GAFA, Geom. funct. anal.
Vol. 12 (2002) 395 – 435
1016-443X/02/020395-41 $ 1.50+0.20/0

GAFA Geometric And Functional Analysis

HYPERSURFACES IN H^n AND THE SPACE OF ITS HOROSPHERES

J.-M. Schlenker

Abstract

A classical theorem, mainly due to Aleksandrov [Al2] and Pogorelov [P], states that any Riemannian metric on S^2 with curvature $K > -1$ is induced on a unique convex surface in H^3. A similar result holds with the induced metric replaced by the third fundamental form. We show that the same phenomenon happens with yet another metric on immersed surfaces, which we call the horospherical metric.

This result extends in higher dimensions, the metrics obtained are then conformally flat. One can also study equivariant immersions of surfaces or the metrics obtained on the boundaries of hyperbolic 3-manifolds. Some statements which are difficult or only conjectured for the induced metric or the third fundamental form become fairly easy when one considers the horospherical metric, which thus provides a good boundary condition for the construction of hyperbolic metrics on a manifold with boundary.

The results concerning the third fundamental form are obtained using a duality between H^3 and the de Sitter space S^3_1. In the same way, the results concerning the horospherical metric are proved through a duality between H^n and the space of its horospheres, which is naturally endowed with a fairly rich geometrical structure.

Convex surfaces in H^3. Let S be a smooth, strictly convex, compact surface in H^3. Then S is diffeomorphic to S^2, and the Gauss formula indicates that its induced metric has curvature $K > -1$. A well-known theorem, to which several mathematicians have contributed (e.g. Weyl, Nirenberg [N], Aleksandrov [Al2], [AlZ] and Pogorelov [P]; see [L1] for a modern approach) is

Theorem 0.1. *Each smooth metric with curvature $K > -1$ on S^2 is induced on a unique convex surface in H^3.*

Note that a similar result holds in $\mathbf{R}^3$, and also in the 3-dimensional sphere S^3. The uniqueness here is of course up to global isometries of H^3.

Although the "usual" way of considering this theorem is as describing surfaces in H^3, it can also be understood as a remarkable statement of existence and uniqueness for a strongly non-linear boundary value problem: finding a hyperbolic metric on the 3-dimensional ball B^3 which induces a given metric on the boundary. When considered in this way a basic question is whether the boundary condition chosen here is the only one possible, or indeed the best. One of the goals of this paper is to show that there is an alternative candidate.

The de Sitter space and the space of horospheres. A striking remark is that an analogue of Theorem 0.1 is also valid with the induced metric replaced by the third fundamental form $I\!I\!I$ of the surface (more details are given in section 1). An explanation for this phenomenon comes from a rather well-known duality between H^3 and the de Sitter space S^3_1, a 3-dimensional Lorentzian space with constant curvature 1. The point is that the third fundamental form of a surface in H^3 is the induced metric on the dual "surface" in S^3_1, so that results on the third fundamental form in H^3 are again isometric embeddings statements, but in S^3_1.

The de Sitter space can be considered as the space of oriented planes in H^3. The main point of this paper is to remark that the space of horospheres in H^3 also has an interesting geometric structure; it carries a degenerate metric – of signature $(2, 0)$ – but one can nonetheless do interesting geometry in it (see section 5). We call this space C^3_+ here. It also has a natural duality with H^3, so that a surface S in H^3 has a dual "surface" – which in general might be singular – in C^3_+. The duality thus defines on S a metric (also the induced metric on the dual surface) which we call the horospherical metric of S, and denote by I^*. The metric I^* has a simple expression in terms of the usual extrinsic invariants of a hypersurface: $I^* = I + 2I\!I + I\!I\!I$, where $I\!I$ is the second fundamental form.

Results. The point is that the horospherical metric provides another good boundary condition for the existence and uniqueness of hyperbolic metrics on B^3. There is class of metrics on S^2, which we call "H-admissible" (resp. "C-admissible"), and which have a rather simple definition (see Definition 6.6); those metrics have curvature $K < 1$ (resp. $K \in (-1, 1)$), and are exactly the horospherical metrics of the H-convex (resp. convex) surfaces in H^3. Here an H-convex surface in H^3 is a surface which remains on one side of all tangent horospheres. An important point is that, given a smooth metric h on S^2, Ch is C-admissible (and thus H-admissible) for C large enough.

Theorem 7.2. *Let h be a smooth metric on S^2. It is the horospherical* *metric I^* of a H-convex immersed sphere S in H^3 if and only if it is H-admissible. It is the horospherical metric of a convex embedded sphere $S \subset H^3$ if and only if it is C-admissible. In each case, S is unique up to the global isometries of H^3.*

In higher dimension, it is not so clear what the metrics induced on e.g. the convex hypersurfaces are. Of course not all metrics are possible, and the conformal flatness of the metrics plays a role [C]. It turns out that the situation is much simpler for the horospherical metric, since here again a simple result holds and is easy to prove.

Theorem 7.1. *Let h be smooth metric on S^{n-1}. h is the horospherical metric I^* of a H-convex immersed sphere S in H^n if and only if:*

- *h is locally conformally flat;*
- *$2ric_h - \frac{S_h}{n-2}h - (n-3)h$ is everywhere negative definite.*

S is then unique up to the isometries of H^n. Moreover, S is convex if and only if all eigenvalues of $2(n-2)ric_h - S_h h$ are in $(-(n-2)(n-3), (n-3)(n-3))$.

Again, there is also a simple characterization of the metrics which are the horospherical metrics of convex hypersurfaces.

Equivariant surfaces. Let Σ be a surface of genus at least two. Although Σ carries many metrics with curvature $K > -1$, they can of course not be induced by an embedding in H^3, since it should then be convex. One needs the slightly refined notion of equivariant embedding. This is a couple (ϕ, ρ), where ϕ is an embedding of the universal cover $\tilde{\Sigma}$ of Σ, and ρ is a morphism from $\pi_1(\Sigma)$ into $\mathrm{Isom}(H^3)$, such that

$$\forall x \in \tilde{\Sigma}\,, \quad \forall \gamma \in \pi_1(\Sigma)\,, \quad \phi(\gamma x) = \rho(\gamma)\phi(x)\,.$$

One can then search for equivariant embeddings inducing a given metric; it turns out that (because of the index theorem) there are too many of those, so that one can impose an additional condition on ρ.

Theorem 0.2 (Gromov [Gro]). *Let Σ be a surface of genus at least 2, and let h be a smooth metric on Σ with curvature $K > -1$. There is an equivariant isometric embedding (ϕ, ρ) of (Σ, h) into H^3 such that ρ fixes a plane.*

A remarkable point is that it is still not known whether the uniqueness holds in the theorem above. On the other hand, an analogous results holds with the induced metric replaced by the third fundamental form:

Theorem 0.3 [LS]. *Let Σ be a surface of genus at least 2, and let h be a smooth metric on Σ with curvature $K < 1$ such that all closed geodesics of $\tilde{\Sigma}$ have length above 2π. There is a unique equivariant embedding (ϕ, ρ) of Σ into H^3 such that the third fundamental form $I\!I\!I$ of ϕ is h and that ρ fixes a plane.*

The uniqueness above is of course up to global isometries of H^3.

Considering the horospherical metric on Σ instead of either the induced metric or the third fundamental form leads to simpler results again. There are simple definitions (see 7.3) of "H-admissible" and "C-admissible" metrics on Σ, which are sub-classes of the metrics with curvature $K < 1$ and $K \in (-1, 1)$ respectively. Then

Theorem 7.4. *A smooth metric h on Σ is the horospherical metric of a H-convex equivariant immersion whose representation fixes a plane if and only if h is H-admissible. It is the horospherical metric of a convex embedding whose representation fixes a plane if and only if h is C-admissible. The equivariant immersion/embedding is then unique up to global isometries.*

Here again the proof is quite simple.

Manifolds with boundaries. As stated above, Theorem 0.1 can (as well as the other results stated above) be considered as a boundary value problem for hyperbolic metrics on the 3-dimensional ball. When considered in this way it should be possible to generalize it to manifolds other than B^3. Such a generalization was proposed in the following conjecture.

CONJECTURE 0.4 (Thurston). *Let M be a 3-dimensional manifold with boundary which admits a complete, convex co-compact metric. Then, for any smooth metric h on ∂M with curvature $K > -1$, there is a unique hyperbolic metric g on M which induces h on the boundary, and for which the boundary is convex.*

The proof of the existence part of the conjecture was obtained by Labourie [L2,3], but the uniqueness remains unknown. Theorem 0.3 also suggests that the same kind of result might hold with the induced metric replaced by the third fundamental form; actually the main point of this paper is that the "horospherical metric" works quite well for this.

In all this paper, we consider a compact 3-manifold with boundary M, which admits a complete, convex co-compact hyperbolic metric. The existence of such a metric can of course be formulated in purely topological terms thanks to the work of Thurston [T]. There are natural classes of

"H-admissible" and "C-admissible" metrics, defined in 8.1, which have curvature $K < 1$, such that

Theorem 8.2. *Let h be a smooth metric on ∂M.*

1. *h is the horospherical metric of ∂M for a hyperbolic metric g on M, such that ∂M is convex, if and only if h is C-admissible. g is then unique.*
2. *h is the horospherical metric of a H-convex immersion ϕ of ∂M in M for a complete hyperbolic metric g on M, such that ϕ can be deformed through immersions to the identity map $\partial M \to \partial_\infty M$, if and only if h is H-admissible. g and ϕ are then unique.*

In this setting again, the proof is easy, although it uses a deep result, the Ahlfors–Bers theorem (seen here as a bijection between conformal structures on ∂M and hyperbolic metrics on M; see [A], [O]). Actually, Theorem 7.4 is a direct consequence of Theorem 8.2; it might still be helpful to some readers to have stated it separately. Section 8 contains some further results concerning the higher dimensional case.

The main point of all this is that some results which are either rather difficult or actually still conjectures for the induced metric or the third fundamental form of (hyper-)surfaces become easy when one considers the horospherical metric instead. Section 9 contains examples of some other areas where this metric might be of interest.

Other applications. The horospherical metric plays an interesting role in other situations, for instance when the surface has constant mean curvature 1. It then corresponds to a well-known metric, which can be obtained (in a rather indirect way) by considering the third fundamental form of the minimal "cousin" of the surface in Euclidean space (see [Br]). The duality between C^3_+ and H^3 can therefore be used to reconstruct a constant mean curvature one surface from its constant curvature one metric. A few additional details on this are given in section 9.

Another viewpoint is in terms of Möbius structures on n-manifolds, or $\mathbf{C}P^1$-structures on surfaces. One can associate canonically to those structures a metric (see [KP]) which turns out to be the horospherical metric of a locally convex surface obtained by a convex hull construction. Although this paper mostly takes a transversal approach, the Möbius structure viewpoint is partly described in section 4, with additional details at the end of section 6.

What follows. Section 1 contains some background material on the de Sitter space and the hyperbolic-de Sitter duality. Section 2 then contains a basic construction of the space of horospheres and its most basic properties. The next section is about the definition of the dual of a hypersurface in H^n. We then pause in section 4 to consider the Möbius structures viewpoint, and to give some relevant examples where the horospherical structure is related to known constructions like the Kulkarni–Pinkall metric. Section 5 contains further geometric properties of the dual of a hypersurface, and section 6 is dedicated to the basic properties of hypersurfaces in C^n_+. The consequences for hypersurfaces in H^n are given in section 7, while section 8 contains the results on hyperbolic manifolds with boundaries. Finally some additional remarks are left for section 9.

1 The Third Fundamental Form and the de Sitter Space

It should be helpful to recall here in short some of the properties of the third fundamental form of a hypersurface in H^n, and of the de Sitter space.

The third fundamental form of a surface. This is a fairly classical bilinear form, called $I\!I\!I$ here, on the tangent space of an immersed surface. Let H be a smooth oriented hypersurface in H^n, and let X and Y be two vector fields on H. Call D the Levi–Cività connection of H^n. Then

$$D_X Y = \overline{D}_X Y + I\!I(X,Y)N\,,$$

where $\overline{D}$ is the Levi–Cività connection of the induced metric I on H, and N is the unit normal vector field on H. $I\!I$ is called the *second fundamental form* of H, it is a symmetric bilinear form on TH. The *Weingarten operator* B of H is then defined by

$$I\!I(X,Y) = I(-BX,Y) = I(X,-BY)\,.$$

The sign convention used here is not so standard but will make things easier because we will want to use the exterior normal of e.g. spheres in H^n. The *third fundamental form* of H is

$$I\!I\!I(X,Y) = I(BX,BY)\,.$$

When H is strictly convex, $I\!I\!I$ is a Riemannian metric; for surfaces in $\mathbf{R}^3$, $I\!I\!I$ is just the pull-back by the Gauss map of the canonical metric on S^2.

