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GAFA Geometric And Functional Analysis

ON THE SPECTRAL SIDE OF THE ARTHUR TRACE FORMULA

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0 Introduction

Let G be a connected reductive algebraic group defined over $\mathbb{Q}$ and let $G(\mathbb{A})$ be the group of points of G with values in the ring of adèles of $\mathbb{Q}$. Then $G(\mathbb{Q})$ embeds diagonally as a discrete subgroup of $G(\mathbb{A})$. Let $G(\mathbb{A})^1$ be the intersection of the kernels of the maps $x \mapsto |\chi(x)|$, $x \in G(\mathbb{A})$, where χ ranges over the group $X(G)_{\mathbb{Q}}$ of characters of G defined over $\mathbb{Q}$. Then the (noninvariant) trace formula of Arthur is an identity

$$\sum_{\mathfrak{o}\in\mathfrak{O}} J_{\mathfrak{o}}(f) = \sum_{\chi\in\mathfrak{X}} J_{\chi}(f)\,, \quad f \in C_0^\infty(G(\mathbb{A})^1)\,,$$

between distributions on $G(\mathbb{A})^1$. The left-hand side is the *geometric side* and the right-hand side the *spectral side* of the trace formula. The distributions $J_{\mathfrak{o}}$ are parametrized by semisimple conjugacy in $G(\mathbb{Q})$ and are closely related to weighted orbital integrals on $G(\mathbb{A})^1$.

In this paper we are concerned with the spectral side of the trace formula. The distribution J_χ are defined in terms of truncated Eisenstein series. They are parametrized by the set of cuspidal data $\mathfrak{X}$ which consists of the Weyl group orbits of pairs (M_B, r_B), where M_B is the Levi component of a standard parabolic subgroup and r_B is an irreducible cuspidal automorphic representation of $M_B(\mathbb{A})^1$. In [A4], Arthur has derived an explicit formula for the distributions J_χ which expresses them in terms of generalized logarithmic derivatives of intertwining operators. So far, the resulting integral-series is only known to converge conditionally. This suffices, for example, for the comparison of trace formulas which, at present, is the main application of the trace formula. However, with regard to potential applications of the trace formula in spectral theory and geometry it would be highly desirable to know that the spectral side of the trace formula is absolutely convergent. This would also simplify the applications of the trace formula in the theory of automorphic forms [L].

The problem of the absolute convergence of the spectral side of the trace formula is the main issue of the present paper. We will not settle the

problem, but we shall reduce it to a question about local components of automorphic representations.

To describe the results in more detail we have to introduce some notation. We fix a Levi component M_0 of a minimal parabolic subgroup P_0 of G. Let P be a parabolic subgroup of G, defined over $\mathbb{Q}$, with unipotent radical N_P. Let M_P be the unique Levi component of P which contains M_0. We denote the split component of the center of M_P by A_P and its Lie algebra by $\mathfrak{a}_P$. For parabolic groups $P \subset Q$ there is a natural surjective map $\mathfrak{a}_P \to \mathfrak{a}_Q$ whose kernel we will denote by $\mathfrak{a}_P^Q$. Let $\mathcal{A}^2(P)$ be the space of square integrable automorphic forms on $N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})$. Let Q be another parabolic subgroup of G, defined over $\mathbb{Q}$, with Levi component M_Q, split component A_Q and corresponding Lie algebra $\mathfrak{a}_Q$. Let $W(\mathfrak{a}_P, \mathfrak{a}_Q)$ be the set of all linear isomorphisms from $\mathfrak{a}_P$ to $\mathfrak{a}_Q$ which are restrictions of elements of the Weyl group $W(A_0)$. The theory of Eisenstein series associates to each $s \in W(\mathfrak{a}_P, \mathfrak{a}_Q)$ an intertwining operator

$$M_{Q|P}(s,\lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(Q)\,, \quad \lambda \in \mathfrak{a}_{P,\mathbb{C}}^*\,,$$

which, for $\mathrm{Re}(\lambda)$ in a certain chamber, can be defined by an absolutely convergent integral and admits an analytic continuation to a meromorphic function of $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$. Set

$$M_{Q|P}(\lambda) := M_{Q|P}(1,\lambda)\,.$$

Let $\Pi(M_P(\mathbb{A})^1)$ be the set of equivalence classes of irreducible unitary representations of $M_P(\mathbb{A})^1$. Let $\chi \in \mathfrak{X}$ and $\pi \in \Pi(M_P(\mathbb{A})^1)$. Then (χ,π) singles out a certain subspace $\mathcal{A}^2_{\chi,\pi}(P)$ of $\mathcal{A}^2(P)$ (see §1.6). Let $\overline{\mathcal{A}}^2_{\chi,\pi}(P)$ be the Hilbert space completion of $\mathcal{A}^2_{\chi,\pi}(P)$ with respect to the canonical inner product. For each $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$ we have an induced representation $\rho_{\chi,\pi}(P,\lambda)$ of $G(\mathbb{A})$ in $\overline{\mathcal{A}}^2_{\chi,\pi}(P)$.

For each Levi subgroup L let $\mathcal{P}(L)$ be the set of all parabolic subgroups with Levi component L. If P is a parabolic subgroup, let Δ_P denote the set of simple roots of (P, A_P). Let L be a Levi subgroup which contains M_P. Set

$$\mathfrak{M}_L(P,\lambda) = \lim_{\Lambda\to 0}\Big(\sum_{Q_1\in\mathcal{P}(L)} \mathrm{vol}\left(\mathfrak{a}_{Q_1}^G/\mathbb{Z}(\Delta_{Q_1}^\vee)\right) M_{Q|P}(\lambda)^{-1}\frac{M_{Q|P}(\lambda+\Lambda)}{\prod_{\alpha\in\Delta_{Q_1}}\Lambda(\alpha^\vee)}\Big),$$

where λ and Λ are constrained to lie in $i\mathfrak{a}_L^*$, and for each $Q_1 \in \mathcal{P}(L)$, Q is a group in $\mathcal{P}(M_P)$ which is contained in Q_1. Then $\mathfrak{M}_L(P,\lambda)$ is an unbounded operator which acts on the Hilbert space $\overline{\mathcal{A}}^2_{\chi,\pi}(P)$. In the special case that $L = M$ and $\dim \mathfrak{a}_L^G = 1$, the operator $\mathfrak{M}_L(P,\lambda)$ has a simple description. Let P be a parabolic subgroup with Levi component M. Let α be the

unique simple root of (P, A_P) and let $\tilde{\omega}$ be the element in $(\mathfrak{a}_M^G)^*$ such that $\tilde{\omega}(\alpha^\vee) = 1$. Let $\overline{P}$ be the opposite parabolic group of P. Then

$$\mathfrak{M}_L(P, z\tilde{\omega}) = -\operatorname{vol}(\mathfrak{a}_M^G/\mathbb{Z}\alpha^\vee) M_{\overline{P}|P}(z\tilde{\omega})^{-1} \cdot \tfrac{d}{dz} M_{\overline{P}|P}(z\tilde{\omega})\,.$$

Let $f \in C_c^\infty(G(\mathbb{A})^1)$. Then Arthur [A4, Theorem 8.2] proved that $J_\chi(f)$ equals the sum over Levi subgroups M containing M_0, over L containing M, over $\pi \in \Pi(M(\mathbb{A})^1)$, and over $s \in W^L(\mathfrak{a}_M)_{\mathrm{reg}}$, a certain subset of the Weyl group, of the product of

$$|W_0^M||W_0|^{-1} \big|\det(s-1)_{\mathfrak{a}_M^L}\big|^{-1} |\mathcal{P}(M)|^{-1},$$

a factor to which we need not pay too much attention , and of

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \sum_{P \in \mathcal{P}(M)} \operatorname{tr}\big(\mathfrak{M}_L(P,\lambda) M_{P|P}(s,0) \rho_{\chi,\pi}(P,\lambda,f)\big) d\lambda\,. \tag{0.1}$$

So far, it is only known that $\sum_{\chi \in \mathfrak{X}} |J_\chi(f)| < \infty$ and the goal is to show that the integral-sum obtained by summing (0.1) over $\chi \in \mathfrak{X}$ and $\pi \in \Pi(M(\mathbb{A})^1)$ is absolutely convergent with respect to the trace norm.

Given $\pi \in \Pi(M(\mathbb{A}))$ with $\pi = \otimes_v \pi_v$, let $J_{Q|P}(\pi_v, \lambda)$ be the local intertwining operator between the induced representations $I_P^G(\pi_{v,\lambda})$ and $I_Q^G(\pi_{v,\lambda})$. By [CLL, §15] and [A7] there exist normalizing factors $r_{Q|P}(\pi_v, \lambda)$ such that the normalized intertwining operators

$$R_{Q|P}(\pi_v, \lambda) = r_{Q|P}(\pi_v, \lambda)^{-1} J_{Q|P}(\pi_v, \lambda)$$

satisfy the conditions of Theorem 2.1 of [A7].

If $v < \infty$, let $K_v \subset G(\mathbb{Q}_v)$ be an open compact subgroup. Denote by $R_{Q|P}(\pi_v, \lambda)_{K_v}$ the restriction of $R_{Q|P}(\pi_v, \lambda)$ to the subspace $\mathcal{H}_P(\pi_v)^{K_v}$ of K_v-invariant vectors in the Hilbert space $\mathcal{H}_P(\pi_v)$ of the induced representation. If $v = \infty$, let $K_\infty \subset G(\mathbb{R})$ be a maximal compact subgroup. Given $\pi \in \Pi(M(\mathbb{R}))$ and $\sigma \in \Pi(K_\infty)$, let $R_{Q|P}(\pi, \lambda)_\sigma$ be the restriction of $R_{Q|P}(\pi, \lambda)$ to the σ-isotypical subspace of $\mathcal{H}_P(\pi)$. Let λ_π and λ_σ denote the Casimir eigenvalues of π and σ, respectively.

For a given place v, let $\Pi_{\mathrm{disc}}(M(\mathbb{Q}_v))$ be the subset consisting of all representations $\pi_v \in \Pi(M(\mathbb{Q}_v))$ such that there exists an automorphic representation π in the discrete spectrum of $M(\mathbb{A})$ whose local component at the place v is π_v. Finally, let $\mathcal{C}^1(G(\mathbb{A})^1)$ be the space of integrable rapidly decreasing functions on $G(\mathbb{A})^1$ (see §1.3 for its definition). Then our main result is the following theorem, which reduces the problem of the absolute convergence of the spectral side of the Arthur trace formula to a problem about local components of automorphic representations.

Theorem 0.1. *Suppose that for every $M \in \mathcal{L}(M_0)$, $Q, P \in \mathcal{P}(M)$ and every place v the following holds.*

1) *If $v < \infty$, then for every open compact subgroup $K_v \subset G(\mathbb{Q}_v)$ and every invariant differential operator D_λ on $i\mathfrak{a}_M^*$ there exists $C > 0$ such that*
$$\big\|D_\lambda R_{Q|P}(\pi_v, \lambda)_{K_v}\big\| \leq C \tag{0.2}$$
for all $\lambda \in i\mathfrak{a}_M^*$ and all $\pi_v \in \Pi_{\mathrm{disc}}(M(\mathbb{Q}_v))$.
2) *If $v = \infty$, then for all invariant differential operators D_λ on $i\mathfrak{a}_M^*$ there exist $C > 0$ and $N \in \mathbb{N}$ such that*
$$\big\|D_\lambda R_{Q|P}(\pi, \lambda)_\sigma\big\| \leq C\big(1 + \|\lambda\| + |\lambda_\pi| + |\lambda_\sigma|\big)^N \tag{0.3}$$
for all $\lambda \in i\mathfrak{a}_M^*$, $\sigma \in \Pi(K_\infty)$ and $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{R}))$.

Then for every $f \in \mathcal{C}^1(G(\mathbb{A})^1)$, the spectral side of the trace formula is absolutely convergent.

We add some comments about the assumptions 1) and 2). It follows from results of Arthur [A5, p. 51] and [A8, Lemma 2.1] that (0.2) and (0.3) hold uniformly for tempered representations π_v. On the other hand, to establish (0.2), (0.3) or (0.4) is not a problem of pure local harmonic analysis. One cannot expect that these estimations will hold uniformly for all $\pi_v \in \Pi(M(\mathbb{Q}_v))$. Let, for example, $\dim \mathfrak{a}_M/\mathfrak{a}_G = 1$ and suppose that for each $\epsilon > 0$ there exists $\pi_v \in \Pi(M(\mathbb{Q}_v))$ such that the normalized intertwining operator $R_{\overline{P}|P}(\pi_v, \lambda)$ has a pole λ_0 with $|\operatorname{Re}(\lambda_0)| \leq \epsilon$. Then it is certainly not possible to obtain uniform estimates for derivatives of $R_{\overline{P}|P}(\pi_v, \lambda)$ along the imaginary axis. An example where this actually occurs is GL_n.

Especially the uniformity in σ in (0.3) seems to be difficult to achieve. Of course, this condition can be relaxed in various ways. If we relax (0.3) by not requesting uniformity in σ, we get the following weaker version of Theorem 0.1 which suffices for many purposes. Let $K = \prod_v K_v$ be a maximal compact subgroup of $G(\mathbb{A})$ which is admissible relative to M_0 (see §1.2).

Theorem 0.2. *Suppose that in Theorem 0.1 in place of condition 2) the following condition holds:*

2′) *If $v = \infty$, then for all invariant differential operators D_λ on $i\mathfrak{a}_M^*$ and all $\sigma \in \Pi(K_\infty)$ there exist $C > 0$ and $N \in \mathbb{N}$ such that*
$$\big\|D_\lambda R_{Q|P}(\pi, \lambda)_\sigma\big\| \leq C\big(1 + \|\lambda\| + |\lambda_\pi|\big)^N \tag{0.4}$$
for all $\lambda \in i\mathfrak{a}_M^*$ and $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{R}))$.

Then for every K-finite $f \in \mathcal{C}^1(G(\mathbb{A})^1)$, the spectral side of the trace formula is absolutely convergent.

At the moment we don't know how to prove any of the conditions (0.2), (0.3) and (0.4) in general. However, for $G = \mathrm{GL}_n$, considered as an algebraic group over a number field, we are able to prove (0.2) and (0.4). The method relies on work of Luo, Rudnick and Sarnak [LuRS] who established nontrivial bounds towards the generalized Ramanujan conjecture. For GL_n any local component of a cuspidal automorphic representation is equivalent to a full induced representation $I_P^G(\tau, \mathbf{s})$ where τ is tempered and the parameters $\mathbf{s} = (s_1, \dots, s_r)$ satisfy $s_1 > s_2 > \cdots > s_r$ and $|s_i| < 1/2$. If π_v is unramified, it follows from Theorem 2 of [LuRS] that the s_i's satisfy the nontrivial bound

$$|s_i| < \frac{1}{2} - \frac{1}{n^2+1}\,, \quad i = 1, \dots, r\,. \tag{0.5}$$

Using the method of [LuRS], one can show that (0.5) holds also for the ramified components. Furthermore, using the work of Mœglin and Waldspurger [MW] on the residual spectrum, one can show that similar nontrivial bounds exist for the continuous parameters of any local component of an automorphic representation in the discrete spectrum of $\mathrm{GL}_m(\mathbb{A})$. These bounds are the essential ingredients in the proof of (0.2) and (0.4) in the case of GL_n. Details will appear in a forthcoming paper with B. Speh [MüS].

Now we shall explain the main steps of the proof of Theorem 0.1. First observe that $M_{P|P}(s, 0)$ is unitary. Therefore, in order to estimate the trace norm of (0.1), it suffices to estimate the integral

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \big\|\mathfrak{M}_L(P, \lambda)\rho_{\chi,\pi}(P, \lambda, f)\big\| d\lambda\,. \tag{0.6}$$

To deal with this integral, we introduce a certain normalization of intertwining operators. For $\pi \in \Pi(M(\mathbb{A}))$ let $\mathcal{A}_\pi^2(P)$ be the space of square integrable automorphic forms of type π (see §1). Let $M_{Q|P}(\pi, \lambda)$ denote the restriction of the intertwining operator $M_{Q|P}(\lambda)$ to $\mathcal{A}_\pi^2(P)$. Let $\pi = \otimes_v \pi_v$ and let $r_{Q|P}(\pi_v, \lambda)$ be the normalizing factor for the local intertwining operator considered above. Suppose $\pi = \otimes \pi_v$ occurs in the discrete spectrum of $M(\mathbb{A})$, which is equivalent to $\mathcal{A}_\pi^2(P) \neq 0$, then the Euler product

$$r_{Q|P}(\pi, \lambda) = \Pi_v r_{Q|P}(\pi_v, \lambda)$$

converges absolutely in some chamber and $r_{Q|P}(\pi, \lambda)$ admits a meromorphic continuation to $\mathfrak{a}_{M,\mathbb{C}}^*$. Using this meromorphic function, we introduce the normalized global intertwining operator by

$$N_{Q|P}(\pi, \lambda) = r_{Q|P}(\pi, \lambda)^{-1} M_{Q|P}(\pi, \lambda)\,. \tag{0.7}$$

By definition, the operator $N_{Q|P}(\pi,\lambda)$ is equivalent to the direct sum of finitely many copies of $\otimes_v R_{Q|P}(\pi_v,\lambda)$.

Let $\mathfrak{M}_L(P,\pi,\lambda)$ be the restriction of $\mathfrak{M}_L(P,\lambda)$ to the subspace $\mathcal{A}^2_\pi(P)$. It follows from Arthur's theory of (G,M) families [A4, p. 1329] that

$$\mathfrak{M}_L(P,\pi,\lambda)=\sum_S \mathfrak{N}'_S(P,\pi,\lambda)\nu_L^S(P,\pi,\lambda)\,,$$

where the sum runs over all parabolic subgroups S containing L, the operator $\mathfrak{N}'_S(P,\pi,\lambda)$ is built out of normalized intertwining operators on the local groups $G(\mathbb{Q}_v)$ and $\nu_L^S(P,\pi,\lambda)$ is a scalar valued function which is defined in terms of normalizing factors. This reduces the estimation of the integral (0.6) to two separate problems, one involving $\mathfrak{N}'_S(P,\pi,\lambda)$ and the other one $\nu_L^S(P,\pi,\lambda)$.

First we are dealing with $\nu_L^S(P,\pi,\lambda)$. By Proposition 7.5 of [A4], $\nu_L^S(P,\pi,\lambda)$ can be expressed in terms of logarithmic derivatives of normalizing factors associated with maximal parabolic subgroups in certain Levi subgroups. Therefore we may assume that $\dim(\mathfrak{a}_P/\mathfrak{a}_G)=1$. Let α be the unique simple root of (P,A). Then there exists a meromorphic function $\widetilde{r}_{\overline{P}|P}(\pi,z)$ of one variable such that $r_{\overline{P}|P}(\pi,\lambda)=\widetilde{r}_{\overline{P}|P}(\pi,\lambda(\alpha^\vee))$, and our problem is to derive estimates, which are uniform with respect to π, of integrals of the form

$$\int_{\mathbb{R}}\left|\widetilde{r}_{\overline{P}|P}(\pi,iu)^{-1}\tfrac{d}{du}\widetilde{r}_{\overline{P}|P}(\pi,iu)\right|(1+u^2)^{-N}du\,. \tag{0.8}$$

To deal with this integral, we note that $\widetilde{r}_{\overline{P}|P}(\pi,z)$ is a meromorphic function of order $n=16\dim G+2$. This follows from (0.7), since by Theorem 0.1 of [Mü3], the matrix coefficients of $M_{Q|P}(\pi,\lambda)$ are meromorphic functions of order $\le n$ and by Theorem 2.1 of [A7], the normalized local intertwining operators $R_{Q|P}(\pi_v,\lambda)$ are rational functions of $q_v^{-\lambda(\alpha^\vee)}$, if $v<\infty$, and of $\lambda(\alpha^\vee)$, if $v=\infty$. Thus there exist entire functions $r_i(\pi,z)$, $i=1,2$, of order $\le n$ such that $\widetilde{r}_{\overline{P}|P}(\pi,z)=r_1(\pi,z)/r_2(\pi,z)$. Using the representation of $r_i(\pi,z)$ as a Weierstraß product, we reduce the estimation of the integral (0.8) to the the estimation of the number of poles, counted with their order, of $\widetilde{r}_{\overline{P}|P}(\pi,z)$ in a circle of radius $R>0$. By (0.7), this problem is closely related to the estimation of the number of poles, counted with their order, of matrix coefficients of $M_{Q|P}(\pi,\lambda)$ in a circle of radius $R>0$. The latter problem has been settled in [Mü3, Proposition 6.6]. Together with Proposition 7.5 of [A4], these estimates imply estimates for the corresponding integrals involving $\nu_L^S(P,\pi,\lambda)$. In this way we get Theorem 5.4 which is our main technical result.

Next consider $\mathfrak{N}'_S(P,\pi,\lambda)$. Given an open compact subgroup K_f of $G(\mathbb{A}_f)$ and $\sigma \in \Pi(K_\infty)$, let $\mathcal{A}^2_\pi(P)_{K_f,\sigma}$ be the subspace of $\mathcal{A}^2_\pi(P)$ consisting of all automorphic forms which are K_f-invariant and transform under K_∞ according to σ. Let $\mathfrak{N}'_S(P,\pi,\lambda)_{K_f,\sigma}$ be the restriction of $\mathfrak{N}'_S(P,\pi,\lambda)$ to the subspace $\mathcal{A}^2_\pi(P)_{K_f,\sigma}$. Now observe that for any $h \in \mathcal{C}^1(G(\mathbb{A})^1)$ there exists an open compact subgroup K_f of $G(\mathbb{A}_f)$ such that h is left and right invariant under K_f. Then the estimation of $\|\mathfrak{N}'_S(P,\pi,\lambda)\rho_{\chi,\pi}(P,\lambda,h)\|_1$ can be reduced to the estimation of $\|\mathfrak{N}'_S(P,\pi,\lambda)_{K_f,\sigma}\|$ where σ runs over $\Pi(K_\infty)$. By Arthur's theory of (G,M)-families, the estimation of the norm of the finite rank operators $\mathfrak{N}'_S(P,\pi,\lambda)_{K_f,\sigma}$ can be reduced to the estimation of derivatives of finitely many normalized local intertwining operators $R_{Q|P}(\pi_v,\lambda)_{K_v}$, $v<\infty$, and $R_{Q|P}(\pi_\infty,\lambda)_\sigma$. Combined with Theorem 5.4 this implies Theorem 0.1. The proof of Theorem 0.2 is similar.

The paper is organized as follows. In §1 we collect some preliminary facts. In §2 we discuss briefly normalized local and global intertwining operators. The local normalizing factors are studied in some detail in §3. We recall the definition of the normalizing factors and we prove some results that we need in the next section. In §4 we investigate the poles of the global normalizing factors. This section is mainly based on results obtained in [Mü3]. In §5 we establish Theorem 5.4 which is the main result about generalized logarithmic derivatives of global normalizing factors. In §6 we study the absolute convergence of the spectral side of the trace formula and we prove our main results, Theorem 0.1 and Theorem 0.2. In §7 we discuss the example of GL_n and we sketch a method to prove (0.2) and (0.4).

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1 Preliminaries

We shall follow partially the notation introduced by Arthur [A1-4].

1.1 Let G be a reductive algebraic group defined over $\mathbb{Q}$. As in [A4,5], we shall fix a subgroup M_0 of G, defined over $\mathbb{Q}$, which is a Levi component of some minimal parabolic subgroup of G defined over $\mathbb{Q}$. In this paper,

a *parabolic subgroup* will mean a parabolic subgroup of G, defined over $\mathbb{Q}$, and a *Levi subgroup* of G will mean a subgroup of G which contains M_0 and is the Levi component of some parabolic subgroup of G. It is a reductive subgroup of G which is defined over $\mathbb{Q}$. If $M \subset L$ are Levi subgroups, we denote the set of Levi subgroups of L which contain M by $\mathcal{L}^L(M)$. Furthermore, let $\mathcal{F}^L(M)$ denote the set of parabolic subgroups of L defined over $\mathbb{Q}$ which contain M, and let $\mathcal{P}^L(M)$ be the set of groups in $\mathcal{F}^L(M)$ for which M is a Levi component. If $L = G$, we shall denote these sets by $\mathcal{L}(M)$, $\mathcal{F}(M)$ and $\mathcal{P}(M)$, respectively. Suppose that $P \in \mathcal{F}^L(M)$. Then

$$P = N_P M_P \,,$$

where N_P is the unipotent radical of P and M_P is the unique Levi component of P which contains M.

Suppose that $M \subset M_1 \subset L$ are Levi subgroups of G. If $Q \in \mathcal{P}^L(M_1)$ and $R \in \mathcal{P}^{M_1}(M)$, there is a unique group $Q(R) \in \mathcal{P}^L(M)$ which is contained in Q and whose intersection with M_1 is R.

