GAFA, Geom. funct. anal.
Vol. 12 (2002) 73 – 79
1016-443X/02/010073-7 $ 1.50+0.20/0

GAFA Geometric And Functional Analysis

STABILITY OF THE LIPSCHITZ EXTENSION PROPERTY UNDER METRIC TRANSFORMS

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Abstract

We show that the linear and nonlinear Lipschitz extension properties of a metric space are not changed when the original metric is replaced by a new metric obtained by composition with an arbitrary concave function.

1. In what follows $(\mathcal{M}, d)$ stands for a metric space and X denotes a Banach space with the norm $\|\cdot\|$. One introduces the Lipschitz space $\mathrm{Lip}(\mathcal{M}, X)$ as a linear space of mappings $f : \mathcal{M} \to X$ defined by finiteness of the seminorm

$$|f|_{\mathrm{Lip}(\mathcal{M},X)} := \sup_{x \neq y} \frac{\|f(x) - f(y)\|}{d(x,y)} .$$

Definition 1. (a) A metric space $\mathcal{M}$ satisfies the *Lipschitz extension property with respect to* X, if there is a constant $\lambda > 0$ such that for every $\mathcal{M}' \subset \mathcal{M}$ there exists an extension operator $E_{\mathcal{M}'} : \mathrm{Lip}(\mathcal{M}', X) \to \mathrm{Lip}(\mathcal{M}, X)$ with the norm $\leq \lambda$.

(b) $\mathcal{M}$ satisfies the *linear Lipschitz extension property with respect to* X, if the operator $E_{\mathcal{M}'}$ can be chosen to be linear.

The optimal choice of λ is denoted by $e(\mathcal{M}, X)$ in case (a) and by $e_l(\mathcal{M}, X)$ in case (b).

Now let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a concave non-decreasing function equaling 0 at 0. (In the sequel ω will stand for such a function.) The operation of superposition $d_\omega := \omega \circ d$ defines a new metric space $(\mathcal{M}, d_\omega)$. This space is called the *metric transform* of $\mathcal{M}$ by ω and denoted by the symbol $\omega(\mathcal{M})$.

Theorem 2. *(a)* $e(\omega(\mathcal{M}), X) \leq C\, e(\mathcal{M}, X)^2$*;*

(b) The same inequality holds for e_l*.*

Hereafter C *stands for an absolute constant.*

The first named author was supported by the Technion V.P.R. Fund.

REMARK 3. A partial case of Theorem 2 (a), for $\omega(t) = t^\alpha$, $0 < \alpha < 1$, and X being a dual Banach space was independently proved by A. Naor [N] in another way.

Theorem 2 plays the role of an "amplifier" which allows us to extend and generalize Lipschitz extension theorems previously known only for special metrics. Thus part (a) of Theorem 2 can be applied to generalize many extension theorems which have been proved since the classical McShane [M] and Kirzsbraun [K] theorems appeared in 1934, see, e.g. [WW] and the recent paper [LS] and references therein. In contrast to this intensive development there are only few results related to the linear extension problem. One of them, which is easily obtained by the Whitney extension method, states that a doubling metric space has the linear Lipschitz extension property, see, e.g. [G, p. 432]. As an example of a non-doubling metric space possessing this property one singles out the classical Beltrami-Lobachevski space $\mathcal{L}_n$. It was proved in [BrS, Section 4] that $e_l(\mathcal{L}_n, \mathbb{R}) \le C(n)$.

2. The crucial step of the proof of Theorem 2 is the following basic result of Interpolation Space Theory (see [BL] or [BrK] for main notions of this theory). Let $\vec{X} := (X_0, X_1)$ be a Banach couple and let $K(\cdot, x; \vec{X})$ be the *K-functional* of an element $x \in \Sigma(\vec{X}) := X_0 + X_1$.