An interesting point is that $I\!I\!I$ provides another good boundary condition for the existence and uniqueness of hyperbolic metrics on B^3:

Theorem 1.1 [S1]. *Let h be a smooth metric on S^2. h is the third fundamental form of a convex surface S in H^3 if and only if it has curvature*

$K<1$ and its closed geodesics have length above 2π. S is then unique up to global isometries.

This result is quite strongly related to analogous polyhedral statements – just like Theorem 0.1 was related to the investigation of polyhedra in H^3 (see [Al1]). See [RH], [R], [S3] for some related questions.

Models of H^n. Recall that the Poincaré model of H^n is a conformal map from H^n to the Euclidean disc D^n (see e.g. [GHL]). Moreover, there is a also a conformal map from the n-sphere S^n minus a point to the Euclidean space $\mathbf{R}^n$. This map can be obtained by stereographic projection. Composing those maps gives us a conformal map from H^n to a geodesic ball in S^n, whose radius can be chosen by choosing the right radius for the image of the Poincaré model of H^n in $\mathbf{R}^n$.

There is also another model of H^{n+1}, called the "Klein" or "projective" model. This is a map from H^{n+1} to D^{n+1} which has the striking property that the geodesics of H^{n+1} are mapped to the segments of D^{n+1}.

It has a natural extension to a projective map from the part of the $n+1$-dimensional de Sitter space S^{n+1}_1 which is on one side of a space-like hyperplane to the complement of $\overline{D^{n+1}}$ in $\mathbf{R}^{n+1}$. Remember that S^{n+1}_1 can also be seen as a quadric in Minkowski $n+2$-space, with the induced metric:

$$S^{n+1}_1 = \left\{x \in \mathbf{R}^{n+2}_1 \;\middle|\; \langle x,x\rangle = 1\right\}.$$

The hyperbolic-de Sitter duality. There is a natural duality between H^{n+1} and S^{n+1}_1, which associates to an oriented totally geodesic hyperplane in H^{n+1} a point in S^{n+1}_1. It can be defined in the Minkowski models of H^{n+1} and S^{n+1}_1 as follows. Remember that H^{n+1} can be seen as

$$H^{n+1} = \left\{x \in \mathbf{R}^{n+2}_1 \;\middle|\; \langle x,x\rangle = -1 \quad \text{and} \quad x_0 > 0\right\}.$$

Given a point $x \in S^{n+1}_1$, let D be the oriented line going through 0 and x in $\mathbf{R}^{n+2}_1$, and let D^d be its orthogonal, which is an oriented space-like hyperplane in $\mathbf{R}^{n+2}_1$. The dual of x is the intersection $x^d := D^d \cap H^{n+1}$. The same works in the opposite direction, from points in H^{n+1} to space-like totally geodesic hyperplanes in S^{n+1}_1.

The dual of an oriented hyperplane $H \in H^{n+1}$ can be constructed geometrically in the Klein model, at the cost of a small loss of information. H corresponds, in the Klein model of H^{n+1}, to the intersection P of a hyperplane with the ball $B^{n+1} \subset \mathbf{R}^{n+1}$; suppose that H does not contain its center $0 \in B^{n+1}$. Let p be a point in $\mathbf{R}^{n+1} \setminus B^{n+1}$ such that all lines going through p and tangent to S^n meet S^n at a point of ∂P. As mentioned above, $\mathbf{R}^{n+1} \setminus B^{n+1}$ carries a projective model of the part of S_1^{n+1} which stands on one side of a space-like hyperplane. In this model, p corresponds to either the dual H^d of H or to the antipodal point.

Given a smooth, oriented, strictly convex hypersurface $S \subset H^{n+1}$, the set of points in S_1^{n+1} which are the duals of the hyperplanes tangent to S is called the dual hypersurface; the notation used here will be S^d. It is a space-like, convex hypersurface in S_1^{n+1}; its induced metric is $I^d = I\!I\!I$, while its third fundamental form is $I\!I\!I^d = I$. Theorem 0.1 can therefore be considered as an isometric embedding theorem in S_1^3 – and indeed that is how it is proved.

See e.g. [S2], [RH] for a detailed construction and some additional remarks (in particular concerning polyhedra) on the projective model.

2 The Space of Horospheres in H^n

We will describe in this section the natural geometric structure on the space of horospheres in H^n. This structure will be a basic tool in the sequel, so it will be important to understand various basic aspects of it, for instance what the "hyperplanes" or the "umbilical hypersurfaces" are, and how the isometries act.

Horospheres in H^n. Recall that horospheres in H^n can be defined as the level set of the Busemann functions. So to each points at infinity $\xi \in \partial_\infty H^n$ is associated a foliation of H^n by horospheres which are the level sets of the Busemann function B_ξ. Two horospheres with the same point at infinity are therefore at a constant distance. Horospheres in H^n are characterized by the equation

$$I\!I\!I = I\!I = I \ ,$$

in particular they are umbilical.

The space of horospheres in H^n. As mentioned above, the Poincaré model of H^n allows us to consider H^n as the interior of an oriented geodesic sphere S_0 in S^n. S^n can, through the Klein model of H^{n+1}, be considered as the boundary at infinity of H^{n+1}. S_0 is the boundary of an oriented totally geodesic hyperplane $H_0 \subset H^{n+1}$; let S_0^* be the point in S_1^{n+1} which is dual of H_0. The horospheres in H^n are then identified with the oriented spheres in S^n which are interior to and tangent to S_0; they are the boundaries of the oriented totally geodesic hyperplanes in H^{n+1} which have exactly one point at infinity in S_0, lie on the "positive" side of H_0, and have a compatible orientation at the intersection point.

Using the geometrical construction of the dual of a hyperplane (in the Klein model), we see that the set of point in S^{n+1}_1 which are the duals of those oriented hyperplanes is included in the cone of lines in $\mathbf{R}^{n+1}$ going through S^*_0 and tangent to S^n; more precisely, it is the set of points of this cone which lie strictly between S^*_0 and S^n, or, in other terms, the positive light-cone of a point in S^{n+1}_1 – whence the notation C^n_+. Note that the term "positive" light-cone is with respect to the time orientation of de Sitter space obtained by deciding that $\partial_\infty H^n$, seen as one connected component of the boundary at infinity of S^n_1, is in the "future" of each point of S^n_1.

We already see that C^n_+ inherits from this construction a degenerate metric – the one induced on the cone by the de Sitter metric – and a foliation by a family of lines – those going through S^*_0. We call those lines "*vertical*". By construction both the metric and the family of vertical lines are independent of the choices made in the construction. The vertical lines are actually characterized as the curves which are everywhere tangent to the kernel of the (degenerate) metric g_0.

Note that C^n_+ has, by construction, a very large group of "isometries" which fix both g_0 and the vertical lines: all isometries of H^n extend to isometries of H^{n+1} and of S^{n+1}_1 which fix S^*_0, and thus they act on C^n_+ fixing g_0 and the vertical lines. This indicates that it is a kind of "degenerate constant curvature space".

A cylindrical model. A slightly different model, which might sometimes be more convenient, is obtained by taking H^n as a hemisphere in S^n; S_0 is then an "equatorial" $(n-1)$-sphere, and its dual point S^*_0 is at infinity, so that C^n_+ is identified with the union of the lines tangent to S^n at a point of S_0, and orthogonal in $\mathbf{R}^{n+1}$ to the hyperplane containing S_0.

The induced structure. As a submanifold of S^{n+1}_1, C^n_+ inherits a degenerate metric g_0, i.e. a bilinear form on the tangent space which is at each point of rank $n-1$. Moreover the kernel of this bilinear form, which at each point is made of a line in the tangent space, integrate as "lines" in C^n_+.

Those lines are the lines in $\mathbf{R}^{n+1}$ which contain S^*_0 and are tangent to S^n. They are therefore light-like geodesics of S^{n+1}_1, and are naturally equipped with a connection; in other terms they have a parametrization by $\mathbf{R}$ which is defined up to an affine transformation. But those lines actually also have a natural parametrization which is defined up to the addition of a constant; namely, it is easy to check that they correspond to

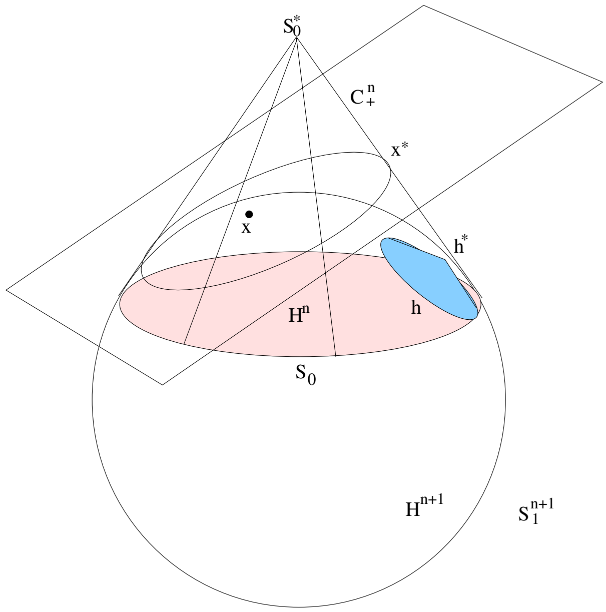

Figure 1: Conical model of C^n_+

the sets of horospheres which have a given focal point at infinity, so that the horospheres corresponding to two points in a given line are equidistant. The distance between them defines the required parametrization.

Note that, from g_0 and this canonical parametrization of the vertical lines, one could define a (family of) Riemannian metrics on C^n_+. But it does not seem very helpful to do this.

Totally geodesic hyperplanes. C^n_+ comes equipped with a collection of hypersurfaces which play a special role, and that will be called "*totally geodesic hyperplanes*". They are the sets of points dual to the horospheres containing a given point in H^n. Section 5 contains details on the geometry of C^n_+, in particular the definition of the second fundamental form of hypersurfaces – this definition is not completely obvious since the "metric" g_0 is degenerate. It should then be clear that those "totally geodesic hyperplanes" are indeed totally geodesic.

PROPOSITION 2.1. *In the cone model described above, the totally geodesic hyperplanes correspond to the intersections of the cone with the hyperplanes of* $\mathbf{R}^{n+1}$ *which are tangent to* S^n *at an interior point of* S_0. *Thus*

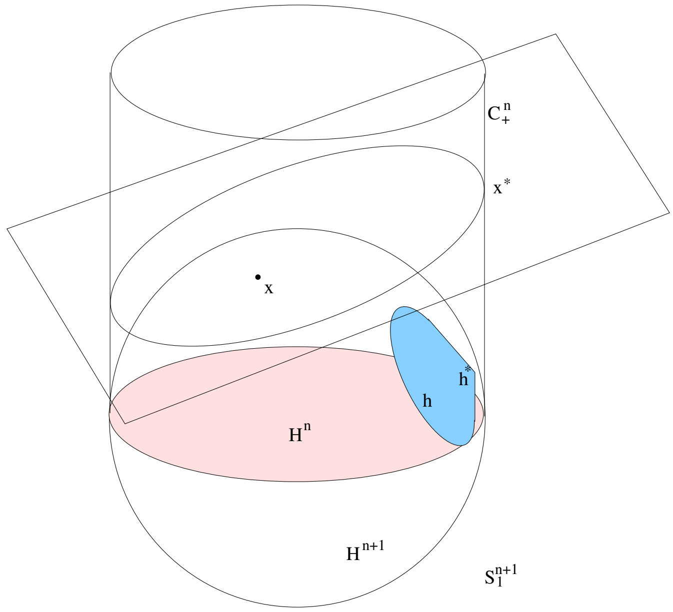

Figure 2: Cylindrical model of C^n_+

the metric induced on those totally geodesic hyperplanes is isometric to the canonical metric on S^{n-1}.

Proof. Let $x_0 \in H^n$. We consider the cone model. The geometric description of the H^{n+1}—S^{n+1}_1 duality given above, extended to the boundary at infinity S^n of H^{n+1}, shows that the dual of x_0 is the hyperplane H_0 in $\mathbf{R}^{n+1}$ which is tangent to S^n at x_0; it is a degenerate totally geodesic hyperplane in S^{n+1}_1. It is also the set of points in S^{n+1}_1 which are duals of hyperplanes in H^{n+1} which contain x_0.

The horospheres in H^n which contain x_0 correspond, in S^n, to the geodesic spheres which contain x_0 and are tangent to S_0. They are the boundaries at infinity of the hyperplanes in H^{n+1} which contain x_0 in their boundary and have exactly one point at infinity in S_0. Therefore, the points which are duals of those hyperplanes are those which are both in H_0 and in the cone C^n_+. This proves the first point.