Let A_P be the split component of the center of M_P. A_P is defined over $\mathbb{Q}$. Let $X(M_P)_{\mathbb{Q}}$ be the group of characters of M_P defined over $\mathbb{Q}$. Then

$$\mathfrak{a}_P = \mathrm{Hom}\big(X(M_P)_{\mathbb{Q}}, \mathbb{R}\big)$$

is a real vector space whose dimension equals that of A_P. Its dual space is

$$\mathfrak{a}_P^* = X(M_P)_{\mathbb{Q}} \otimes \mathbb{R} \,.$$

We shall often denote A_P, $\mathfrak{a}_P$ and $\mathfrak{a}_P^*$ by A_M, $\mathfrak{a}_M$ and $\mathfrak{a}_M^*$, respectively, since they depend only on M. Also, we shall write $A_0 = A_{M_0}$, $\mathfrak{a}_0 = \mathfrak{a}_{M_0}$ and $\mathfrak{a}_0^* = \mathfrak{a}_{M_0}^*$.

Let $P \in \mathcal{F}(M_0)$. We shall denote the roots of (P, A_P) by Φ_P, the reduced roots by Σ_P^r, and the simple roots by Δ_P. They are elements in $X(A_P)_{\mathbb{Q}}$ and are canonically embedded in $\mathfrak{a}_P^*$.

Let P and Q be groups in $\mathcal{F}(M_0)$ with $P \subset Q$. Then there is a canonical surjection $\mathfrak{a}_P \to \mathfrak{a}_Q$ and a canonical injection $\mathfrak{a}_Q^* \hookrightarrow \mathfrak{a}_P^*$. The kernel of the first map will be denoted by $\mathfrak{a}_P^Q$. Then $\mathfrak{a}_P^Q$ is dual to $\mathfrak{a}_P^*/\mathfrak{a}_Q^*$.

The group $M_Q \cap P$ is a parabolic subgroup of M_Q with unipotent radical

$$N_P^Q = N_P \cap M_Q \,.$$

Let Δ_P^Q be the set of simple roots of $(M_Q \cap P, A_P)$. Then Δ_P^Q is a subset of Δ_P. We may identify $\mathfrak{a}_Q$ with the subspace

$$\big\{H \in \mathfrak{a}_P \mid \alpha(H) = 0,\ \alpha \in \Delta_P^Q\big\} \,.$$

Furthermore, to Δ_P^Q one can associate a basis $\{\alpha^\vee \mid \alpha \in \Delta_P^Q\}$ of $\mathfrak{a}_P^Q$ and $\hat{\Delta}_P^Q$ is defined to be the corresponding dual basis of $(\mathfrak{a}_P^Q)^*$ [A2]. Then Δ_P^Q

and $\hat{\Delta}_P^Q$ are naturally embedded subsets of $\mathfrak{a}_0^*$. Let

$$\mathfrak{a}_P^+ = \left\{ H \in \mathfrak{a}_P \;\middle|\; \alpha(H) > 0 \quad \text{for all } \alpha \in \Delta_P \right\},$$

and

$$(\mathfrak{a}_P^*)^+ = \left\{ \Lambda \in \mathfrak{a}_P^* \mid \Lambda(\alpha^\vee) > 0 \quad \text{for all } \alpha \in \Delta_P \right\}.$$

We shall denote the restricted Weyl group of (G, A_0) by W_0. It acts on $\mathfrak{a}_0$ and $\mathfrak{a}_0^*$ in the usual way. For every $s \in W_0$ we shall fix a representative w_s in the intersection of $G(\mathbb{Q})$ with the normalizer of A_0. w_s is determined modulo $M_0(\mathbb{Q})$. If P_1 and P_2 are parabolic subgroups, let $W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_2})$ denote the set of distinct isomorphisms from $\mathfrak{a}_{P_1}$ onto $\mathfrak{a}_{P_2}$ obtained by restricting elements of W_0 to $\mathfrak{a}_{P_1}$. P_1 and P_2 are said to be *associated* if $W(\mathfrak{a}_{P_1}, \mathfrak{a}_{P_2})$ is not empty.

1.2 We fix an embedding of G into GL_n, defined over $\mathbb{Q}$. For a given place v of $\mathbb{Q}$, let $G(\mathbb{Q}_v)$ be the group of $\mathbb{Q}_v$-rational points of G. Let $\mathbb{A}$ be the ring of adèles of $\mathbb{Q}$ and let $G(\mathbb{A})$ be the corresponding adèle-valued group. If f stands for the set of finite places of $\mathbb{Q}$ and $\mathbb{A}_f$ is the corresponding ring of finite adèles, then

$$G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f).$$

For any prime p, let

$$G(\mathbb{Z}_p) = \mathrm{GL}_n(\mathbb{Z}_p) \cap G(\mathbb{Q}_p).$$

This is an open compact subgroup of $G(\mathbb{Q}_p)$. We shall fix a maximal compact subgroup

$$K = \prod_v K_v$$

of $G(\mathbb{A})$ which is admissible relative to M_0 in the sense of [A5]. For any such K the following properties hold:

1) For almost all primes p, one has $K_p = G(\mathbb{Z}_p)$.
2) For every finite p, K_p is a special maximal compact subgroup. This implies that $G(\mathbb{Q}_p) = P_0(\mathbb{Q}_p) \cdot K_p$ for all $P_0 \in \mathcal{F}(M_0)$.
3) The Lie algebras of $K_{\mathbb{R}}$ and $A_0(\mathbb{R})$ are orthogonal with respect to the Killing form.

Given $P \in \mathcal{F}(M_0)$, let

$$M_P(\mathbb{A})^1 = \bigcap_{\chi \in X(M_P)_{\mathbb{Q}}} \ker(|\chi|).$$

This is a closed subgroup of $M_P(\mathbb{A})$, and $M_P(\mathbb{A})$ is the direct product of $M_P(\mathbb{A})^1$ and $A_P(\mathbb{R})^0$. By the assumptions on K, $G(\mathbb{A}) = P(\mathbb{A})K$. Therefore, any $x \in G(\mathbb{A})$ can be written as

$$x = namk\,,\ n \in N_P(\mathbb{A})\,,\ m \in M(\mathbb{A})^1\,,\ a \in A_P(\mathbb{R})^0\,,\ k \in K\,. \tag{1.1}$$

Let

$$H_P\colon G(\mathbb{A}) \to \mathfrak{a}_P$$

be the associated height function as defined in [A2]. Then $H_P(x)$ is the image of $a \in A_P(\mathbb{R})^0$ in the decomposition (1.1) with respect to the isomorphism $A_P(\mathbb{R})^0 \cong \mathfrak{a}_P$.

We shall fix a Euclidean norm $\|\cdot\|$ on $\mathfrak{a}_0$ which is invariant under the action of the Weyl group of (G, A_0). On each space $\mathfrak{a}_P^Q$, $P \subset Q$, we take as Haar measure the Euclidean measure associated to the restriction of $\|\cdot\|$ to $\mathfrak{a}_P^Q$. We then normalize the Haar measures on K, $G(\mathbb{A})$, $N_P(\mathbb{A})$, $M_P(\mathbb{A})$, $A_P(\mathbb{R})^0$, $M_P(\mathbb{A})^1$, etc. as in [A2].

1.3 Let Ξ and σ be the functions that enter the definition of Harish-Chandra's Schwartz space on $G(\mathbb{R})$ [W2, p. 156] and extend them to functions on $G(\mathbb{A})$ in the obvious way. For any place v, let $G(\mathbb{Q}_v)^1$ denote the intersection of $G(\mathbb{Q}_v)$ with $G(\mathbb{A})^1$. Let $\mathcal{U}(\mathfrak{g}(\mathbb{R})^1 \otimes \mathbb{C})$ be the universal enveloping algebra of the complexification of the Lie algebra of $G(\mathbb{R})^1$. Let K_f be an open compact subgroup of $G(\mathbb{A}_f)^1$. Then the double coset space $K_f\backslash G(\mathbb{A})^1/K_f$ is a discrete union of countably many copies of $G(\mathbb{R})^1$. In particular it is a differentiable manifold. Suppose that Ω is a subset of $G(\mathbb{A})^1$ such that $K_f \cdot \Omega \cdot K_f = \Omega$ and $K_f\backslash\Omega/K_f$ is the disjoint union of finitely many copies of $G(\mathbb{R})^1$. Let $\mathcal{C}^1(G(\mathbb{A})^1; \Omega, K_f)$ be the space of all functions $h\colon G(\mathbb{A})^1 \to \mathbb{C}$ satisfying the following conditions:

1. h is bi-invariant under K_f, $\operatorname{supp} h \subset \Omega$, and $h : K_v\backslash\Omega/K_f \to \mathbb{C}$ is a smooth function.
2. For all $D_1, D_2 \in \mathcal{U}(\mathfrak{g}(\mathbb{R})^1 \otimes \mathbb{C})$ and all $r \in \mathbb{N}$, we have
$$\|h\|_{D_1,D_2,r} := \sup_{x \in G(\mathbb{A})^1} \big((1+\sigma(x))^r \Xi^{-2}(x) |D_1 * h * D_2(x)|\big) < \infty\,.$$

Let $\mathcal{C}^1(G(\mathbb{A})^1; \Omega, K_f)$ be equipped with the topology defined by the seminorms $\|\cdot\|_{D_1,D_2,r}$. Let $\mathcal{C}^1(G(\mathbb{A})^1)$ be the topological direct limit over all pairs (Ω, K_f) of the spaces $\mathcal{C}^1(G(\mathbb{A})^1; \Omega, K_f)$.

1.4 Let H be any algebraic group over $\mathbb{Q}$ and let F be a local field. We shall denote by $\Pi(H(\mathbb{A}))$ (resp. $\Pi(H(F))$, $\Pi(K)$, etc.) the set of equivalence classes of irreducible unitary representations of $H(\mathbb{A})$ (resp. $H(F)$, K, etc.).

1.5 Given a unitary character ξ of $A_P(\mathbb{R})^0$, let $L^2(M_P(\mathbb{Q})\backslash M_P(\mathbb{A}))_\xi$ be the space of all measurable functions ϕ on $M_P(\mathbb{Q})\backslash M_P(\mathbb{A})$ such that $\phi(xm) = \xi(x)\phi(m)$ for all $x \in A_P(\mathbb{R})^0$, $m \in M_P(\mathbb{A})$, and ϕ is square integrable on $M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1$. Let $\Pi_{\text{disc}}(M_P(\mathbb{A}))_\xi$ denote the subspace of all $\pi \in \Pi(M_P(\mathbb{A}))$ which are equivalent to a subrepresentation of the

regular representation of $M_P(\mathbb{A})$ on $L^2(M_P(\mathbb{Q})\backslash M_P(\mathbb{A}))_\xi$. Set

$$\Pi_{\mathrm{disc}}(M_P(\mathbb{A})) = \bigcup_{\xi\in\Pi(A_P(\mathbb{R})^0)} \Pi_{\mathrm{disc}}(M_P(\mathbb{A}))_\xi\,.$$

Recall that $\Pi(M_P(\mathbb{A})^1)$ can be canonically identified with the set of orbits under the action of $i\mathfrak{a}_P^*$ defined by

$$\pi \to \pi_\lambda = e^{\lambda(H_P(\cdot))}\pi\,, \quad \pi\in\Pi(M_P(\mathbb{A}))\,, \quad \lambda\in i\mathfrak{a}_P^*\,.$$

Since $M_P(\mathbb{A})$ is the direct product of $M_P(\mathbb{A})^1$ and $A_P(\mathbb{R})^0$, any representation of $M_P(\mathbb{A})^1$ corresponds to a representation of $M_P(\mathbb{A})$ which is trivial on $A_P(\mathbb{R})^0$. We identify these two representations and in this way we obtain an embedding of $\Pi(M_P(\mathbb{A})^1)$ in $\Pi(M_P(\mathbb{A}))$.

Given $\pi\in\Pi(M_P(\mathbb{A}))$ with $\pi=\otimes_v\pi_v$, set $\pi_f=\otimes_{v<\infty}\pi_v$. For an open compact subgroup $K_f\subset G(\mathbb{A}_f)$, let

$$K_{M,f} = M_P(\mathbb{A}_f)\cap K_f\,.$$

Set

$$\Pi_{\mathrm{disc}}\big(M_P(\mathbb{A});K_f\big) = \big\{\pi\in\Pi_{\mathrm{disc}}(M_P(\mathbb{A}))\;\big|\;\pi_f^{K_{M,f}}\neq\{0\}\big\}\,. \tag{1.2}$$

Let $\Pi_{\mathrm{disc}}(M_P(\mathbb{A})^1;K_f)$ be the intersection of $\Pi_{\mathrm{disc}}(M_P(\mathbb{A});K_f)$ with the subspace $\Pi_{\mathrm{disc}}(M_P(\mathbb{A})^1)$ of $\Pi_{\mathrm{disc}}(M_P(\mathbb{A}))$.

1.6 Let $P=NM$ be a parabolic subgroup and let ϕ be a measurable, locally integrable function on $N(\mathbb{Q})\backslash G(\mathbb{A})$. Then the constant term ϕ_P of ϕ along P is defined for almost every g by

$$\phi_P(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})}\phi(ng)dn\,. \tag{1.3}$$

This is a measurable, locally integrable function on $N(\mathbb{A})\backslash G(\mathbb{A})$.

1.7 Let P be a parabolic subgroup. Then we denote by $\mathcal{A}^2(P)$ the space of automorphic forms on $N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})$ which are square integrable on $M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1\times K$. This is the space of smooth functions

$$\phi\colon N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A})\to\mathbb{C}$$

which satisfy the following conditions:

i) The span of the set of functions

$$x\mapsto(z\phi)(xk)\,,\quad x\in G(\mathbb{A})\,,$$

indexed by $k\in K$ and $z\in Z(\mathfrak{g}_{\mathbb{C}})$, is finite dimensional.

ii)

$$\|\phi\|^2 = \int_K\int_{M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1}|\phi(mk)|^2dm\,dk<\infty\,.$$

Furthermore, an automorphic form $\phi\in\mathcal{A}^2(P)$ is called cuspidal, if the following additional condition holds:

iii) For all standard parabolic subgroups $Q \subsetneqq P$, $\phi_Q = 0$.

The subspace of all cuspidal automorphic forms in $\mathcal{A}^2(P)$ will be denoted by $\mathcal{A}^2_0(P)$.

1.8 Given $\pi \in \Pi_{\text{disc}}(M_P(\mathbb{A}))_\xi$, let $\mathcal{A}^2_\pi(P)$ be the subspace of $\mathcal{A}^2(P)$ consisting of all functions ϕ such that for every $x \in G(\mathbb{A})$, the function

$$\phi_x(m) = \phi(mx) \quad m \in M_P(\mathbb{A})\,,$$

belongs to the π-isotypical subspace of $L^2(M_P(\mathbb{Q})\backslash M_P(\mathbb{A}))_\xi$. If $\pi \in \Pi(M_P(\mathbb{A}))$ is not contained in $\Pi_{\text{disc}}(M_P(\mathbb{A}))$, we put $\mathcal{A}^2_\pi(P) = 0$. Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$. Then we denote by $\mathcal{A}^2_\pi(P)_{K_f}$ the subspace of all K_f-invariant functions in $\mathcal{A}^2_\pi(P)$. Furthermore, if $\sigma \in \Pi(K_\infty)$, then we denote by $\mathcal{A}^2_\pi(P)_{K_f,\sigma}$ the σ-isotypical subspace of $\mathcal{A}^2_\pi(P)_{K_f}$.

1.9 Let $\mathfrak{X}$ be the set of W_0 conjugacy classes of pairs (M_B, r_B), where B is a parabolic subgroup and r_B is an irreducible cuspidal automorphic representation of $M_B(\mathbb{A})^1$. Let

$$L^2\big(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1\big) = \bigoplus_{\chi\in\mathfrak{X}} L^2\big(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1\big)_\chi$$

be the decomposition of $L^2(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1)$ introduced by Arthur in [A2, Section 3]. Given $\chi \in \mathfrak{X}$, let $\mathcal{A}^2_{\chi,\pi}(P)$ be the subspace of $\mathcal{A}^2_\pi(P)$ consisting of all function ϕ such that for each $x \in G(\mathbb{A})$, the restriction of ϕ_x to $M_P(\mathbb{A})^1$ belongs to $L^2(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1)_\chi$.

If we identify $\Pi(M_P(\mathbb{A})^1)$ with a subset of $\Pi(M_P(\mathbb{A}))$, then $\mathcal{A}^2_{\chi,\pi}(P)$ is well defined for any $\pi \in \Pi(M_P(\mathbb{A})^1)$. This is a space of functions on $N_P(\mathbb{A})M_P(\mathbb{A})A_P(\mathbb{R})^0\backslash G(\mathbb{A})$. The direct sum

$$\bigoplus_{\pi\in\Pi(M_P(\mathbb{A})^1)} \mathcal{A}^2_\chi(P,\pi)$$

is the space that was denoted by $A^2(P,\chi)$ in [Mü3].

1.10 Let $\overline{\mathcal{A}}^2(P)$ be the Hilbert space completion of $\mathcal{A}^2(P)$. For any $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ we have an induced representation $\rho(P,\lambda)$ of $G(\mathbb{A})$ on $\overline{\mathcal{A}}^2(P)$ which is defined by

$$\big(\rho(P,\lambda,y)\phi\big)(x) = \phi(xy)e^{(\lambda+\rho_P)(H_P(xy))}e^{-(\lambda+\rho_P)(H_P(x))}, \tag{1.4}$$

for elements $x, y \in G(\mathbb{A})$ and $\phi \in \overline{\mathcal{A}}^2(P)$. The Hilbert space completions $\overline{\mathcal{A}}^2_\pi(P)$ and $\overline{\mathcal{A}}^2_{\chi,\pi}(P)$ of the subspaces $\mathcal{A}^2_\pi(P)$ and $\mathcal{A}^2_{\chi,\pi}(P)$, respectively, are invariant under $\rho(P,\lambda)$ and we shall denote the restriction of $\rho(P,\lambda)$ to $\mathcal{A}^2_\pi(P)$ (resp. $\mathcal{A}^2_{\chi,\pi}(P)$) by $\rho_\pi(P,\lambda)$ (resp. $\rho_{\chi,\pi}(P,\lambda)$).

1.11 Given any irreducible unitary representation π of $M_B(\mathbb{A})^1$, let λ_π be the eigenvalue of the Casimir operator of $M_B(\mathbb{R})$, acting in the Gårding space $\mathcal{H}_{\pi_\infty}^\infty$ of the Archimedean constituent π_∞ of π. For $\chi \in \mathfrak{X}$ and $(M_B, r_B) \in \chi$, the Casimir eigenvalue λ_{r_B} depends only on the class χ and we denote it by λ_χ.

2 Normalized Intertwining Operators

Let $M, M_1 \in \mathcal{L}(M_0)$, $P \in \mathcal{P}(M)$ and $P_1 \in \mathcal{P}(M_1)$. For each $s \in W(\mathfrak{a}_M, \mathfrak{a}_{M_1})$, $\phi \in \mathcal{A}^2(P)$, and $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$ such that $\mathrm{Re}(\lambda) \in (\mathfrak{a}_P^*)^+ + \rho_P$, let $M_{P_1|P}(s,\lambda)\phi$ be defined by

$$M_{P_1|P}(s,\lambda)\phi(x) = e^{-(s\lambda+\rho_{P_1})(H_{P_1}(x))} \cdot \int_{N_1(\mathbb{A})\cap w_s N(\mathbb{A}) w_s^{-1}\backslash N_1(\mathbb{A})} \phi(w_s^{-1}n_1 x) e^{(\lambda+\rho_P)(H_P(w_s^{-1}n_1 x))} dn_1 \quad (2.1)$$

for $x \in G(\mathbb{A})$. The integral is absolutely convergent for λ as above and admits an analytic continuation to a meromorphic function of $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$ with values in the space of linear operators from $\mathcal{A}^2(P)$ to $\mathcal{A}^2(P_1)$. This operator is the global intertwining operator

$$M_{P_1|P}(s,\lambda)\colon \mathcal{A}^2(P) \to \mathcal{A}^2(P_1)\,.$$

Let $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ and $\chi \in \mathfrak{X}$. Then $M_{P_1|P}(s,\lambda)$ maps the subspace $\mathcal{A}_\pi^2(P)$ (resp. $A_{\chi,\pi}^2(P)$) to $A_{s\pi}^2(P_1)$ (resp. $\mathcal{A}_{\chi,s\pi}^2(P_1)$). The main functional equations are

$$M_{P_2|P}(ts,\lambda) = M_{P_2|P_1}(t,s\lambda)M_{P_1|P}(s,\lambda) \quad (2.2)$$

for $t \in W(\mathfrak{a}_1, \mathfrak{a}_2)$ and $s \in W(\mathfrak{a}, \mathfrak{a}_1)$.

By (1.4) and (1.5) of [A4], most of the considerations concerning intertwining operators can be reduced to the case where P_1 and P have the same Levi component M, and $s = 1$.

Thus, from now on we shall assume that $P, Q \in \mathcal{P}(M)$ and we put

$$M_{Q|P}(\lambda) := M_{Q|P}(1,\lambda)\,, \quad \lambda \in \mathfrak{a}_{M,\mathbb{C}}^*\,.$$

Given $\pi \in \Pi(M(\mathbb{A}))$, let

$$M_{Q|P}(\pi,\lambda)\colon \mathcal{A}_\pi^2(P) \to \mathcal{A}_\pi^2(Q)$$

be the restriction of $M_{Q|P}(\lambda)$ to $\mathcal{A}_\pi^2(P)$. We shall now express this operator in terms of local intertwining operators. Let π_λ be the representation of $P(\mathbb{A})$ which is defined by

$$\pi_\lambda(nm) = e^{\lambda(H_M(m))}\pi(m)\,, \quad n \in N_P(\mathbb{A})\,,\ m \in M_P(\mathbb{A})\,.$$

Let $(I_P^G(\pi_\lambda), \mathcal{H}_P(\pi))$ be the induced representation of $G(\mathbb{A})$. Similarly let $(I_Q^G(\pi_\lambda), \mathcal{H}_Q(\pi))$ be the representation of $G(\mathbb{A})$ induced from $Q(\mathbb{A})$. Let ξ

be a unitary character of $A_M(\mathbb{R})^0$ and suppose that $\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))_\xi$. We extend ξ by 1 to a character of $M(\mathbb{Q})A_M(\mathbb{R})^0$. Then there is a canonical isomorphism

$$j_P : \mathcal{H}_P(\pi) \otimes \text{Hom}_{M(\mathbb{A})}\left(\pi, I_{M(\mathbb{Q})A_M(\mathbb{R})^0}^{M(\mathbb{A})}(\xi)\right) \to \overline{\mathcal{A}}_\pi^2(P) \tag{2.3}$$

of $G(\mathbb{A})$-modules where $G(\mathbb{A})$ acts on the left by $I_P^G(\pi_\lambda) \otimes \text{Id}$. A similar isomorphism j_Q exists with respect to Q. Let $\mathcal{H}_P^0(\pi)$ (resp. $\mathcal{H}_Q^0(\pi)$) be the subspace of elements which are right K-finite and left $Z(\mathfrak{g}_\mathbb{C})$-finite. Using (2.3), it follows that $M_{Q|P}(\pi, \lambda)$ induces an intertwining operator

$$J_{Q|P}(\pi, \lambda) \colon \mathcal{H}_P^0(\pi) \to \mathcal{H}_Q^0(\pi)$$

such that

$$j_Q \circ \left(J_{Q|P}(\pi, \lambda) \otimes \text{Id}\right) = M_{Q|P}(\pi, \lambda) \circ j_P \,.$$

It follows from (2.1) that for $\text{Re}(\lambda) \in (\mathfrak{a}_P^*)^+ + \rho_P$, this operator is defined by the following convergent integral

$$J_{Q|P}(\pi, \lambda)\phi(x) = e^{-(\lambda+\rho_Q)(H_Q(x))} \cdot \int_{N_Q(\mathbb{A})\cap N_P(\mathbb{A})\backslash N_Q(\mathbb{A})} \phi(nx) e^{(\lambda+\rho_P)(H_P(nx))} dn \,. \tag{2.4}$$

where $x \in G(\mathbb{A})$ and $\phi \in \mathcal{H}_P^0(\pi)$.

Let v be any place of $\mathbb{Q}$ and let $(\pi_v, V_v) \in \Pi(M(\mathbb{Q}_v))$. Given $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$, let $\pi_{v,\lambda}$ be the representation of $P(\mathbb{Q}_v)$ on V_v defined by

$$\pi_{v,\lambda}(n_v m_v) = \pi_v(m_v) e^{\lambda(H_M(m_v))}, \quad n_v \in N(\mathbb{Q}_v)\,,\ m_v \in M(\mathbb{Q}_v)\,.$$

Let $(I_P^G(\pi_{v,\lambda}), \mathcal{H}_P(\pi_v))$ denote induced representation. The Hilbert space is the space of measurable functions

$$\phi_v : N(\mathbb{Q}_v) \setminus G(\mathbb{Q}_v) \to V_v$$

such that

1. $\phi_v(m_v x_v) = \pi(m_v)\phi_v(x_v)$, $m_v \in M(\mathbb{Q}_v)$, $x_v \in G(\mathbb{Q}_v)$;
2. $\|\phi_v\|^2 = \int_{K_v} \|\phi_v(k)\|_{V_v}^2 dk < \infty$.