Theorem 4 (on K-divisibility of $\vec{X}$, see [BrK, Theorem 3.2.7]). *Assume that ω is a sum of concave functions $\omega_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i \in \mathbb{N}$,*

$$\omega = \sum_{i=1}^{\infty} \omega_i \quad (\textit{pointwise convergence}) \tag{1}$$

and for every $t > 0$ the following inequality

$$K(t, x; \vec{X}) \le \omega(t) \tag{2}$$

is true. Then there exists a decomposition $x = \sum_{i=1}^{\infty} x_i$ (convergence in $\Sigma(\vec{X})$) such that $K(t, x_i; \vec{X}) \le \gamma\omega_i(t)$ for all $t > 0$ and $i \in \mathbb{N}$. Here $\gamma > 0$ is an absolute constant.

For the formulation of a "linear" version of this result recall the following:

DEFINITION 5. A Banach couple $\vec{X}$ is said to be *K-linearizable* if there exist a positive constant λ and two families of linear bounded operators $\{U_t : \Sigma(\vec{X}) \to X_0 : \ t > 0\}$ and $\{V_t : \Sigma(\vec{X}) \to X_1 : \ t > 0\}$ such that $U_t + V_t = \mathrm{Id}_{\Sigma(\vec{X})}$ for all $t > 0$ and for every $x \in \Sigma(\vec{X})$ and $t > 0$

$$\|U_t x\|_{X_0} + t\|V_t x\|_{X_1} \le \lambda K(t, x; \vec{X})\,.$$

We set $\lambda(\vec{X}) := \inf \lambda$.

Theorem 6. *Suppose $\vec{X}$ is K-linearizable and condition* (1) *holds. Then there exists a family of linear operators $\{T_i : \Sigma(\vec{X}) \to \Sigma(\vec{X}) : \ i \in \mathbb{N}\}$ such that*

$$\mathrm{Id}_{\Sigma(\vec{X})} = \sum_{i=1}^{\infty} T_i \qquad (\textit{pointwise convergence})$$

and for every element $x \in \Sigma(\vec{X})$ satisfying (2) *and every $t > 0$, $i \in \mathbb{N}$*

$$K(t, T_i x; \vec{X}) \le \gamma \lambda(\vec{X})\, \omega_i(t)\,.$$

The proof of this result literally follows the proof of Theorem 3.2.7 in [BrK]. The only change is required in the definition of elements $x_0(t), x_1(t)$ of that proof, see (3.2.30), p. 327 of [BrK]. Instead of that we now use operators U_t, V_t of Definition 5 to set $x_0(t) := U_t x$ and $x_1(t) := V_t x$. □

3. We apply these results to the Banach couple $\vec{L}:=(l_\infty(\mathcal{M},X),\mathrm{Lip}(\mathcal{M},X))$. Here $l_\infty(\mathcal{M}, X)$ is the space of bounded mappings $f : \mathcal{M} \to X$ equipped with the norm $\|f\|_{l_\infty(\mathcal{M},X)} := \sup\{\|f(m)\| : \ m \in \mathcal{M}\}$. The K-functional of this couple is related to the *modulus of continuity* of a mapping $f : \mathcal{M} \to X$, i.e.

$$\Omega(t, f) := \sup\left\{\|f(m) - f(m')\| : d(m, m') \le t\right\}.$$

The least concave majorant of Ω will be denoted by $\widehat{\Omega}$.

Proposition 7. *For every $f \in \Sigma(\vec{L})$ and $t > 0$*

$$\tfrac{1}{2}\, \widehat{\Omega}\, (t, f) \le K(t, f; \vec{L}) \le 3e(\mathcal{M}, X)\, \widehat{\Omega}\, (t, f)\,. \tag{3}$$

Proof. Given $\varepsilon > 0$ and $t > 0$ choose a decomposition $f = f_0 + f_1$ such that

$$\|f_0\|_{l_\infty} + t\|f_1\|_{\mathrm{Lip}} \le K(t, f; \vec{L}) + \varepsilon\,.$$

Here and below we omit $\mathcal{M}$ and X from notations writing, e.g. $\|f\|_{l_\infty}$ instead of $\|f\|_{l_\infty(\mathcal{M},X)}$. Then we have

$$\Omega(t, f) \le \Omega(t, f_0) + \Omega(t, f_1) \le 2\|f_0\|_{l_\infty} + t|f_1|_{\mathrm{Lip}} \le 2K(t, f; \vec{L}) + 2\varepsilon\,.$$

Since the K-functional is concave and $\varepsilon > 0$ is arbitrary, this implies the left inequality in (3).