The second point is a direct consequence; since H_0 is a degenerate hyperplane in S^{n+1}_1, all the spheres which are "around" the singular point are isometric to (S^n, can). □

It should be clear from the description below that they are the only space-like hypersurfaces in C^n_+ with an induced metric isometric to (S^{n-1}, can). By definition, the set of those totally geodesic hyperplanes is an n-dimensional manifold – it is parametrized by H^n.

LEMMA 2.2. *Let $x \in C^n_+$, and let $P \subset T_x C^n_+$ be a hyperplane which is transverse to the vertical line at x. There is a unique totally geodesic hyperplane H_0 in C^n_+ which is tangent to P at x.*

Proof. Consider the cylindrical model of C^n_+ described above. P corresponds to an $(n-1)$-plane in $\mathbf{R}^{n+1}$ which is disjoint from S^n. There are two hyperplanes containing P which are tangent to S^n, and one of them is tangent to C^n_+ along a line; so there is a unique hyperplane $\overline{P}$ which contains P, is transverse to C^n_+, and is tangent to S^n. $\overline{P}$ intersects C^n_+ along an $(n-1)$-dimensional manifold which, by construction, is a totally geodesic hyperplane in C^n_+. □

Parallel transport along the vertical lines. In the cone model above, the tangent space to C^n_+ is parallel (in S^{n+1}_1) along the "vertical lines" (which are the lines in C^n_+ which are tangent to S^n at the points of S_0). Therefore, the restriction of the Levi–Cività connection of S^{n+1}_1 defines a connection along the vertical lines in C^n_+, and thus also a natural notion of parallel transport along those lines. We call this induced connection D^v.

A kind of connection. Now let $x_0 \in C^n_+$, and let H be a hyperplane in $T_{x_0} C^n_+$ which is transverse to the vertical direction. We can define a kind of connection, which we call D^H, along the vectors tangent to H at x_0. Note that it depends on the choice of H ! It is defined as follows. Call H_0 the totally geodesic hyperplane tangent to H at x_0, let $X \in H$, and let Y be a vector field defined in a neighborhood of x_0, which is tangent to H_0; then define

$$D^H_X Y = D^0_X Y\,,$$

where D^0 is the Levi–Cività connection of H_0 for the induced metric. Moreover, if T is the vector field everywhere parallel to the vertical lines, and with length given by the natural parametrization of those lines, then we decide that, for any function f on C^n_+,

$$D^H_X fT = df(X)T\,.$$

This clearly defines $D^H_X Y$ by linearity for any vector field Y on C^n_+. Moreover, its definition shows that it is compatible with g_0.

Note that, on the other hand, we do not define a canonical connection on C^n_+ – and we will not really need one here. Of course g_0 has no Levi–Cività connection since it is degenerate.

The definition of D^H can also be obtained in an extrinsic way as follows. For x_0 and H chosen as above, there is a unique hyperplane of $\mathbf{R}^{n+1}$ which is transverse to C^n_+, tangent to S^n, and contains H. This plane contains a unique light-like line D' containing x_0. Now choose $X \in H$, and let Y be a vector field defined on C^n_+ in a neighborhood of x_0. One can project on $T_{x_0}C^n_+$ along D' the vector $D^{S^{n+1}_1}_X Y$, where $D^{S^{n+1}_1}$ is the Levi–Cività connection of S^{n+1}_1. The reader might want to check that this indeed defines the same vector as $D^H_X Y$. Of course the point is that the result depends on D', and therefore on H.

3 The Geometry of C^n_+.

This section contains some elementary remarks about the dual, in C^n_+, of some hypersurfaces in H^n. They are then used to give an intrinsic, and quite simple, expression of the metric on C^n_+, and this leads to some remarks on the geometry of C^n_+.

H-convex hypersurfaces. The following notion of convexity is important in our context. Note that horospheres are convex hypersurfaces in H^n, so they bound two domains of H^n, one of which is convex.

Definition 3.1. *Let S be an oriented hypersurface in H^n, let $x \in S$, and let h be a horosphere in H^n. We say that h is* tangent *to S at x if h is tangent to S at x in the usual sense, and moreover the convex side of h is on the "positive" side of S.*

Definition 3.2. *Let S be an oriented hypersurface in H^n. S is* H-convex *if, at each point $x \in S$, S remains on the concave side of the horosphere tangent to S at x. S is* strictly H-convex *if, moreover, the distance between S and that horosphere does not vanish up to the second order in any direction at x.*

We now define the dual of a hypersurface S in H^n; it will be a smooth hypersurface in C^n_+ when S is strictly H-convex.

Definition 3.3. *Let S be a hypersurface in H^n. We denote by S^* the set of points in C^n_+ which are dual to the horospheres tangent to S.*

Almost all the hypersurfaces in that we will consider in C^n_+, in particular the duals of hypersurfaces in H^n, have the following property.

Definition 3.4. *Let S be a smooth hypersurface in C^n_+. We say that S is* space-like *if S is everywhere transverse to the vertical lines.*

An alternate formulation is that the restriction to the tangent plane of the (degenerate) metric of C^n_+ is positive definite.

Note that, if S is a compact space-like hypersurface in C^n_+, then the projection from S to any totally geodesic hypersphere along the vertical lines is a diffeomorphism; therefore, any compact space-like hypersurface surface is topologically a sphere.

We will often implicitly identify a hypersurface $S \subset H^n$ with its dual, using the natural map sending a point $x \in S$ to the dual $h^* \in C^n_+$ of the horosphere $h \subset H^n$ tangent to S at x. This is done for instance in the next proposition, where it allows us to compare metrics on S and on S^*.

LEMMA 3.5. *If S is a hypersurface in H^n such that its principal curvatures are nowhere equal to -1, then S^* is an immersed space-like surface in C^n_+. This happens in particular when S is strictly H-convex. The metric induced by g_0 on S^* is*

$$I^* := I + 2I\!I + I\!I\!I \ ,$$

where $I\!I$ and $I\!I\!I$ are the second and third fundamental forms of S respectively.

The metric on S which appears here is the main object of study of this paper.

DEFINITION 3.6. *The metric $I^* = I + 2I\!I + I\!I\!I$ is called the* horospherical metric *of S.*

The proof of Lemma 3.5 will use the cylindrical model of C^n_+ in an explicit way. Consider a strictly H-convex hypersurface H in H^n, and let $x \in H$. We will use the cylindrical model of C^n_+, with x located at the "north pole" of S^n; this is possible since its isometry group acts transitively on H^n. The dual of the horosphere h which is tangent to H at x is then a point h^* of the intersection of C^n_+ (seen as a cylinder) with the hyperplane in $\mathbf{R}^{n+1}$ which is tangent to S^n at x.

The tangent space to H at x is identified with an affine $(n-1)$-dimensional subspace V of $\mathbf{R}^{n+1}$, and the tangent space to C^n_+ at h^* can be seen as an n-dimensional affine subspace W of $\mathbf{R}^{n+1}$ which contain an $(n-1)$-plane parallel to V. We call ϕ the duality map from H to H^*, sending a point y in H to the dual of the horosphere tangent to H at y, and we consider $d\phi$ as a map from V to W, where $W \supset V$. Then

PROPOSITION 3.7. *The linearized map at x is $T_x\phi = E + B$, where E is the identity map on $V = T_xH$ and B is the Weingarten operator of H.*

Proof. Let $v \in T_xH$; call v^* the vector in W corresponding to the variation of the dual point to the horosphere tangent to H at a point which moves in the direction of v on H. v^* is the sum of a term v_1^* corresponding to the displacement of x (with a parallel transport of the tangent hyperplane) and a term v_2^* corresponding to the variation of the tangent hyperplane, while x doesn't move. Using the cylindrical model, one checks that $v_1^* = v$ (with both terms seen as in W) while $v_2^* = Bv$. □

Proof of Lemma 3.5. The previous proposition shows that S^* is smooth and space-like except maybe when B has -1 as one of its eigenvalues.

Moreover, the bilinear form induced on W by g_0 (i.e. by the de Sitter metric on the outside of the ball) is a degenerate metric which coincides, on the parallel transport of V, with the metric induced on V by H^n. Therefore, if $v, v' \in T_xH$, we have that $v^*, v'^* \in W$ and:

$$\langle v + Bv, v' + Bv' \rangle = \langle v, v' \rangle + \langle Bv, v' \rangle + \langle v, Bv' \rangle + \langle Bv, Bv' \rangle \,,$$

so that

$$\langle v^*, v'^* \rangle = I(v, v') + 2I\!I(v, v') + I\!I\!I(v, v') \,,$$

and the result follows. □

Example. Let M be a complete, convex co-compact hyperbolic manifold. Let C be the convex core of M; Thurston [T] has shown that the induced metric on its boundary hyperbolic metric. ∂C also carries a measured lamination μ describing its bending. μ determines a distance on ∂C, with the length of a segment transverse to μ equal to the integral over it of the transverse measure of μ. Adding this distance to the induced metric – a process sometimes known as "grafting" – determines a metric on ∂C, which is none other than its horospherical metric. More generally, in higher dimension, the horospherical metric can be used to recover the metric introduced by Kulkarni and Pinkall. This is shown in the next section.

A duality. An important point is that the map sending a hypersurface S in H^n to its dual S^* in C_+^n is a real duality, in the following sense. First remark that to each totally geodesic hyperplane H_0 in C_+^n is associated a point in H^n, namely the intersection of all the horospheres which are dual to the points of H_0. We call this point the dual of H_0, and denote it by H_0^*. Then we have

Lemma 3.8. *If S^* is smooth, then S is the set of points in H^n which are the duals of the totally geodesic hyperplanes tangent to S^* in C_+^n.*

Proof. This follows again from Proposition 3.7, and from the correspondence between vectors on S and on S^*. □

An intrinsic definition of the metric g_0. Lemma 3.5 can be used to give a simple form of the metric on C^n_+; using it will relieve us from the constant use of the cone model, the de Sitter space and so on.

LEMMA 3.9. *There exists an isometry Φ from C^n_+ to $S^{n-1}\times\mathbf{R}$ with the (degenerate) metric*

$$g_0 \simeq e^{2t}\mathrm{can}_{S^{n-1}} ,$$

where $\mathrm{can}_{S^{n-1}}$ is the canonical metric on S^{n-1}. Moreover the vertical lines are sent to the lines $\{s\}\times\mathbf{R}$, for $s\in S^{n-1}$, with the same parametrization.

Proof. Let $x_0\in H^n$. For $t\in\mathbf{R}\setminus\{0\}$, call S_t the geodesic sphere of radius $|t|$ centered at x_0, with the normal oriented towards the exterior for $t>0$ and towards the interior for $t<0$. Define a map Ψ from $S^{n-1}\times(\mathbf{R}\setminus\{0\})$ to C^n_+ sending (s,t) to the horosphere tangent to S_t at the point $\exp_{x_0}(ts)$, where s is considered as a unit vector in $T_{x_0}H^n$. Ψ can then be extended by continuity to a map from $S^{n-1}\times\mathbf{R}$ to C^n_+. Φ is the inverse of Ψ.

By Lemma 3.5, the metric induced on S^*_t is

$$\begin{aligned} I^*_t &= I_t + 2I\!I_t + I\!I\!I_t \\ &= \sinh^2(t)\mathrm{can}_{S^{n-1}}(1+2\coth(t)+\coth^2(t)) \\ &= \big(\sinh^2(t)+2\sinh(t)\cosh(t)+\cosh^2(t)\big)\mathrm{can}_{S^{n-1}} \\ &= e^{2t}\mathrm{can}_{S^{n-1}} . \end{aligned}$$

Now, using e.g. the cylindrical model described above, with x_0 as the "north pole" in S^n, shows that the hypersurfaces S^*_t are the intersections of C^n_+ (seen as a cylinder in R^{n+1}) with the horizontal hyperplanes, i.e. the hyperplanes in $\mathbf{R}^{n+1}$ which are parallel to the hyperplane containing S_0. Therefore the lines $\{s\}\times\mathbf{R}$ are in the kernel of g_0, and moreover they correspond to the vertical lines. Finally, by definition of their parametrization (by the distance between equidistant horospheres) it is the same as the one they have in $S^{n-1}\times\mathbf{R}$. □

A conformal map. Now we remark that all the space-like hypersurfaces in C^n_+ can be naturally identified in a conformal way; they are moreover all naturally conformal to the boundary at infinity of H^n. Call Π_0 the map from C^n_+ to $\partial_\infty H^n$ sending a horosphere to its point at infinity. Then

LEMMA 3.10. *1. Let H_1 and H_2 be two compact space-like hypersurfaces in C^n_+. The projection from H_2 to H_1 along the vertical lines is conformal for the induced metrics on H_1 and H_2.*

2. For each space-like hypersurface $H_1\subset C^n_+$, the restriction of Π_0 to H_1 is conformal for the induced metric on H_1 and the usual conformal *structure on $\partial_\infty H^n$.*

Proof. The first point is a direct consequence of Lemma 3.9 above. For the second point remark that, if x_0 is the point in H^n which is the intersection of the horospheres in H_1, then the map sending a horosphere $h \in H_1$ to its point at infinity is by construction an isometry between H_1 with its induced metric and $\partial_\infty H^n$ with the visual metric at x_0. It is therefore a conformal map. □

Let H be an oriented hypersurface in H^n; there is a natural map from H to $\partial_\infty H^n$, which sends a point $x \in H$ to the end point of the ray starting at x in the direction of the oriented normal vector to H at x. We call this "Gauss map" G (see e.g. [L3] for some applications of this map). As a consequence of Lemma 3.10 we obtain the following:

LEMMA 3.11. *If H is a strictly H-convex hypersurface in H^n, the conformal structure obtained on H as the pull-back by G of the conformal structure on $\partial_\infty H^n$ is the conformal structure of I^*.*

Umbilical hyperplanes. Some hypersurfaces in C^n_+ play a special role and have a very simple geometry; they are the hypersurfaces H^*, where H is an umbilical hypersurface in H^n. By Lemma 3.5, H^* is then homothetic to H. This is specially interesting when H is a totally geodesic hyperplane in H^n, since then H^* is isometric to H. We call those hypersurfaces "*dual hyperplanes*". It is not difficult to check that the image of a dual hyperplane by the projection on a totally geodesic hyperplane along the vertical lines is a hemisphere.