Let $\mathcal{H}_P^0(\pi_v) \subset \mathcal{H}_P(\pi_v)$ be the subspace of K_v-finite functions. Then the local intertwining operator

$$J_{Q|P}(\pi_v, \lambda) : \mathcal{H}_P^0(\pi_v) \to \mathcal{H}_Q^0(\pi_v)$$

is defined by

$$J_{Q|P}(\pi_v, \lambda)\phi_v(x_v) = e^{-(\lambda+\rho_Q)(H_Q(x_v))} \cdot \int_{N_Q(\mathbb{Q}_v)\cap N_P(\mathbb{Q}_v)\backslash N_Q(\mathbb{Q}_v)} \phi_v(n_v x_v) e^{(\lambda+\rho_P)(H_P(n_v x_v))} dn_v \,. \tag{2.5}$$

The integral converges absolutely for $\mathrm{Re}(\lambda) \in (\mathfrak{a}_P^*)^+ + \rho_P$ and can be continued to a meromorphic function of $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ with values in the space of linear operators from $\mathcal{H}_P^0(\pi_v)$ to $\mathcal{H}_Q^0(\pi_v)$ [Sh1].

Now let $\pi \in \Pi(M(\mathbb{A}))$. Then π is a restricted tensor product

$$\pi = \otimes_v \pi_v$$

where almost all (π_v, V_v) are unramified, i.e. $\dim V_v^{K_v \cap M(\mathbb{Q}_v)} = 1$ for almost all v. Moreover, we have

$$\big(I_P^G(\pi_\lambda), \mathcal{H}_P(\pi)\big) \cong \big(\otimes_v I_P^G(\pi_{v,\lambda}), \otimes_v \mathcal{H}_P(\pi_v)\big)\,.$$

Let $\phi \in \mathcal{H}_P^0(\pi)$ and suppose that $\phi = \otimes_v \phi_v$. Observe that each ϕ_v belongs to $\mathcal{H}_P^0(\pi_v)$ and for almost all v, ϕ_v is K_v–invariant. Comparing (2.4) and (2.5), it follows that

$$J_{Q|P}(\pi, \lambda)\phi = \otimes_v \big(J_{Q|P}(\pi_v, \lambda)\phi_v\big) \tag{2.6}$$

whenever the product on the right converges.

It is possible to normalize local intertwining operators. Let v be any valuation of $\mathbb{Q}$ and let $\pi_v \in \Pi(M(\mathbb{Q}_v))$. It is proved in [A7], [CLL] that there exist scalar valued meromorphic functions $r_{Q|P}(\pi_v, \lambda)$ of $\lambda \in \mathfrak{a}_{P,\mathbb{C}}^*$ such that the normalized intertwining operators

$$R_{Q|P}(\pi_v, \lambda) = r_{Q|P}(\pi_v, \lambda)^{-1} J_{Q|P}(\pi_v, \lambda) \tag{2.7}$$

satisfy the conditions (R_1)–(R_8) of Theorem 2.1 of [A7]. We recall some of the properties satisfied by the normalized intertwining operators.

(R.1) If $S \in \mathcal{P}(M)$, then

$$R_{S|P}(\pi_v, \lambda) = R_{S|Q}(\pi_v, \lambda) R_{Q|P}(\pi_v, \lambda)\,. \tag{2.8}$$

(R.2)

$$R_{Q|P}(\pi_v, \lambda)^* = R_{P|Q}(\pi_v, -\overline{\lambda})\,. \tag{2.9}$$

(R.3) Let $L \in \mathcal{L}(M)$, $S \in \mathcal{P}(L)$, and $Q, Q' \in \mathcal{P}^L(M)$. Then

$$\big(R_{S(Q')|S(Q)}(\pi_v, \lambda)\phi\big)_k = R_{Q'|Q}(\pi_v, \lambda)\phi_k \tag{2.10}$$

for any $\phi \in \mathcal{H}_P^0(S(R)(\mathbb{Q}_v))$ and $k \in K_v$.

(R.4) Let v be a finite place. Suppose that π_v is unramified, and that K_v is very special. Let $\phi_v \in \mathcal{H}_P(\pi_v)$ be a function such that $\phi_v(k) = \phi_v(1)$ for all $k \in K_v$. Then in the compact picture of the induced representation, one has

$$R_{Q|P}(\pi_v, \lambda)\phi_v = \phi_v\,. \tag{2.11}$$

The functions $r_{Q|P}(\pi_v, \lambda)$ are called *normalizing factors*. They satisfy similar properties. We recall some of them. Given $P \in \mathcal{P}(M)$, let Σ_P^r be the set of reduced roots of (P, A_M).

(r.1) For $\beta \in \Sigma_P^r$, let $M_\beta \in \mathcal{L}(M)$ be such that

$$\mathfrak{a}_{M_\beta} = \big\{ H \in \mathfrak{a}_M \;\big|\; \beta(H) = 0 \big\}$$

and let P_β be the unique group in $\mathcal{P}^{M_\beta}(M)$ whose simple root is β. Then

$$r_{Q|P}(\pi_v, \lambda) = \prod_{\beta \in \Sigma_Q^r \cap \Sigma_{\overline{P}}^r} r_{\overline{P}_\beta | P_\beta}(\pi_v, \lambda)\,, \tag{2.12}$$

Note that each $r_{\overline{P}_\beta|P_\beta}(\pi_v, \lambda)$ depends only on the projection $\lambda(\beta^\vee)$.

(r.2) If π_v is an irreducible constituent of an induced representation $I_R^M(\sigma_v)$, where $\sigma_v \in \Pi_2(M_1(\mathbb{Q}_v))$, $R \in \mathcal{P}^M(M_1)$, and $M_1 \subset M$, then

$$r_{Q|P}(\pi_v, \lambda) = r_{Q(R)|P(R)}(\sigma_v, \lambda)\,. \tag{2.13}$$

(r.3)

$$r_{Q|P}(\pi_v, \lambda) r_{P|Q}(\pi_v, \lambda) = J_{Q|P}(\pi_v, \lambda) J_{P|Q}(\pi_v, \lambda)\,. \tag{2.14}$$

(r.4)

$$\overline{r_{Q|P}(\pi_v, \lambda)} = r_{P|Q}(\pi_v, -\overline{\lambda})\,. \tag{2.15}$$

(r.5) If v is a finite place of $\mathbb{Q}$, then $r_{Q|P}(\pi_v, \lambda)$ is a rational function in the variables $\{q_v^{-\lambda(\widetilde{\beta})} \mid \beta \in \Sigma_Q^r \cap \Sigma_{\overline{P}}^r\}$, where the $\widetilde{\beta}$'s are suitably normalized "coroots". If $v = \infty$, then $r_{Q|P}(\pi_v, \lambda)$ is a rational function in the variables $\{\lambda(\beta^\vee) \mid \beta \in \Sigma_Q^r \cap \Sigma_{\overline{P}}^r\}$.

Now we return to global intertwining operators. Let $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$. For $\phi \in \mathcal{H}_P^0(\pi)$ with $\phi = \otimes_v \phi_v$ set

$$R_{Q|P}(\pi, \lambda)\phi = \otimes_v \big(R_{Q|P}(\pi_v, \lambda)\phi_v \big)\,. \tag{2.16}$$

Since ϕ_v is K_v- invariant for almost all v, it follows from (2.11) that the right-hand side is actually a finite product and therefore, it converges for all $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ which are not poles of the local intertwining operators. In this way we get a a meromorphic operator valued function

$$R_{Q|P}(\pi, \lambda) : \mathcal{H}_P^0(\pi) \to \mathcal{H}_Q^0(\pi)$$

of $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$. Using the isomorphism (2.3) and the corresponding one for Q, we obtain a meromorphic operator valued function

$$N_{Q|P}(\pi, \lambda) \colon \mathcal{A}_\pi^2(P) \to \mathcal{A}_\pi^2(Q)$$

of $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ such that

$$j_Q \circ N_{Q|P}(\pi, \lambda) = \big(R_{Q|P}(\pi, \lambda) \otimes \mathrm{Id} \big) \circ j_P\,. \tag{2.17}$$

Furthermore, put

$$r_{Q|P}(\pi, \lambda) = \prod_v r_{Q|P}(\pi_v, \lambda)\,. \tag{2.18}$$

By (R.4) it follows that for ϕ as above, we have

$$J_{Q|P}(\pi_v, \lambda)\phi_v = r_{Q|P}(\pi_v, \lambda)\phi_v$$

for almost all v. Therefore, the infinite product (2.18) converges in the domain of absolute convergence of the infinite product (2.6) and for λ in this domain we have

$$M_{Q|P}(\pi,\lambda) = r_{Q|P}(\pi,\lambda) N_{Q|P}(\pi,\lambda)\,. \tag{2.19}$$

Since both $M_{Q|P}(\pi,\lambda)$ and $N_{Q|P}(\pi,\lambda)$ are meromorphic functions of $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$, it follows that $r_{Q|P}(\pi,\lambda)$ admits a meromorphic continuation to $\mathfrak{a}^*_{M,\mathbb{C}}$. The meromorphic function $r_{Q|P}(\pi,\lambda)$ is the global normalizing factor and $N_{Q|P}(\pi,\lambda)$ is the normalized global intertwining operator.

Using (2.12), (2.14), (2.15) and the functional equations (2.2), it follows that $r_{Q|P}(\pi,\lambda)$ has the following properties

1.
$$r_{Q|P}(\pi,\lambda) r_{P|Q}(\pi,\lambda) = 1\,. \tag{2.20}$$

2.
$$\overline{r_{Q|P}(\pi,\lambda)} = r_{P|Q}(\pi,-\overline{\lambda})\,. \tag{2.21}$$

3. For each $\beta \in \Sigma^r_Q \cap \Sigma^r_{\overline{P}}$ let P_β be as in (2.12). Then

$$r_{Q|P}(\pi,\lambda) = \prod_{\beta \in \Sigma^r_Q \cap \Sigma^r_{\overline{P}}} r_{\overline{P}_\beta|P_\beta}(\pi,\lambda)\,. \tag{2.22}$$

Note that $r_{\overline{P}_\beta|P_\beta}(\pi,\lambda)$ depends only on the projection $\lambda(\beta^\vee)$.

3 Local Normalizing Factors

In this section we shall investigate the local normalizing factors in more detail. In particular, we shall study their logarithmic derivatives. To begin with, we recall the construction of the normalizing factors.

First assume that v is a finite valuation. Then the existence of normalizing factors such that Theorem 2.1 of [A7] holds has been verified by Langlands in [CLL, Lecture 15]. Let $\pi_v \in \Pi(M(\mathbb{Q}_v))$. The local normalizing factors $r_{Q|P}(\pi_v,\lambda)$ have to satisfy (2.1)–(2.3) in [A7]. Therefore, it suffices to define them when $\dim(\mathfrak{a}_M/\mathfrak{a}_G) = 1$ and π_v is square integrable modulo A_G. Assume for the moment that these conditions are satisfied. Let $P \in \mathcal{P}(M)$ and let α be the unique simple root of (P, A_M). Then Langlands has shown that there exists a rational function $V_P(\pi_v, z)$ of one variable such that

$$r_{\overline{P}|P}(\pi_v,\lambda) = V_P(\pi_v, q_v^{-\lambda(\widetilde{\alpha})})\,, \tag{3.1}$$

where $\widetilde{\alpha} \in \mathfrak{a}_M$ is independent of π_v. We recall the definition of $V_P(\pi_v, z)$. Suppose that P_v is a parabolic subgroup of G defined over $\mathbb{Q}_v$ and let M_v

be a Levi component of P_v over $\mathbb{Q}_v$. Denote by A_{M_v} the split component of the center of M_v. Set

$$\mathfrak{a}_{M_v} = \mathrm{Hom}\left(X(M_v)_{\mathbb{Q}_v}, \mathbb{R}\right)$$

and

$$\mathfrak{a}_{M_v}^* = X(M_v)_{\mathbb{Q}_v} \otimes \mathbb{R}\ .$$

Let

$$H_{M_v} : M_v(\mathbb{Q}_v) \to \mathfrak{a}_{M_v}$$

be defined by

$$q_v^{\langle H_{M_v}(m_v),\chi\rangle} = |\chi(m_v)|_v\ , \quad \chi \in X(M_v)_{\mathbb{Q}_v}\ ,\ m_v \in M_v(\mathbb{Q}_v)\ .$$

Given $\pi \in \Pi(M_v(\mathbb{Q}_v))$ and $\lambda \in i\mathfrak{a}_{M_v}^*$, let π_λ denote the representation defined by

$$\pi_\lambda(m_v) = \pi(m_v)e^{\lambda(H_{M_v}(m_v))}\ , \quad m_v \in M_v(\mathbb{Q}_v)\ .$$

Let

$$\mathfrak{a}_{M_v,\pi}^\vee = \{\lambda \in i\mathfrak{a}_{M_v}^* \mid \pi_\lambda \cong \pi\}$$

denote the stabilizer of π with respect to this action of $i\mathfrak{a}_{M_v}^*$. Then $\mathfrak{a}_{M_v,\pi}^\vee$ is a lattice in $i\mathfrak{a}_{M_v}^*$ and the orbit $\mathfrak{o}_\pi$ of π is equal to $i\mathfrak{a}_{M_v}^*/\mathfrak{a}_{M_v,\pi}^\vee$. Let

$$\mathfrak{a}_{M_v,\mathbb{Q}_v} = H_{M_v}(M_v(\mathbb{Q}_v))\ , \quad \widetilde{\mathfrak{a}}_{M_v,\mathbb{Q}_v} = H_{M_v}(A_{M_v}(\mathbb{Q}_v))\ .$$

Then $\mathfrak{a}_{M_v,\mathbb{Q}_v}$ and $\widetilde{\mathfrak{a}}_{M_v,\mathbb{Q}_v}$ are lattices in $\mathfrak{a}_{M_v}$. Given a real vector space V and a closed subgroup V_1 of V, let us agree to set

$$V_1^\vee = \mathrm{Hom}(V_1, 2\pi i\mathbb{Z}) \subset iV^*.$$

Let $\mathfrak{a}_{M_v,\pi} \subset \mathfrak{a}_{M_v}$ be the dual lattice to $\mathfrak{a}_{M_v,\pi}^\vee$. Then

$$\widetilde{\mathfrak{a}}_{M_v,\mathbb{Q}_v} \subset \mathfrak{a}_{M_v,\pi} \subset \mathfrak{a}_{M_v,\mathbb{Q}_v}.$$

Set

$$L_{M_v} = (\mathfrak{a}_{M_v,\mathbb{Q}_v} + \mathfrak{a}_{G_v})/\mathfrak{a}_{G_v}\ , \quad \widetilde{L}_{M_v} = (\widetilde{\mathfrak{a}}_{M_v,\mathbb{Q}_v} + \mathfrak{a}_{G_v})/\mathfrak{a}_{G_v},$$

and

$$L(\pi) = (\mathfrak{a}_{M_v,\pi} + \mathfrak{a}_{G_v})/\mathfrak{a}_{G_v}\ .$$

Then L_{M_v}, $\widetilde{L}_{M_v}$, and $L(\pi)$ are lattices in $\mathfrak{a}_{M_v}^{G_v} = \mathfrak{a}_{M_v}/\mathfrak{a}_{G_v}$.

Suppose that P_v is a maximal parabolic subgroup, that is $\dim \mathfrak{a}_{M_v}^{G_v} = 1$. Then there exists $\alpha(\pi) \in \mathfrak{a}_{M_v}$ such that

$$L(\pi) = \frac{\log q}{2\pi}\mathbb{Z}(\alpha(\pi))\,.$$

In [Si1] Silberger has shown that for a supercuspidal representation π there exists a rational function $\widetilde{U}_{P_v}(\pi, z)$ such that the Plancherel measure $\mu(\pi,\lambda)$ satisfies

$$\mu(\pi,\lambda) = \widetilde{U}_{P_v}\left(\pi, q^{-\lambda(\alpha(\pi))}\right). \tag{3.2}$$

Let $\widetilde{\alpha} \in \mathfrak{a}_{M_v}$ be such that

$$L_{M_v} = \frac{\log q}{2\pi}\mathbb{Z}(\widetilde{\alpha})\,.$$

Since $L(\pi) \subset L_{M_v}$, there exists $k(\pi) \in \mathbb{Z}$ such that $\alpha(\pi) = k(\pi)\widetilde{\alpha}$. Let

$$U_{P_v}(\pi, z) = \widetilde{U}_{P_v}\big(\pi, z^{k(\pi)}\big)\,. \tag{3.3}$$

Then

$$\mu(\pi, \lambda) = U_{P_v}\big(\pi, q^{-\lambda(\widetilde{\alpha})}\big)\,.$$

Now suppose that P_v is arbitrary, but π is still supercuspidal. For each reduced root $\alpha \in \Sigma_r(P_v, A_v)$ let A_α denote the largest subtorus of A_v which lies in the kernel of the root character of α. Let M_α denote the centralizer of A_α. Let ${}^*P_\alpha = P_v \cap M_\alpha$. Then ${}^*P_\alpha = M_v N_\alpha$. Let $\mu_\alpha(\pi, \lambda)$ be the Plancherel measure with respect to $(M_\alpha, {}^*P_\alpha)$. According to [H3, Theorem 24], there exist constants $\gamma = \gamma(G/M)$ and $\gamma_\alpha = \gamma(M_\alpha/M)$, $\alpha \in \Sigma_r(P_v, A_v)$, such that

$$\gamma^{-2}\mu(\pi, \lambda) = \prod_{\alpha \in \Sigma_r(P_v, A_v)} \gamma_\alpha^{-2}\mu_\alpha(\pi, \lambda)\,, \tag{3.4}$$

Hence if $\{\widetilde{\alpha} \mid \alpha \in \Delta_{P_v}\}$ is a set of generators of the lattice L_{M_v}, then $\mu(\pi, \lambda)$ is a rational function in the variables $\{q^{-\lambda(\widetilde{\alpha})} \mid \alpha \in \Delta_{P_v}\}$. Finally, by Theorem 1 of [Si2], this can be extended to all discrete series representations of $M_v(\mathbb{Q}_v)$.

Now let $P = MN$ be a maximal parabolic subgroup of G defined over $\mathbb{Q}$. Then $X(M)_{\mathbb{Q}} \subset X(M)_{\mathbb{Q}_v}$ induces an embedding $\mathfrak{a}_M^* \subset \mathfrak{a}_{M_v}^*$ and by the above, it follows that there exists $\widetilde{\alpha} \in \mathfrak{a}_M$ and a rational function $U_P(\pi, z)$ such that

$$\mu(\pi, \lambda) = U_P(\pi, q^{-\lambda(\widetilde{\alpha})})\,, \tag{3.5}$$

for all $\pi \in \Pi_2(M(\mathbb{Q}_v))$, $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$. As shown by Langlands [CLL], the rational function $U_P(\pi, z)$ has the form

$$U_P(\pi, z) = a\frac{\prod_{i=1}^r (1 - \alpha_i z)(1 - \overline{\alpha}_i^{-1} z)}{\prod_{i=1}^r (1 - \beta_i z)(1 - \overline{\beta}_i^{-1} z)}\,,$$

where the α_i's and β_i's satisfy $|\alpha_i| \le 1$, $|\beta_i| \le 1$, $i = 1, \dots, r$, and a is a certain constant. Then the rational function $V_P(\pi, z)$ in (3.1) is defined by

$$V_P(\pi, z) = b\frac{\prod_{i=1}^r (1 - \beta_i z)}{\prod_{i=1}^r (1 - \alpha_i z)} \tag{3.6}$$

for a suitable constant b. In particular, it follows that $2r$ is the number of poles of $U_P(\pi, z)$. For our applications we need a bound for r. This is done in the following lemma.

LEMMA 3.1. *Let $M \in \mathcal{L}(M_0)$ be such that $\dim(\mathfrak{a}_M/\mathfrak{a}_G) = 1$. There exists $C > 0$ such that for all $P \in \mathcal{P}(M)$ and all $\pi \in \Pi_2(M(\mathbb{Q}_v))$ the number of poles of the rational function $V_P(\pi, z)$ is less than or equal to C.*

Proof. Let P_v be a maximal parabolic subgroup of G defined over $\mathbb{Q}_v$ and let $\pi \in \Pi(M_v(\mathbb{Q}_v))$ be supercuspidal. By Theorem 1.6 of [Si1], the rational function $\widetilde{U}_{P_v}(\pi, z)$ in (3.2) has at most 4 poles. Now observe that $\widetilde{L}_{M_v} \subset L(\pi) \subset L_{M_v}$ and $L_{M_v}/\widetilde{L}_{M_v}$ is finite. This implies that the number of poles of the rational function $U_{P_v}(\pi, z)$ defined by (3.3) is bounded by a constant which is independent of π. The general case is reduced to this one using the product formula (3.4) and Theorem 1 of [Si2]. □

Using (2.1)–(2.3) of [A7], the local normalizing factors can be defined for all $M \in \mathcal{L}(M_0)$, $P, Q \in \mathcal{P}(M)$ and $\pi_v \in \Pi(M(\mathbb{Q}_v))$.

Next suppose that $v = \infty$. In this case the existence of normalizing factors such that Theorem 2.1 in [A7] holds has been established by Arthur [A7]. The definition is as follows. Let ${}^L M$ be the L-group of M and let $\rho = \tilde{\rho}_{Q|P}$ be the contragredient representation of the adjoint representation $\rho_{Q|P}$ of ${}^L M$ on the complex vector space ${}^L\mathfrak{n}_Q \cap {}^L\mathfrak{n}_P \backslash {}^L\mathfrak{n}_Q$. Let $L(s, \pi, \rho)$ be the L-factor attached to π and $\rho = \tilde{\rho}_{Q|P}$. Then Arthur has shown in [A7] that the functions

$$r_{Q|P}(\pi, \lambda) := \frac{L(0, \pi_\lambda, \rho)}{L(1, \pi_\lambda, \rho)} \tag{3.7}$$

satisfy all properties required by normalizing factors. We briefly recall the definition of the L-function and refer to [A7, p. 33–35] for more details.