Fix now $t > 0$ and choose a maximal t-net N_t in $\mathcal{M}$, i.e. (i) $d(x, y) \ge t$ for different x, y in N_t, and (ii) for every $x \in \mathcal{M}$ there is $\hat{x} \in N_t$ such that $d(x, \hat{x}) < t$.

Estimate the norm of $f|_{N_t}$ in $\mathrm{Lip}(N_t, X)$. For every $x, y \in N_t, x \ne y$ we have

$$\left\|f|_{N_t}(x) - f|_{N_t}(y)\right\| = \left\|f(x) - f(y)\right\| \le \widehat{\Omega}\, \left(d(x, y), f\right).$$

Since $\widehat{\Omega}$ is concave, the function $\widehat{\Omega}\,(t)/t$ is non-increasing so that by (i)

$$|f|_{N_t}|_{\mathrm{Lip}(N_t,X)} \le \sup\left\{\frac{\widehat{\Omega}\,(d(x,y),f)}{d(x,y)} : x,y \in N_t\,,\ x\ne y\right\} \le \frac{\widehat{\Omega}\,(t,f)}{t}\,.$$

Now let $e(\mathcal{M},X)<\infty$. Given $\varepsilon>0$ choose an extension $f_1:\mathcal{M}\to X$ of $f|_{N_t}$ such that

$$|f_1|_{\mathrm{Lip}} \le \big(e(\mathcal{M},X)+\varepsilon\big)\frac{\widehat{\Omega}\,(t,f)}{t}\,. \tag{4}$$

Set $f_0 := f - f_1$ and estimate $\|f_0\|_{l_\infty}$. To this end given $x\in\mathcal{M}$ choose $\hat{x}\in N_t$ satisfying $d(x,\hat{x})<t$. Since

$$\|f_0(x)\| \le \big\|f(x)-f(\hat{x})\big\| + \big\|f(\hat{x})-f_1(x)\big\|$$

and $f(\hat{x}) = f_1(\hat{x})$, one gets for the left-hand side the bound $\Omega(d(x,\hat{x}),f) + |f_1|_{\mathrm{Lip}}\, d(x,\hat{x})$. Together with (4) this implies

$$\|f_0\|_{l_\infty} \le \widehat{\Omega}\,(t,f) + \big(e(\mathcal{M},X)+\varepsilon\big)\,\widehat{\Omega}\,(t,f) \le \big(2e(\mathcal{M},X)+\varepsilon\big)\,\widehat{\Omega}\,(f,t)\,. \tag{5}$$

From this inequality and (4) it follows that

$$\begin{aligned} K(t,f;\vec{L}) &\le \|f_0\|_{l_\infty} + t|f_1|_{\mathrm{Lip}} \\ &\le \big(2e(\mathcal{M},X)+\varepsilon\big)\,\widehat{\Omega}\,(t,f) + \big(e(\mathcal{M},X)+\varepsilon\big)\,\widehat{\Omega}\,(t,f)\,. \qquad \square \end{aligned}$$

In the "linear" case the following is true.