4 The Horospherical Metric and Möbius Structures

We develop here some relations between the horospherical metric and Möbius or $\mathbf{C}P^1$-structures, starting with some remarks on the isometries of C^n_+.

Isometries. Let γ be an isometry of H^n. Consider the cone model of C^n_+ described in section 2. Then γ acts on S^n as a Möbius transformation leaving S_0 stable. Therefore it acts as an isometry on H^{n+1}, seen as the interior of S^n, and therefore also as an isometry on the de Sitter space which lies on the outside of S^n, leaving invariant the cone made of the (light-like) lines tangent to S^n along S_0 and containing S^*_0. So, by construction, γ also acts on C^n_+ without changing its metric or its vertical lines.

Note that if γ has no parabolic fixed point in $\partial_\infty H^n$, then it has no fixed point in C^n_+ – since an isometry fixing a horosphere should fix its point at infinity. This strongly contrasts with the H^n/S^n_1 duality, where all isometries of H^n without fixed point in H^n or parabolic fixed point in $\partial_\infty H^n$ have at least one fixed point in S^n_1.

The isometries of C^n_+ can be characterized in the following simple ways.

LEMMA 4.1. *1. Let H be a totally geodesic hyperplane in C^n_+. For any isometry γ of H^n, (the extension to C^n_+ of) γ, composed with the projection on H along the vertical lines, is a Möbius transformation of H.*

2. Moreover, any global conformal transformation of H corresponds in this way to a unique isometry.

3. Let D be a dual hyperplane in C^n_+. Each isometry of D extends in exactly two ways as an isometry of C^n_+, one of which preserves orientation.

Proof. Isometries correspond by definition to isometries of H^n, which act conformally on $\partial_\infty H^n$, and thus on H by Lemma 3.10; point (1) follows. Conversely, any conformal transformation of H defines by Lemma 3.10 a conformal transformation of $\partial_\infty H^n$, and therefore an isometry of H^n, and also an isometry of C^n_+. This proves point (2).

For point (3), let D^* be the dual hyperplane of D, i.e. the oriented totally geodesic hyperplane in H^n such that D corresponds to the set of horospheres tangent to D^*. Let γ be an isometry of D. By construction, D^* is isometric to D, so that γ defines an isometry γ^* of D^*. Since D^* is an hyperplane in H^n, γ^* has two extensions as an isometry of H^n, one of which preserves orientation. We call this orientation preserving extension $\overline{\gamma}^*$. $\overline{\gamma}^*$ defines a unique isometry $\overline{\gamma}$ of C^n_+, which leaves D stable by construction. The same works for the other extension of γ^*. □

A quasi-fuchsian example. Let $\Gamma \subset SO(3,1)$ be a quasi-fuchsian group, which is not fuchsian. Let $\Lambda \subset \partial_\infty H^3$ be the limit set of Γ, and let C be its convex hull. Then ∂C has two connected components Σ_+ and Σ_-. Although those surfaces are not smooth, they are convex (but not strictly convex), and thus H-convex. Let g_+ and g_- be the induced metrics on Σ_+ and Σ_-. Then both (Σ_+, g_+) and (Σ_-, g_-) are isometric to H^2.

In addition, Σ_+ and Σ_- carry measured laminations μ_+ and μ_- describing their bending (see [T]). Those "bending laminations" can be "added" to $g_\pm$ to obtain metrics $h_\pm$ on $\Sigma_\pm$ as follows: the length of a segment is defined as the sum of its length for $g_\pm$ and the integral over it of the measure on $\mu_\pm$.

This is simpler to understand when $\mu_\pm$, considered on $\Sigma_\pm/\Gamma$, have support on a finite set of closed curves. Then $h_\pm$ on $\Sigma_\pm/\Gamma$ is obtained by cutting open $(\Sigma_\pm, g_\pm)$ along each of those curves (which are geodesics) and gluing in a flat band of width equal to the bending angle at the curve. A moment of thought shows that this operation is indeed the same as the one used to define $h_\pm$ above; it is called "grafting" $g_\pm$ along $\mu_\pm$.

One would like to consider the horospherical metric of $\Sigma_\pm/\Gamma$, but some care is necessary since Σ_+ and Σ_- are not C^1 smooth. To define it, define first, for $\epsilon > 0$, the surfaces $\Sigma_{+,\epsilon}$ and $\Sigma_{-,\epsilon}$, which are the surfaces at constant distance ϵ from $\Sigma_+/\Gamma, \Sigma_-/\Gamma$ on the side opposite to C/Γ; $\Sigma_{\pm,\epsilon}$ are convex and $C^{1,1}$ smooth. One can thus consider the horospherical metrics $I^*_{\pm,\epsilon}$ on them.

PROPOSITION 4.2. *As $\epsilon \to 0$, $(I^*_{\pm,\epsilon})$ converges to the metric $h_\pm$ on $\Sigma_\pm/\Gamma$.*

Note that, as $\epsilon \to 0$, a degeneration occurs: different points of $\Sigma_{\pm,\epsilon}$ might collapse to the same point of $\Sigma_\pm$. So the limit metric I^* is not really a metric on $\Sigma_\pm$, but rather on the set of unit normals $U\Sigma_\pm$ of the oriented support planes of $\Sigma_\pm$.

We will not prove this statement here, since it is a consequence of the more general Proposition 4.3 below. But we can explain the proof in the simple case where $\mu_\pm$ have support on a finite set of closed curves. For simplicity we restrict our attention to Σ_+. Note that there is a natural projection π from $\Sigma_{+,\epsilon}$ to Σ_+, sending each point in $\Sigma_{+,\epsilon}$ to the closest point in Σ_+.

For each $\epsilon > 0$, $\Sigma_{+,\epsilon}$ can be decomposed as the union of two closed subsets, $\Sigma_{+,\epsilon} = \Sigma^f_{+,\epsilon} \cup \Sigma^b_{+,\epsilon}$, corresponding respectively to points project to a "flat" point of Σ_+ or to a point which is on a bending geodesic. An elementary computation, using the fact that $I^* = I + 2I\!I + I\!I\!I$, shows that

- $\Sigma^f_{+,\epsilon}$ is umbilical, with principal curvatures equal to $\tanh(\epsilon)$. So the horospherical metric is $(1 + \tanh\epsilon)^2$ times the induced metric, which converges as $\epsilon \to 0$ to the pull-back metric by π.
- at each point of $\Sigma^b_{+,\epsilon}$, there are two principal curvatures, equal respectively to $\tanh(\epsilon)$ and to $\mathrm{cotanh}(\epsilon)$, with associated principal directions the inverse image by π of the direction of the bending geodesic and the orthogonal direction, respectively. Thus, as $\epsilon \to 0$, the horospherical metric converges to a flat metric, which is the product of the bending geodesic g by an interval of length equal to the bending angle at g.

This shows Proposition 4.2 when the bending lamination has support on a finite set of closed curves.

$\mathbf{C}P^1$-structures on surfaces. The construction above can be stated in terms of $\mathbf{C}P^1$-structure on surfaces; recall (see [T]) that the boundary at infinity of an end of a convex co-compact manifolds carries a canonical $\mathbf{C}P^1$-structure, which completely determines the end.

Moreover, a $\mathbf{C}P^1$-structure on a surface is the same as a Möbius structure. Seen in this light, Proposition 4.2 appears as a special case of the construction of the next subsection, concerning higher dimensional manifolds with Möbius structures.

The Kulkarni–Pinkall metric. The same can be done in higher dimension (and in a slightly more general setting). Start with a hyperbolic Möbius structure on a compact $(n-1)$-dimensional manifold M. Let $\mathrm{dev} : \tilde{M} \to S^{n-1}$ be the associated developing map. One can associate to M a convex hull construction in H^n (S^{n-1} is seen as the boundary at infinity of H^n), see [KP]. There is thus a natural equivariant Lipschitz map from $\tilde{M}$ to a locally convex (non-smooth) immersed hypersurface $\Sigma \subset H^n$, which is defined as the inverse of the "normal exponential" map N from Σ to $\partial_\infty H^n$ – note that N is in general multi-valued since it depends on the support plane of Σ which is chosen at each point. It can be defined as a function on the set of couples (x, N), where $x \in \Sigma$ and N is the unit normal vector of an oriented support plane of Σ at x. Σ is invariant under the action of $\pi_1 M$ on H^n induced by its Möbius action on $\tilde{M}$.

Since Σ lacks smoothness we can not *a priori* define its horospherical metric; this can be done, however, by considering the equidistant surfaces Σ_t at distance t on the concave side of Σ, and taking the limit as $t \to \infty$. It is not difficult to check that

1. for $t > 0$, Σ_t is $C^{1,1}$ smooth, in particular it has at each point a unique normal vector;
2. the normal exponential map N_t from Σ_t to $\partial_\infty H^n$ defines an equivariant diffeomorphism from Σ_t to $\tilde{M}$;
3. if I^*_t is the horospherical metric of Σ_t, $(N_{t*} I^*_t)$ converges as $t \to 0$ to a $C^{1,1}$ equivariant metric I^*;
4. I^* is compatible with the Möbius structure we started with (or more precisely with its conformal structure).

Of course, as above I^* is in fact not a metric on Σ, but rather on the set $U\Sigma$ of unit normals of the oriented support planes of Σ.

The properties of I^* in this case can be seen as consequences of

Proposition 4.3. *The metric I^* on $\tilde{M}$ is the same as the metric defined by Kulkarni and Pinkall in [KP].*

Before proving the proposition we have to recall the definition of the Kulkarni–Pinkall metric. They consider the universal cover $\tilde{M}$ of an n-dimensional manifold M with a Möbius structure, and the developing map $\mathrm{dev} : \tilde{M} \to S^n$. Then they consider the maximal balls in the image. Each such ball B carries a hyperbolic metric which is compatible with the restriction of the Möbius structure, and its boundary intersects $\partial\,\mathrm{dev}(\tilde{M})$. Kulkarni and Pinkall then introduce the hyperbolic convex hull $C(B)$ in B of $\partial B \cap \partial\,\mathrm{dev}(\tilde{M})$, and they show that the $C(B)$, when B ranges over all maximal balls in $\mathrm{dev}(\tilde{M})$, define a stratification of $\tilde{M}$, which by construction is equivariant under the action of $\pi_1 M$.

So each $x \in \tilde{M}$ is in $C(B)$ for a maximal ball B, and the conformal factor corresponding to the hyperbolic metric on B determines a conformal factor at x. The metrics on the $C(B)$ for different choices of B glue well, and they are by construction all compatible with the conformal structure underlying the Möbius structure on $\tilde{M}$.

We shall consider a slightly modified (but equivalent) definition. Consider the same hypersurface Σ as above. Let $U\Sigma$ the set of unit normal vectors of the oriented support hyperplanes of Σ, and let N be the "hyperbolic Gauss map", which sends a unit vector $v \in U\Sigma$ to the endpoint on $\partial_\infty H^n$ of the geodesic ray starting from v. The local convexity of Σ shows that N determines a bijection from $U\Sigma$ to $\tilde{M}$.