To any $\pi \in \Pi(M(\mathbb{R}))$ and $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$, there corresponds a map

$$\phi_\lambda : W_{\mathbb{R}} \to {}^L M$$

from the Weil group of $\mathbb{R}$ to the L-group of M, which is uniquely determined by π_λ up to conjugation by ${}^L M^0$ [La3]. Let

$$\rho \cdot \phi_\lambda = \bigoplus_\tau \tau_\lambda \tag{3.8}$$

be the decomposition of $\rho \cdot \phi_\lambda$ into irreducible representations of $W_{\mathbb{R}}$. Then by definition

$$L(s, \pi_\lambda, \rho) = L(s, \rho \cdot \phi_\lambda) = \prod_\tau L(s, \tau_\lambda)\,.$$

So it remains to describe the L-factors $L(0, \tau_\lambda) L(1, \tau_\lambda)^{-1}$. To this end let $T \subset M$ be a maximal torus over $\mathbb{R}$ whose real split component is A_M. Let $\langle \lambda, \lambda^\vee \rangle$ denote the canonical pairing $X^*(T) \times X_*(T) \to \mathbb{Z}$ between the space $X^*(T)$ of characters and the space $X_*(T)$ of one-parameter subgroups of T. Let $\Sigma_P(G, T)$ be the set of roots of (G, T) which restrict to roots of (P, A_M). The Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\Sigma_P(G, T)$. Let $\overline{\sigma}$ be the action of the nontrivial element of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. The eigenspaces of $\tilde{\rho}_{Q|P}(\phi_\lambda(\mathbb{C}^*))$ are the

root spaces of $\{-\alpha^\vee | \alpha \in \Sigma_P(G,T)\}$ and the irreducible constituents τ_λ of $\tilde{\rho}_{Q|P} \cdot \phi_\lambda$ correspond to orbits of $\overline{\sigma}$ in $\Sigma_P(G,T)$. Furthermore, the map $\phi\colon W_{\mathbb{R}} \to {}^L M$ determines elements $\mu, \nu \in X^*(T) \otimes \mathbb{C}$ with $\mu - \nu \in X^*(T)$. Let

$$\Gamma_{\mathbb{C}}(z) := 2(2\pi)^{-z}\Gamma(z) \text{ and } \Gamma_{\mathbb{R}}(z) = \pi^{-z/2}\Gamma(z/2)\,.$$

If a two-dimensional constituent τ_λ corresponds to a pair $\{\alpha, \overline{\sigma}\alpha\}$ of complex roots, then τ_λ is induced from the quasi-character

$$z \mapsto z^{\langle \mu+\lambda, \alpha^\vee\rangle}\overline{z}^{\langle \nu+\lambda, \alpha^\vee\rangle}$$

of $\mathbb{C}^*$. Replacing $\alpha^\vee$ by $\overline{\sigma}\alpha^\vee$ if necessary, we can assume that $\langle \overline{\sigma}\mu - \mu, \alpha^\vee\rangle$ is a nonpositive integer. Then

$$\frac{L(0,\tau_\lambda)}{L(1,\tau_\lambda)} = \frac{\Gamma_{\mathbb{C}}(\langle \mu+\lambda, \alpha^\vee\rangle)}{\Gamma_{\mathbb{C}}(\langle \mu+\lambda, \alpha^\vee\rangle + 1)}\,. \tag{3.9}$$

The one-dimensional constituents τ_λ correspond to real roots α_0 in $\Sigma_P(G,T)$. There is at most one of these. If α_0 exists, then τ_λ is induced from the quasi-character of $\mathbb{R}^*$

$$x \mapsto \left(\tfrac{x}{|x|}\right)^{-N_0}|x|^{\langle \mu+\lambda, \alpha_0^\vee\rangle}\,,$$

where $N_0 \in \{0,1\}$. In this case

$$\frac{L(0,\tau_\lambda)}{L(1,\tau_\lambda)} = \frac{\Gamma_{\mathbb{R}}(\langle \mu+\lambda, \alpha_0^\vee\rangle + N_0)}{\Gamma_{\mathbb{R}}(\langle \mu+\lambda, \alpha_0^\vee\rangle + N_0 + 1)}\,. \tag{3.10}$$

REMARK. It has been conjectured by Langlands [La1, p. 282] that for any local field, intertwining operators can be normalized by L-functions. For $\mathrm{GL}(n)$ this was proved by Shahidi [Sh2]. Namely, let P be a standard maximal parabolic subgroup of $\mathrm{GL}(n)$. Then $\mathfrak{a}_{P,\mathbb{C}}^* \cong \mathbb{C}^2$ and $M_P = \mathrm{GL}(n_1) \times \mathrm{GL}(n_2)$ for some decomposition $n = n_1 + n_2$. Let F be a local non-Archimedean field and let ψ be a non-trivial additive character of F. Let $\pi_1 \otimes \pi_2$ be an irreducible unitary representation of $M_P(F) = \mathrm{GL}(n_1, F) \times \mathrm{GL}(n_2, F)$. Let $L(z, \pi_1 \times \check{\pi}_2)$ and $\varepsilon(z, \pi_1 \times \check{\pi}_2, \psi)$ be the Rankin–Selberg L-function and the ε-factor defined by Jacquet, Piatetski-Shapiro and Shalika [JPS]. Then the normalizing factor $r_{\overline{P}|P}(\pi, s)$, $s = (s_1, s_2)$, can be chosen to be

$$r_{\overline{P}|P}(\pi, s) = \frac{L(s_1 - s_2, \pi_1 \times \check{\pi}_2)}{\varepsilon(s_1 - s_2, \pi_1 \times \check{\pi}_2, \psi)L(1 + s_1 - s_2, \pi_1 \times \check{\pi}_2)}\,.$$

In [Sh4], this has been generalized to quasi-split groups and generic representations π.

We can now estimate the logarithmic derivatives of the normalizing factors. First we consider the case of a finite place v. Let q_v be the number of elements of the residue field of $\mathbb{Q}_v$ (which in our case is a prime number).

LEMMA 3.2. *Let $\alpha \in \overline{(\mathfrak{a}_M^*)^+}$. There exist $C, c > 0$ such that for every finite valuation v and every $\pi_v \in \Pi(M(\mathbb{Q}_v))$ we have*

$$\left|r_{Q|P}(\pi_v, z\alpha)^{-1} \cdot \tfrac{d}{dz} r_{Q|P}(\pi_v, z\alpha)\right| \leq C q_v^{-2} \tag{3.11}$$

for $\mathrm{Re}(z) \geq c$.

Proof. First we assume that $\dim \mathfrak{a}_M/\mathfrak{a}_G = 1$ and $\pi_v \in \Pi(M(\mathbb{Q}_v))$ is square integrable modulo $A_M(\mathbb{Q}_v)$. Let α be the unique simple root of (P, A). Then $r_{\overline{P}|P}(\pi_v, \lambda)$ is given by (3.1). Let $\lambda = z\alpha$, $z \in \mathbb{C}$. Then $\lambda(\alpha^\vee) = z$ and by (3.6), it follows that

$$r_{\overline{P}|P}(\pi_v, z\alpha)^{-1} \frac{d}{dz} r_{\overline{P}|P}(\pi_v, z\alpha) = \log(q_v)\, q_v^{-z} \cdot \sum_{i=1}^{r} \left\{ \frac{\beta_i}{1 - \beta_i q_v^{-z}} - \frac{\alpha_i}{1 - \alpha_i q_v^{-z}} \right\}.$$

Recall that the α_i's and β_i's satisfy $|\alpha_i| \leq 1$, $|\beta_i| \leq 1$, $i = 1, \dots, r$. Moreover, by Lemma 3.1 there exists $C_1 \geq 0$, which is independent of $\pi_v \in \Pi(M(\mathbb{Q}_v))$, such that $r \leq C_1$. Therefore, for $\mathrm{Re}(z) > 3$ we obtain

$$\left| r_{\overline{P}|P}(\pi_v, z\alpha)^{-1} \frac{d}{dz} r_{\overline{P}|P}(\pi_v, z\alpha) \right| \leq C_1 \frac{\log(q_v) q_v^{-\mathrm{Re}(z)}}{1 - q_v^{-\mathrm{Re}(z)}} < C_2 q_v^{-2}\,. \tag{3.12}$$

Now let $M \in \mathcal{L}(M_0)$ be arbitrary, but still assume that π_v is square integrable modulo $A_M(\mathbb{Q}_v)$. Let $P, Q \in \mathcal{P}(M)$. For each $\beta \in \Sigma_P^r$ let $M_\beta \in \mathcal{L}(M)$ be such that

$$\mathfrak{a}_{M_\beta} = \left\{ H \in \mathfrak{a}_M \;\middle|\; \beta(H) = 0 \right\}.$$

Then $\dim \mathfrak{a}_M/\mathfrak{a}_{M_\beta} = 1$. Let P_β be the unique group in $\mathcal{P}^{M_\beta}(M)$ whose simple root is β. Furthermore, let $\alpha \in (\mathfrak{a}_M^*)^+$, $\nu \in \mathfrak{a}_{M,\mathbb{C}}^*$ and $z \in \mathbb{C}$. Then by (2.12) we get

$$\begin{aligned} &r_{Q|P}(\pi_v, z\alpha + \nu)^{-1} \cdot \tfrac{d}{dz} r_{Q|P}(\pi_v, z\alpha + \nu) \\ &= \sum_{\beta \in \Sigma_P^r \cap \Sigma_{\overline{Q}}^r} r_{\overline{P}_\beta|P_\beta}\big(\pi_v, \langle z\alpha + \nu, \beta^\vee\rangle \beta\big)^{-1} \cdot \tfrac{d}{dz} r_{\overline{P}_\beta|P_\beta}\big(\pi_v, \langle z\alpha + \nu, \beta^\vee\rangle \beta\big)\,. \end{aligned}$$

By assumption we have $\langle \alpha, \beta^\vee\rangle \geq 0$ for every $\beta \in \Sigma_P^r$. If $\langle \alpha, \beta^\vee\rangle = 0$, then the corresponding logarithmic derivative vanishes. Suppose that $a := \langle \alpha, \beta^\vee\rangle > 0$. Let $c_0 > 0$ be such that $\|\beta\| \leq c_0$ for all $\beta \in \Sigma_P^r$. Then

$$\mathrm{Re}\left(z\langle \alpha, \beta^\vee\rangle + \langle \nu, \beta^\vee\rangle\right) \geq a\,\mathrm{Re}(z) - c_0\|\nu\|$$

and it follows from (3.7) that there exist $C, c > 0$, depending on α, c_0 and $\|\nu\|$, such that

$$\left|r_{Q|P}(\pi_v, z\alpha + \nu)^{-1} \cdot \tfrac{d}{dz} r_{Q|P}(\pi_v, z\alpha + \nu)\right| \leq C q_v^{-2} \tag{3.13}$$

for all $\pi_v \in \Pi_2(M(\mathbb{Q}_v))$ and $\mathrm{Re}(z) \geq c$.

Next assume that π_v is tempered. Then π_v is an irreducible constituent of an induced representation $I_R^M(\tau_v)$, where $R \in \mathcal{P}^M(M_1)$, $M_1 \subset M$ and

$\tau_v \in \Pi_2(M_R(\mathbb{Q}_v))$. Then $I_P^G(\pi_{v,\lambda})$ is canonically isomorphic to a subrepresentation of $I_{P(R)}^G(\tau_{v,\lambda})$ and by (2.13) we have

$$r_{Q|P}(\pi_v, \lambda) = r_{Q(R)|P(R)}(\tau_v, \lambda)\,,$$

where $P(R) \subset P$, $Q(R) \subset Q$. Now recall that there is a canonical inclusion $\mathfrak{a}_P^* \subset \mathfrak{a}_{P(R)}^*$ and with respect to this inclusion, we have $(\mathfrak{a}_P^*)^+ \subset \overline{(\mathfrak{a}_{P(R)}^*)^+}$. Thus α can be identified with an element of $\overline{(\mathfrak{a}_{P(R)}^*)^+}$. Hence (3.11) holds for all tempered $\pi_v \in \Pi(M(\mathbb{Q}_v))$.

Now let π_v be an arbitrary representation in $\Pi(M(\mathbb{Q}_v))$. Then π_v is the Langlands quotient $J_R^M(\tau_v, \mu)$ of a representation $I_R^M(\tau_v, \mu)$, where M_R is an admissible Levi subgroup of M, $\tau_v \in \Pi(M_R(\mathbb{Q}_v))$ is a tempered representation, and μ is a point in the chamber of $(\mathfrak{a}_R^M)^* = \mathfrak{a}_R^*/\mathfrak{a}_M^*$ attached to R [Si3]. Set $\Lambda = \mu + \lambda$. Then, as explained in [A7, p. 30], we have

$$r_{Q|P}(\pi_v, \lambda) = r_{Q(R)|P(R)}(\tau_v, \Lambda)\,. \tag{3.14}$$

Let $\rho_v \in \mathfrak{a}_M^*$ be defined by

$$\delta_P(a)^{1/2} = q_v^{\rho_v(H(a))}, \quad a \in A_M\,.$$

Then it follows from Theorem 3.3 of Chapter XI of [BW] that

$$\langle \mu, \beta^\vee \rangle \le \langle \rho_v, \beta^\vee \rangle\,, \quad \beta \in \Phi(R, A_R)\,.$$

Since μ belongs to $\mathfrak{a}_R^*/\mathfrak{a}_M^*$, it follows that $\|\mu\| \le \|\rho_v\|$. Let $\alpha \in (\mathfrak{a}_P^*)^+$. As observed above, α can be identified with an element of $\overline{(\mathfrak{a}_{P(R)}^*)^+}$. Hence, combining (3.13) and (3.14) the desired estimation (3.11) follows. □

Next we consider the infinite place. Let $\pi \in \Pi(M(\mathbb{R}))$ and let $\phi\colon W_{\mathbb{R}} \to {}^L M$ be the map associated to π. Let $\mu, \nu \in X^*(T) \otimes \mathbb{C}$ be the elements determined by the map ϕ (see [La3], [A7, p. 34]). To indicate the dependence on π, we shall write μ_π and ν_π. We note that there is a canonical injection of the space

$$\mathfrak{a}_{M,\mathbb{C}}^* = X^*(M)_{\mathbb{Q}} \otimes \mathbb{C}$$

into $X^*(T) \otimes \mathbb{C}$.

LEMMA 3.3. *Let $\beta \in (\mathfrak{a}_M^*)^+$. There exist $C, c > 0$ such that*

$$\left| r_{Q|P}(\pi, z\beta)^{-1} \cdot \tfrac{d}{dz} r_{Q|P}(\pi, z\beta) \right| \le C \tag{3.15}$$

for all $\pi \in \Pi(M(\mathbb{R}))$ and all $z \in \mathbb{C}$ with $\mathrm{Re}(z) \ge c$.

Proof. First assume that $\pi \in \Pi(M(\mathbb{R}))$ is tempered. As explained above, the normalizing factor $r_{Q|P}(\pi, \lambda)$ is a product of finitely many meromorphic functions each of which is either of the form (3.9) or (3.10). So it suffices to consider the logarithmic derivative of the Gamma factors. Recall that for $\mathrm{Re}(z) > 0$ the following formula holds

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} = \frac{1}{2z} + \log z - \int_0^\infty \left\{ \frac{1}{2} - \frac{1}{u} + \frac{1}{e^u - 1} \right\} e^{-uz}\, du$$

[Wh, p. 248]. Let $0 \le a \le 1$ and $\mathrm{Re}(z) > 2$. Then we get

$$\left|\frac{\Gamma'(z)}{\Gamma(z)} - \frac{\Gamma'(z+a)}{\Gamma(z+a)}\right| \le \frac{2}{\mathrm{Re}(z)} + \log\left|1 + \frac{a}{z-1}\right| + \frac{\pi}{2} + 2\int_0^\infty \left|\frac{1}{2} - \frac{1}{u} + \frac{1}{e^u - 1}\right| e^{-u\,\mathrm{Re}(z)/2} du\,.$$

Hence there exists $C > 0$ such that

$$\left|\frac{\Gamma'(z)}{\Gamma(z)} - \frac{\Gamma'(z+a)}{\Gamma(z+a)}\right| \le C \quad \text{for } \mathrm{Re}(z) > 2\,. \tag{3.16}$$

Let $\beta \in (\mathfrak{a}_M^*)^+$, $\nu \in \mathfrak{a}_{M,\mathbb{C}}^*$ and $\alpha \in \Sigma_P(G,T)$. Since $\alpha \restriction \mathfrak{a}_M$ is a root of (P,A), it follows that $\langle \beta, \alpha^\vee \rangle > 0$. Let $\alpha \in \Sigma_P(G,T)$ be such that $\langle \overline{\sigma}\mu_\pi - \mu_\pi, \alpha^\vee \rangle \le 0$. Then we have $\mathrm{Re}\langle \mu_\pi, \alpha^\vee \rangle \ge 0$. Hence

$$\mathrm{Re}\langle \mu_\pi + z\beta + \nu, \alpha^\vee \rangle \ge \langle \beta, \alpha^\vee \rangle\, \mathrm{Re}(z) - \|\alpha\| \cdot \|\nu\|\,.$$

Using (3.16) together with (3.9) and (3.10), it follows that there exist constants $C, c > 0$ such that

$$\left|\frac{L(1, \tau_{z\beta+\nu})}{L(0, \tau_{z\beta+\nu})} \cdot \frac{d}{dz}\left(\frac{L(0, \tau_{z\beta+\nu})}{L(1, \tau_{z\beta+\nu})}\right)\right| \le C \tag{3.17}$$

for $\mathrm{Re}(z) \ge c(1 + \|\nu\|)$ and all $\pi \in \Pi(M(\mathbb{R}))$, where τ_λ and π_λ are related by (3.8).

Now let π be an arbitrary representation in $\Pi(M(\mathbb{R}))$. Then there exist a parabolic subgroup R of M, a tempered representation τ of $M_R(\mathbb{R})$ and a point ξ in the positive chamber of $(\mathfrak{a}_R^*/\mathfrak{a}_M^*)$ attached to R such that π is equivalent to the Langlands quotient $J_R(\tau, \xi)$ [La3]. Set $\Lambda = \xi + \lambda$. Then, as explained in [A7, p. 30], we have

$$r_{Q|P}(\pi, \lambda) = r_{Q(R)|P(R)}(\tau, \Lambda)\,.$$

Moreover, by Theorem 5.2 of Chapter IV of [BW] it follows that

$$\left|\mathrm{Re}\langle \xi, \alpha^\vee \rangle\right| \le 4\|\rho_P\| \quad \text{for all } \alpha \in \Delta_P\,.$$

Together with (3.17) this implies the claimed result. □

4 Poles of Global Normalizing Factors

Let $M \in \mathcal{L}(M_0)$ and $P, Q \in \mathcal{P}(M)$. Let $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ with $\pi = \otimes_v \pi_v$. Then by §2 the infinite product

$$r_{Q|P}(\pi, \lambda) = \prod_v r_{Q|P}(\pi_v, \lambda)$$

is absolutely convergent in some chamber and admits an analytic extension to a meromorphic function of $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$. In this section we shall study the poles of $r_{Q|P}(\pi, \lambda)$.

Recall that a function $f\colon \mathbb{C}^N \to \mathbb{C}$ is called a meromorphic function of order $p \geq 0$, if f can be written as a quotient $f = g_1/g_2$ of two entire functions $g_i : \mathbb{C}^N \to \mathbb{C}$, $i = 1, 2$, satisfying

$$|g_i(z)| \leq Ce^{c\|z\|^p}, \quad z \in \mathbb{C}^N, \ i = 1, 2,$$

for certain constants $C, c > 0$. With this definition we have the following proposition.

PROPOSITION 4.1. *Let $n = \dim G(\mathbb{R})/K_\infty$. For all $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$, the normalizing factor $r_{Q|P}(\pi, \lambda)$ is a meromorphic function of $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ of order $\leq n + 2$.*

Proof. By (2.22) we may assume that $\dim(\mathfrak{a}_M/\mathfrak{a}_G) = 1$. Let $P \in \mathcal{P}(M)$ and let α be the unique simple root of (P, A_M). Let $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$. Then $\mathcal{A}^2_\pi(P) \neq \{0\}$ and we have to consider the intertwining operator $M_{\overline{P}|P}(\pi, \lambda)$. Recall that $M_{\overline{P}|P}(\pi, \lambda)$ is unitary for $\lambda \in i\mathfrak{a}^*_M$. In particular, $M_{\overline{P}|P}(\pi, \lambda)$ is regular at $\lambda = 0$. Put

$$M(\pi, \lambda) = M_{P|\overline{P}}(\pi, 0)M_{\overline{P}|P}(\pi, \lambda), \quad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}}.$$

Next consider the normalized intertwining operator $N_{\overline{P}|P}(\pi, \lambda)$ which is defined by (2.19). It follows from (2.8), (2.9), (2.16) and (2.17) that

$$N_{P|\overline{P}}(\pi, 0)^* N_{P|\overline{P}}(\pi, 0) = \mathrm{Id}.$$

Hence $N_{P|\overline{P}}(\pi, \lambda)$ is regular at $\lambda = 0$ and $N_{P|\overline{P}}(\pi, 0)$ is invertible. Put

$$N(\pi, \lambda) = N_{P|\overline{P}}(\pi, 0)N_{\overline{P}|P}(\pi, \lambda), \quad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}}.$$

Furthermore by (2.20) and (2.21) we get

$$\left|r_{\overline{P}|P}(\pi, \lambda)\right| = 1, \quad \lambda \in i\mathfrak{a}^*_M. \tag{4.1}$$

Thus $r_{\overline{P}|P}(\pi, \lambda)$ is also regular at $\lambda = 0$ and $r_{\overline{P}|P}(\pi, 0) \neq 0$. By (2.19) we get

$$M(\pi, \lambda) = r_{P|\overline{P}}(\pi, 0)r_{\overline{P}|P}(\pi, \lambda)N(\pi, \lambda). \tag{4.2}$$

Now observe that there exists an open compact subgroup $K_f \subset G(\mathbb{A}_f)$ such that $\mathcal{A}^2_\pi(P)_{K_f} \neq \{0\}$. Hence there exists $\sigma \in \Pi(K_\infty)$ such that $\mathcal{A}^2_\pi(P)_{K_f,\sigma} \neq \{0\}$ (cf. section 1.8 for the definition). Put

$$d = \dim \mathcal{A}^2_\pi(P)_{K_f,\sigma}$$

and

$$c(\pi, \sigma) = r_{P|\overline{P}}(\pi, 0)^{-d}.$$

Then $|c(\pi, \sigma)| = 1$. Let $M(\pi, \lambda)_{K_f,\sigma}$ (resp. $N(\pi, \lambda)_{K_f,\sigma}$) denote the restriction of $M(\pi, \lambda)$ (resp. $N(\pi, \lambda)$) to the subspace $\mathcal{A}^2_\pi(P)_{K_f,\sigma}$. Then we have $\det N(\pi, \lambda)_{K_f,\sigma} \not\equiv 0$ and by (4.2) we get

$$r_{\overline{P}|P}(\pi, \lambda)^d = c(\pi, \sigma)\,\frac{\det M(\pi, \lambda)_{K_f,\sigma}}{\det N(\pi, \lambda)_{K_f,\sigma}}. \tag{4.3}$$

Thus it suffices to prove that both the numerator and the denominator on the right-hand side are meromorphic functions of order $\leq n+2$. As for the numerator, it follows from Theorem 0.1 of [Mü3] that $\det M(\pi,\lambda)_{K_f,\sigma}$ is a meromorphic function of $\lambda \in \mathfrak{a}^*_{M,\mathbb{C}}$ of order $\leq n+2$. In fact, in [Mü3] we only dealt with the case of the trivial character ξ. However, all the results of [Mü3] can be extend without any difficulty to the case of a nontrivial character ξ. It remains to consider the denominator. By (2.11), (2.16) and (2.17) there exists a finite set S_π of finite places of $\mathbb{Q}$ such that

$$\det N(\pi,\lambda)_{K_f,\sigma} = \det\left(R_{P|\overline{P}}(\pi_\infty,0)_\sigma R_{\overline{P}|P}(\pi_\infty,\lambda)_\sigma\right) \cdot \prod_{v\in S_\pi} \det\left(R_{P|\overline{P}}(\pi_v,0)_{K_v} R_{\overline{P}|P}(\pi_v,\lambda)_{K_v}\right), \quad (4.4)$$

where $R_{\overline{P}|P}(\pi_\infty,\lambda)_\sigma$ denotes the restriction of $R_{\overline{P}|P}(\pi_\infty,\lambda)$ to the σ-isotypical subspace $\mathcal{H}_P(\pi_\infty)_\sigma$ of $\mathcal{H}_P(\pi_\infty)$ and $R_{\overline{P}|P}(\pi_v,\lambda)_{K_v}$ denotes the restriction of $R_{\overline{P}|P}(\pi_v,\lambda)$ to the subspace $\mathcal{H}_P(\pi_v)^{K_v}$ of K_v-invariant functions. By Theorem 2.1 of [A7], $R_{\overline{P}|P}(\pi_\infty,\lambda)$ is a rational function of $\lambda(\alpha^\vee)$ and for each finite place v, $R_{\overline{P}|P}(\pi_v,\lambda)$ is a rational function of $q_v^{-\lambda(\alpha^\vee)}$. Therefore $\det(R_{P|\overline{P}}(\pi_\infty,0)_\sigma R_{\overline{P}|P}(\pi_\infty,\lambda)_\sigma)$ is a rational functions of $\lambda(\alpha^\vee)$ and for each $v<\infty$, $\det(R_{P|\overline{P}}(\pi_v,0)_{K_v} R_{\overline{P}|P}(\pi_v,\lambda)_{K_v})$ is a rational function of $q_v^{-\lambda(\alpha^\vee)}$. Since the function $z\in\mathbb{C}\mapsto q^{-z}$ is entire and of order 1, it follows that $\det N(\pi,\lambda)_{\sigma,K_f}$ is a meromorphic function of $\lambda\in\mathfrak{a}^*_{M,\mathbb{C}}$ of order ≤ 1. By (4.3) it follows that $r_{\overline{P}|P}(\pi,\lambda)^d$ and hence $r_{\overline{P}|P}(\pi,\lambda)$ is a meromorphic function of $\lambda\in\mathfrak{a}^*_{M,\mathbb{C}}$ of order $\leq n+2$. □

Remark. Assume that G is a quasi-split connected reductive group over a number field F with ring of adèles $\mathbb{A}_F$. Let $P=MN$ be a maximal parabolic subgroup of G. Let π be a globally generic cuspidal representation of $M(\mathbb{A}_F)$. Then it follows from [Sh4] that the intertwining operator $M_{\overline{P}|P}(\pi,\lambda)$ can be normalized by automorphic L-functions. Furthermore in [GS], Gelbart and Shahidi proved that the L-functions occurring in the normalizing factor are meromorphic functions of order 1. Therefore, one should expect that the normalizing factor $r_{Q|P}(\pi,\lambda)$ is of order 1 in general.

Now assume that $\dim \mathfrak{a}_M/\mathfrak{a}_G = 1$. Our next goal is to estimate the number of poles of $r_{\overline{P}|P}(\pi,\lambda)$ in a circle of radius $R>0$. For this purpose we have to introduce some notation.

