we have the following relations:

$$\begin{aligned}
&\lambda = 1 - a\,,\\
&\lambda a = a - \frac{1}{q+1}(qb+1)\,,\\
&\lambda b = b - \frac{1}{q+1}(qc+b)\,,\\
&\lambda c = c - b\,.
\end{aligned}$$

This implies that $\lambda \in \left\{0, 1-\frac{\sqrt{q}}{q+1}, 1+\frac{\sqrt{q}}{q+1}, 2\right\}$. □

By Theorem 1 this gives the following estimation for Kazhdan constants:

$$\frac{2}{\sqrt{3}}\left(2 - \frac{1}{\lambda_1(L)}\right) = \frac{2}{\sqrt{3}}\left(2 - \frac{q+1}{(q+1-\sqrt{q})}\right).$$

6 Generic Hyperbolic Groups

In this section we prove a result concerning genericity of hyperbolic groups.

One of the possible definitions of the hyperbolicity is that all Dehn diagrams satisfy a linear isoperimetric inequality (for the relevant definitions and statements see [A et al.], [GhH], [Gr1] and [LyS]). But in order to prove that a given group is hyperbolic it is enough to prove the linear isoperimetric inequality for a finite number of Dehn diagrams (see [Gr1] and [Pap]).

Let us be more precise. We consider a group Γ given by a finite presentation $\langle S;R\rangle$ where $S=\{s_1,\dots,s_s\}$, $R=\{r_1,\dots,r_r\}$. A Dehn diagram D over the group Γ is a finite, planar, connected and simply connected 2-complex such that every 2-cell is labeled with a cyclic permutation of some relation $r_i^{\pm1}\in R$. A word w in the alphabet S represents the identity in Γ if and only if there is a Dehn diagram D over Γ such that w is the boundary of D. The area of D is the number of faces of D and the area of the word w representing the trivial element is the minimal area of the Dehn diagram with the boundary w. The area $A(w)$ of the word w is also equal to the smallest n such that $w=\prod_{i=1}^n u_i r_i^{\pm1} u_i^{-1}$ where u_i are words in the free group $F(S)$ on S, $r_i\in R$ and the equality is in $F(S)$. The group Γ is hyperbolic if there exists a positive constant c such that for every word $w\in F(S)$ which represents the trivial element in Γ we have $A(w)\le c|w|$. But in fact it is enough to check this inequality for finitely many $w\in F(S)$. More precisely we have:

PROPOSITION 5 (Gromov [Gr1], Papasoglu [Pap]). *Let $\langle S;R\rangle$ be a triangular presentation of the group Γ. Assume that for some integer $K>0$ every diagram A with the area $K^2/2\le|A|\le 240K^2$ satisfies $|A|\le\frac{1}{20000}|\partial A|^2$, where $|\partial A|$ is the length of the boundary ∂A with respect to S. Then every diagram A over Γ satisfies $A\le K^2|\partial A|$.*

Proof. For a fixed $d<\frac12$ let $K=\frac{20000}{25(\frac12-d)}$, $\varepsilon=\frac12(\frac12-d)$ and $C=240K^2$.

Let us estimate the number of presentations of density d with m generators, for which there exists a Dehn diagram D of a given combinatorial type which consists of c cells, where $c\le C$. By a given combinatorial type we mean that we prescribe which cells have common edges. Suppose that the lengths of words over which the cells meet sum up to L.

Suppose that in the given diagram there are n_1 relators r_1, n_2 relators $r_2,\dots,n_k$ relators r_k and that $n_1\ge n_2\ge\dots\ge n_k$. Thus $n_1+\dots+n_k=c$.

First we put in the diagram n_1 relators r_1. If they have some edges in common, denote by l_1 the length of the longest common sequence, i.e. $0\le l_1\le 3$. Then let us put in the diagram n_2 relators r_2. And let l_2 denote the longest sequence that the relation r_2 has in common with the relations which are in the diagram so far, i.e. the relators r_1 and r_2. We continue the process and in the same manner we define $l_3,\dots,l_k$. In particular

$$L\le n_1l_1+\dots+n_kl_k\,.$$

If we consider the first i relators $r_1,\dots,r_i$ the number of choices of such i relators is equal to

$$(2m-1)^{3i-(l_1+\dots+l_i)}$$

As we consider the presentations with $(2m-1)^{3d}$ relations for the other relations we have
$$\big((2m-1)^3\big)^{(2m-1)^{3d}-i}$$
choices.