PROPOSITION 8. *If $e_l(\mathcal{M},X)<\infty$, then the couple $\vec{L}$ is K-linearizable and $\lambda(\vec{L}) \le 6e_l(\mathcal{M},X)$.*

Proof. Let N_t be as in Proposition 7. Consider a *linear* extension operator E_t from $\mathrm{Lip}(N_t,X)$ into $\mathrm{Lip}(\mathcal{M},X)$ with the norm $\|E_t\|\le e_l(\mathcal{M},X)+\varepsilon$. Now define the required operator V_t of Definition 5 by $V_t f := E_t(f|_{N_t})$. Similarly to (4) and (5) we then obtain the inequalities

$$|V_t f|_{\mathrm{Lip}} \le \big(e_l(\mathcal{M},X)+\varepsilon\big)\frac{\widehat{\Omega}\,(t,f)}{t}\,, \quad \|f - V_t f\|_{l_\infty} \le \big(2e_l(\mathcal{M},X)+\varepsilon\big)\,\widehat{\Omega}\,(f;t)\,.$$

Using these and the left inequality in (3) we get

$$\begin{aligned} \|f - V_t f\|_{l_\infty} + t|V_t f|_{\mathrm{Lip}} &\le \big(3e_l(\mathcal{M},X)+2\varepsilon\big)\,\widehat{\Omega}\,(f,t) \\ &\le 2\big(3e_l(\mathcal{M},X)+2\varepsilon\big)K(t,f;\vec{L})\,. \end{aligned}$$

Hence $\lambda(\vec{L}) \le 6e_l(\mathcal{M},X)$. □

As a consequence of Theorem 4 and Proposition 7 one has

COROLLARY 9. *Let $e(\mathcal{M},X)<\infty$ and ω,ω_i be as in Theorem 4. If $f\in\mathrm{Lip}(\omega(\mathcal{M}),X)$, then there exists a sequence of mappings $f_i\in\mathrm{Lip}(\omega(\mathcal{M}),X)$, $i\in\mathbb{N}$, such that $f=\sum_{i=1}^{\infty} f_i$ (pointwise convergence) and, in addition,*

$$|f_i|_{\mathrm{Lip}(\omega_i(\mathcal{M}),X)} \le C e(\mathcal{M},X)|f|_{\mathrm{Lip}(\omega(\mathcal{M}),X)}\,. \tag{6}$$

Proof. Since by (3)

$$K(t,f;\vec{L}) \le 3e(\mathcal{M},X)\ \widehat{\Omega}\ (t,f) \le 3e(\mathcal{M},X)|f|_{\mathrm{Lip}(\omega(\mathcal{M}),X)}\omega(t)\,,$$

one can apply Theorem 4 to $\vec{X}{=}\vec{L}$ and ω replaced by $3e(\mathcal{M}{,}X)|f|_{\mathrm{Lip}(\omega(\mathcal{M}),X)}\omega$. By this theorem there exists a decomposition $f=\sum_{i=1}^{\infty} f_i$ with convergence in $\Sigma(\vec{L}) = l_\infty(\mathcal{M},X)/\{const\}$ such that for every $t>0$ and $i\in\mathbb{N}$

$$K(t,f_i;\vec{L}) \le Ce(\mathcal{M},X)|f|_{\mathrm{Lip}(\omega(\mathcal{M}),X)}\omega_i(t)\,.$$

From here it easily follows that $\sum_{i=1}^{\infty} f_i$ converges to $f+c$ where c is a suitable constant. Thus we can redefine f to preserve the equality $f = \sum_{i=1}^{\infty} f_i$ with the pointwise convergence. According to the left inequality in (3) this implies (6). □

Now applying Theorem 6 and Proposition 8 we obtain in the very same fashion the following

Corollary 10. *Let $e_l(\mathcal{M},X)<\infty$, and ω,ω_i be as in Theorem 4. Then there exists a family $\{T_i : \mathrm{Lip}(\omega(\mathcal{M}),X) \to \mathrm{Lip}(\omega_i(\mathcal{M}),X) :\ i\in\mathbb{N}\}$ such that*

$$\mathrm{Id}_{\mathrm{Lip}(\omega(\mathcal{M}),X)} = \sum_{i=1}^{\infty} T_i \quad (\textit{pointwise convergence})$$

and, in addition, $\sup_i \|T_i\| \le Ce_l(\mathcal{M},X)$.