Let $\Pi_{\mathrm{disc}}(M(\mathbb{A});K_f)$ be the space of representations defined by (1.2). For every $\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A});K_f)$ we have $\mathcal{A}^2_\pi(P)^{K_f}\neq\{0\}$.

Let π be an irreducible unitary representation of $M(\mathbb{R})$ and let $I_P^G(\pi)$

be the induced representation of $G(\mathbb{R})$. Recall that among all K_∞-types $\tau_{\Lambda'}$ occurring in $I_P^G(\pi)$, the minimal K_∞-types of $I_P^G(\pi)$ are those τ_Λ for which $|\Lambda' + 2\rho_K|^2$ is minimizing at $\Lambda' = \Lambda$. Let $W_P(\pi)$ be the set of minimal K_∞-types of $I_P^G(\pi)$. Then $W_P(\pi)$ is a non empty finite subset of $\Pi(K_\infty)$. Let λ_π be the Casimir eigenvalues of π and for any $\tau \in \Pi(K_\infty)$, let λ_τ be the Casimir eigenvalue of τ. Put

$$\Lambda_\pi := \min_{\tau \in W_P(\pi)} \sqrt{\lambda_\pi^2 + \lambda_\tau^2}\,. \tag{4.5}$$

If $\pi \in \Pi(M(\mathbb{A}))$, put

$$\Lambda_\pi := \Lambda_{\pi_\infty}.$$

For a given pole η of $r_{\overline{P}|P}(\pi, \lambda)$, let $m(\eta)$ denote its order. Set

$$n_P(\pi, R) = \sum_{|\eta| \le R} m(\eta)\,,$$

where the sum runs over all poles of $r_{\overline{P}|P}(\pi, \lambda)$.

PROPOSITION 4.2. *Let $m = \dim G$ and let K_f be an open compact subgroup of $G(\mathbb{A}_f)$. There exists $C > 0$ such that for all $R > 0$ and all $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}); K_f)$ we have*

$$n_P(\pi, R) \le C(1 + R^2 + \Lambda_\pi^2)^{8m}.$$

Proof. Let $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}); K_f)$. Then there exists $\sigma \in \Pi(K_\infty)$ such that $\mathcal{A}_\pi^2(P)_{K_f,\sigma} \neq \{0\}$. Put

$$\overline{N}(\pi, \lambda) := N_{P|\overline{P}}(\pi, \lambda) N_{\overline{P}|P}(\pi, 0)\,, \quad \lambda \in \mathfrak{a}_{M,\mathbb{C}}^*\,.$$

Then

$$\overline{N}(\pi, \lambda) N(\pi, \lambda) = \mathrm{Id}$$

and by (4.3) we get

$$r_{\overline{P}|P}(\pi, \lambda)^d = c(\pi, \sigma) \det\left(M(\pi, \lambda)_{K_f,\sigma}\right) \cdot \det\left(\overline{N}(\pi, \lambda)_{K_f,\sigma}\right). \tag{4.6}$$

Thus it suffices to estimate the number of poles of the functions occurring on the right. It follows from Proposition 6.6 and Lemma 6.1 of [Mü3], that the number of poles, counted with their order, of $\det M(\pi, \lambda)_{K_f,\sigma}$ in the disc $|\lambda| \le R$ is bounded by

$$C(1 + R^2 + \lambda_\pi^2 + \lambda_\sigma^2)^{8m},$$

where $C > 0$ is independent of π and σ. As noted above, in [Mü3] we only dealt with the case of the trivial character ξ. However, everything can be extended to a nontrivial character ξ without any difficulty.

It remains to consider $\det \overline{N}(\pi, \lambda)_{K_f,\sigma}$. For any place v let

$$\overline{R}(\pi_v, \lambda) = R_{P|\overline{P}}(\pi_v, \lambda) R_{\overline{P}|P}(\pi_v, 0)\,.$$

By (2.16) and (2.17) we have

$$\overline{N}(\pi,\lambda)\circ j_P = j_P\circ\left((\otimes_v \overline{R}(\pi_v,\lambda))\otimes \mathrm{Id}\right)$$

and there exists a finite set S_π of finite places, which depends only on π and K_f, such that

$$\overline{R}(\pi_v,\lambda)_{K_v} = \mathrm{Id}$$

for all $v\notin S_\pi\cup\{\infty\}$. Thus

$$\det\left(\overline{N}(\pi,\lambda)_{K_f,\sigma}\right) = \det\left(\overline{R}(\pi_\infty,\lambda)_\sigma\right)\cdot\prod_{v\in S_\pi}\det\left(\overline{R}(\pi_v,\lambda)_{K_v}\right). \tag{4.7}$$

Let $n_P(\pi_v,R)$, $v<\infty$, (resp. $n_P(\pi_\infty,R)$) denote the number of poles, counted with the order, of $\det(\overline{R}(\pi_v,\lambda)_{K_v})$ (resp. $\det(\overline{R}(\pi_\infty,\lambda)_\sigma)$) in the disc $|\lambda|<R$. Then we have to estimate $n_P(\pi_v,R)$ for any $v\le\infty$.

Let $v<\infty$ and let π_v be any irreducible unitary representation of $M(\mathbb{Q}_v)$. Let $\widetilde{\alpha}\in\mathfrak{a}_M$ be as (3.5). By Theorem 2.2.2 of [Sh1] there exists a polynomial $Q_v(z)$, $Q_v(0)=1$, such that

$$Q_v(q_v^{-\lambda(\widetilde{\alpha})})J_{P|\overline{P}}(\pi_v,\lambda)$$

is a holomorphic and non-zero operator. Moreover, the degree of the polynomial Q_v is independent of π_v. Let

$$d_v = \dim\mathcal{H}_P(\pi_v)^{K_v}.$$

Then it follows from (2.7) and the definition of $\overline{R}(\pi_v,\lambda)$ that

$$r_{P|\overline{P}}(\pi_v,\lambda)^{d_v}Q_v(q_v^{-\lambda(\widetilde{\alpha})})^{d_v}\det\left(\overline{R}(\pi_v,\lambda)_{K_v}\right)$$

is a holomorphic function on $\mathfrak{a}_{M,\mathbb{C}}^*$. By (r.5) there exist polynomials $P_1(z)$ and $P_2(z)$ such that

$$r_{P|\overline{P}}(\pi_v,\lambda) = \frac{P_1(q_v^{-\lambda(\widetilde{\alpha})})}{P_2(q_v^{-\lambda(\widetilde{\alpha})})}.$$

Thus it suffices to estimate the number of zeros of $P_1(q_v^{-\lambda(\widetilde{\alpha})})$ and $Q_v(q_v^{-\lambda(\widetilde{\alpha})})$, respectively, in a circle of radius $R>0$. First observe that for every $z\in\mathbb{C}$ the number of solutions of $q_v^{-s}=z$ in the disc $|s|\le R$ is bounded by $1+(2\pi)^{-1}\log(q_v)R$. Furthermore, the degree of the polynomial Q_v is bounded by some constant $c_v>0$ which is independent of π_v. Using Lemma 3.1 and (2.1)–(2.3) of [A7], it follows that the degree of the polynomial $P_1(z)$ is also bounded by a constant which is independent of π_v. This implies that there exists $C_v>0$ such that

$$n_P(\pi_v,R)\le C_v\dim\left(\mathcal{H}_P(\pi_v)^{K_v}\right)(1+R)$$

for all $\pi_v\in\Pi(M(\mathbb{Q}_v))$ and $R>0$. It remains to estimate the dimension of $\mathcal{H}_P(\pi_v)^{K_v}$. Suppose that π_v is the component at v of a representation $\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A}));K_f)$. Then there exists $\xi\in\Pi(A_M(\mathbb{R})^0)$ such that

$\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))_\xi$. Let $\mathcal{H}_P(\pi)_\sigma^{K_f}$ be the σ-isotypical subspace of $\mathcal{H}_P(\pi)^{K_f}$. By (2.3) it follows that

$$\mathcal{H}_P(\pi)_\sigma^{K_f} \otimes \mathrm{Hom}_{M(\mathbb{A})}\left(\pi, I_{M(\mathbb{Q})AM(\mathbb{R})^0}^{M(\mathbb{A})}(\xi)\right) \cong \mathcal{A}_\pi^2(P)_{K_f,\sigma}\,.$$

Moreover we have

$$\mathcal{H}_P(\pi)_\sigma^{K_f} \cong \mathcal{H}_P(\pi_\infty)_\sigma \otimes \bigotimes_{v<\infty} \mathcal{H}_P(\pi_v)^{K_v}$$

and $\dim \mathcal{H}_P(\pi_v)^{K_v} = 1$ for $v \notin S_\pi$. Thus it follows that

$$\dim \mathcal{H}_P(\pi_v)^{K_v} \le \dim A_\pi^2(P)_{K_f,\sigma}\,.$$

The right-hand side can be estimated by Lemma 6.1 of [Mü3]. It follows that

$$n_P(\pi_v, R) \le C_v(1+\lambda_\pi^2+\lambda_\sigma^2)^{3m}(1+R)\,. \tag{4.8}$$

Now let $v=\infty$ and let $\pi_\infty \in \Pi(M(\mathbb{R}))$. Set

$$\overline{J}(\pi_\infty,\lambda) = J_{P|\overline{P}}(\pi_\infty,\lambda)J_{\overline{P}|P}(\pi_\infty,0)\,.$$

Then

$$\overline{R}(\pi_\infty,\lambda) = \left(r_{P|\overline{P}}(\pi_\infty,\lambda)r_{\overline{P}|P}(\pi_\infty,0)\right)^{-1}\overline{J}(\pi_\infty,\lambda)\,. \tag{4.9}$$

Let $K_{M,\infty} = K_\infty \cap M(\mathbb{R})$ and let

$$\sigma|_{K_{M,\infty}} = \bigoplus_{\tau\in\Pi(K_{M,\infty})} n_\tau \tau\,.$$

Set

$$[\sigma : \pi_\infty] = \sum_{\tau\in\Pi(K_{M,\infty})} n_\tau[\tau : \pi_\infty|_{K_{M,\infty}}]\,.$$

By Corollary 4.7 of [VW], there exist complex numbers $a_i(\pi_\infty)$, $i=1,\dots,r$, and $b_i(\pi_\infty,\sigma)$, $i=1,\dots,r[\sigma:\pi_\infty]$, with $r=r(\pi_\infty)$ depending only on π_∞, and a constant $C\in\mathbb{C}$, such that

$$\det \overline{J}(\pi_\infty,\lambda) = C\,\frac{\prod_{i=1}^{r(\pi_\infty)}\Gamma(\langle\lambda,\alpha^\vee\rangle/(4\langle\rho_P,\alpha^\vee\rangle)-a_i(\pi_\infty))^{[\sigma:\pi_\infty]}}{\prod_{i=1}^{r(\pi_\infty)[\sigma:\pi_\infty]}\Gamma(\langle\lambda,\alpha^\vee\rangle/(4\langle\rho_P,\alpha^\vee\rangle)-b_i(\pi_\infty))}\,. \tag{4.10}$$

LEMMA 4.3. *There exists $c>0$ such that $r(\tau)\le c$ for all $\tau\in\Pi(M(\mathbb{R}))$.*

Proof. Let b_τ be the polynomial which is associated to τ by Theorem 1.5 of [VW]. Then $r(\tau)$ is the degree of b_τ [VW, p. 228]. So we have to estimate the degree of b_τ. The polynomial b_τ is obtained from a more general polynomial $b_{\tau,\lambda}$ occurring in Theorem 2.2 of [VW] by choosing $\lambda = 4\rho_P$. The polynomial $b_{\tau,\lambda}$ is associated to τ and a finite dimensional representation (η, F) of G satisfying the conditions (1)–(3) in [VW, p. 210]. Then λ is the action of $\mathfrak{a}_M$ on $F^{\mathfrak{n}}$. It follows from the constructions on p. 217–219 in [VW], that the polynomial $b_{\tau,\lambda}$ is the product of the denominator of β and the denominator of the element $\check{z}_\nu$, defined on p. 219. Let $\Omega\in Z(\mathfrak{g}_\mathbb{C})$ be

the Casimir element, and let χ_Λ be the infinitesimal character of τ. Then it follows that $b_{\tau,\lambda}$ equals

$$b_{\tau,\lambda}(\nu) = \prod_{\mu\in\Pi(F)-\{\lambda\}} \big(\chi_{\Lambda+\nu}(\Omega) - \chi_{\Lambda+\nu+\lambda-\mu}(\Omega)\big)^{r(\mu)} \cdot \prod_{\mu\in\Pi(F)-\{\lambda\}} \big(\chi_{\Lambda+\nu+\lambda}(\Omega) - \chi_{\Lambda+\nu+\mu}(\Omega)\big)^{r(\mu)}\,.$$

Here $\Pi(F)$ denotes the set of weights of F with respect to a fixed Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and $r(\mu)$ is the multiplicity of a given weight μ. From this description of $b_{\tau,\lambda}$ it follows that

$$r(\tau) \le 2(\dim F - 1)\,.$$

Finally, b_τ is obtained by choosing F to be the representation described in example 2.1 of [VW]. □

Now recall that the poles of the Gamma function $\Gamma(z)$ are simple poles at $z = 0, -1, -2, \dots$ and $1/\Gamma(z)$ is entire. Then it follows from (4.10) together with Lemma 4.3 that there exists a constant $C_1 > 0$, independent of π_∞, such that the number of zeros, counted with their order, of $\det \overline{J}(\pi_\infty, \lambda)$ in the disc $|\lambda| \le R$ is bounded by

$$C_1[\sigma : \pi_\infty](1 + R)\,. \tag{4.11}$$

By Theorem 8.1 of [K] and remark 1 following Theorem 8.4 in [K], we have

$$[\sigma : \pi_\infty] \le \sum_{\tau\in\Pi(K_{M,\infty})} n_\tau \dim\tau \le \dim\sigma\,.$$

Furthermore, by Weyl's dimension formula, there exists $C_2 > 0$ such that $\dim\sigma \le C_2(1 + \lambda_\sigma^2)^p$, where $p = 1/2 \dim K$. Thus (4.11) is bounded by

$$C_2(1 + \lambda_\sigma^2)^p(1 + R)\,.$$

It remains to consider the normalizing factor $r_{P|\overline{P}}(\pi_\infty, \lambda)$. It is given by formula (3.7). Let $\phi_\lambda : W_{\mathbb{R}} \to {}^L M$ be the map associated to $(\pi_\infty)_\lambda$. Let q be the number of irreducible constituents occurring in the decomposition (3.8) of $\rho \cdot \phi_\lambda$. Then q is bounded independently of π and it follows from the description of the L-factors in §3 that $r_{P|\overline{P}}(\pi_\infty, \lambda)$ is a product of q meromorphic functions of the form (3.9) or (3.10). From the form of these functions it follows immediately that the number of poles, counted with their order, of $r_{P|\overline{P}}(\pi_\infty, \lambda)$ in the disc $|\lambda| \le R$ is bounded by $C(1 + R)$. Putting our estimated together, we have proved that there exists $C > 0$, depending on K_f, such that

$$n_P(\pi, R) \le C(1 + R^2 + \lambda_\pi^2 + \lambda_\sigma^2)^{8m} \tag{4.12}$$

for all $R \geq 0$, and all $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ and $\sigma \in \Pi(K_\infty)$ such that $A^2_\pi(P)_{K_f,\sigma} \neq \{0\}$.

Let $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}); K_f)$. Let τ be a minimal K_∞-type of $I_P^G(\pi_\infty)$. Choose $\sigma \in \Pi(K_\infty)$ such that $\sigma = \tau$. Then (4.12) applied to σ together with the definition of Λ_π implies the proposition. □

COROLLARY 4.4. *Let $m = \dim G$ and $n = 16m + 2$. There exists $C > 0$, depending on K_f, such that for each $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}); K_f)$ we have*

$$\sum_{\rho \neq 0} \frac{m(\rho)}{|\rho|^n} \leq C(1 + \Lambda_\pi^2)^{8m},$$

where ρ runs over the poles of $r_{\overline{P}|P}(\pi, \lambda)$.

5 Logarithmic Derivatives of Global Normalizing Factors

In this section we shall study generalized logarithmic derivatives of the global normalizing factors. First we assume that $M \in \mathcal{L}(M_0)$ is such that $\dim \mathfrak{a}_M / \dim \mathfrak{a}_G = 1$. Let $P \in \mathcal{P}(M)$ and let α be the unique simple root of (P, A). Let $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ with $\pi = \otimes_v \pi_v$. By property (r.5) satisfied by the local normalizing factors, it follows that for each place v, there exists a meromorphic function $\tilde{r}_{\overline{P}|P}(\pi_v, z)$ of one complex variable z such that the local normalizing factor $r_{\overline{P}|P}(\pi_v, \lambda)$ is given by

$$r_{\overline{P}|P}(\pi_v, \lambda) = \tilde{r}_{\overline{P}|P}\big(\pi_v, \lambda(\alpha^\vee)\big).$$

Let

$$\tilde{r}_{\overline{P}|P}(\pi, z) := \prod_v \tilde{r}_{\overline{P}|P}(\pi_v, z).$$

The infinite product is absolutely convergent in the half-plane $\mathrm{Re}(z) > \rho_P(\alpha^\vee)$, admits a meromorphic continuation to $\mathbb{C}$ and the global normalizing factor is given by

$$r_{\overline{P}|P}(\pi, \lambda) = \tilde{r}_{\overline{P}|P}\big(\pi, \lambda(\alpha^\vee)\big), \quad \lambda \in \mathfrak{a}^*_{M,\mathbb{C}}.$$

Our present goal is to estimate the logarithmic derivative of $\tilde{r}_{\overline{P}|P}(\pi, z)$ along the imaginary axis.

To begin with, observe that by Lemma 3.2 and Lemma 3.3 there exist $C, c > 0$ such that

$$\left|\tilde{r}_{\overline{P}|P}(\pi, z)^{-1} \tfrac{d}{dz} \tilde{r}_{\overline{P}|P}(\pi, z)\right| \leq \sum_{v \leq \infty} \left|\tilde{r}_{\overline{P}|P}(\pi_v, z)^{-1} \tfrac{d}{dz} \tilde{r}_{\overline{P}|P}(\pi_v, z)\right| \leq C \quad (5.1)$$

for all $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ and $\mathrm{Re}(z) \geq c$. Using (2.20) and (2.21), it follows that the function $\widetilde{r}_{\overline{P}|P}(\pi, z)$ satisfies

$$\widetilde{r}_{\overline{P}|P}(\pi, z)\overline{\widetilde{r}_{\overline{P}|P}(\pi, -\overline{z})} = 1. \quad (5.2)$$

Hence we get

$$\left|\tilde{r}_{\overline{P}|P}(\pi,z)^{-1}\frac{d}{dz}\tilde{r}_{\overline{P}|P}(\pi,z)\right| = \left|\tilde{r}_{\overline{P}|P}(\pi,-\overline{z})^{-1}\left(\frac{d\tilde{r}_{\overline{P}|P}}{dz}\right)(\pi,-\overline{z})\right|,$$

and together with (5.1) we obtain the following proposition.

PROPOSITION 5.1. *There exist $C, c > 0$ such that*

$$\left|\tilde{r}_{\overline{P}|P}(\pi,z)^{-1}\tfrac{d}{dz}\tilde{r}_{\overline{P}|P}(\pi,z)\right| \le C$$

for all $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ and all $z \in \mathbb{C}$ with $|\operatorname{Re}(z)| \ge c$.

In order to get estimates for the logarithmic derivative on the imaginary axis, we shall use the partial fraction decomposition of the meromorphic function $\widetilde{r}_{\overline{P}|P}(\pi,z)^{-1}(d/dz)(\widetilde{r}_{\overline{P}|P}(\pi,z))$, which allows us to treat the sum of the principal parts separately. Let $n = 16\dim G + 2$. Then it follows from Corollary 4.4 that $\widetilde{r}_{\overline{P}|P}(\pi,z)$ is a meromorphic function of order $\le n$. Thus there exist entire functions $r_1(\pi,z)$ and $r_2(\pi,z)$ of order $\le n$ such that

$$\widetilde{r}_{\overline{P}|P}(\pi,z) = \frac{r_1(\pi,z)}{r_2(\pi,z)}.$$

Furthermore, observe that by (5.2) a complex number η is a zero of $\widetilde{r}_{\overline{P}|P}(\pi,z)$ if and only if $-\overline{\eta}$ is a pole of $\widetilde{r}_{\overline{P}|P}(\pi,z)$. Thus by Hadamard's factorization theorem there exists a polynomial $Q(z)$ of degree $\le n$ such that

$$\widetilde{r}_{\overline{P}|P}(\pi,z) = e^{Q(z)}\frac{\prod_\eta\left[(1-\frac{z}{\eta})\exp\left(\sum_{k=1}^n\frac{1}{k}\left(\frac{z}{\eta}\right)^k\right)\right]^{a(\eta)}}{\prod_\eta\left[(1+\frac{z}{\overline{\eta}})\exp\left(\sum_{k=1}^n\frac{1}{k}\left(-\frac{z}{\overline{\eta}}\right)^k\right)\right]^{a(\eta)}}, \tag{5.3}$$

where η runs over all zeros of $\widetilde{r}_{\overline{P}|P}(\pi,z)$ and $a(\eta)$ denotes the order of the zero η.

Let $D(\pi)$ denote the set of all poles and zeros of $\widetilde{r}_{\overline{P}|P}(\pi,z)$. Given $\eta \in D(\pi)$, we denote by $m(\eta)$ the order of η, i.e. $m(\eta)$ is the integer such that $(z-\eta)^{-m(\eta)}\,\widetilde{r}_{\overline{P}|P}(\pi,z)$ is holomorphic in a neighborhood of η and does not vanish at $z=\eta$. For $\eta \in \mathbb{C}^\times$ we define the function $h_\eta(z)$ by

$$h_\eta(z) = -\frac{1}{\eta}\sum_{k=0}^{n-1}\left(\frac{z}{\eta}\right)^k, \quad z \in \mathbb{C}.$$

Then it follows from Corollary 4.4 that the series

$$f(\pi,z) = \sum_{\substack{\eta\in D(\pi)\\ |\eta|\le 1}}\frac{m(\eta)}{z-\eta} + \sum_{\substack{\eta\in D(\pi)\\ |\eta|>1}} m(\eta)\left\{\frac{1}{z-\eta} - h_\eta(z)\right\} \tag{5.4}$$

is absolutely convergent on compact subsets of $\mathbb{C} \setminus D(\pi)$ and the resulting function $f(\pi, z)$ is a meromorphic function on $\mathbb{C}$ whose set of poles equals $D(\pi)$. Differentiating (5.3), we get

$$\widetilde{r}_{\overline{P}|P}(\pi, z)^{-1} \tfrac{d}{dz} \widetilde{r}_{\overline{P}|P}(\pi, z) = f(\pi, z) - \sum_{\substack{\eta \in D(\pi) \\ |\eta| \leq 1}} m(\eta) h_\eta(z) + Q'(z)\,.$$

Thus there is a polynomial $g(\pi, z)$ of degree $\leq n-1$ such that

$$\widetilde{r}_{\overline{P}|P}(\pi, z)^{-1} \tfrac{d}{dz} \widetilde{r}_{\overline{P}|P}(\pi, z) = f(\pi, z) + g(\pi, z)\,. \tag{5.5}$$

We begin with the investigation of $g(\pi, z)$.

PROPOSITION 5.2. *Let $m = \dim G$. There exist $C, c > 0$ such that*

$$\big|g(\pi, z)\big| \leq C\big(1 + |z|^2 + \Lambda_\pi^2\big)^{18m}$$

for all $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}), K_f)$ and all $z \in \mathbb{C}$ with $|\operatorname{Re}(z)| \leq c$.