Our i relations do not have to be the first ones so we have to include the number of possible permutations of these i relations among all $(2m-1)^{3d}$ relations, which can be bounded by
$$\big((2m-1)^{3d}\big)^i.$$
Thus the number of presentations which give such a diagram can be bounded by
$$(2m-1)^{3di}\big((2m-1)^3\big)^{(2m-1)^{3d}-i}(2m-1)^{3i-(l_1+\ldots+l_i)}$$
$$=(2m-1)^{3di-(l_1+\ldots+l_i)+3(2m-1)^{3d}}.$$
If we divide it by the number of all presentations we get
$$(2m-1)^{3di-(l_1+\ldots+l_i)}$$
which tends to zero when m tends to infinity if $3di-(l_1+\ldots+l_i)<-\varepsilon/C$.

In our considerations the number of diagrams we consider is finite. Thus when m tends to infinity for most presentations, for the diagrams we consider, we have for every $1\le i\le k$
$$3di-(l_1+\ldots+l_i)\ge -\tfrac{\varepsilon}{C}\,.$$
If we sum the above inequalities with the i-th inequality multiplied by the positive coefficient n_i-n_{i+1} for $i=1,\ldots,k-1$ and the coefficient n_k for $i=k$ we get
$$3dc-(n_1l_1+\ldots+n_kl_k)\ge -\varepsilon.$$
This implies that most presentations will give rise to diagrams such that
$$3dc-L\ge -\varepsilon\,.$$

By the above, when m tends to infinity for almost all presentations with relations of length 3 and density d, all diagrams D with at most $C=240K^2$ cells satisfy
$$3dc-L>-\varepsilon=\tfrac12\left(d-\tfrac12\right).$$
But
$$3c-2L=|\partial D|$$
where $|\partial D|$ is the sum of the lengths of edges in the boundary ∂D of D. As $c=|D|$ this implies
$$|\partial D|=3c-2L>3|D|-2\varepsilon-6|D|d=6|D|\left(\tfrac12-d\right)-\left(\tfrac12-d\right)\ge 5|D|\left(\tfrac12-d\right)$$
which gives for $|D|\ge K^2/2$
$$\frac{1}{20000}|\partial D|^2\ge\frac{1}{20000}25|D|^2\left(\frac12-d\right)\ge |D|\frac{1}{20000}25|K|\left(\frac12-d\right)=|D|.$$

By Proposition 5 this implies that Γ is hyperbolic and infinite because we considered all and not only minimal diagrams. $\square$

7 Property (T) and Generic Presentations

In this section we show that a simple combinatorial condition which implies property (T) (see Proposition 6) is satisfied for generic groups for $d > 1/3$. We start by analyzing the situation for $d < 1/3$.

Proof of Theorem 5. The number of presentations without two last generators from m generators is equal to

$$\big((2m-5)^3\big)^{m^{3d}}.$$

The probability that we will get such a presentation is equal to

$$\frac{((2m-5)^3)^{m^{3d}}}{((2m-1)^3)^{3d}} \geq \left(\frac{2m-5}{2m-1}\right)^{3m^{3d}}$$

and the last term tends to 1 when m tends to infinity as $3d < 1$. $\square$

The rest of this section concerns the proof of Theorem 4.

7.1 Property (T) and presentations. Let Γ be a group given by a presentation

$$\Gamma = \big\langle s_1, \dots, s_k; R_1, \dots, R_n, R'_1, R'_2, \dots \big\rangle \tag{9}$$

where $s_1, \dots, s_k$ are generators, the relations $R_1, \dots, R_n$ are words with $s_1, \dots, s_k, s_1^{-1}, \dots, s_k^{-1}$ of length 3 and the relations $R'_1, R'_2, \dots$ are arbitrary words with $s_1, \dots, s_k, s_1^{-1}, \dots, s_k^{-1}$ and their number does not have to be finite. We define the graph $L'(S)$ as follows. The vertices of the graph $L'(S)$ are generators $s_1, \dots, s_k$ and their inverses $s_1^{-1}, \dots, s_k^{-1}$. For every relation $R \in \{R_1, \dots, R_n\}$, say $R = s_x s_y s_z$, we add to the graph the edges (s_x^{-1}, s_y), (s_y^{-1}, s_z) and (s_z^{-1}, s_x). Thus in the graph $L'(S)$ we can have multiple edges. The definition of the graph $L'(S)$ is slightly different from the definition of the graph $L(S)$. Still we can prove an analogue of Theorem 1.