Let $x \in \Sigma$, and let P be an oriented support hyperplane of Σ at x. The boundary at infinity ∂P of P is then the boundary of a maximal ball B in $\mathrm{dev}(\tilde{M})$. There is natural hyperbolic metric on B coming from identifying B with P, and the convex hull $C(B)$ of $\partial B \cap \Lambda$ then corresponds to $P \cap \Sigma$. Let u_P be the map sending a point $x \in P \cap \Sigma$ to the unit normal vector to P at x, which is in $U\Sigma$. An elementary translation shows that an equivalent definition of the Kulkarni–Pinkall metric g_{KP} is that it is the metric on $U\Sigma$ which

- is compatible with the pull-back by N of the conformal structure on $\mathrm{dev}\,\tilde{M}$;
- for each support plane P of Σ, is equal on $u_P(P \cap \Sigma)$ to the induced metric on $P \cap \Sigma$.

Now it is a consequence of the previous section that, for each $t > 0$, the horospherical metric I^*_t on Σ_t is compatible with the conformal structure on $\partial_\infty H^n$; more precisely, the "hyperbolic Gauss map" N_t, which sends a point $x \in \Sigma_t$ to the endpoint of the geodesic ray normal to Σ_t at x, is a conformal map between (Σ_t, I^*_t) and $\partial_\infty H^n$ with its canonical conformal structure. So the horospherical metric I^* defined above on $U\Sigma$ is conformal to g_{KP}. Moreover, for each $v \in U\Sigma$, the oriented hyperplane P which is orthogonal to v is by construction a support plane of Σ, and $P \cap \Sigma$ contains a geodesic γ. Since γ is a geodesic contained in Σ and Σ is locally convex, both $I\!I$ and $I\!I\!I$ vanish on γ, so that $I^*_{|\gamma} = I_{|\gamma}$, while the alternate definition of g_{KP} above shows that $g_{KP|u_P(\gamma)}$ is the induced metric on γ. This shows that the conformal factor between g_{KP} and I^* at v is 1, and proves Proposition 4.3.

An indirect consequence of this identification is that some of the properties of the Kulkarni–Pinkall metric – for instance the fact that its curvature is well defined almost everywhere and between -1 and 1, and between -1 and 0 for surfaces – can be seen as a consequence of the statements that follow concerning the horospherical metric of convex hypersurfaces in H^n.

5 Hypersurfaces in C^n_+

We now come back to the geometry of C^n_+, and in particular of its hypersurfaces.

Second fundamental forms in C^n_+. Let H be a hypersurface in C^n_+. Let $x \in H$, and call $H_0 \in T_x C^n_+$ the totally geodesic hyperplane tangent to H at x. Let X and Y be vector fields on H. Locally (in the neighborhood of x) H intersects exactly once each vertical line; therefore, the "vertical connection" D^v defined in section 2 allows us to extend X and Y as vector fields on a neighborhood of x in C^n_+ by parallel transport along the vertical lines. We can then use the kind of connection defined in section 2 to define a "second fundamental form" of H at x.

Definition 5.1. *The second fundamental form of H at x is defined as*

$$I\!I^*(X,Y) := \Pi(D^{H_0}_X Y)\,,$$

for the extended vector fields, where Π is the "length" of the projection on the vertical direction in $T_x C^n_+$ along the direction of H_0.

Note that the "length" of vertical vectors is well defined since, as mentioned above, the vertical lines have a canonical parametrization defined up to addition of a constant.

Lemma 5.2. *1. $I\!I^*$ defines a symmetric bilinear form on H_0.*

2. If P_0 is the (unique) totally geodesic hyperplane in C^n_+ which is tangent to H_0 at x, then H is locally the graph of a function u above P_0; $I\!I^*$ is then the Hessian of u at x for the metric induced on P_0.

3. $I\!I^*$ is also the Hessian at x of u, seen as a function on H, for the induced metric I^* on H.

In the second part of this lemma, u is the function such that, at a point $y \in P_0$ near x, $u(y)$ is the "oriented distance" from y to the intersection of H with the vertical line through y, for the natural parametrization of that vertical line.

Proof. The first point is obviously a consequence of the others. For the second point note that, in the neighborhood of x, the extended vector field Y is, up to the first order, of the form:

$$Y = Y_0 + du(Y_0)T \,,$$

with Y_0 tangent to P_0. Therefore the definition of D^{H_0} shows that

$$D^{H_0}_X Y = D^0_X Y_0 + \big(X.du(Y_0)\big)T \,,$$

where D^0 is the Levi–Cività connection of the induced metric g_0 on P^0, and the result follows since $du = 0$ at x.

For the third point note that, by Lemma 3.9, $I^* = e^{2u}g_0$, so that the Levi–Cività connection D^* of I^* is given by

$$D^*_X Y = D^0_X Y + du(X)Y + du(Y)X - g_0(X,Y)D^0 u \,,$$

where vector fields on H and P_0 are identified through the projection along the vertical lines. Therefore (by the usual conformal transformation formulas, see e.g. [B, chapter 1]),

$$(D^*du)(X,Y) = (D^0du)(X,Y) - 2du(X)du(Y) + g_0(X,Y)\|du\|^2_{g_0} \,,$$

so that $D^*du = D^0du$ at x since $du = 0$ at x. □

We then use $I\!I^*$ to define the "Weingarten operator" of a hypersurface H in C^n_+.

Definition 5.3. *If H is a space-like hypersurface in C^n_+ and $x \in H$, the "Weingarten operator" of H at x is the linear map B^* from T_xH to T_xH, self-adjoint for I^*, defined by*

$$I\!I^*(X,Y) = I^*(B^*X,Y) = I^*(X,B^*Y) \,.$$

The mean curvature of H at x is $H^* := tr(B^*)/2$; the third fundamental form of H at x is defined by

$$I\!I\!I^*(X,Y) := I^*(B^*X,B^*Y) \,.$$

Lemma 5.4. *If H is space-like, then the induced metric on the dual hypersurface in H^n is $I = I\!I\!I^*$.*

Proof. The proof can be done using the same setting as in the proof of Lemma 3.5, or in Proposition 3.7; one needs to consider the opposite map, and to remark that the displacement of the point in H^n which is dual to a tangent hyperplane in C^n_+ is given by the third fundamental form in C^n_+. □

An inversion formula. We have already seen in Lemma 3.5 that

$$I^*(X,Y) = I((E+B)X,(E+B)Y) \ .$$

Together with the previous lemma, we get

LEMMA 5.5. *If S is a hypersurface in H^n with no principal curvature equal to -1 at any point, then:*

$$B^* = (E+B)^{-1} \, .$$

Convex hypersurfaces. Using the previous definition, we can define a convex hypersurface in C^n_+:

DEFINITION 5.6. *Let H be a space-like hypersurface in C^n_+. We say that H is* convex *if $I\!I^*$ is positive semi-definite, and that H is* strictly convex *if $I\!I^*$ is positive definite at each point of H. H is* tamely convex *if all eigenvalues of B^* are in $(0,1)$ at each point.*

The point is that convex hypersurfaces in C^n_+ have a smooth and H-convex dual hypersurface in H^n, and that tamely convex hypersurfaces have convex duals. More precisely:

LEMMA 5.7. *Let H be a hypersurface in C^n_+ such that B^* is nowhere degenerate. Then H^* is smooth, and its induced metric is*

$$I(X,Y) = I^*(B^*X, B^*Y) \, .$$

H is strictly convex if and only if H^* is strictly H-convex. H is tamely convex if and only if H^* is strictly convex.

Proof. This follows again from Lemma 5.5. □

6 Isometric Embeddings in C^n_+

The point of this section is to give an elementary study of the induced metrics on hypersurfaces in C^n_+, like the one which can be found in elementary differential geometry books for hypersurfaces in e.g. $\mathbf{R}^n$. The results are a little different, however, due to the degeneracy of the metric.

The Gauss formula. The curvature tensor of the induced metric on a hypersurface in C^n_+ is determined by the following analogue of the Gauss formula:

LEMMA 6.1. *Let H be a space-like hypersurface in C^n_+. Let $x \in H$, call P_0 the (unique) totally geodesic hyperplane in C^n_+ which is tangent to H at x. Let X, Y, Z be three vector fields on H. The Riemann curvature tensor R^* of the induced metric I^* on H is given by*

$$R^*_{X,Y}Z = R^0_{X,Y}Z + I\!I^*(X,Z)Y - I\!I^*(Y,Z)X - I^*(Y,Z)B^*X + I^*(X,Z)B^*Y \ ,$$

where R^0 is the curvature tensor of P_0.

Note that this formula differs from the Euclidean one, in particular because it is linear in B^* instead of quadratic.

Proof. We also call X, Y and Z the projections of the vector fields on P_0, and g_{P_0} its metric, which has constant curvature 1. The metric on H is then the pull-back of $e^{2u}g_{P_0}$ under the projection of H to P_0 along the vertical lines. Therefore, the Levi–Cività connection $\overline{D}$ of I^* is (see e.g. [B, chap. 1]):

$$\overline{D}_X Y = D_X Y + du(X)Y + du(Y)X - g_{P_0}(X,Y)Du\,,$$

where D is the Levi–Cività connection of g_{P_0}. Thus, using the fact that $du = 0$ at x, we find that, still at x,

$$\begin{aligned}
R^*_{X,Y}Z &= \overline{D}_X\overline{D}_Y Z - \overline{D}_Y\overline{D}_X Z - \overline{D}_{[X,Y]}Z \\
&= D_X\overline{D}_Y Z - D_Y\overline{D}_X Z - D_{[X,Y]}Z \\
&= D_X(D_Y Z + du(Y)Z + du(Z)Y - g_{P_0}(Y,Z)Du) - \\
&\quad - D_Y\big(D_X Z + du(X)Z + du(Z)X - g_{P_0}(X,Z)Du\big) - D_{[X,Y]}Z \\
&= R^0_{X,Y}Z + (D_X du)(Y)Z + (D_X du)(Z)Y - \\
&\quad -(D_Y du)(X)Z - (D_Y du)(Z)X - I^*(Y,Z)D_X Du + I^*(X,Z)D_Y Du,
\end{aligned}$$

and the result follows. □

Consequences for surfaces. To simplify somewhat the exposition, we first concentrate on surfaces, i.e. the $n = 3$ case. The above formula becomes, for the Gauss curvature of a surface,

$$K^* = 1 - \mathrm{tr}(B^*)\,.$$

From Lemma 5.5, this can be translated as

$$K^* = 1 - \mathrm{tr}\big((E+B)^{-1}\big) = 1 - \frac{\mathrm{tr}(E+B)}{\det(E+B)}\,,$$

so that

$$K^* = \frac{\det(E+B) - \mathrm{tr}(E+B)}{\det(E+B)} = \frac{\det(B) - 1}{1 + \mathrm{tr}(B) + \det(B)}\,,$$

and, since $\det(B) - 1$ is the Gauss curvature K of the dual surface (in H^3), by the (usual) Gauss formula in H^3,

$$K^* = \frac{K}{K + 2H + 2}\,,$$

where H is the mean curvature of the dual surface in H^3. Therefore, when $K \neq 0$, we have

$$K^* = \frac{1}{1 + 2(H+1)/K}\,. \tag{1}$$

This has interesting consequences for constant mean curvature 1 surfaces in the hyperbolic 3-space, see section 9.

Higher dimensions. Now consider a space-like hypersurface H in C^n_+. Choose $x \in H$, and let $(e_i)_{1\leq i\leq n-1}$ be an orthonormal frame of T_xH for the induced metric on H in which B^* is diagonal. From Lemma 6.1 we see that the scalar curvature S^* of the induced metric on H is

$$\begin{aligned} S^* &= \sum_{i\neq j} I^*(R^*_{e_i,e_j}e_j, e_i) \\ &= \sum_{i\neq j} I^*(R^0_{e_i,e_j}e_j, e_i) - I\!I^*(e_j,e_j)I^*(e_i,e_i) - I^*(e_j,e_j)I^*(B^*e_j,e_j) \\ &= (n-1)(n-2) - 2(n-2)\mathrm{tr}(B^*)\,. \end{aligned}$$

Thus we see that the constant scalar curvature metrics on H are exactly the metrics induced on the "minimal" (i.e. mean curvature 0) hypersurfaces in C^n_+.

The Codazzi theorem. Another basic point is that, just as for hypersurfaces in Euclidean space, we have

LEMMA 6.2. *Let H be a space-like hypersurface in C^n_+ with a smooth dual hypersurface; let D^* be the Levi–Cività connection of its induced metric, and let D be the Levi–Cività connection of the metric induced on its dual (in H^n). Then, for any vector fields X, Y on H,*

$$D^*_XY = B^*D_X(B^{*-1}Y)\,,$$

and

$$(D^*_XB^*)Y = (D^*_YB^*)X\,.$$

Proof. For the first part of the lemma, we want to show that the connection (again called D^*) defined by

$$D^*_XY = (E+B)^{-1}D_X\big((E+B)Y\big)$$

is torsion-free and compatible with I^*. But it is torsion-free because

$$\begin{aligned} D^*_XY - D^*_YX &= (E+B)^{-1}\big(D_X((E+B)Y) - D_Y(E+B)X\big) \\ &= (E+B)^{-1}\big((E+B)(D_XY - D_YX) + \\ &\quad + (D_XE)Y - (D_YE)X + (D_XB)Y - (D_YB)X\big) \\ &= D_XY - D_YX\,, \end{aligned}$$

the last step using the Codazzi equation on the dual hypersurface. Therefore $D^*_XY - D^*_YX = [X,Y]$, and D^* is torsion-free.