Proof. Let $c > 0$ be the constant occurring in Proposition 5.1. First assume that $|\operatorname{Re}(z)| = c$. By Proposition 5.1 it suffices to estimate $f(\pi, z)$. Referring again to Proposition 5.1, it follows that $D(\pi)$ is contained in the strip $|\operatorname{Re}(z)| \leq c$. Hence we may assume that $c > 0$ has been chosen so that for all $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}), K_f)$, the zeros and poles of $\widetilde{r}_{\overline{P}|P}(\pi, z)$ are contained in the strip $|\operatorname{Re}(z)| < c - \delta$, where $\delta > 0$ is independent of π. Hence the poles of $f(\pi, z)$ are contained in $|\operatorname{Re}(z)| < c - \delta$. Let $\eta \in \mathbb{C}$ be a pole of $f(\pi, z)$. Then for $|\operatorname{Re}(z)| \geq c$ we get

$$|z - \eta| \geq \big|\operatorname{Re}(z - \eta)\big| \geq c - \big|\operatorname{Re}(\eta)\big| \geq \delta\,. \tag{5.6}$$

Furthermore, from the definition of $f(\pi, z)$ it follows that

$$\begin{aligned}\big|f(\pi, z)\big| \leq \sum_{|\eta| \leq 2|z|} \frac{|m(\eta)|}{|z - \eta|} + \sum_{1 \leq |\eta| \leq 2|z|} |m(\eta)||h_\eta(z)| \\ + \sum_{|\eta| > 2|z|} |m(\eta)| \left| \frac{1}{z - \eta} - h_\eta(z) \right|.\end{aligned}$$

Using (5.6) and Proposition 4.2, we can estimate the first sum as follows

$$\begin{aligned}\sum_{|\eta| \leq 2|z|} \frac{|m(\eta)|}{|z - \eta|} \leq \frac{1}{\delta} \sum_{|\eta| \leq 2|z|} |m(\eta)| \leq \frac{2}{\delta} n_P\big(\pi, 2|z|\big) \\ \leq C\big(1 + |z|^2 + \Lambda_\pi^2\big)^{8m}.\end{aligned}$$

Again by Proposition 4.2, we obtain for the second sum

$$\begin{aligned}\sum_{1 \leq |\eta| \leq 2|z|} |m(\eta)||h_\eta(z)| &\leq \sum_{1 \leq |\eta| \leq 2|z|} |m(\eta)| \sum_{k=0}^{n-1} \frac{|z|^k}{|\eta|^{k+1}} \\ &\leq C|z|^{n-1} n_P\big(\pi, 2|z|\big) \\ &\leq C_1 |z|^{n-1} \big(1 + |z|^2 + \Lambda_\pi^2\big)^{8m}.\end{aligned} \tag{5.7}$$

Finally, by Corollary 4.4 we get

$$\begin{aligned}\sum_{|\eta|>2|z|} |m(\eta)|\left|\frac{1}{z-\eta} - h_\eta(z)\right| &\le 2|z|^n \sum_{|\eta|>2|z|} \frac{|m(\eta)|}{|\eta|^{n+1}} \\ &\le C(1+\Lambda_\pi^2)^{8m}|z|^n \\ &\le C\big(1+|z|^2+\Lambda_\pi^2\big)^{8m}.\end{aligned} \tag{5.8}$$

Putting our estimates together, it follows that there exists $C > 0$ such that

$$\big|f(\pi,z)\big| \le C\big(1+|z|^2+\Lambda_\pi^2\big)^{18m}$$

for $|\operatorname{Re}(z)| \ge c$. Hence by Proposition 5.1, there exists $C > 0$ such that

$$\big|g(\pi,z)\big| \le C\big(1+|z|^2+\Lambda_\pi^2\big)^{18m} \tag{5.9}$$

for all $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}), K_f)$ and all $z \in \mathbb{C}$ with $|\operatorname{Re}(z)| = c$. Now the proposition follows from the Phragmen–Lindelöf theorem. □

Note that Proposition 5.2 gives an upper bound for $g(\pi, z)$ on the imaginary axis.

We shall now investigate $f(\pi, z)$. From the definition of $f(\pi, z)$ by (5.3) it is clear that the growth of $f(\pi, z)$ along the imaginary axis depends on the distance of the poles and zeros of $\tilde{r}_{\overline{P}|P}(\pi, z)$ from the imaginary axis. Therefore, without any further information about the distribution of the poles and zeros we cannot expect to get any estimates for $f(\pi, i\lambda)$ as $|\lambda| \to \infty$. However, what we can hope for is to obtain estimates for integrals involving $f(\pi, i\lambda)$.

To this end, we decompose $f(\pi, z)$ as follows

$$\begin{aligned}f(\pi,z) = \sum_{\substack{\eta\in D(\pi)\\ |\eta|\le 2|z|}} m(\eta)\frac{1}{z-\eta} - \sum_{\substack{\eta\in D(\pi)\\ 1\le|\eta|\le 2|z|}} m(\eta)h_\eta(z) \\ + \sum_{\substack{\eta\in D(\pi)\\ |\eta|>2|z|}} m(\eta)\left\{\frac{1}{z-\eta} - h_\eta(z)\right\}.\end{aligned}$$

As for the second and the third sum, we observe that the estimations (5.7) and (5.8) are uniform in $z \in \mathbb{C}$. It remains to consider the first sum which we denote by $f_1(\pi, z)$. Let

$$D_\pm(\pi) = \big\{\eta \in D(\pi) \mid \pm m(\eta) > 0\big\}.$$

Then the map $\eta \to -\overline{\eta}$ is a bijection of $D_+(\pi)$ onto $D_-(\pi)$, and therefore, $f_1(\pi, z)$ can be written as

$$f_1(\pi,z) = \sum_{\substack{\eta\in D_+(\pi)\\ |\eta|\le 2|z|}} m(\eta)\left\{\frac{1}{z-\eta} - \frac{1}{z+\overline{\eta}}\right\}.$$

In particular, for $\lambda \in \mathbb{R} \setminus \{0\}$ we get

$$f_1(\pi, i\lambda) = -\sum_{\substack{\eta \in D_+(\pi) \\ |\eta| \le 2|\lambda|}} m(\eta) \frac{2\operatorname{Re}(\eta)}{\operatorname{Re}(\eta)^2 + (\lambda - \operatorname{Im}(\eta))^2}\,.$$

Let $\zeta \in C^\infty(\mathbb{R})$ be such that $0 \le \zeta \le 1$, $\zeta(u) = 0$ for $|u| \ge 3$ and $\zeta(u) = 1$ for $|u| \le 2$. Then it follows that

$$\begin{aligned}\big|f_1(\pi, i\lambda)\big| &\le \sum_{\substack{\eta \in D_+(\pi) \\ |\eta| \le 2|\lambda|}} m(\eta) \frac{2|\operatorname{Re}(\eta)|}{\operatorname{Re}(\eta)^2 + (\lambda - \operatorname{Im}(\eta))^2} \\ &\le \sum_{\eta \in D_+(\pi)} \zeta\left(\frac{|\eta|}{|\lambda|}\right) m(\eta) \frac{2|\operatorname{Re}(\eta)|}{\operatorname{Re}(\eta)^2 + (\lambda - \operatorname{Im}(\eta))^2}\,.\end{aligned}$$

Thus we have proved that for $\lambda \in \mathbb{R} \setminus \{0\}$ the following inequality holds

$$\begin{aligned}\big|f(\pi, i\lambda)\big| \le \sum_{\eta \in D_+(\pi)} \zeta\left(\frac{|\eta|}{|\lambda|}\right) m(\eta) \frac{2|\operatorname{Re}(\eta)|}{\operatorname{Re}(\eta)^2 + (\lambda - \operatorname{Im}(\eta))^2} \\ + C(1 + \lambda^2 + \Lambda_\pi^2)^{8m}. \quad (5.10)\end{aligned}$$

Put

$$F(\lambda) := \begin{cases} \sum_{\eta \in D_+(\pi)} \zeta\left(\frac{|\eta|}{|\lambda|}\right) m(\eta) \frac{2|\operatorname{Re}(\eta)|}{\operatorname{Re}(\eta)^2 + (\lambda - \operatorname{Im}(\eta))^2}\,, & \lambda \ne 0\,; \\ 0\,, & \lambda = 0\,. \end{cases}$$

Note that $0 \notin D(\pi)$. Therefore, on any finite interval $[-a, a]$, $F(\lambda)$ is the sum of finitely many smooth and non-negative functions. Hence $F(\lambda)$ is a smooth and non-negative function. We shall now estimate the integral of $F(u)$ over a finite interval. Using Proposition 4.2 and the properties of ζ, we obtain

$$\begin{aligned}\int_0^\lambda F(u)du &\le \sum_{\substack{\eta \in D_+(\pi) \\ |\eta| \le 3\lambda}} m(\eta) \int_0^\lambda \frac{2|\operatorname{Re}(\eta)|}{\operatorname{Re}(\eta)^2 + (u - \operatorname{Im}(\eta))^2}\, du \\ &\le 2\pi n_P(\pi, 3\lambda) \\ &\le C(1 + \lambda^2 + \Lambda_\pi^2)^{8m}. \quad (5.11)\end{aligned}$$

Let $N \ge 8m + 2$ and $R > 0$. Using integration by parts and (5.11), we obtain

$$\begin{aligned}\left|\int_{-R}^R F(u)(1+u^2)^{-N} du\right| &= \left|\int_{-R}^R \left(\int_0^u F(t)\, dt\right) \tfrac{d}{du}(1+u^2)^{-N} du\right| \\ &\quad + \left|\int_{-R}^R F(t)\, dt\right| (1+R^2)^{-N}\end{aligned}$$

$$\leq C(1+\Lambda_\pi^2)^{8m}\int_{\mathbb{R}}(1+u^2)^{8m}\left|\tfrac{d}{du}(1+u^2)^{-N}\right|du$$
$$\leq C_N(1+\Lambda_\pi^2)^{8m}.$$

Here we have used that by (5.11) the boundary term is bounded by a constant independent of R. Since $F\geq 0$, this inequality implies that $F(u)$ is integrable with respect to the measure $(1+u^2)^{-N}du$. Putting our estimates together, we obtain the following theorem.

Theorem 5.3. *Let $M\in\mathcal{L}(M_0)$ and assume that $\dim\mathfrak{a}_M/\mathfrak{a}_G=1$. Let $P\in\mathcal{P}(M)$ and let $m=\dim G(\mathbb{R})$. For every $N\geq 8m+2$ there exists $C_N>0$ such that for all $\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A});K_f)$ the following inequality holds*

$$\int_{\mathbb{R}}\left|\tilde{r}_{\overline{P}|P}(\pi,iu)^{-1}\tfrac{d}{du}\tilde{r}_{\overline{P}|P}(\pi,iu)\right|(1+u^2)^{-N}du\leq C_N(1+\Lambda_\pi^2)^{18m}.$$

Now suppose that $M\in\mathcal{L}(M_0)$ is arbitrary. Then we have to consider the multidimensional logarithmic derivatives of the normalizing factors defined by Arthur in [A4]. For this purpose we will use the notion of a (G,M) family introduced by Arthur in Section 6 of [A5]. For the convenience of the reader we recall the definition of a (G,M) family and explain some of its properties.

For each $P\in\mathcal{P}(M)$, let $c_P(\lambda)$ be a smooth function on $i\mathfrak{a}_M^*$. Then the set

$$\{c_P(\lambda)\mid P\in\mathcal{P}(M)\}$$

is called a (G,M) family if the following holds: Let $P,P'\in\mathcal{P}(M)$ be adjacent parabolic groups and suppose that λ belongs to the hyperplane spanned by the common wall of the chambers of P and P'. Then

$$c_P(\lambda)=c_{P'}(\lambda)\,.$$

Let

$$\theta_P(\lambda)=\mathrm{vol}\left(\mathfrak{a}_P^G/\mathbb{Z}(\Delta_P^\vee)\right)^{-1}\prod_{\alpha\in\Delta_P}\lambda(\alpha^\vee)\,,\quad\lambda\in i\mathfrak{a}_P^*\,,\tag{5.12}$$

where $\mathbb{Z}(\Delta_P^\vee)$ is the lattice in $\mathfrak{a}_P^G$ generated by the co-roots

$$\{\alpha^\vee\mid\alpha\in\Delta_P\}\,.$$

Let $\{c_P(\lambda)\}$ be a (G,M) family. Then by Lemma 6.2 of [A5], the function

$$c_M(\lambda)=\sum_{P\in\mathcal{P}(M)}c_P(\lambda)\theta_P(\lambda)^{-1},\tag{5.13}$$

which is defined on the complement of a finite set of hyperplanes, extends to a smooth function on $i\mathfrak{a}_M^*$. The value of $c_M(\lambda)$ at $\lambda=0$ is of particular importance in connection with the spectral side of the trace formula. It

can be computed as follows. Let $p = \dim(A_M/A_G)$. Set $\lambda = t\Lambda$, $t \in \mathbb{R}$, $\Lambda \in \mathfrak{a}_M^*$, and let t tend to 0. Then

$$c_M(0) = \tfrac{1}{p!} \sum_{P \in \mathcal{P}(M)} \Big(\lim_{t \to 0} \big(\tfrac{d}{dt}\big)^p c_P(t\Lambda) \Big) \theta_P(\Lambda)^{-1} \tag{5.14}$$

[A5, (6.5)]. This expression is of course independent of Λ.

For any (G, M) family $\{c_P(\lambda) \mid P \in \mathcal{P}(M)\}$ and any $L \in \mathcal{L}(M)$ there is associated a natural (G, L) family which is defined as follows. Let $Q \in \mathcal{P}(L)$ and suppose that $P \subset Q$. The function

$$\lambda \in i\mathfrak{a}_L^* \mapsto c_P(\lambda)$$

depends only on Q. We will denote it by $c_Q(\lambda)$. Then

$$\big\{c_Q(\lambda) \mid Q \in \mathcal{P}(L)\big\}$$

is a (G, L) family. We write

$$c_L(\lambda) = \sum_{Q \in \mathcal{P}(L)} c_Q(\lambda) \theta_Q(\lambda)^{-1}$$

for the corresponding function (5.13).

Let $Q \in \mathcal{P}(L)$ be fixed. If $R \in \mathcal{P}^L(M)$, then $Q(R)$ is the unique group in $\mathcal{P}(M)$ such that $Q(R) \subset Q$ and $Q(R) \cap L = R$. Let c_R^Q be the function on $i\mathfrak{a}_M^*$ which is defined by

$$c_R^Q(\lambda) = c_{Q(R)}(\lambda)\,.$$

Then $\{c_R^Q(\lambda) \mid R \in \mathcal{P}^L(M)\}$ is an (L, M) family. Let $c_M^Q(\lambda)$ be the function (5.13) associated to this (L, M) family.

We consider now special (G, M) families defined by the global normalizing factors. Fix $P \in \mathcal{P}(M)$, $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$ and $\lambda \in i\mathfrak{a}_M^*$. Define

$$\nu_Q(P, \pi, \lambda, \Lambda) := r_{Q|P}(\pi, \lambda)^{-1} r_{Q|P}(\pi, \lambda + \Lambda)\,, \quad Q \in \mathcal{P}(M)\,. \tag{5.15}$$

This set of functions is a (G, M) family [A4, p. 1317]. It is of a special form. Given $\beta \in \Sigma_P^r \cap \Sigma_{\overline{Q}}^r$, set

$$r_\beta(\pi, z) = \tilde{r}_{\overline{P}_\beta | P_\beta}(\pi, z)\,, \quad z \in \mathbb{C}\,.$$

Then by (2.22) we have

$$\nu_Q(P, \pi, \lambda, \Lambda) = \prod_{\beta \in \Sigma_Q^r \cap \Sigma_{\overline{P}}^r} r_\beta\big(\pi, \lambda(\beta^\vee)\big)^{-1} r_\beta\big(\pi, \lambda(\beta^\vee) + \Lambda(\beta^\vee)\big)\,.$$

Suppose that $L \in \mathcal{L}(M)$, $L_1 \in \mathcal{L}(L)$ and $S \in \mathcal{P}(L_1)$. Let

$$\big\{\nu_{Q_1}^S(P, \pi, \lambda, \Lambda) \;\big|\; Q_1 \in \mathcal{P}^{L_1}(L)\big\}$$

be the associated (L_1, L) family and let $\nu_L^S(P, \pi, \lambda, \Lambda)$ be the function (5.13) defined by this family. Set

$$\nu_L^S(P, \pi, \lambda) := \nu_L^S(P, \pi, \lambda, 0)\,.$$

If β is any root in $\Sigma^r(G,A_M)$, let $\beta_L^\vee$ denote the projection of $\beta^\vee$ onto $\mathfrak{a}_L$. If F is a subset of $\Sigma^r(G,A_M)$, let $F_L^\vee$ be the disjoint union of all the vectors $\beta_L^\vee$, $\beta\in F$. Then by Proposition 7.5 of [A4] we have

$$\nu_L^S(P,\pi,\lambda)=\sum_F \operatorname{vol}\big(\mathfrak{a}_L^{L_1}/\mathbb{Z}(F_L^\vee)\big) \cdot\Big(\prod_{\beta\in F} r_\beta\big(\pi,\lambda(\beta^\vee)\big)^{-1} r'_\beta(\pi,\lambda(\beta^\vee))\Big), \quad (5.16)$$

where F runs over all subsets of $\Sigma^r(L_1,A_M)$ such that $F_L^\vee$ is a basis of $\mathfrak{a}_L^{L_1}$. Let $N\in\mathbb{N}$. Then by (5.16) we get

$$\int_{i\mathfrak{a}_L^*/\mathfrak{a}_G^*} \big|\nu_L^S(P,\pi,\lambda)\big|\big(1+\|\lambda\|^2\big)^{-N}d\lambda \le \sum_F \operatorname{vol}\big(\mathfrak{a}_L^{L_1}/\mathbb{Z}(F_L^\vee)\big) \cdot\int_{i\mathfrak{a}_L^*/\mathfrak{a}_G^*}\prod_{\beta\in F}\big|r_\beta(\pi,\lambda(\beta^\vee))^{-1}r'_\beta(\pi,\lambda(\beta^\vee))\big|\big(1+\|\lambda\|^2\big)^{-N}d\lambda\,.$$

Here F runs over all subsets of $\Sigma^r(L_1,A_M)$ such that $F_L^\vee$ is a basis of $\mathfrak{a}_L^{L_1}$. Fix such a subset F. Let

$$\{\tilde\omega_\beta \mid \beta\in F\}$$

be the basis of $(\mathfrak{a}_L^{L_1})^*$ which is dual to $F_L^{L_1}$. We can write $\lambda\in i\mathfrak{a}_L^*/i\mathfrak{a}_G^*$ as

$$\lambda=\sum_{\beta\in F} z_\beta\tilde\omega_\beta+\lambda_1\,,\quad z_\beta\in i\mathbb{R}\,,\quad \lambda_1\in i\mathfrak{a}_{L_1}^*/i\mathfrak{a}_G^*\,.$$

Observe that $\lambda(\beta^\vee)=z_\beta$. Suppose that $N>2\dim(A_{L_1}/A_G)+2$. Then there exists $C_N>0$, independent of π, such that

$$\int_{i\mathfrak{a}_L^*/\mathfrak{a}_G^*}\prod_{\beta\in F}\big|r_\beta(\pi,\lambda(\beta^\vee))^{-1}r'_\beta(\pi,\lambda(\beta^\vee))\big|\big(1+\|\lambda\|^2\big)^{-N}d\lambda \le C_N\prod_{\beta\in F}\int_{i\mathbb{R}}\big|r_\beta(\pi,z_\beta)^{-1}r'_\beta(\pi,z_\beta)\big|\big(1+|z_\beta|^2\big)^{-N/2}dz_\beta\,.$$

Combined with Theorem 5.3 we obtain

Theorem 5.4. *Let $M\in\mathcal{L}(M_0)$, $L\in\mathcal{L}(M)$, $L_1\in\mathcal{L}(L)$ and $S\in\mathcal{P}(L_1)$. Let $m=\dim G(\mathbb{R})/K_\infty$. For every $N\ge 8m+2$ there exists $C_N>0$ such that*

$$\int_{i\mathfrak{a}_L^*/\mathfrak{a}_G^*}\big|\nu_L^S(P,\pi,\lambda)\big|\big(1+\|\lambda\|^2\big)^{-N}d\lambda\le C_N(1+\lambda_\pi^2+\lambda_\sigma^2)^{8m^2}$$

for all $\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A}),K_f)$ and any minimal K_∞-type σ of $I_P^G(\pi_\infty)$.

6 Absolute Convergence of the Spectral Side

In this section we prove Theorem 0.1 and Theorem 0.2. For this purpose we have to study the multidimensional logarithmic derivatives of the global intertwining operators that are the main ingredients of the spectral side. First we explain the structure of the spectral side in more detail. Let $M \in \mathcal{L}(M_0)$. Fix $P \in \mathcal{P}(M)$ and $\lambda \in i\mathfrak{a}_M^*$. For $Q \in \mathcal{P}(M)$ and $\Lambda \in i\mathfrak{a}_M^*$ define

$$\mathfrak{M}_Q(P,\lambda,\Lambda) = M_{Q|P}(\lambda)^{-1} M_{Q|P}(\lambda+\Lambda)\,.$$

Then

$$\big\{\mathfrak{M}_Q(P,\lambda,\Lambda) \;\big|\; \Lambda \in i\mathfrak{a}_M^*\,,\ Q \in \mathcal{P}(M)\big\} \tag{6.1}$$

is a (G,M) family with values in the space of operators on $\mathcal{A}^2(P)$ [A4, p. 1310].

Let $L \in \mathcal{L}(M)$. Then as above, the (G,M) family (6.1) has an associated (G,L) family

$$\big\{\mathfrak{M}_{Q_1}(P,\lambda,\Lambda) \;\big|\; \Lambda \in i\mathfrak{a}_L^*\,,\ Q_1 \in \mathcal{P}(L)\big\}$$

and

$$\mathfrak{M}_L(P,\lambda,\Lambda) = \sum_{Q_1 \in \mathcal{P}(L)} \mathfrak{M}_{Q_1}(P,\lambda,\Lambda)\theta_{Q_1}(\Lambda)^{-1}$$

extends to a smooth function on $i\mathfrak{a}_L^*$. Put

$$\mathfrak{M}_L(P,\lambda) = \mathfrak{M}_L(P,\lambda,0)\,.$$

For $s \in W(\mathfrak{a}_M^*)$ let

$$M(P,s) = M_{P|P}(s,0)\,.$$

The spectral side is a sum of distributions

$$\sum_{\chi \in \mathfrak{X}} J_\chi$$

on $G(\mathbb{A})^1$. By Theorem 8.2 of [A4], the distribution J_χ can be described as follows. Let $\chi \in \mathfrak{X}$, $\pi \in \Pi(M(\mathbb{A})^1)$ and $h \in C_c^\infty(G(\mathbb{A})^1)$. Note that $\mathfrak{M}_L(P,\lambda)$ and $\rho_{\chi,\pi}(P,\lambda,h)$ both act in the Hilbert space $\mathcal{H}_P(\pi)_\chi$. Let $W^L(\mathfrak{a}_M)_{\mathrm{reg}}$ be the set of elements $s \in W(\mathfrak{a}_M)$ such that $\{H \in \mathfrak{a}_M \mid sH = H\}$ $= \mathfrak{a}_L$. Then $J_\chi(f)$ equals the sum over $M \in \mathcal{L}(M_0)$, $L \in \mathcal{L}(M)$, $\pi \in \Pi(M(\mathbb{A})^1)$ and $s \in W^L(\mathfrak{a}_M)_{\mathrm{reg}}$ of the product of

$$|W_0^M|\,|W_0|^{-1}|\det(s-1)_{\mathfrak{a}_M^L}|^{-1}$$

with

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} |\mathcal{P}(M)|^{-1} \sum_{P \in \mathcal{P}(M)} \mathrm{tr}\left(\mathfrak{M}_L(P,\lambda)M(P,s)\rho_{\chi,\pi}(P,\lambda,h)\right)d\lambda\,.$$

Our goal is to determine the conditions under which the integral-series obtained by summing this expression over $\chi \in \mathfrak{X}$, is absolutely convergent.

Since $M(P,s)$ is unitary, we have to estimate the integral

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \big\|\mathfrak{M}_L(P,\lambda)\rho_{\chi,\pi}(P,\lambda,h)\big\|_1 d\lambda\,, \tag{6.2}$$

where $\|\cdot\|_1$ denotes the trace norm.

We shall now assume that $h\in\mathcal{C}^1(G(\mathbb{A})^1)$. Let $N_{Q|P}(\pi,\lambda)$, $P,Q\in\mathcal{P}(M)$, be the normalized intertwining operator which by (2.19) is defined as

$$N_{Q|P}(\pi,\lambda) := r_{Q|P}(\pi,\lambda)^{-1} M_{Q|P}(\pi,\lambda)\,,\quad \lambda\in\mathfrak{a}_{M,\mathbb{C}}^*\,,$$

Let $P\in\mathcal{P}(M)$ and $\lambda\in i\mathfrak{a}_M^*$ be fixed. For $Q\in\mathcal{P}(M)$ and $\Lambda\in i\mathfrak{a}_M^*$ define

$$\mathfrak{N}_Q(P,\pi,\lambda,\Lambda) = N_{Q|P}(\pi,\lambda)^{-1} N_{Q|P}(\pi,\lambda+\Lambda)\,, \tag{6.3}$$

Then as functions of $\Lambda\in i\mathfrak{a}_M^*$,

$$\big\{\mathfrak{N}_Q(P,\pi,\lambda,\Lambda)\;\big|\; Q\in\mathcal{P}(M)\big\}$$

is a (G,M) family. The verification is the same as in the case of the unnormalized intertwining operator [A4, p. 1310]. For $L\in\mathcal{L}(M)$, let

$$\big\{\mathfrak{N}_{Q_1}(P,\pi,\lambda,\Lambda)\;\big|\;\Lambda\in i\mathfrak{a}_L^*,\ Q_1\in\mathcal{P}(L)\big\}$$

be the associated (G,L) family.