Proposition 6. *Let Γ be a group given by a presentation* (9). *If the graph $L'(S)$ is connected and*

$$\lambda_1(L'(S)) > \tfrac{1}{2}$$

then Γ has property (T).

Proof. For a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H}_\pi)$ of Γ let us consider a self-adjoint operator $M : \mathcal{H}_\pi \to \mathcal{H}_\pi$ defined as follows

$$Mv = \frac{1}{DEG(S)}\big(\pi(s_1)v\cdot\deg(s_1)+\ldots+\pi(s_k)v\cdot\deg(s_k)$$
$$+\pi(s_1^{-1})v\cdot\deg(s_1^{-1})+\ldots\pi(s_k^{-1})v\cdot\deg(s_k^{-1})\big)$$

where $DEG(S)=\deg(s_1)+\ldots+\deg(s_k)+\deg(s_1^{-1})+\ldots+\deg(s_k^{-1})$. Let us suppose that the representation π almost has invariant vectors but does not have any non-zero invariant vector.

This means that for any $\varepsilon>0$ there exists a positive λ such that $1>\lambda>(1-\varepsilon)$ and there exist vectors u and $u_\lambda\in\mathcal{H}_\pi$ such that

$$Mu=\lambda u+u_\lambda$$

where $\|u_\lambda\|<\frac{1}{2}(1-\lambda)\,\|u\|$. On the vertices of the graph $L'(S)$ let us define a function $f:L'(S)\to\mathcal{H}_\pi$ as follows:

$$f(s_i^{\pm1})=\pi(s_i^{\pm1})u-u$$

for every generator $s_i^{\pm1}$. Then we have

$$\langle\Delta f,f\rangle_{l^2(L'(S),\deg)}=\sum_{(s,s')\in\text{ oriented edges in }L'(S)}\big\langle f(s')-f(s),f(s')-f(s)\big\rangle$$
$$=\sum_{i=1;R_i=s_{x_i}s_{y_i}s_{z_i}}^{n}\big(\langle f(s_{x_i}^{-1})-f(s_{y_i}),f(s_{x_i}^{-1})-f(s_{y_i})\rangle$$
$$+\langle f(s_{y_i}^{-1})-f(s_{z_i}),f(s_{y_i}^{-1})-f(s_{z_i})\rangle$$
$$+\langle f(s_{z_i}^{-1})-f(s_{x_i}),f(s_{z_i}^{-1})-f(s_{x_i})\rangle\big)$$
$$=\sum_{i=1;R_i=s_{x_i}s_{y_i}s_{z_i}}^{n}\big(\langle\pi(s_{x_i}^{-1})u-u-(\pi(s_{y_i})u-u),\pi(s_{x_i}^{-1})u-u-(\pi(s_{y_i})u-u)\rangle$$
$$+\langle\pi(s_{y_i}^{-1})u-u-(\pi(s_{z_i})u-u),\pi(s_{y_i}^{-1})u-u-(\pi(s_{z_i})u-u)\rangle$$
$$+\langle\pi(s_{z_i}^{-1})u-u-(\pi(s_{x_i})u-u),\pi(s_{z_i}^{-1})u-u-(\pi(s_{x_i})u-u)\rangle\big)$$
$$=\sum_{i=1;R_i=s_{x_i}s_{y_i}s_{z_i}}^{n}\big(\langle\pi(s_{y_i}^{-1}s_{x_i}^{-1})u-u,\pi(s_{y_i}^{-1}s_{x_i}^{-1})u-u\rangle$$
$$+\langle\pi(s_{z_i}^{-1}s_{y_i}^{-1})u-u,\pi(s_{z_i}^{-1}s_{y_i}^{-1})u-u\rangle$$
$$+\langle\pi(s_{x_i}^{-1}s_{z_i}^{-1})u-u,\pi(s_{x_i}^{-1}s_{z_i}^{-1})u-u\rangle\big)$$
$$=\sum_{i=1;R_i=s_{x_i}s_{y_i}s_{z_i}}^{n}\big(\langle\pi(s_{z_i})u-u,\pi(s_{z_i})u-u\rangle$$
$$+\langle\pi(s_{x_i})u-u,\pi(s_{x_i})u-u\rangle+\langle\pi(s_{y_i})u-u,\pi(s_{y_i})u-u\rangle\big)$$