To check that D^* is compatible with I^* is even easier. If X, Y, Z are vector fields on H, then:

$$X.I^*(Y,Z) = X.I\big((E+B)Y, (E+B)Z\big)$$

$$= I\big(D_X((E+B)Y),(E+B)Z\big) + I\big((E+B)Y, D_X((E+B)Z)\big)$$
$$= I^*(D^*_X Y, Z) + I^*(Y, D^*_X Z)\,.$$

The second point of the lemma is easy to prove using the first; if X and Y are vector fields on H, then

$$\begin{aligned}(D^*_X B^*)Y - (D^*_Y B^*)X &= D^*_X(B^*Y) - D^*_Y(B^*X) - B^*(D^*_X Y - D^*_Y X)\\ &= B^*(D_X Y - D_Y X) - B^*(D^*_X Y - D^*_Y X)\\ &= B^*[X,Y] - B^*[X,Y]\\ &= 0\,. \qquad \square\end{aligned}$$

Remark. Lemma 6.2 provides another proof of the formulas given above, relating K to K^* for surfaces in H^3 and in C^3_+. Indeed, let (e_1,e_2) be an orthonormal frame on a surface $S \subset H^3$; then, by definition of I^*, $(\overline{e}_1,\overline{e}_2) := ((E+B)^{-1}e_1,(E+B)^{-1}e_2)$ is an orthonormal frame for I^* on S^*. Moreover, the connection 1-forms ω and $\overline{\omega}$ of those frames are the same:

$$\begin{aligned}\omega(u) &:= I(D_u e_1, e_2)\\ &= I^*((E+B)^{-1}D_u e_1,(E+B)^{-1}e_2)\\ &= I^*(D^*_u((E+B)^{-1}e_1),(E+B)^{-1}e_2)\\ &= I^*(D^*_u\overline{e}_1,\overline{e}_2)\\ &=: \overline{\omega}(u)\,.\end{aligned}$$

Therefore, the curvatures on S and S^* differ only by the same factor as the area forms, so that

$$K^* = \frac{K}{\det(E+B)}\,.$$

Isometric embeddings – higher dimensions. Here we take $n \geq 4$, the next paragraph will center on $n = 3$. Let h be a smooth metric on S^{n-1}, we have the following elementary characterization of whether h can be obtained as the induced metric on a space-like hypersurface in C^n_+.

Theorem 6.3. *(S^{n-1},h) admits a space-like isometric embedding into C^n_+ if and only if h is locally conformally flat. In this case the embedding is unique up to the isometries of C^n_+.*

Proof. Let P_0 be any totally geodesic hyperplane in C^n_+. If (S^{n-1},h) has a space-like isometric embedding in C^n_+, then the projection from the image to P_0 along the vertical lines is conformal by Lemma 3.10. Therefore h is conformal to $\mathrm{can}_{S^{n-1}}$. Conversely, if h is locally conformally flat, it is conformal to $\mathrm{can}_{S^{n-1}}$, so there exists a function $u : S^{n-1} \to \mathbf{R}$ such that

$h = e^{2u}\text{can}_{S^{n-1}}$; then the graph of u above P_0 is, by Lemma 3.9, isometric to h. □

A more interesting – but still easy – question is to determine when h is induced on a strictly convex or tamely convex hypersurface in C^n_+. We call S_h the scalar curvature of h.

Theorem 6.4. *h is induced on a strictly convex space-like hypersurface H in C^n_+ if and only if h is locally conformally flat and it satisfies condition* (H)*:*

$$2ric_h - \frac{S_h}{n-2}h - (n-3)h \text{ is everywhere negative definite.} \qquad \text{(H)}$$

H is then unique up to isometries of C^n_+. H is tamely convex if and only if it satisfies condition (C)*:*

$$\text{all eigenvalues of } 2(n-2)ric_h - S_h h \text{ are in } (-(n-2)(n-3), (n-2)(n-3)). \qquad \text{(C)}$$

We will say that h is *H-admissible* if it satisfies condition (H), and *C-admissible* if it satisfies condition (C).

Proof. Let $(e_i)_{1\leq i\leq n-1}$ be an orthonormal frame for I^* which diagonalizes B^*, and let $(k_i)_{1\leq i\leq n-1}$ be the associated eigenvalues of B^*. Call $K_{i,j}$ the sectional curvature of h on the 2-plane generated by e_i and e_j. Then, by Lemma 6.1,

$$K_{i,j} = 1 - k_i - k_j \ ,$$

so that

$$\text{ric}_h(e_i, e_i) = \sum_{j\neq i} K_{i,j} = (n-2) - (n-3)k_i - \sum_j k_j \ ,$$

and

$$S_h = \sum_i \text{ric}_h(e_i, e_i) = (n-1)(n-2) - 2(n-2)\sum_j k_j \ ,$$

so that

$$\begin{aligned} k_i &= \frac{S_h + (n-2)(n-3) - 2(n-2)\text{ric}_h(e_i, e_i)}{2(n-2)(n-3)} \\ &= \frac{S_h - 2(n-2)\text{ric}_h(e_i, e_i)}{2(n-2)(n-3)} + \frac{1}{2} \ , \end{aligned}$$

and both results follow. □

Isometric embeddings of surfaces. The analogue of Theorem 6.3 is even simpler in dimension $n = 3$, since in that case all metrics on S^2 are conformal to the canonical metric. Therefore:

Theorem 6.5. *For any smooth metric h on S^2, (S^2, h) admits a unique (up to the isometries of C^3_+) space-like isometric embedding in C^3_+.*

To understand the metrics induced on convex surfaces we have to introduce a definition (which is also a lemma).

DEFINITION 6.6. *Let h be a smooth metric on S^2. Let $x \in S^2$. There is a unique function u_x on S^2 such that the metric $e^{-2u_x}h$ has constant curvature 1 and that $u_x(x) = du_x(x) = 0$. We say that h is* H-admissible *if, for each $x \in S^2$, the Hessian of u_x at x is positive definite, and that h is* C-admissible *if, for each x, all eigenvalues of the Hessian of u_x at x are in $(0, 1)$.*

Proof. We have to prove the existence and uniqueness of u_x.

h is conformal to can_{S^2}, so there exists a function $u : S^2 \to \mathbf{R}$ such that $e^{2u}\mathrm{can}_{S^2} = h$. Choose a totally geodesic plane $P_0 \subset C^3_+$, and let S be the graph of u above P_0. Then, by Lemma 3.9, the metric induced on S is h.

Now let $x \in S$. By Lemma 2.2, there exists a unique totally geodesic plane P_1 in C^3_+ which is tangent to S at x. P_1 is the graph above S of a function v on S. Then $e^{-2v}h$ is the metric induced on P_1, and is isometric to can_{S^2}, so v satisfies the conditions set on u_x.

Conversely, if $w : S \to \mathbf{R}$ satisfies those conditions, then the graph P of w above S has as induced metric can_{S^2}, so it is a totally geodesic plane, and moreover it is tangent to S at x. Thus, by Lemma 2.2, $P = P_1$, and $w = v$. □

Now we can state

Theorem 6.7. *Let h be a smooth metric on S^2. h is induced on a strictly convex surface in C^3_+ if and only if h is H-admissible. h is induced on a tamely convex surface if and only if h is C-admissible.*

Proof. Since h is conformal to can_{S^2}, there exists a function $u : S^2 \to \mathbf{R}$ such that $e^{2u}\mathrm{can}_{S^2}$ is isometric to h. If P_0 is any totally geodesic plane in C^3_+, the graph S of u above P_0 has h as its induced metric. Moreover, by the previous definition, h is H-admissible if and only if S has positive definite second fundamental form $I\!I^*$, so if and only if S is strictly convex. And h is C-admissible if and only if B^* has its eigenvalues in $(0, 1)$, so if and only if S is tamely convex. □

REMARK. H-admissible metrics on S^2 have curvature $K < 1$, while C-admissible metrics on S^2 have curvature in $(-1, 1)$. The converse, however, is not true.

Proof. Theorem 6.7 shows that any H-admissible metric is induced on a strictly convex surface in C^3_+, and Lemma 6.1 then indicates that it has curvature strictly below 1. Similarly C-admissible metrics are induced on tamely convex surfaces, which have curvature $K \in (-1, 1)$ by Lemma 6.1. □

Möbius structures. It might be helpful to point out that the natural setting for the definition of H-admissible and C-admissible metrics uses the notion of Möbius structure, as described in section 4, rather than conformally flat structure. In dimension at least 3 (i.e. for $n \geq 4$) it is basically the same thing, but not in dimension 2. On surfaces, Möbius structures are the same as $\mathbf{C}\mathrm{P}^1$ structures, and a $\mathbf{C}\mathrm{P}^1$ structure contains in general – for instance on a surface of genus $g \geq 2$ – much more information than its conformal metric. On the other hand, on the sphere, the two notions are basically the same, and this explains why Definition 6.6 can be given as it is. The same holds in the other situations described below, where in each case a $\mathbf{C}\mathrm{P}^1$-structure is picked out by the context: for Definition 7.3 it is the one such that the universal cover of Σ is isomorphic (as a complex projective surface) to an hemisphere, while in Definition 8.1 it is related to the magic of the Ahlfors–Bers theorem.

The analogue of Definition 6.6 for a surface Σ endowed with a $\mathbf{C}\mathrm{P}^1$ structure σ is simple. Note that σ induces a conformal structure on Σ. Choose $x \in \Sigma$ and a metric h on Σ whose conformal structure is the same as that of σ. σ has a developing map $\mathrm{dev} : \tilde{\Sigma} \to S^2$ which is well defined up to a Möbius transformation of S^2. So there is a unique metric h_0 on Σ which

- has constant curvature 1,
- has the same conformal structure as h and σ,
- has the same developing map in S^2 as σ,
- coincides with h on $T_x\Sigma$.

In the neighborhood of x, there exists a function u such that $h = e^{2u}h_0$; the conditions of Definition 6.6 then apply to u.

The point of course is that the conditions in Definition 6.6, which are global when referring to a surface with a conformal metric, are local when a $\mathbf{C}\mathrm{P}^1$ structure is given.

7 Hypersurfaces in H^n

We will use in this section the results concerning the metrics on convex hypersurfaces to understand the dual metrics on H-convex spheres in H^n, and then on equivariant hypersurfaces.

Compact surfaces in H^n, $n \geq 4$. As a consequence of Theorems 6.3 and 6.4, we have for $n \geq 4$:

Theorem 7.1. *Let h be smooth metric on S^{n-1}. h is the horospherical metric I^* of a strictly H-convex sphere S in H^n if and only if h is H-admissible, in the sense that it is locally conformally flat and satisfies condition* (H). *S is then unique up to the isometries of H^n. Moreover, H is tamely convex if and only if h satisfies condition* (C).

Compact surfaces in H^3. The same theorem holds in H^3 with the adequate notion of H-convexity; it is a consequence of Theorems 6.5 and 6.7.

Theorem 7.2. *Let h be a smooth metric on S^2. It is the horospherical metric I^* of a strictly H-convex immersed sphere S in H^3 if and only if it is H-admissible. It is the horospherical metric of a strictly convex embedded sphere $S \subset H^3$ if and only if it is C-admissible. In each case, S is unique up to the global isometries of H^3.*

Equivariant surfaces. We consider now a surface Σ of genus at least 2. First we introduce a class of metrics on Σ in the following way – this definition is also a lemma.

DEFINITION 7.3. *Let h be a smooth metric on Σ; we also call h the pull-back metric on the universal cover $\tilde{\Sigma}$ of Σ. For each $x \in \tilde{\Sigma}$, there is a unique function $u_x : \tilde{\Sigma} \to \mathbf{R}$ such that $e^{-2u_x}h$ is isometric to a hemisphere of (S^2, can), and that $u_x(x) = du_x(x) = 0$. h is* H-admissible *if, for each x, the Hessian of u_x is positive definite at x. h is* C-admissible *if, for each x, all eigenvalues of the Hessian at x of u_x are in* $(0, 1)$.

Note that, here again, H-admissible metrics have $K < 1$, and C-admissible metrics have $K \in (-1, 1)$, while the converse is false. Any metric can be "scaled up" to a metric which is C-admissible, and thus H-admissible.

Proof. We only have to prove the existence and uniqueness of u_x.