4 Proof of Theorem 2.

According to Lemma 3.2.8 in [BrK]

$$\omega(t) \approx \sum_{i=1}^{\infty} \min(\lambda_i,\mu_i t)\,, \quad t\in\mathbb{R}_+ \tag{7}$$

with suitable $\lambda_i,\mu_i>0$ and absolute constants of equivalence. Thus it is natural to prove first the required result for "atoms" $\min(\lambda_i,\mu_i t)$. In turn, the latter follows from the corresponding result for $\alpha(t):=\min(1,t)$, $t\in\mathbb{R}_+$.

Lemma 11. *(a) $e(\alpha(\mathcal{M}),X) \le 3\,e(\mathcal{M},X)$; (b) The same is true for e_l.*

Proof. Let $\mathcal{M}'\subset\mathcal{M}$ and let $f\in\mathrm{Lip}(\alpha(\mathcal{M}'),X)$ be such that $|f|_{\mathrm{Lip}(\alpha(\mathcal{M}'),X)} \le 1$. Denote $A=\{x\in\mathcal{M} : d(x,\mathcal{M}')\ge 1\}$. Then fix $x_0\in\mathcal{M}'$ and define a mapping $g:\mathcal{M}'\cup A\to X$ by setting $g(x):=f(x)$, $x\in\mathcal{M}'$ and $g(x):=f(x_0)$, $x\in A$.

Clearly, $|g|_{\mathrm{Lip}(\mathcal{M}'\cup A,X)} \le 1$, so that given $\varepsilon>0$ we can extend g to a Lipschitz mapping $\widetilde{g}:\mathcal{M}\to X$ with constant $K:=e(\mathcal{M},X)+\varepsilon$. In particular, $\widetilde{g}$ extends f. If $x,y\in\mathcal{M}\setminus A$, then we can find $x',y'\in\mathcal{M}'$ with $d(x,x')\le 1$ and $d(y,y')\le 1$. Hence

$$\|\widetilde{g}(x)-\widetilde{g}(y)\| \le \|\widetilde{g}(x)-\widetilde{g}(x')\| + \|f(x')-f(y')\| + \|\widetilde{g}(y')-\widetilde{g}(y)\|$$

$$\leq Kd(x,x') + \alpha\big(d(x',y')\big) + Kd(y,y') \leq 2K+1\,.$$

Similarly, if $x \in A$ and $y \in \mathcal{M} \setminus A$ find $y' \in \mathcal{M}'$ with $d(y,y') \leq 1$. Hence

$$\big\|\widetilde{g}(x) - \widetilde{g}(y)\big\| \leq \big\|f(x_0) - f(y')\big\| + \big\|\widetilde{g}(y') - \widetilde{g}(y)\big\| \leq 1 + K\,.$$

Thus for all $x, y \in \mathcal{M}$

$$\begin{aligned}\big\|\widetilde{g}(x) - \widetilde{g}(y)\big\| &\leq \min\big\{Kd(x,y), 2K+1\big\} \leq (2K+1)\alpha\big(d(x,y)\big)\\ &\leq \big(3\,e(\mathcal{M},X) + 2\varepsilon\big)\alpha\big(d(x,y)\big)\end{aligned}$$

and the statement (a) follows. The corresponding modification of the proof for the "linear" case is obvious. □

Let $\mathcal{M}' \subset \mathcal{M}$ and $f \in \mathrm{Lip}(\omega(\mathcal{M}'), X)$. We have to find an extension $\widetilde{f} \in \mathrm{Lip}(\omega(\mathcal{M}), X)$ of f satisfying

$$|\widetilde{f}|_{\mathrm{Lip}(\omega(\mathcal{M}),X)} \leq Ce(\mathcal{M},X)^2\,|f|_{\mathrm{Lip}(\omega(\mathcal{M}'),X)}\,. \tag{8}$$