$$
\begin{aligned}
&= \sum_{i=1;R_i=s_xs_ys_z}^{n} \big(\langle f(s_{z_i}), f(s_{z_i})\rangle + \langle f(s_{x_i}), f(s_{x_i})\rangle + \langle f(s_{y_i}), f(s_{y_i})\rangle\big) \\
&= \tfrac{1}{2} \sum_{i=1;R_i=s_xs_ys_z}^{n} \big(\langle f(s_{z_i}), f(s_{z_i})\rangle + \langle f(s_{z_i}^{-1}), f(s_{z_i}^{-1})\rangle \\
&\quad + \langle f(s_{x_i}), f(s_{x_i})\rangle + \langle f(s_{x_i}^{-1}), f(s_{x_i}^{-1})\rangle \\
&\qquad + \langle f(s_{y_i}), f(s_{y_i})\rangle + \langle f(s_{y_i}^{-1}), f(s_{y_i}^{-1})\rangle\big) = \tfrac{1}{2}\langle f, f\rangle_{l^2(L'(S),\deg)}\,.
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
\big|\langle f, 1\rangle_{l^2(L'(S),\deg)}\big| &= \Big| \sum_{s=s_1,\dots,s_k,s_1^{-1},\dots,s_k^{-1}} \langle \pi(s)u - u, 1\rangle_{\mathcal{H}_\pi} \deg(s)\Big| \\
&= \big|\langle DEG(S)(Mu - u), 1\rangle_{\mathcal{H}_\pi}\big| \\
&= DEG(S)\big|\langle (\lambda-1)u + u_\lambda, 1\rangle_{\mathcal{H}_\pi}\big| \\
&\le DEG(S)\big(|\lambda - 1|\|u\| + \|u_\lambda\|\big) \\
&\le DEG(S)2|\lambda-1|\|u\|\,.
\end{aligned}
$$

This implies

$$
\begin{aligned}
\|f\|_{l^2(L'(S),\deg)} &= \sqrt{\sum_{s=s_1,\dots,s_k,s_1^{-1},\dots,s_k^{-1}} \langle f(s), f(s)\rangle \deg(s)} \\
&= \sqrt{\sum_{s=s_1,\dots,s_k,s_1^{-1},\dots,s_k^{-1}} \langle \pi(s)u - u, \pi(s)u - u\rangle \deg(s)} \\
&= \sqrt{\sum_{s=s_1,\dots,s_k,s_1^{-1},\dots,s_k^{-1}} 2\langle u, u\rangle \deg(s) - 2\big\langle u, \pi(s)u\big\rangle \deg(s)} \\
&= \sqrt{2\langle u, u - Mu\rangle DEG(S)} \\
&= \sqrt{2\big\langle u, (1-\lambda)u + u_\lambda\big\rangle DEG(S)} \\
&= \sqrt{2(1-\lambda)\langle u, u\rangle DEG(S) + 2\langle u, u_\lambda\rangle DEG(S)} \\
&\ge \sqrt{2(1-\lambda)\langle u, u\rangle DEG(S) - 2\tfrac{1}{2}(1-\lambda)\langle u, u\rangle DEG(S)} \\
&= \sqrt{1-\lambda}\sqrt{DEG(S)}\|u\| \\
&\ge \sqrt{1-\lambda}\sqrt{DEG(S)}\frac{|\langle f, 1\rangle|}{DEG(S)2(1-\lambda)} \\
&= \langle f, 1\rangle \frac{1}{2\sqrt{\lambda-1}\sqrt{DEG(S)}}\,.
\end{aligned}
$$