It is well known that there exists a unique hyperbolic metric in the conformal class of h, i.e. a unique function $u : \Sigma \to \mathbf{R}$ such that $e^{-2u}h$ has constant curvature -1. We also call u the induced function on $\tilde{\Sigma}$. Then $(\tilde{\Sigma}, e^{-2u}h)$ is isometric to H^2, and this defines a function u on H^2 which is invariant under an action of $\pi_1(\Sigma)$ by isometries.

Now choose a dual plane $P_0 \subset C^3_+$. Its induced metric is isometric to that of H^2; choose an isometry between P_0 and $(\tilde{\Sigma}, e^{-2u}h)$. This defines a function u on P_0, and by construction and Lemma 3.9, the graph of u above P_0 is isometric to $(\tilde{\Sigma}, h)$. We identify $\tilde{\Sigma}$ with this graph.

Now choose $x \in \tilde{\Sigma}$, and let P_1 be the totally geodesic plane tangent to $\tilde{\Sigma}$ at x. P_0 is a graph above a hemisphere $P_{1,+}$ of P_1, thus $\tilde{\Sigma}$ is also the graph above $P_{1,+}$ of a function v; by construction, v satisfies the conditions on u_x.

Conversely, if w is a function satisfying those conditions, then the graph of $-w$ above $\tilde{\Sigma}$ is a hemisphere of a totally geodesic plane which is tangent to $\tilde{\Sigma}$ at x, so $w = v$. □

This leads to a characterization of the metrics induced on equivariant surfaces in H^3 as follows.

Theorem 7.4. *A smooth metric h on Σ is the horospherical metric of a strictly H-convex equivariant immersion whose representation fixes a plane if and only if h is H-admissible. It is the horospherical metric of a strictly convex equivariant embedding whose representation fixes a plane if and only if h is C-admissible. The equivariant immersion/embedding is then unique up to global isometries.*

Proof. First note that any metric h on Σ has an equivariant isometric embedding into C^3_+ whose representation fixes a dual plane. Indeed, there is a unique function $u : \Sigma \rightarrow \mathbf{R}$ such that $e^{-2u}h$ is hyperbolic; u can then be identified with an equivariant function defined on a dual plane $P_0 \subset C^3_+$, and then $(\tilde{\Sigma}, h)$ is isometric to the graph of u above P_0.

The previous proof then indicates that $\tilde{\Sigma} \subset C^3_+$ is strictly convex if and only if h is H-admissible, and tamely convex if and only if h is C-admissible. Therefore, the dual immersion in H^3 is strictly H-convex if and only if h is H-admissible, and strictly convex if and only if h is C-admissible. In this last case, the convexity implies that the immersion is an embedding. □

Some kind of analogous results in higher dimension might hold, but they could be less interesting since the metrics obtained are conformally flat, which is a fairly strong condition. On the other hand they might be used to put special (e.g. hyperbolic) metrics on conformally flat manifolds, through deformations of equivariant sub-manifolds of H^n or C^n_+.

8 Hyperbolic Manifolds with Boundaries

Why do all this? As pointed out in the introduction, a natural question along the lines of Conjecture 0.4 is to find the right boundary condition necessary to obtain a unique hyperbolic metric on a given 3-manifold with boundary. While Conjecture 0.4 strongly suggests that one should consider the metric induced on the boundary, Theorem 0.3 indicates that the third fundamental form of the boundary could be another choice.

The same question can be asked in higher dimensions, with hyperbolic metrics replaced by Einstein metrics of negative curvature. A basic step is taken in [S4], where it is shown that any small deformation of the metric induced on the boundary of a hyperbolic ball can be "followed" by an (essentially unique) Einstein deformation of the metric in the interior. However, in this case again it is not completely obvious whether the induced metric on the boundary is the right object to consider.

It appears clearly from recent work (see e.g. [GrL], [GrW], [W], [A1,2]) that, when one considers complete, conformally compact manifolds instead of metrics for which the boundary is at finite distance, then the conformal class of the boundary is what one needs. This does not indicate in any clear way what one should use when the boundary is at finite distance, because, in a conformally compact manifold, the hypersurfaces which are "close" to the boundary in the conformal compact model are "almost umbilical", so that the conformal class of the induced metric is also (asymptotically) the conformal class of the second or third fundamental forms.

The solution advocated here is that the horospherical metric might be the right thing to consider; the main argument is that, for hyperbolic 3-manifolds, one can then obtain a satisfying existence and uniqueness result in a very simple way. Of course the real challenge will be to obtain similar results in higher dimensions, for Einstein manifolds with boundary, or in other settings.

H-admissible metrics. We consider now a geometrically finite 3-manifold with boundary $(M, \partial M)$ which admits a complete convex co-compact hyperbolic metric. Then the universal cover of M is the 3-dimensional ball B^3. If h is a Riemannian metric on ∂M, then h defines a complete metric on an open dense subset of S^2, which is invariant under a conformal action of $\pi_1 M$. Moreover, its conformal structure defines a conformal structure on an open dense set of S^2, which extends to a conformal metric on S^2, and the universal cover $\widetilde{\partial M}$ of ∂M has a unique conformal embedding into S^2 whose image is an open dense set (see e.g. [AB], [A], [O]).

We can thus define a proper class of metrics on ∂M.

Definition 8.1. *Let h be a smooth metric on ∂M, and let $x \in \widetilde{\partial M}$. There exists a unique function u_x on $\widetilde{\partial M}$ such that $e^{-2u_x} h$ extends to a constant curvature* 1 *metric on S^2, and such that $u_x(x) = du_x(x) = 0$. We say that h is* H-admissible *if, for all x, the Hessian of u_x at x is positive* *definite.* h *is* C-admissible *if, for all* x*, the eigenvalues of the Hessian of* u_x *at* x *are in* $(0,1)$.

Note that this definition coincides with Definitions 6.6 and 7.3 in the corresponding special cases. Again as above, H-admissible metrics have curvature $K < 1$, and C-admissible metrics have curvature $K \in (-1,1)$. Moreover, here again, any metric can be scaled up to one which is C-admissible.

Proof. Again we only have to prove the existence and uniqueness of u_x.

We know that there exists a function u on the universal cover of ∂M such that $e^{-2u}h$ is isometric to an open dense subset of S^2. This defines $(\tilde{\Sigma}, h)$ as the graph of u above an open dense subset of a totally geodesic plane P_0 (with the induced metric).

The rest of the proof is just like for Definitions 6.6 and 7.3, and uses the uniqueness of the conformal change of metric. □

Existence and uniqueness. We can now state the analogue of Conjecture 0.4 for the horospherical metric.

Theorem 8.2. *Let* h *be a smooth metric on* ∂M.

1. h *is the horospherical metric of* ∂M *for a hyperbolic metric* g *on* M*, such that* ∂M *is strictly convex, if and only if* h *is C-admissible.* g *is then unique.*
2. h *is the horospherical metric of a strictly H-convex immersion* ϕ *of* ∂M *in* M *for a complete hyperbolic metric* g *on* M*, such that* ϕ *can be deformed through immersions to the identity map* $\partial M \to \partial_\infty M$*, if and only if* h *is H-admissible.* g *and* ϕ *are then unique.*

Proof. We already know from the proof of Definition 8.1 that $(\widetilde{\partial M}, h)$ is isometric to the graph of a unique (up to global isometries) graph above a totally geodesic plane P_0.

Taking the dual surface in H^3 gives an immersion ϕ of $\widetilde{\partial M}$ in H^3 which is strictly H-convex if h is H-admissible, and strictly convex if h is C-admissible.

Moreover, $\pi_1 M$ acts by conformal transformations on P_0, so, by Lemma 4.1, by isometries on H^3. By construction, $\phi(\widetilde{\partial M})$ is invariant under those isometries. Thus $(\partial M, h)$ is isometric to the quotient by $\pi_1 M$ of the image of ϕ with its horospherical metric. □

Note that, if h is only strictly H-convex, we only obtain a priori an immersion of ∂M in M, which can be deformed through immersions to an embedding. If h is C-admissible, on the other hand, ∂M is obtained as a strictly convex surface in M, so it is embedded (and it bounds a strictly convex domain in M).

It should be pointed out that Theorem 7.4 is a direct consequence of Theorem 8.2. Indeed consider the manifold $(M, \partial M) = (\Sigma \times [-1,1], \Sigma \times \{-1,1\})$, where Σ is a surface of genus at least 2, and take on ∂M a metric which is identical on both copies of Σ. Then the uniqueness statement in Theorem 8.2 implies that the metric g obtained will have a $\mathbf{Z}/2\mathbf{Z}$ symmetry, which will exchange the two connected components of ∂M. Therefore $\widetilde{\partial M}$ will be immersed/embedded in H^3 as two equivariant surfaces, symmetric with respect to a plane which is fixed by both representations.

Higher dimensions. We briefly discuss here some properties appearing in higher dimensions.

Let Σ be a compact $n-1$-dimensional manifold, with a smooth, conformally flat Riemannian metric h. h defines on Σ a Möbius structure, and the corresponding developing map $\mathrm{dev} : \tilde{\Sigma} \to S^{n-1}$. If g_0 is the canonical metric on S^{n-1}, $\mathrm{dev}^* g_0$ is conformal to h, so there exists a unique function $u : \tilde{\Sigma} \to \mathbf{R}$ such that $h = e^{2u} \mathrm{dev}^* g_0$.

Now identify isometrically S^{n-1} with a totally geodesic plane $H_0 \subset C^n_+$, and let ϕ be the function sending $x \in \Sigma$ to the point at distance $u(x)$ from $\mathrm{dev}(x)$ along the vertical line going through $\mathrm{dev}(x)$. By construction, ϕ is an equivariant isometric embedding of $(\tilde{\Sigma}, h)$ into C^n_+. Moreover, it should be clear to the reader that its image is tamely convex (resp. convex) if and only if h satisfies condition (C) (resp. (H)). In this case, the dual of $\phi(\Sigma)$ will be a smooth, convex (resp. H-convex), equivariant, immersed hypersurface in H^n, whose horospherical metric is h.

This picture is more precise with additional assumptions. Suppose for instance that the conformal class of h has positive scalar curvature (i.e. that there is a metric, conformal to h, with positive scalar curvature). Then, by the results of Schoen and Yau [ScY], the developing map $\mathrm{dev} : \Sigma \to S^n$ is injective, and its image is a simply connected, dense subset of S^{n-1}. If moreover h verifies condition (C), then the hypersurface in H^n which is dual to $\phi(\Sigma)$ will be convex and embedded. Taking the quotient of H^n by the representation $\pi_1(\Sigma) \to \mathrm{Isom}(H^n)$ defined by the developing map dev, we see that $\phi(\Sigma)$ appears as the convex boundary of a compact hyperbolic n-orbifold.

9 Moreover

Hypersurfaces in S^n_1. Note that if $S \subset H^n$ is a convex surface, and if S^d is the dual surface in S^n_1 (which is also a convex surface, and moreover is space-like) then the first, second and third fundamental forms on S^d are $I^d = I\!I\!I$, $I\!I^d = I\!I$, and $I\!I\!I^d = I$ respectively. Therefore:

$$I + 2I\!I + I\!I\!I = I^d + 2I\!I^d + I\!I\!I^d \ ,$$

so that most of the themes described in this paper for hypersurfaces in H^n are also valid for convex hypersurfaces in S^n_1, and can be proved by considering the dual surface in H^n. Presumably a weaker hypothesis than convexity could be used (like the H-convexity condition in H^n); it should be possible to repeat some of the arguments below without reference to the dual hypersurface in H^n, by replacing the horospheres in H^n by their dual hypersurfaces in S^n_1.

An elementary approach. A large part of what we have described here can be reduced essentially to a simple (but remarkable) property. Let H be a complete oriented hypersurface in H^n, which is "uniformly H-convex" in the most natural sense. Let u be a function on h, with a differential which is "small". For each point $x \in H$, consider the horosphere h_x tangent to H at x, and its equidistant horosphere h'_x at distance $u(x)$. Then let H' be the envelope of the horospheres h'_x, and let ϕ be the map sending $x \in H$ to the point $\phi(x) \in H'$ where h'_x is tangent to H' (this is well defined if u and H are well behaved). Then ϕ is an isometry between $(H, e^{2u} I^*_H)$ and $(H', I^*_{H'})$.

Of course this is basically a translation, in purely hyperbolic terms, of the basic properties of the metric on C^n_+, as described in Lemma 3.9, and of the duality between H^n and C^n_+. Moreover the statement is quite imprecise concerning the precise conditions on u; of course things are clear in C^n_+, the point is only that u has to be such that the graph of u above H^* (which will be the dual of H') remains convex, so that H' remains smooth and H-convex. More generally, I guess that some of the results obtained here could be achieved without using C^n_+, but I doubt whether it could improve the clarity of this matter.