In the second part of the theorem we also have to define $\widetilde{f}$ linearly depending on f. Using decomposition (7) and Corollary 9 with $\omega_i(t) := \min(\lambda_i, \mu_i t)$, $i \in \mathbb{N}$, we represent f as a sum $f = \sum_{i=1}^{\infty} f_i$ (pointwise convergence) with $f_i : \mathcal{M}' \to X$ satisfying

$$|f_i|_{\mathrm{Lip}(\omega_i(\mathcal{M}'),X)} \leq Ce(\mathcal{M},X)|f|_{\mathrm{Lip}(\omega(\mathcal{M}'),X)}\,, \quad i \in \mathbb{N}\,.$$

Using this and Lemma 11, (a) we now obtain an extension $\widetilde{f}_i : \mathcal{M} \to X$ satisfying

$$|\widetilde{f}_i|_{\mathrm{Lip}(\omega_i(\mathcal{M}),X)} \leq 3Ce(\mathcal{M},X)^2|f|_{\mathrm{Lip}(\omega(\mathcal{M}'),X)}\,, \quad i \in \mathbb{N}\,. \tag{9}$$

Now set $\widetilde{f} := \sum_{i=1}^{\infty} \widetilde{f}_i$ and show that $\widetilde{f}$ is well defined and satisfies (8). Given $x \in \mathcal{M}$ we choose a point $x_0 \in \mathcal{M}'$ and write

$$\Big\|\sum_{i=m}^{n} \widetilde{f}_i(x)\Big\| \leq \sum_{i=m}^{n} \big\|\widetilde{f}_i(x) - \widetilde{f}_i(x_0)\big\| + \Big\|\sum_{i=m}^{n} \widetilde{f}_i(x_0)\Big\|.$$

Since $\widetilde{f}_i(x_0) = f_i(x_0)$, $i \in \mathbb{N}$, this and (9) imply

$$\Big\|\sum_{i=m}^{n} \widetilde{f}_i(x)\Big\| \leq 3Ce(\mathcal{M},X)^2|f|_{\mathrm{Lip}(\omega(\mathcal{M}'),X)} \sum_{i=m}^{n} \omega_i\big(d(x,x_0)\big) + \Big\|\sum_{i=m}^{n} f_i(x_0)\Big\|.$$

But $\sum_{i=1}^{\infty} f_i(x_0)$ and $\sum_{i=1}^{\infty} \omega_i(d(x,x_0))$ are convergent, therefore the right-hand side of this inequality tends to 0 as $m, n \to \infty$. Thus f is well defined. It remains to note that by (9) and (7)

$$\begin{aligned}\big\|\widetilde{f}(x) - \widetilde{f}(y)\big\| &\leq \sum_{i=1}^{\infty} \big\|\widetilde{f}_i(x) - \widetilde{f}_i(y)\big\|\\ &\leq 3Ce(\mathcal{M},X)^2|f|_{\mathrm{Lip}(\omega(\mathcal{M}'),X)}\bigg(\sum_{i=1}^{\infty} \omega_i\big(d(x,y)\big)\bigg)\\ &\leq C_1 e(\mathcal{M},X)^2|f|_{\mathrm{Lip}(\omega(\mathcal{M}'),X)}\omega\big(d(x,y)\big)\,, \quad x, y \in \mathcal{M}\,,\end{aligned}$$

with suitable C_1. Thus part (a) of Theorem 2 is proved.

As for part (b) one should repeat word for word the proof of part (a) replacing Corollary 9 by Corollary 10 and statement (a) of Lemma 11 by statement (b). If in this case E_i denotes the corresponding linear extension operator from $\mathrm{Lip}(\omega_i(\mathcal{M}'), X)$ into $\mathrm{Lip}(\omega(\mathcal{M}), X)$ for which $\|E_i\| \le 3\, e_l(\mathcal{M}, X)$, then the required extension operator is defined by $E = \sum_{i=1}^{\infty} E_i T_i$. Here T_i are linear operators of Corollary 10.

The proof of Theorem 2 is complete. □

Acknowledgement. We are very grateful to A. Naor for a simplification of the proof of Lemma 11 and helpful discussions.

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Submitted: January 2001
Revised version: February 2001