Let $\mathfrak{M}_{Q_1}(P,\pi,\lambda,\Lambda)$ be the restriction of $\mathfrak{M}_{Q_1}(P,\lambda,\Lambda)$ to $\mathcal{H}_P(\pi)_\chi$. Then by (2.19) and (5.15) it follows that

$$\mathfrak{M}_{Q_1}(P,\pi,\lambda,\Lambda) = \mathfrak{N}_{Q_1}(P,\pi,\lambda,\Lambda)\nu_{Q_1}(P,\pi,\lambda,\Lambda) \tag{6.4}$$

for all $\Lambda\in i\mathfrak{a}_L^*$ and all $Q_1\in\mathcal{P}(L)$.

For $Q\supset P$ let $\hat{L}_P^Q\subset\mathfrak{a}_P^Q$ be the lattice generated by $\{\tilde{\omega}^\vee\mid\tilde{\omega}\in\hat{\Delta}_P^Q\}$. Define

$$\hat{\theta}_P^Q(\lambda) = \mathrm{vol}(\mathfrak{a}_P^Q/\hat{L}_P^Q)^{-1}\prod_{\tilde{\omega}\in\hat{\Delta}_P^Q}\lambda(\tilde{\omega}^\vee)\,.$$

For $S\in\mathcal{F}(L)$ put

$$\begin{aligned}&\mathfrak{N}_S'(P,\pi,\lambda)\\ &\quad=\lim_{\Lambda\to0}\sum_{\{R|R\supset S\}}(-1)^{\dim(A_S/A_R)}\hat{\theta}_S^R(\Lambda)^{-1}\mathfrak{N}_R(P,\pi,\lambda,\Lambda)\theta_R(\Lambda)^{-1}.\end{aligned} \tag{6.5}$$

Let $\mathfrak{M}_L(P,\pi,\lambda)$ be the restriction of $\mathfrak{M}_L(P,\lambda)$ to $\mathcal{H}_P(\pi)_\chi$. Then by Lemma 6.3 of [A5] we have

$$\mathfrak{M}_L(P,\pi,\lambda) = \sum_{S\in\mathcal{F}(L)}\mathfrak{N}_S'(P,\pi,\lambda)\nu_L^S(P,\pi,\lambda)\,. \tag{6.6}$$

Hence the integral (6.2) can be estimated by

$$\sum_{S\in\mathcal{F}(L)}\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*}\big\|\mathfrak{N}_S'(P,\pi,\lambda)\rho_{\chi,\pi}(P,\lambda,h)\big\|_1\big|\nu_L^S(P,\pi,\lambda)\big|d\lambda\,.$$

We shall now study the integral in more detail. Let Ω and Ω_K be the Casimir operators of $G(\mathbb{R})$ and K_∞ respectively. Set

$$\Delta = \mathrm{Id} - \Omega + 2\Omega_K\,.$$

Then Δ acts on $\mathcal{A}^2_{\chi,\pi}(P)$ through each of the representations $\rho_{\chi,\pi}(P,\lambda)$. Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$ and let $\sigma \in \Pi(K_\infty)$. Then the operators

$$\rho_{\chi,\pi}(P,\lambda,\Delta)\,, \quad \lambda \in i\mathfrak{a}_P^*\,,$$

have $\mathcal{A}^2_{\chi,\pi}(P)_{K_f}$ and $\mathcal{A}^2_{\chi,\pi}(P)_{K_f,\sigma}$ as invariant subspaces. We shall denote the restriction of $\rho_{\chi,\pi}(P,\lambda,\Delta)$ to $\mathcal{A}^2_{\chi,\pi}(P)_{K_f}$ and $\mathcal{A}^2_{\chi,\pi}(P)_{K_f,\sigma}$, respectively, by $\rho_{\chi,\pi}(P,\lambda,\Delta)_{K_f}$ and $\rho_{\chi,\pi}(P,\lambda,\Delta)_{K_f,\sigma}$, respectively. Recall that by (2.3), $\rho_{\chi,\pi}(P,\lambda)$ is equivalent to $I_P^G(\pi_\lambda)\otimes \mathrm{Id}$. Let λ_π and λ_σ denote the Casimir eigenvalues of π_∞ and σ, respectively. Then it follows from Proposition 8.22 of [K] that

$$\rho_{\chi,\pi}(P,\lambda,\Delta)_{K_f,\sigma} = \big(1 + \|\lambda\|^2 - \lambda_\pi + 2\lambda_\sigma\big)\,\mathrm{Id}\,. \tag{6.7}$$

To estimate the right-hand side we use the following lemma.

LEMMA 6.1. *For all $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A})^1; K_f)$ and $\sigma \in \Pi(K_\infty)$, one has*

$$-\lambda_\pi + \lambda_\sigma \geq 0 \quad \textit{if} \quad \dim \mathcal{A}^2_\pi(P)_{K_f,\sigma} \neq \{0\}\,. \tag{6.8}$$

Proof. The lemma is a consequence of a more general result. Let $\pi_\infty \in \Pi(M(\mathbb{R}))$ and suppose that $\sigma \in \Pi(K_\infty)$ occurs in $I^{G(\mathbb{R})}_{P(\mathbb{R})}(\pi_\infty)|_{K_\infty}$. Let

$$\pi_\infty|_{K_\infty\cap M(\mathbb{R})} = \sum_{\omega\in\Pi(K_\infty\cap M(\mathbb{R}))} n_\omega\omega\,.$$

Then

$$\big[I^{G(\mathbb{R})}_{P(\mathbb{R})}(\pi_\infty)|_{K_\infty} : \sigma\big] = \sum_{\omega\in\Pi(K_\infty\cap M(\mathbb{R}))} n_\omega\big[\sigma|_{K_\infty\cap M(\mathbb{R})} : \omega\big]$$

[K, p. 208]. Hence there exists $\omega \in \Pi(K_\infty \cap M(\mathbb{R}))$ such that

$$[\sigma|_{K_\infty\cap M(\mathbb{R})} : \omega] > 0 \quad \text{and} \quad [\pi_\infty|_{K_\infty\cap M(\mathbb{R})} : \omega] > 0\,.$$

By [Mü2, (5.15)], the first condition implies that the Casimir eigenvalues λ_ω of ω and λ_σ of σ satisfy $\lambda_\omega \leq \lambda_{\sigma_\infty}$. On the other hand, since ω occurs in $\pi_\infty|_{K_\infty\cap M(\mathbb{R})}$ it follows that $-\lambda_{\pi_\infty} + \lambda_\omega \geq 0$ [DH, Lemma 2.6]. This completes the proof. □

Using (6.7) and (6.8), it follows that

$$\begin{aligned}\big\|\rho_{\chi,\pi}(P,\lambda,\Delta)_{K_f,\sigma}\big\|^2 &\geq \big(1+\|\lambda\|\big)^2 + (-\lambda_\pi + 2\lambda_\sigma)^2\\ &\geq \tfrac{1}{4}\big(1+\|\lambda\|^2+\lambda_\pi^2+\lambda_\sigma^2\big)\,.\end{aligned} \tag{6.9}$$

Let $S \in \mathcal{F}(L)$ be fixed. Given an open compact subgroup K_f of $G(\mathbb{A}_f)$ and $\sigma \in \Pi(K_\infty)$, let $\mathfrak{N}'_S(P,\pi,\lambda)_{K_f,\sigma}$ denote the restriction of $\mathfrak{N}'_S(P,\pi,\lambda)$ to $\mathcal{A}^2_\pi(P)_{K_f,\sigma}$.

LEMMA 6.2. *Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$ and let $h \in \mathcal{C}^1(G(\mathbb{A})^1)$ be bi-invariant under K_f. Suppose that there exist $N \in \mathbb{N}$ and $C > 0$ such that*

$$\big\|\mathfrak{N}'_S(P,\pi,\lambda)_{K_f,\sigma}\big\| \le C\big(1 + \|\lambda\|^2 + \lambda_\pi^2 + \lambda_\sigma^2\big)^N \tag{6.10}$$

for all $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}), K_f)$, $\sigma \in \Pi(K_\infty)$ and $\lambda \in i\mathfrak{a}_L^*$. Then for every $k \in \mathbb{N}$ there exists $C_k > 0$ such that

$$\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \big\|\mathfrak{N}'_S(P,\pi,\lambda)\rho_{\chi,\pi}(P,\lambda,h)\big\|_1 \big|\nu_L^S(P,\pi,\lambda)\big| d\lambda \le C_k(1+\Lambda_\pi)^{-k}$$

for all $\chi \in \mathfrak{X}$ and $\pi \in \Pi(M(\mathbb{A})^1)$.

Proof. Since h is bi-invariant under K_f, $\rho_{\chi,\pi}(P,\lambda,h)$ maps the Hilbert space $\overline{\mathcal{A}}^2_{\chi,\pi}(P)$ into the subspace $\overline{\mathcal{A}}^2_{\chi,\pi}(P)_{K_f}$. Moreover $\overline{\mathcal{A}}^2_{\chi,\pi}(P)_{K_f}$ is an invariant subspace for $\rho_{\chi,\pi}(P,\lambda,h)$. Hence $\rho_{\chi,\pi}(P,\lambda,h) = 0$, unless π belongs to $\Pi_{\mathrm{disc}}(M(\mathbb{A}), K_f)$. So we may assume that π belongs to $\Pi_{\mathrm{disc}}(M(\mathbb{A}), K_f)$. Then for each $k \in \mathbb{N}$ we get

$$\begin{aligned}
&\big\|\mathfrak{N}'_S(P,\pi,\lambda)\rho_{\chi,\pi}(P,\lambda,h)\big\|_1 \\
&\qquad = \big\|\mathfrak{N}'_S(P,\pi,\lambda)_{K_f}\rho_{\chi,\pi}(P,\lambda,h)_{K_f}\big\|_1 \\
&\qquad \le \big\|\mathfrak{N}'_S(P,\pi,\lambda)_{K_f}\rho_{\chi,\pi}(P,\lambda,\Delta^{2k})^{-1}_{K_f}\big\|_1 \\
&\qquad\qquad \cdot \big\|\rho_{\chi,\pi}(P,\lambda,\Delta^{2k}h)\big\|\,.
\end{aligned} \tag{6.11}$$

Furthermore, using (6.9) and (6.10) we get

$$\begin{aligned}
&\big\|\mathfrak{N}'_S(P,\pi,\lambda)_{K_f}\rho_{\chi,\pi}(P,\lambda,\Delta^{2k})^{-1}_{K_f}\big\|_1 \\
&\quad \le \sum_{\sigma\in\Pi(K_\infty)} \big\|\mathfrak{N}'_S(P,\pi,\lambda)_{K_f,\sigma}\big\| \cdot \big\|\rho_{\chi,\pi}(P,\lambda,\Delta^{2k})^{-1}_{K_f,\sigma}\big\|_1 \\
&\quad \le 2C \sum_{\sigma\in\Pi(K_\infty)} \frac{\dim \mathcal{A}^2_{\chi,\pi}(P)_{K_f,\sigma}}{(1+\|\lambda\|^2+\lambda_\pi^2+\lambda_\sigma^2)^{k-N}}\,.
\end{aligned} \tag{6.12}$$

By Lemma 6.1 of [Mü3] there exist $C_1 > 0$ and $N_1 \in \mathbb{N}$ such that

$$\dim \mathcal{A}^2_\chi(P)_{K_f,\sigma} \le C_1(1+\lambda_\chi^2+\lambda_\sigma^2)^{N_1}$$

for all $\chi \in \mathfrak{X}$ and $\sigma \in \Pi(K_\infty)$. Actually in [Mü3] we considered the space $\mathcal{A}^2(P,\chi,\sigma)$, where σ is an irreducible representation of K. The two spaces are not equal, but they are closely related. Moreover λ_χ was denoted by μ_χ in [Mü3]. If $\mathcal{A}^2_{\chi,\pi}(P) \ne 0$, it follows from Langlands' construction of $\mathcal{A}^2_\chi(P)$ in terms of iterated residues of cuspidal Eisenstein series that

$$|\lambda_\chi - \lambda_\pi| \le c \tag{6.13}$$

with $c > 0$ independent of χ and π (see (4.21) of [Mü3]). Hence there exist $C_2 > 0$ and $N_1 \in \mathbb{N}$ such that

$$\dim \mathcal{A}^2_{\chi,\pi}(P)_{K_f,\sigma} \le C_2(1+\lambda_\pi^2+\lambda_\sigma^2)^{N_1} \tag{6.14}$$

for all $\chi \in \mathfrak{X}$, $\pi \in \Pi_{\rm disc}(M(\mathbb{A}), K_f)$ and $\sigma \in \Pi(K_\infty)$. Set

$$N_2 = \tfrac{1}{2}(N + N_1)\,.$$

Now observe that there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{\sigma_\infty \in \Pi(K_\infty)} (1 + \lambda_\sigma)^{-n} < \infty\,.$$

the right-hand side is finite for $n \geq n_0$. Let Λ_π be the number defined by (4.5). Then by (6.12) and (6.14) it follows that for every $k > 2(n_0 + N_2)$ there exists $C_k > 0$ such that

$$\begin{aligned} \big\|\mathfrak{N}'_S(P,\pi,\lambda)_{K_f}\rho_{\chi,\pi}(P,\lambda,\Delta^{4k})^{-1}_{K_f}\big\|_1 & \\ \leq C_k\big(1 + \|\lambda\|^2 + \Lambda_\pi^2\big)^{N_2-k} & \qquad (6.15)\\ \leq C_k\big(1 + \|\lambda\|^2\big)^{(N_2-k)/2}(1+\Lambda_\pi^2)^{(N_2-k)/2} & \end{aligned}$$

for all $\pi \in \Pi_{\rm disc}(M(\mathbb{A}), K_f)$ and $\chi \in \mathfrak{X}$. Next observe that for $\lambda \in i\mathfrak{a}_L^*$ the operator $\rho_{\chi,\pi}(P,\lambda,g)$ is unitary. Hence it follows that

$$\big\|\rho_{\chi,\pi}(P,\lambda,\Delta^{4k}h)\big\| \leq \|\Delta^{4k}h\|_{L^1(G(\mathbb{A})^1)} \qquad (6.16)$$

for all $\pi \in \Pi_{\rm disc}(M(\mathbb{A}), K_f)$ and $\chi \in \mathfrak{X}$. Combing (6.11), (6.15) and (6.16), it follows that for every $n \in \mathbb{N}$ there exists $C_n > 0$ such that

$$\big\|\mathfrak{N}'_S(P,\pi,\lambda)\rho_{\chi,\pi}(P,\lambda,h)\big\|_1 \leq C_n\big(1 + \|\lambda\|^2\big)^{-n}(1+\Lambda_\pi^2)^{-n}$$

for all $\chi \in \mathfrak{X}$ and $\pi \in \Pi_{\rm disc}(M(\mathbb{A}), K_f)$. Combined with Theorem 5.4 the claimed estimation of the integral follows. □

Proof of Theorem 0.1. Let $h \in \mathcal{C}^1(G(\mathbb{A})^1)$ be bi-invariant under K_f. As observed in the proof of Lemma 6.2, it follows that $\rho_{\chi,\pi}(P,\lambda,h) = 0$, unless $\pi \in \Pi_{\rm disc}(M(\mathbb{A}), K_f)$. Let $L^2_{\rm disc}(M(\mathbb{Q})A_P(\mathbb{R})^0\backslash M(\mathbb{A}))$ be the largest closed subspace of the Hilbert space $L^2(M(\mathbb{Q})A_P(\mathbb{R})^0\backslash M(\mathbb{A}))$ which decomposes discretely under the regular representation of $M(\mathbb{A})$. Then

$$L^2_{\rm disc}\big(M(\mathbb{Q})A_P(\mathbb{R})^0\backslash M(\mathbb{A})\big) = \bigoplus_{\pi \in \Pi(M(\mathbb{A}))} m(\pi)\mathcal{H}_\pi\,,$$

and each multiplicity $m(\pi)$ is finite. Thus, if the assumption (6.10) of Lemma 6.2 is satisfied, it follows from Lemma 6.2 that for every $n \in \mathbb{N}$ there exists $C_n > 0$ such that

$$\begin{aligned} \sum_{\chi \in \mathfrak{X}} \sum_{\pi \in \Pi(M(\mathbb{A})^1)} \int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \big\|\mathfrak{N}'_S(P,\pi,\lambda)\rho_{\chi,\pi}(P,\lambda,h)\big\|_1 \big|\nu_L^S(P,\pi,\lambda)\big| d\lambda & \\ \leq C_n \sum_{\pi \in \Pi_{\rm disc}(M(\mathbb{A})^1, K_f)} m(\pi)(1+\Lambda_\pi)^{-n}\,. & \qquad (6.17) \end{aligned}$$

It remains to investigate the sum on the right-hand side. Let

$$K_{M,f} = K_f \cap M(\mathbb{A}_f)\,.$$

Then there exist arithmetic subgroups $\Gamma_{M,i} \subset M(\mathbb{R})$, $i = 1, \dots, l$, such that

$$M(\mathbb{Q})\backslash M(\mathbb{A})/K_{M,f} \cong \bigcup_{i=1}^{l} \big(\Gamma_{M,i}\backslash M(\mathbb{R})\big)$$

(cf. Section 9 of [Mü1]). Therefore we get

$$L^2\big(A_M(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A})\big)^{K_{M,f}} \cong \bigoplus_{i=1}^{l} L^2\big(A_M(\mathbb{R})^0\Gamma_{M,i}\backslash M(\mathbb{R})\big) \tag{6.18}$$

as $M(\mathbb{R})$-modules. For each i, $i = 1, \dots, l$, let $L^2_{\mathrm{disc}}(A_P(\mathbb{R})^0\Gamma_{M,i}\backslash M(\mathbb{R}))$ be the discrete subspace of the regular representation of $M(\mathbb{R})$ in $L^2(A_P(\mathbb{R})^0\Gamma_{M,i}\backslash M(\mathbb{R}))$. Then it follows from (6.18) that

$$L^2_{\mathrm{disc}}\big(A_P(\mathbb{R})^0 M(\mathbb{Q})\backslash M(\mathbb{A})\big)^{K_{M,f}} \cong \bigoplus_{i=1}^{l} L^2_{\mathrm{disc}}\big(A_P(\mathbb{R})^0\Gamma_{M,i}\backslash M(\mathbb{R})\big) \tag{6.19}$$

as $M(\mathbb{R})$ modules. For i, $1 \le i \le l$, and $\pi_\infty \in \Pi(M(\mathbb{R}))$ denote by $m_{\Gamma_{M,i}}(\pi_\infty)$ the multiplicity of π_∞ in the regular representation of $M(\mathbb{R})$ in $L^2_{\mathrm{disc}}(A_P(\mathbb{R})^0\Gamma_{M,i}\backslash M(\mathbb{R}))$. Then by (6.19) we get

$$\sum_{\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A})^1, K_f)} m(\pi)(1+\Lambda_\pi)^{-n}$$

$$\le \sum_{i=1}^{l} \sum_{\pi_\infty \in \Pi(M(\mathbb{R}))} m_{\Gamma_{M,i}}(\pi_\infty)(1+\Lambda_{\pi_\infty})^{-n} \tag{6.20}$$

Let $\sigma \in \Pi(K_\infty)$ be a minimal K_∞-type occurring in $I_P^G(\pi_\infty)$ with Casimir eigenvalue λ_σ. Let $K_{M,\infty} = M(\mathbb{R}) \cap K_\infty$. By (5.15) of [Mü2] we have that $\lambda_\sigma \ge \lambda_\tau$ for any irreducible constituent $\tau \in \Pi(K_{M,\infty})$ of $\sigma_\infty | K_{M,\infty}$. Thus the right-hand side of (6.20) is bounded by

$$\sum_{i=1}^{l} \sum_{\tau \in \Pi(K_{M,\infty})} \sum_{\pi_\infty \in \Pi(M(\mathbb{R}))} m_{\Gamma_{M,i}}(\pi_\infty) \frac{\dim(\mathcal{H}(\pi_\infty) \otimes V_\tau)^{K_{M,\infty}}}{(1+\lambda^2_{\pi_\infty}+\lambda^2_\tau)^{n/2}}\,.$$

By Corollary 0.3 of [Mü2] this sum is finite for n sufficiently large. Thus we proved

PROPOSITION 6.3. *Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$ and let $h \in \mathcal{C}^1(G(\mathbb{A})^1)$ be bi-invariant under K_f. Suppose that there exist $N \in \mathbb{N}$ and $C > 0$ such that*

$$\big\|\mathfrak{N}'_S(P,\pi,\lambda)_{K_f,\sigma}\big\| \le C\big(1+\|\lambda\|^2+\lambda_\pi^2+\lambda_\sigma^2\big)^N \tag{6.21}$$

for all $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}), K_f)$, $\sigma \in \Pi(K_\infty)$ and $\lambda \in i\mathfrak{a}_L^*$. Then

$$\sum_{\chi \in \mathfrak{X}} \sum_{\pi \in \Pi(M(\mathbb{A})^1)} \int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \big\|\mathfrak{M}_L(P,\lambda)\rho_{\chi,\pi}(P,\lambda,h)\big\|_1 d\lambda < \infty\,. \tag{6.22}$$

Let $h \in \mathcal{C}^1(G(\mathbb{A})^1)$. Then there exists an open compact subgroup K_f of $G(\mathbb{A}_f)$ such that h is bi-invariant under K_f. Using the observations made at the beginning of this section, it follows that (6.22) implies that the spectral side of the trace formula is absolutely convergent.