Symmetric spaces and dualities. Given a symmetric space G/K, there is a quite general way of constructing other spaces (of the form G/H, for various choices of $H \subset G$) which are in "duality" with G/K – see e.g. [H1,2]. The duality between H^n and C^n_+ can be seen as a special case of this (with $G = \mathrm{SO}(n,1)$, $K = \mathrm{SO}(n)$ and $H = \mathrm{Isom}(\mathbf{R}^{n-1})$), just like the duality between H^n and S^n_1 (with $G = \mathrm{SO}(n,1)$, $K = \mathrm{SO}(n)$ and $H = \mathrm{SO}(n-1,1)$). In this general setting there is a natural – and well understood – duality between the functions or distributions on a space and on its dual. The duality between the hypersurfaces can be put in this context by replacing a hypersurface by some measure which it defines, the dual hypersurface is then the support of the dual measure.

In the cases which we have described, however, one should not use the measure associated to the area form on the hypersurfaces, since the duality would then act with a factor equal to the Gauss–Kronecker curvature of the hypersurfaces (in the case of the H^n/S^n_1 duality) or the determinant of $E+B$ (in the H^n/C^n_+ duality). Rather one should normalize this area measure by a factor $1/\sqrt{\det(B)}$ or $1/\sqrt{\det(E+B)}$ in H^n, and $1/\sqrt{\det(B^*)}$ in S^n_1 or C^n_+.

A natural question is to understand to what extend the duality properties of hypersurfaces in other symmetric spaces extend from the hyperbolic setting described above, and what one could get out of it.

Constant mean curvature one surfaces. Equation (1) shows that the constant mean curvature -1 surface in H^3 are characterized as those whose dual has constant curvature 1 (of course the minus sign is just a question of orientation). As pointed out in the introduction – and as the reader can readily check – the horospherical metric of those surfaces corresponds to the third fundamental form of the minimal cousin surface in $\mathbf{R}^3$ (see [Br]).

This metric, along with some additional information in some cases, can be used to reconstruct a constant mean curvature one surface in H^3 from a constant curvature one metric on a surface. Several interesting results were obtained in this direction in particular by Umehara and Yamada, see e.g. [UY1,2]. We will only outline here how the H^3—C^3_+ duality provides a geometric way of understanding this correspondence.

A first remark, which is not too difficult to check, is that the catenoid-like ends of constant mean curvature one surfaces induce conical points on the dual surface, with an angle which depends on the behavior at infinity in H^3. Moreover, umbilical points of the surfaces correspond to singular, ramifications points on the dual surface.

On the other hand, the problem of finding a constant curvature one metric on a sphere with prescribed conical singularities is now rather well understood (see for instance [Tr] and [LuT]). Given such a metric, it is induced on a unique surface in C^3_+ (see Theorem 6.4 below). It is a simple matter to prove (by using the same arguments as above backwards) that the dual surface in H^3 has constant mean curvature one.

When one considers surfaces of higher genus, things are slightly more complicated because such surfaces have more than one conformal structure. Thus one needs to prescribe, in addition to the position of the conical points and their cone angle, the conformal structure of the surface, or alternatively a map from the surface to S^2. The ramification points will be associated to the umbilical points of the constant curvature one surface in H^3. But the main idea remains valid: the conformal structure, along with the conical singularities, determines (under some conditions) a constant curvature one metric (with some singularities). This metric is induced on a unique surface in C^3_+ (with prescribed projection to a totally geodesic S^2), and the dual surface in H^3 has constant mean curvature one.

Induced metrics and third fundamental forms. One striking feature of the results above is that they are simpler to obtain – and more powerful in some cases – that the corresponding results obtained for convex (hyper-)surfaces when one considers on them the induced metrics or third fundamental forms. This leads to the idea that those results could be used as a tool to obtain results on the induced metrics or third fundamental form; for this one should obtain rigidity results on the way the induced metric (resp. third fundamental form) varies when a deformation changes the horospherical metric.

The horospherical metric of a polyhedron. Let P be a convex polyhedron in H^n. P has a dual polyhedron P^d in the de Sitter space S^n_1 (see e.g. [RH]); each (closed) k-face F of P has a dual $(n-1-k)$-face F^d in S^n_1, which is a face of P^d and carries a metric with constant curvature one.

P also has a dual object P^* in C^n_+; it is the set of support horospheres of P. P^* has an induced metric, and it is not too difficult to understand its relationship with the metrics on P and P^d. Namely, P^*, with its induced metric, is isometric to the object obtained by gluing the "faces" $\phi(F) := F \times F^d$, where F is any k-face of P (with $0 \leq k \leq n-1$) with its induced metric in H^n, and F^d is the dual $(n-1-k)$-face with its induced metric in S^n_1.

The fact that those "faces" can be glued is an interesting (although not too surprising) fact; for each face F of P, $\phi(F)$ is glued to the $\phi(F')$, where F' is a face of P which either is contained in F or contains F.

Einstein manifolds, etc. The most natural framework in which Conjecture 0.4 could be extended is the theory of negatively curved Einstein manifolds with boundaries; indeed, in dimension 3, negatively curved Einstein metrics are the same as hyperbolic metrics.

An elementary (and far too restricted) first step was taken in this direction in [S4] (see also [RS1,2] for some strikingly related rigidity results obtained by very different methods). The outstanding problem there, however, is that the infinitesimal rigidity result which is needed – stating that an infinitesimal deformation of the interior metric induces a non-trivial deformation of the boundary metric – is only obtained when the boundary is umbilical.

A natural question is therefore whether an analogue of the horospherical metric (maybe defined as $I + 2I\!I + I\!I\!I$) could lead to some infinitesimal rigidity result for Einstein manifolds with boundary; this would open the door to possible results on the existence and/or uniqueness of Einstein metrics inducing a given horospherical metric on the boundary.

Note that the theory concerning complete metrics is rather more advanced; in that case one only prescribes the conformal structure on the boundary at infinity, and the Einstein metrics are required to be conformally compact. In dimension 3 it is just the classical Ahlfors–Bers theorem, while in higher dimension the theory seems to be advancing (see the previous section for references).

Acknowledgements. This paper has been greatly improved thanks to constructive remarks from an anonymous referee.

I would also like to thank Pascal Collin for some useful remarks and references concerning constant mean curvature one surfaces in hyperbolic 3-space.

References

[AB] L. Ahlfors, L. Bers, Riemann's mapping theorem for variable metrics, Ann. of Math. (2), 72 (1960), 385–404.

[A] L.V. Ahlfors, Lectures on Quasiconformal Mappings, D. Van Nostrand Co., Inc., Toronto, Ont.-New York-London, 1966. Manuscript prepared with the assistance of Clifford J. Earle, Jr. Van Nostrand Mathematical Studies, No. 10.

[Al1] A.D. Aleksandrov, Convex Polyhedra, GITTL, 1951 (in Russian; English translation to appear, Springer).

[Al2] A.D. Aleksandrov, Selected Works. Part I, Selected Scientific Papers, Translated from Russian by P.S.V. Niadu (Preface by and edited by Yu.G. Reshetnyak and S.S. Kutateladze), Gordon and Breach Publishers, Amsterdam, 1996.

[AlZ] A.D. Aleksandrov, V.A. Zalgaller, Intrinsic Geometry of Surfaces 15, Translations of Mathematical Monographs, AMS, 1967.

[An1] M.T. Anderson, Closedness of the space of AHE metrics on 4-manifolds, math.DG/0012167.

[An2] M.T. Anderson, L^2 curvature and volume renormalization of AHE metrics on 4-manifolds, math.DG/0011051.

[B] A. Besse, Einstein Manifolds, Springer, 1987.

[Br] R.L. Bryant, Surfaces of mean curvature one in hyperbolic space, Astérisque (154-155):12, 321–347, 353 (1988), 1987. Théorie des variétés minimales et applications (Palaiseau, 1983–1984).

[C] E. Cartan, La déformation des hypersurfaces dans l'espace euclidien réel à n dimensions, Bull. Soc. Math. France 44 (1916), 65–99.

[GHL] S. Gallot, D. Hulin, J. Lafontaine, Riemannian Geometry, Springer, 1987.

[GrL] C.R. Graham, J.M. Lee, Einstein metrics with prescribed conformal infinity on the ball, Adv. Math. 87 (1991), 186–225.

[GrW] C.R. Graham, E. Witten, Conformal anomaly of submanifold observables in AdS/CFT correspondence, hep-th/9901021.

[Gro] M. Gromov, Partial Differential Relations, Springer, 1986.

[H1] S. Helgason, Geometric Analysis on Symmetric Spaces, American Mathematical Society, Providence, RI, 1994.

[H2] S. Helgason, Groups and Geometric Analysis, American Mathematical Society, Providence, RI, 2000. Integral geometry, invariant differential operators, and spherical functions, Corrected reprint of the 1984 original.

[KP] R.S. Kulkarni, U. Pinkall, A canonical metric for Möbius structures and its applications, Math. Z. 216:1 (1994), 89–129.

[L1] F. Labourie, Immersions isométriques elliptiques et courbes pseudo-holomorphes, J. Differential Geom. 30 (1989), 395–424.

[L2] F. Labourie, Métriques prescrites sur le bord des variétés hyperboliques de dimension 3, J. Differential Geom. 35 (1992), 609–626.

[L3] F. Labourie, Surfaces convexes dans l'espace hyperbolique et CP1-structures, J. London Math. Soc. Ser. II 45 (1992), 549–565.

[LS] F. Labourie, J.-M. Schlenker, Surfaces convexes fuchsiennes dans les espaces lorentziens à courbure constante, Math. Annalen 316 (2000), 465–483.

[LuT] Feng Luo, Gang Tian, Liouville equation and spherical convex polytopes, Proc. Amer. Math. Soc. 116:4 (1992), 1119–1129.

[N] L. Nirenberg, The Weyl and Minkowski problem in differential geometry in the large, Comm. Pure Appl. Math. 6 (1953), 337–394.

[O] J.-P. Otal, Le théorème d'hyperbolisation pour les variétés fibrées de dimension 3, Astérisque 235, 1996.

[P] A.V. Pogorelov, Extrinsic Geometry of Convex Surfaces. Translations of Mathematical Monographs 35, American Mathematical Society, 1973.

[R] I. Rivin, A characterization of ideal polyhedra in hyperbolic 3-space, Annals of Math. 143 (1996), 51–70.

[RH] I. Rivin, C.D. Hodgson, A characterization of compact convex polyhedra in hyperbolic 3-space, Invent. Math. 111 (1993), 77–111.

[RS1] I. Rivin, J.-M. Schlenker, The Schläfli formula in Einstein manifolds with boundary, Electronic Research Announcements of the AMS 5 (1999), 18–23.

[RS2] I. Rivin, J.-M. Schlenker, The Schläfli formula and Einstein manifolds, preprint math. DG/0001176, 2000.

[S1] J.-M. Schlenker, Surfaces convexes dans des espaces lorentziens à courbure constante, Commun. Anal. and Geom. 4 (1996), 285–331.

[S2] J.-M. Schlenker, Métriques sur les polyèdres hyperboliques convexes, J. Differential Geom. 48:2 (1998), 323–405.

[S3] J.-M. Schlenker, Dihedral angles of convex polyhedra, Discrete Comput. Geom. 23:3 (2000), 409–417.

[S4] J.-M. Schlenker, Einstein manifolds with convex boundaries, Commentarii Math. Helvetici, 76:1 (2001), 1–28.

[ScY] R. Schoen, S.-T. Yau, Conformally flat manifolds, Kleinian groups and scalar curvature, Invent. Math. 92:1 (1988), 47–71.

[T] W.P. Thurston, Three-dimensional geometry and topology, Available on the MSRI website, at: http://www.msri.org/publications/books/gt3m/; recent version of the 1980 notes, 1997.

[Tr] M. Troyanov, Prescribing curvature on compact surfaces with conical singularities, Trans. Amer. Math. Soc. 324:2 (1991), 793–821.

[UY1] M. Umehara, K. Yamada, Complete surfaces of constant mean curvature 1 in the hyperbolic 3-space, Ann. of Math. (2) 137:3 (1993), 611–638.

[UY2] M. Umehara, K. Yamada, Surfaces of constant mean curvature c in $H^3(-c^2)$ with prescribed hyperbolic Gauss map, Math. Ann. 304:2 1996), 203–224.

[W] E. Witten, Anti de Sitter space and holography, Adv. Theor. Math. Phys. 2 (1998), 253–291.

Jean-Marc Schlenker, Laboratoire Emile Picard, UMR CNRS 5580, Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse Cedex 4, France
schlenker@picard.ups-tlse.fr; http://picard.ups-tlse.fr/~schlenker.

Submitted: March 2001
Revision: November 2001