Because

$$\left\langle f - \frac{\langle f,1\rangle}{DEG(S)}, 1 \right\rangle = 0\,,$$

by definition of λ_1 we have

$$\begin{aligned}
\lambda_1(L'(S)) &\le \frac{\langle \Delta f, f\rangle}{\langle f - \frac{\langle f,1\rangle}{DEG(S)}, f - \frac{\langle f,1\rangle}{DEG(S)}\rangle} \\
&\le \frac{\langle \Delta f, f\rangle}{\big(\|f\| - \|\frac{\langle f,1\rangle}{DEG(S)}\|\big)^2} = \frac{\langle \Delta f, f\rangle}{(\|f\| - |\langle f,1\rangle|)^2} \\
&\le \frac{\frac{1}{2}\|f\|^2}{(\|f\| - 2\sqrt{(1-\lambda)DEG(S)}\|f\|)^2} \le \frac{1}{2}\frac{1}{1-2\sqrt{\varepsilon DEG(S)}}\,.
\end{aligned}$$

As ε can be arbitrary small this implies that

$$\lambda_1(L'(S)) \le \tfrac{1}{2}.$$

This gives us a desired contradiction and finishes the proof of Proposition 6. □

7.2 Random graphs. Let $L(n,k)$ be the set of finite graphs of degree $k = 2d$ with n vertices. We consider the following model for $L(n,k)$ [Bo2], [Fr]. We take d permutations $\pi_1, \dots, \pi_d$ on n letters. These permutations give rise to the graph with vertices $V = \{1, \dots, n\}$ and the unoriented edges $E = \{(i, \pi_j(i))\}$ for $i = 1, \dots, n$, $j = 1, \dots, d$. Two graphs are considered in this model to be different if the corresponding permutations are different.

In [Fr], it was proven that

Proposition 7 [Fr]. *There exists a positive constant c, independent of k, such that*

$$\lim_{n\to+\infty} \frac{\#\big\{L \in L(n,k);\ \lambda_1(L) \ge 1 - \big(\frac{\sqrt{2k-1}}{k} + \frac{\log(k)}{k} + \frac{c}{k}\big)\big\}}{\#L(n,k)} = 1\,. \tag{10}$$

7.3 Random groups. We consider the following model $\mathcal{F}$ for random groups. Let v be a natural number and let $\{\pi_1^1, \pi_2^1\}, \dots, \{\pi_1^v, \pi_2^v\}$ be v couples of permutations on $2m$ letters $s_1, \dots, s_m, s_1^{-1}, \dots, s_m^{-1}$. These couples of permutations give rise to the following $2mv$ relations for $i = 1, \dots, v$, $j = 1, \dots, m$

$$s_j^{\pm 1}\pi_1^i(s_j^{\pm 1})\pi_2^i(s_j^{\pm 1})\,.$$

Let $P_{\mathcal{F}}(m,v)$ be the set of presentations with m generators $s_1, \dots, s_m$ and the above relations given by v couples of permutations. Two presentations are different if the permutations are different.

Theorem 9. *For any fixed $v \geq 1$ one has*

$$\lim_{m\to+\infty} \frac{\#\{P \in P_{\mathcal{F}}(m,v); \Gamma(P) \text{ is infinite and hyperbolic}\}}{\#P_{\mathcal{F}}(m,v)} = 1\,.$$

Theorem 9 can be proved in the same way as Theorem 3.

Theorem 10. *For a fixed v which is sufficiently large one has*

$$\lim_{m\to+\infty} \frac{\#\{P \in P_{\mathcal{F}}(m,v); \Gamma(P) \text{ has property (T)}\}}{\#P_{\mathcal{F}}(m,v)} = 1\,.$$

The density of relations in the model $\mathcal{F}$ is 1/3. In particular for $d > 1/3$ the groups in the model $\mathcal{M}$ are quotients of groups in the model $\mathcal{F}$ and Theorem 4 follows from Theorem 10.