We shall now continue by investigating condition (6.21) in detail. To calculate $\mathfrak{N}'_S(P,\pi,\lambda)$, let $\Lambda \in i\mathfrak{a}_M^*$. By [A5, p. 37] $\mathfrak{N}'_S(P,\pi,\lambda)$ equals

$$\tfrac{1}{q!} \sum_{\{R|R\supset S\}} (-1)^q \hat{\theta}_S^R(\Lambda)^{-1} \Big(\lim_{t\to 0} \big(\tfrac{d}{dt}\big)^q \mathfrak{N}_R(P,\pi,\lambda,t\Lambda)\Big)\, \theta_R(\Lambda)^{-1},$$

where $q = \dim(A_S/A_R)$. Since $N_{Q|P}(\pi,\lambda)$ is unitary for $\lambda \in i\mathfrak{a}_M^*$, it follows from (6.3) that we have to estimate the norm of

$$\lim_{t\to 0} \big(\tfrac{d}{dt}\big)^q N_{Q|P}(\pi,\lambda+t\Lambda)_\sigma\,, \quad \lambda \in i\mathfrak{a}_M^*\,. \tag{6.23}$$

To this end, we may use (2.16) and (2.17) to replace $N_{Q|P}(\pi,\lambda)$ by

$$R_{Q|P}(\pi,\lambda) = \otimes_v R_{Q|P}(\pi_v,\lambda)\,.$$

Next note that any compact open subgroup $K_f = \prod_{v<\infty} K_v$ of $G(\mathbb{A}_f)$ is such that K_v is a hyperspecial compact subgroup for almost all v. Hence, by (2.11) there exists a finite set of places S_0, including the Archimedean one, such that we have

$$R_{Q|P}(\pi_v,\lambda)_{K_v} = \mathrm{Id}\,, \quad v \notin S_0\,,\ \pi \in \Pi_{\mathrm{disc}}\big(M(\mathbb{A}),K_f\big)\,.$$

Let D_Λ denote the directional derivative on $i\mathfrak{a}_M^*$ in the direction of Λ. Then it follows that there exists $C>0$ such that

$$\begin{aligned}\big\|\mathfrak{N}'_S(P,\pi,\lambda)_{K_f,\sigma}\big\| \le C\Bigg(&\sum_{v\in S_0\setminus\{\infty\}} \sum_{k=1}^{q} \big\|D_\Lambda^k R_{Q|P}(\pi_v,\lambda)_{K_v}\big\| \\ &+ \sum_{k=1}^{q} \big\|D_\Lambda^k R_{Q|P}(\pi_\infty,\lambda)_\sigma\big\|\Bigg)\end{aligned} \tag{6.24}$$

for all $\lambda \in i\mathfrak{a}_M^*$, $\sigma \in \Pi(K_\infty)$ and $\pi \in \Pi(M(\mathbb{A}))$. Together with Proposition 6.3 this implies Theorem 0.1. □

Proof of Theorem 0.2. The proof of Theorem 0.2 is similar to the proof of Theorem 0.1. We only have to modify some of the arguments. Given an open compact subgroup K_f of $G(\mathbb{A}_f)$ and $\sigma \in \Pi(K_\infty)$, let $\Pi_{K_f,\sigma}$ denote the orthogonal projection of the Hilbert space $\overline{\mathcal{A}}^2_{\chi,\pi}(P)$ onto the finite dimensional subspace $\mathcal{A}^2_{\chi,\pi}(P)_{K_f,\sigma}$. Let $h \in \mathcal{C}^1(G(\mathbb{A})^1)$ be K-finite. Then there exists an open compact subgroup K_f of $G(\mathbb{A}_f)$ such that h is left and right invariant under K_f. Furthermore, there exist $\sigma_1,\dots,\sigma_m \in \Pi(K_\infty)$ such that

$$\rho_{\chi,\pi}(P,\lambda,h) = \sum_{i,j=1}^{m} \Pi_{K_f,\sigma_i} \circ \rho_{\chi,\pi}(P,\lambda,h) \circ \Pi_{K_f,\sigma_j} \tag{6.25}$$

for all $\pi \in \Pi(M(\mathbb{A})^1)$ and $\chi \in \mathfrak{X}$. Let $k \in \mathbb{N}$. Then by (6.25) we get

$$\begin{aligned}\big\|\mathfrak{R}'_S(P,\pi,\lambda)\rho_{\chi,\pi}(P,\lambda,h)\big\|_1 \\ \leq \sum_{i=1}^{m} \big\|\mathfrak{R}'_S(P,\pi,\lambda)_{K_f,\sigma_i}\big\| \cdot \big\|\rho_{\chi,\pi}(P,\lambda,\Delta^{2k})^{i-1}_{K_f,\sigma_i}\big\|_1 \\ \cdot \big\|\rho_{\chi,\pi}(P,\lambda,\Delta^{2k}h)\big\|. \quad (6.26)\end{aligned}$$

Here we assume, of course, that $\mathcal{A}^2_{\chi,\pi}(P)_{K_f,\sigma_i} \neq 0$, $i = 1,\dots,m$. Then it follows from (6.9) that

$$\big\|\rho_{\chi,\pi}(P,\lambda,\Delta^{2k})^{-1}_{K_f,\sigma_i}\big\|_1 \leq C\frac{\dim \mathcal{A}^2_{\chi,\pi}(P)_{K_f,\sigma_i}}{(1+\|\lambda\|^2+\lambda_\pi^2)^k} \quad (6.27)$$

for $i = 1,\dots,m$. Given $\sigma \in \Pi(K_\infty)$, let

$$\Pi_{\mathrm{disc}}(M(\mathbb{A})^1)_{K_f,\sigma} = \big\{\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A})^1;K_f) \;\big|\; [I_P^G(\pi_\infty)|_{K_\infty} : \sigma] > 0\big\}.$$

Then we proceed as above to show that for every $n \in \mathbb{N}$ there exists $C_n > 0$ such that

$$\begin{aligned}\sum_{\chi\in\mathfrak{X}}\sum_{\pi\in\Pi(M(\mathbb{A})^1)}\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*} \big\|\mathfrak{R}'_S(P,\pi,\lambda)\rho_{\chi,\pi}(P,\lambda,h)\big\|_1 \big|\nu_L^S(P,\pi,\lambda)\big|d\lambda \\ \leq C_n\sum_{i=1}^{m}\sum_{\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A})^1)_{K_f,\sigma_i}} m(\pi)(1+\lambda_\pi^2)^{-n}. \quad (6.28)\end{aligned}$$

To estimate the right-hand side, we fix $\sigma \in \Pi(K_\infty)$. Then as in (6.20) we get

$$\begin{aligned}\sum_{\pi\in\Pi_{\mathrm{disc}}(M(\mathbb{A})^1)_{K_f,\sigma}} \frac{m(\pi)}{(1+\lambda_\pi^2)^n} \\ \leq \sum_{i=1}^{l}\sum_{\pi_\infty\in\Pi(M(\mathbb{R}))} m_{\Gamma_{M,i}}(\pi_\infty)\frac{\dim(\mathcal{H}(\pi_\infty)\otimes V_\sigma)^{K_{M,\infty}}}{(1+\lambda_{\pi_\infty}^2)^n}.\end{aligned}$$

It follows from Theorem 0.1 of [Mü1] that for sufficiently large n, this series is convergent. This completes the proof of Theorem 0.2. □

We observe that for tempered representations, the existence of estimates like (0.2), (0.3) and (0.4) follows from results of Arthur [A5, p. 51] and [A8, Lemma 2.1]. Let $\Pi_{\mathrm{temp}}(M(\mathbb{A})^1)$ be the subspace of all π in $\Pi(M(\mathbb{A})^1)$ such that the local constituents π_v of π are tempered for all v. Then we obtain

PROPOSITION 6.4. *For every $M \in \mathcal{L}(M_0)$, $L \in \mathcal{L}(M)$ and $P \in \mathcal{P}(M)$ we*

have

$$\sum_{\chi\in\mathfrak{X}}\sum_{\pi\in\Pi_{\mathrm{temp}}(M(\mathbb{A})^1)}\int_{i\mathfrak{a}_L^*/i\mathfrak{a}_G^*}\left\|\mathfrak{M}_L(P,\pi,\lambda)\rho_{\chi,\pi}(P,\lambda,h)\right\|_1 d\lambda<\infty\,.$$

7 The Example of GL_n

In this section we shall briefly discuss the case where $G=\mathrm{GL}_n$. Let P_0 be the subgroup of upper triangular matrices of G. This is the minimal standard parabolic subgroup of G. Its Levi subgroup M_0 is the group of diagonal matrices. Let P be a parabolic subgroup of G defined over $\mathbb{Q}$, and let M be the unique Levi component of P which contains M_0. Then

$$M\cong\mathrm{GL}_{n_1}\times\cdots\times\mathrm{GL}_{n_r}\,.$$

We shall identify $\mathfrak{a}_M$ with $\mathbb{R}^r$. Let $e_1,\dots,e_r$ denote the standard basis of $(\mathbb{R}^r)^*$. Then the roots Σ_P are given by

$$\Sigma_P=\{e_i-e_j\mid 1\le i<j\le r\}\,.$$

Let v be a place of $\mathbb{Q}$. Fix a nontrivial continuous character ψ_v of the additive group $\mathbb{Q}_v^+$ of $\mathbb{Q}_v$ and equip $\mathbb{Q}_v$ with the Haar measure which is selfdual with respect to ψ_v. Given irreducible unitary representations $\pi_{1,v}$ and $\pi_{2,v}$ of $\mathrm{GL}_{n_1}(\mathbb{Q}_v)$ and $\mathrm{GL}_{n_2}(\mathbb{Q}_v)$, respectively, let $L(s,\pi_{1,v}\times\pi_{2,v})$ and $\epsilon(s,\pi_{1,v}\times\pi_{2,v},\psi_v)$ denote the Rankin–Selberg L-factor and the ϵ-factor defined by Jacquet, Piatetski-Shapiro, and Shalika [JPS], [JS1].

Let $P_1,P_2\in\mathcal{P}(M)$. Then there exist permutations $\sigma_1,\sigma_2\in S_r$ such that the set of roots of (P_i,A_i) is given by

$$\Sigma_{P_k}=\left\{e_i-e_j\;\middle|\;\sigma_k(i)<\sigma_k(j)\right\}.$$

Put

$$I(\sigma_1,\sigma_2)=\left\{(i,j)\;\middle|\;1\le i,j\le r\,,\ \sigma_1(i)<\sigma_1(j)\,,\ \sigma_2(i)>\sigma_2(j)\right\}.$$

Then

$$\Sigma_{P_1}\cap\Sigma_{\overline{P}_2}=\left\{e_i-e_j\;\middle|\;(i,j)\in I(\sigma_1,\sigma_2)\right\}.$$

Let $\pi_v=\pi_{1,v}\otimes\cdots\otimes\pi_{r,v}$, where $\pi_{i,v}\in\Pi(\mathrm{GL}_{n_i}(\mathbb{Q}_v))$, $i=1,\dots,r$. Given $\mathbf{s}=(s_1,\dots,s_r)\in\mathbb{C}^r$, set

$$r_{P_2|P_1}(\pi_v,\mathbf{s}):=\prod_{(i,j)\in I(\sigma_1,\sigma_2)}\frac{L(s_i-s_j,\pi_{i,v}\times\widetilde{\pi}_{j,v})}{L(1+s_i-s_j,\pi_{i,v}\times\widetilde{\pi}_{j,v})\epsilon(s_i-s_j,\pi_{i,v}\times\widetilde{\pi}_{j,v},\psi_v)}\,.$$

As explained in [AC, p. 87], the meromorphic functions $r_{P_2|P_1}(\pi,\mathbf{s})$ satisfy all the properties of Theorem 2.1 of [A7] and they are the natural choice of normalizing factors in the case of GL_n. We note that they do not coincide with the normalizing factors used in the previous sections. They differ, however, only by a factor which can be expressed in terms of the ϵ-factors.

Now let π_1 and π_2 be automorphic representations of $\mathrm{GL}_{n_1}(\mathbb{A})$ and $\mathrm{GL}_{n_2}(\mathbb{A})$, respectively. Then the global Rankin–Selberg L-function $L(s, \pi_1 \times \pi_2)$ is defined by

$$L(s, \pi_1 \times \pi_2) = \prod_v L(s, \pi_{1,v} \times \pi_{2,v}),$$

where the product is over all places v of $\mathbb{Q}$ and $\pi_i = \otimes_i \pi_{i,v}$. The product converges absolutely in a half-plane $\mathrm{Re}(s) \gg 0$. If π_1 and π_2 belong to the discrete spectrum of $\mathrm{GL}_{n_1}(\mathbb{A})$ and $\mathrm{GL}_{n_2}(\mathbb{A})$, respectively, then $L(s, \pi_1 \times \pi_2)$ admits a meromorphic extension to the whole complex plane.

To define the global ϵ-factor $\epsilon(s, \pi_1 \times \pi_2)$ one has to pick a non-trivial continuous character $\psi : \mathbb{A}^+ \to \mathbb{C}^\times$ of the additive group $\mathbb{A}^+$ of $\mathbb{A}$. Then $\psi = \otimes_v \psi_v$ and $\epsilon(s, \pi_{1,v} \times \pi_{2,v}, \psi_v) = 1$ for almost all places v. Hence the product

$$\epsilon(s, \pi_1 \times \pi_2, \psi) = \prod_v \epsilon(s, \pi_{1,v} \times \pi_{2,v}, \psi_v)$$

exists for all $s \in \mathbb{C}$ and defines an entire function. The global ϵ-factor is independent of ψ and therefore, will be denoted by $\epsilon(s, \pi_1 \times \pi_2)$.

Let $\pi \in \Pi_{\mathrm{disc}}(M(\mathbb{A}))$. Then $\pi = \pi_1 \otimes \cdots \otimes \pi_r$ with $\pi_i \in \Pi_{\mathrm{disc}}(\mathrm{GL}_{n_i}(\mathbb{A}))$ and for $\mathbf{s} \in \mathbb{C}^r$, the global normalizing factor is defined by

$$r_{P_2|P_1}(\pi, \mathbf{s}) := \prod_{(i,j) \in I(\sigma_1, \sigma_2)} \frac{L(s_i - s_j, \pi_i \times \widetilde{\pi}_j)}{L(1 + s_i - s_j, \pi_i \times \widetilde{\pi}_j) \epsilon(s_i - s_j, \pi_i \times \widetilde{\pi}_j)}.$$

Theorem 5.3 is closely related to the estimation of the winding numbers

$$\int_1^\lambda \frac{L'(1 + it, \pi_1 \times \widetilde{\pi}_2)}{L(1 + it, \pi_1 \times \widetilde{\pi}_2)}\, dt$$

with upper bounds depending on the Casimir eigenvalues of $\pi_{1,\infty}$ and $\pi_{2,\infty}$ in the same way as in Theorem 5.3. In the present case, such estimates can be obtained using standard methods of analytic number theory. In fact, the bounds can be improved considerably.

Next we discuss the conditions (0.2) and (0.4). As mentioned in the introduction, for GL_n it is possible to prove that (0.2) and (0.4) hold. We shall briefly indicate the main steps of the proof. Let ρ be a representation of $\mathrm{GL}_m(\mathbb{Q}_v)$ and $s \in \mathbb{C}$. Then we denote by $\rho[s]$ the representation of $\mathrm{GL}_m(\mathbb{Q}_v)$ defined by

$$\rho[s](g) = |\det g|^s \rho(g), \quad g \in \mathrm{GL}_m(\mathbb{Q}_v).$$

Let π be a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A})$. Then it is known that each local component π_v of π is generic [Sha] and therefore, by [JS2] it follows that π_v is equivalent to a full induced representation, i.e.

$$\pi_v \cong I_P^{\mathrm{GL}_m}\big(\tau_1[t_1] \otimes \cdots \otimes \tau_r[t_r]\big), \tag{7.1}$$

where P is a standard parabolic subgroup of GL_m with Levi component $\mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_r}$, τ_i is a tempered representation of $\mathrm{GL}_{m_i}(\mathbb{Q}_v)$ and the t_i's are real numbers satisfying

$$t_1 > t_2 > \cdots > t_r\,, \quad |t_i| < 1/2\,,\ i = 1, \ldots, r\,.$$

For π_v unramified, Luo, Rudnick and Sarnak [LuRS] proved that the parameters t_i satisfy the following nontrivial bound:

$$\max_i |t_i| < \frac{1}{2} - \frac{1}{m^2+1}\,. \tag{7.2}$$

Using the same method, one can show that (7.2) holds at all places. Now let P be a standard parabolic subgroup of GL_n with Levi component $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$ and let π_v be the local v-component of a cuspidal automorphic representation of $M(\mathbb{A})$. Then $\pi_v = \otimes_i \pi_{i,v}$ and each $\pi_{i,v}$ is a full induced representation of the form (7.1) with parameters t_{ij} satisfying (7.2). Using induction in stages, it follows that for each i there exist a parabolic subgroup R_i of $\mathrm{GL}_{n_i}(\mathbb{Q}_v)$ of type $(n_{i1}, \ldots, n_{il_i})$, a discrete series representation $\delta_{i,v}$ of $M_{R_i}(\mathbb{Q}_v)$ and $\mathbf{t}_i = (t_{i1}, \ldots, t_{il_i}) \in \mathbb{R}^{l_i}$ satisfying

$$\max_i |t_{ij}| < \frac{1}{2} - \frac{1}{n_i^2+1}\,, \tag{7.3}$$

such that $\pi_{i,v} \cong I_{R_i}^M(\delta_{i,v}, \mathbf{t}_i)$. Put $l = l_1 + \cdots + l_r$,

$$\delta_v = \otimes_i \delta_{i,v}\,, \quad \mathbf{t} = (t_{11}, \ldots, t_{1l_1}, \ldots, t_{r1}, \ldots, t_{rl_r})\,.$$

Generalizing property (R.2) of [A8, p. 172], we get

$$R_{Q|P}(\pi_v, \mathbf{s}) = R_{Q(R)|P(R)}(\delta_v, \mathbf{s} + \mathbf{t})\,, \quad \mathbf{s} \in \mathbb{C}^l\,, \tag{7.4}$$

where $\mathbf{s}$ is identified with an element in $\mathbb{C}^l$ with respect to the embedding which corresponds to the canonical embedding $\mathfrak{a}_M^* \subset \mathfrak{a}_{P(R)}^*$. This leads to an immediate reduction of the problem. We can assume that π_v is square integrable. However, now we have to estimate the norm of the derivatives of $R_{Q|P}(\pi_v, \mathbf{s})_{K_v}$ (resp. $R_{Q|P}(\pi_v, \mathbf{s})_{\sigma_v}$) in the domain

$$\left\{\mathbf{s} \in \mathbb{C}^r \;\middle|\; |\operatorname{Re}(s_i)| < 1/2 - 1/(n^2+1)\,,\ i = 1, \ldots, r\right\}.$$

The important point is that for π_v square integrable, $R_{Q|P}(\pi_v, \mathbf{s})$ is holomorphic in the domain

$$\left\{\mathbf{s} \in \mathbb{C}^r \;\middle|\; \operatorname{Re}(s_i - s_j) > -1\,,\ 1 \le i < j \le r\right\}$$

[MW]. Using the product formula for normalized intertwining operators, the above problem can be further reduced to the case where P is maximal parabolic and $Q = \overline{P}$. Then $M = \mathrm{GL}_{n_1} \times \mathrm{GL}_{n_2}$, $\pi_v = \pi_{1,v} \otimes \pi_{2,v}$, and we may regard the intertwining operator as a function $R_{\overline{P}|P}(\pi_v, s)$ of one complex variable. Now we distinguish two cases.

1. $v < \infty$.

Let $K_v \subset \mathrm{GL}_n(\mathbb{Q}_v)$ be an open compact subgroup. We may assume that K_v is a congruence subgroup. Then we have to estimate the norm of derivatives of $R_{\overline{P}|P}(\pi_v, s)_{K_v}$ in the strip $|\operatorname{Re}(s)| < 1 - 2/(n^2+1)$. Let

$$K_{M,v} = K_v \cap M(\mathbb{Q}_v)\,.$$

Then $K_{M,v}$ is an open compact subgroup of $M(\mathbb{Q}_v)$. Let $\mathbf{1}$ denote the trivial representation of $K_{M,v}$ and let $\Pi_2(M(\mathbb{Q}_v); K_{M,v})$ be the set of all $\pi_v \in \Pi_2(M(\mathbb{Q}_v))$ such that $[\pi_v|_{K_{M,v}} : \mathbf{1}] > 0$. By Theorem 10 of [H2], $\Pi_2(M(\mathbb{Q}_v); K_{M,v})$ is a compact subset of $\Pi_2(M(\mathbb{Q}_v))$. Furthermore, $\mathfrak{a}_M^* \cong \mathbb{R}^2$ acts on $\Pi_2(M(\mathbb{Q}_v))$ by

$$\pi_{1,v} \otimes \pi_{2,v} \mapsto \pi_{1,v}[iu_1] \otimes \pi_{2,v}[iu_2]\,, \quad (u_1, u_2) \in \mathbb{R}^2\,.$$

The stabilizer of a given representation π_v is a lattice $L \subset \mathbb{R}^2$ so that the orbit $\mathfrak{o}_{\pi_v}$ of π_v is a compact torus $\mathbb{R}^2/L$. Thus there exist $\delta_1, \dots, \delta_l \in \Pi_2(M(\mathbb{Q}_v); K_{M,v})$ such that

$$\Pi_2\big(M(\mathbb{Q}_v); K_{M,v}\big) = \mathfrak{o}_{\delta_1} \sqcup \cdots \sqcup \mathfrak{o}_{\delta_l}\,.$$

Since

$$R_{\overline{P}|P}\big(\pi_{1,v}[iu_1] \otimes \pi_{2,v}[iu_2], s\big) = R_{\overline{P}|P}\big(\pi_{1,v} \otimes \pi_{2,v}, s + i(u_1+u_2)\big)\,,$$

it suffices to consider a fixed discrete series representation π_v. Now recall that $R_{\overline{P}|P}(\pi_v, s)$ is holomorphic in the strip $|\operatorname{Re}(s)| < 1$. Furthermore by Theorem 2.1 of [A7], $R_{\overline{P}|P}(\pi_v, s)_{K_v}$ is a finite rank matrix whose entries are rational functions of p_v^{-s}. Hence for every $u \in \mathbb{R}$, $R_{\overline{P}|P}(\pi_v, u+iw)_{K_v}$ is a periodic function of $w \in \mathbb{R}$. From these observations it follows immediately that for every $k \in \mathbb{N}_0$ there exists $C > 0$ such that

$$\big\|D_s^k R_{\overline{P}|P}(\pi_v, s)_{K_v}\big\| \le C \tag{7.5}$$

for all $s \in \mathbb{C}$ in the strip $|\operatorname{Re}(s)| \le 1 - 2/(n^2+1)$.

2. $v = \infty$.

Let $\sigma_v \in \Pi(\mathrm{O}(n))$. Then we have to estimate the norm of derivatives of $R_{\overline{P}|P}(\pi_v, s)_{\sigma_v}$ in the strip $|\operatorname{Re}(s)| < 1 - 2/(n^2+1)$. First note that

$$M(\mathbb{R}) \cong (\mathbb{R}^*)^2 \times \big(\mathrm{SL}_{n_1}(\mathbb{R}) \times \mathrm{SL}_{n_2}(\mathbb{R})\big)\,.$$

Furthermore, the set of discrete series representations of $\mathrm{SL}_{n_i}(\mathbb{R})$ containing a fixed $\mathrm{SO}(n_i)$-type is finite [W2, p. 398]. Hence in the same way as above, it follows that we can fix the discrete series representation π_v. Again $R_{\overline{P}|P}(\pi_v, s)$ is holomorphic in the strip $|\operatorname{Re}(s)| < 1$ and by Theorem 2.1 of [A7], $R_{\overline{P}|P}(\pi_v, s)_{\sigma_v}$ is a rational function of $s \in \mathbb{C}$. This implies that for every $k \in \mathbb{N}_0$ there exist $C > 0$ and $N \in \mathbb{N}$ such that

$$\big\|D_s^k R_{\overline{P}|P}(\pi_v, s)_{\sigma_v}\big\| \le C\big(1 + |s|\big)^N \tag{7.6}$$

for all $s \in \mathbb{C}$ with $|\operatorname{Re}(s)| < 1 - 2/(n^2+1)$.

Combining (7.5) and (7.6) with the various steps of the reduction it follows that (0.2) and (0.4) hold for all local components π_v of cuspidal automorphic representations.

It remains to deal with local components of automorphic forms in the residual spectrum. For this purpose we use the description of the residual spectrum given by Mœglin and Waldspurger [MW]. First we recall the notion of a Speh representation [MW, I.5]. Let $k|m$, $d = m/k$ and R a standard parabolic subgroup of GL_m of type $(d, \dots, d)$. Let δ be a discrete series representation of $\mathrm{GL}_d(\mathbb{Q}_v)$. Then the induced representation

$$I_R^{\mathrm{GL}_m}\big(\delta[(k-1)/2] \otimes \delta[(k-3)/2 \otimes \cdots \otimes \delta[(1-k)/2]\big)$$

has a unique irreducible quotient which we denote by $J(\delta, k)$. It follows from Theorem D of [T] and [V] that for every $\pi_v \in \Pi(\mathrm{GL}_m(\mathbb{Q}_v))$ there exist a standard parabolic subgroup P of type $(m_1, \dots, m_r)$, $k_i|m_i$, discrete series representations δ_i of $\mathrm{GL}_{d_i}(\mathbb{Q}_v)$, $d_i = m_i/k_i$, and real numbers $t_1, \dots, t_r$ satisfying $|t_i| < 1/2$ such that

$$\pi_v \cong I_P^{\mathrm{GL}_m}\big(J(\delta_1, k_1)[t_1] \otimes \cdots \otimes J(\delta_r, k_r)[t_r]\big)\,.$$

Now suppose that π_v is a local component of an automorphic representation π in the residual spectrum of $\mathrm{GL}_m(\mathbb{A})$. By [MW] there exist a standard parabolic subgroup Q of GL_m of type $(d, \dots, d)$ and a cuspidal automorphic representation μ of $\mathrm{GL}_d(\mathbb{A})$ such that π_v is the unique irreducible quotient of the induced representation

$$I_Q^{\mathrm{GL}_m}\big(\mu_v[(k-1)/2] \otimes \mu_v[(k-3)/2] \otimes \cdots \otimes \mu_v[(1-k)/2]\big)\,,$$

where μ_v is the v-component of μ. As explained above, μ_v is equivalent to an induced representation of the form (7.1) with parameters t_i satisfying (7.2). Using induction in stages, it follows that

$$\mu_v \cong I_R^{\mathrm{GL}_d}\big(\delta_1[t_1] \otimes \cdots \otimes \delta_r[t_r]\big)\,,$$

where R is a standard parabolic subgroup of GL_d of type $(d_1, \dots, d_r)$, δ_i is a discrete series representations of $\mathrm{GL}_{d_i}(\mathbb{Q}_v)$, $i = 1, \dots, r$, and the parameters t_i satisfy $t_1 \geq t_2 \geq \cdots \geq t_r$ and (7.2). Then it follows from Proposition I.9 and Lemma I.8 of [MW] that there is a standard parabolic subgroup P of GL_m of type $(kd_1, \dots, kd_r)$ such that

$$\pi_v \cong I_P^{\mathrm{GL}_m}\big(J(\delta_1, k)[t_1] \otimes \cdots \otimes J(\delta_r, k)[t_r]\big) \tag{7.7}$$

and

$$\max_i |t_i| < \frac{1}{2} - \frac{1}{m^2+1}\,. \tag{7.8}$$

This is the extension of the results of [LuRS] to local components of automorphic representations in the discrete spectrum.

Now we can proceed in the same way as in the cuspidal case. The only difference is that we have to deal with the slightly more general Speh representations in place of the discrete series representations. In this way one can establish (0.2) and (0.4). This implies that for GL_n the spectral side of the Arthur trace formula is absolutely convergent. Details will appear in [MüS].

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