Proof of Theorem 10. Let us consider a presentation $P \in P_{\mathcal{F}}(m,v)$. Let Γ be a group defined by P. The graph L' associated to P has $2m$ vertices. The relation $s_j^{\pm1}\pi_1^i(s_j^{\pm1})\pi_2^i(s_j^{\pm1})$ corresponds to the edges $((s_j^{\pm1})^{-1}, \pi_1^i(s_j^{\pm1}))$, $((\pi_1^i(s_j^{\pm1}))^{-1}, \pi_2^i(s_j^{\pm1}))$ and $((\pi_2^i(s_j^{\pm1}))^{-1}, s_j^{\pm1})$ in L'. Let us denote by L_1', L_2' and L_3' the graphs with the same vertices as L' and such that for a relation $s_j^{\pm1}\pi_1^i(s_j^{\pm1})\pi_2^i(s_j^{\pm1})$ we put the edge $((s_j^{\pm1})^{-1}, \pi_1^i(s_j^{\pm1}))$ in L_1', the edge $((\pi_1^i(s_j^{\pm1}))^{-1}, \pi_2^i(s_j^{\pm1}))$ in L_2' and the edge $((\pi_2^i(s_j^{\pm1}))^{-1}, s_j^{\pm1})$ in the graph L_3'. The graph L' has degree $6v$ and the graphs L_1', L_2' and L_3' have degree $2v$. According to Proposition 6 in order to show that Γ has property (T) we need to show that $\lambda_1(L') > 1/2$.

Lemma 6. *If for $i = 1,2,3$ $\lambda_1(L_i') > 1/2$ then*

$$\lambda_1(L') > \tfrac{1}{2}\,.$$

Proof. Suppose that $\lambda_1(L') \leq 1/2$. Then there exists $f \in l_0^2(L', \deg)$ such that $\langle \Delta f, f\rangle \leq \frac{1}{2}\langle f, f\rangle$. Let f_i denote the restriction of the function f to the graph L_i'. Because the graphs L' and L_i' are regular, we have $f_i \in l_0^2(L_i', \deg)$. Furthermore

$$\begin{aligned}\sum_{i=1,2,3} \langle \Delta f_i, f_i\rangle &= \sum_{i=1,2,3} \langle df_i, df_i\rangle = \langle df, df\rangle = \langle \Delta f, f\rangle \\ &\leq \tfrac{1}{2}\langle f, f\rangle = \sum_{i=1,2,3} \tfrac{1}{2}\langle f_i, f_i\rangle\,,\end{aligned}$$

where for an edge (s, s'), $df(s,s') = f(s) - f(s')$. Thus for some i we have $\langle \Delta f_i, f_i\rangle \leq \frac{1}{2}\langle f_i, f_i\rangle$. By the definition of λ_1 this implies $\lambda_1(L_i') \leq \frac{1}{2}$ and gives a desired contradiction. □

The graphs L_1', L_2' and L_3' were obtained in a similar manner. Thus if we show that for most presentations $\lambda_1(L_1') > 1/2$ the same is true for L_2' and L_3'.

The construction of the graph L_1' corresponds exactly to the model of random graphs $L(n,k)$ described in section 7.2. For most of these graphs by Proposition 7 we have

$$\lambda_1 \geq 1 - \left(\frac{\sqrt{4v-1}}{2v} + \frac{\log(2v)}{2v} + \frac{c}{2v} \right)$$

where c is a constant independent of v. Thus for v sufficiently large $\lambda_1 > 1/2$, which ends the proof of Theorem 10. $\square$

8 Stability of Property (T)

In this section we prove a result concerning stability of property (T).

Proof of Theorem 6. We have to consider the case when Γ satisfies condition (2) of Theorem 1. We can suppose that the presentation of Γ consists of relations of length three. Removing one relation, for instance the relation $s_1s_2s_3 = e$, corresponds to removing the edges (s_1^{-1}, s_2), (s_2^{-1}, s_3) and (s_3^{-1}, s_1) from $L(S)$. Let us see how this can change $\lambda_1(L(S))$. Suppose that in the graph $(L(S), \deg)$ we removed an edge (s_a, s_b) and we obtained the graph $(L'(S), \deg')$. Let us compare $\lambda_1(L'(S))$ to $\lambda_1(L(S))$.

For $f \in l^2(L(S), \deg)$ let $\mathcal{E}_{L(S)}(f)$ be its Dirichlet form and let $\mathrm{var}_{L(S)}$ be its variation, i.e.

$$\mathcal{E}_{L(S)}(f) = \langle \Delta f, f \rangle_{l^2(L(S),\deg)} = \sum_{s,s' \in L(S), s \sim s'} \left| f(s) - f(s') \right|^2,$$

$$\mathrm{var}_{L(S)}(f) = \min_c \|f - c\|^2_{l^2(L(S),\deg)} .$$

It is not difficult to see that the minimum in the definition of $\mathrm{var}_{L(S)}(f)$ is attained when c is equal to the mean value of f, i.e.

$$c = \frac{\langle f, 1 \rangle_{l^2(L(S),\deg)}}{\langle 1, 1 \rangle_{l^2(L(S),\deg)}} .$$

So $\lambda_1(L(S))$ can be defined as

$$\lambda_1(L(S)) = \sup\big\{ r \in \mathbb{R}; r\, \mathrm{var}_{L(S)}(f) \leq \mathcal{E}_{L(S)}(f) \ \text{ for any } f \in l^2(L(S), \deg) \big\}.$$

Thus

$$\begin{aligned}
\mathrm{var}_{L'(S)}(f) &= \min_c \|f - c\|^2_{l^2(L'(S),\deg')} \\
&\leq \left\| f - \frac{\langle f, 1 \rangle_{l^2(L(S),\deg)}}{\langle 1, 1 \rangle_{l^2(L(S),\deg)}} \right\|^2_{l^2(L'(S),\deg')} \\
&\leq \max_{s \in L(S)} \left\{ \frac{\deg'(s)}{\deg(s)} \right\} \left\| f - \frac{\langle f, 1 \rangle_{l^2(L(S),\deg)}}{\langle 1, 1 \rangle_{l^2(L(S),\deg)}} \right\|^2_{l^2(L(S),\deg)}
\end{aligned}$$

$$\leq \mathrm{var}_{L(S)}(f)\,.$$

The last inequality follows from the fact that $\deg'(s) \leq \deg(s)$ for any vertex s.

Now

$$\begin{aligned}\mathcal{E}_{L'(S)}(f) &= \mathcal{E}_{L(S)}(f) - \left|f(s_a) - f(s_b)\right|^2 \\ &\geq \lambda_1(L(S))\mathrm{var}_{L(S)}(f) - \left|f(s_a) - f(s_b)\right|^2 \\ &\geq \lambda_1(L(S))\mathrm{var}_{L(S)}(f) - \frac{2}{\min\{\deg(s_a), \deg(s_b)\}}\mathrm{var}_{L(S)}(f)\,.\end{aligned}$$

As $\mathrm{var}_{L(S)}(f) \geq \mathrm{var}_{L'(S)}(f)$ and as $\mathcal{E}_{L(S)}(f)$ is positive, we get

$$\mathcal{E}_{L'(S)}(f) \geq \left(\lambda_1(L(S)) - \frac{2}{\min\{\deg(s_a), \deg(s_b)\}}\right)\mathrm{var}_{L(S)}(f)$$

from which follows

$$\lambda_1(L'(S)) \geq \lambda_1(L(S)) - \frac{2}{\min\{\deg(s_a), \deg(s_b)\}}\,.$$

Let us recall that removing one relation from the presentation of the group Γ corresponds to removing three edges from the graph $L(S)$. Thus from the above inequality it follows that if we remove t relations from the presentation of the group Γ, then for the graph $L''(S)$ associated to this new group we have

$$\lambda_1(L''(S)) > \lambda_1(L(S)) - \frac{6t}{\deg(S) - 3t}\,. \tag{11}$$

In order to show that the group obtained after removing t relations has property (T) if suffices by Theorem 1 to show that $\lambda_1(L''(S)) > 1/2$. By (11) this is true for t such that

$$\lambda_1(L(S)) - \frac{6t}{\deg(S) - 3t} \geq \frac{1}{2}\,.$$

The above inequality is satisfied by any $t \in \mathbb{N}$ such that

$$t \leq \tfrac{1}{11}\left(\lambda_1(L(S)) - \tfrac{1}{2}\right)\deg(S)\,,$$

which ends the proof of Theorem 6. □

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