GAFA, Geom. funct. anal.
Vol. 11 (2001) 1229 – 1281
1016-443X/01/0601229-53 $ 1.50+0.20/0

© Birkhäuser Verlag, Basel 2001

GAFA Geometric And Functional Analysis

GLUING THEOREMS FOR COMPLETE ANTI-SELF-DUAL SPACES

A. Kovalev and M. Singer

1 Introduction

1.1 Summary. One of the special features of 4-dimensional differential geometry is the existence of objects with self-dual (SD) or anti-self-dual (ASD) curvature. The objects in question can be connections in an auxiliary bundle over a 4-manifold, leading to the study of instantons in Yang–Mills theory [DK], or as in this paper, Riemannian metrics or conformal structures. Although such *ASD conformal structures* give absolute minima of the functional $c \mapsto \|W(c)\|_2^2$, where $W(c)$ denotes the Weyl tensor of the conformal structure c, variational methods are not well suited to the study of this problem, essentially because of its conformal invariance. For this reason, *gluing theorems* provide a very important source of information about ASD conformal structures. Our purpose in this paper is to give some new and rather general gluing theorems for ASD and Hermitian-ASD conformal structures, following the method suggested by Floer in [F]. The prototypical gluing theorem takes a pair (X_j, c_j) $(j = 1, 2)$ of compact conformally ASD 4-manifolds and analyzes the problem of finding an ASD conformal structure c on $X = X_1 \sharp X_2$ that is 'close to' c_j in suitable subsets $X_j \backslash B_j \subset X_1 \sharp X_2$. In this situation there exist finite-dimensional vector spaces (the *obstruction spaces*) $H^2_{c_j}(X_j)$ whose vanishing is sufficient to guarantee the existence of c with the desired properties. (If $H^2_{c_j}(X_j) \neq 0$, then the gluing theorem yields a map from another finite-dimensional vector space into $H^2_{c_1}(X_1) \oplus H^2_{c_2}(X_2)$, the zeroes of which yield ASD conformal structures on $X_1 \sharp X_2$.)

The result just stated (gluing for compact conformally ASD spaces) was proved by Donaldson and Friedman [DF] and in a very special case by Floer [F]. The approach of [DF] was to exploit the twistor description [P], [AtHS] of conformally ASD spaces, to translate the gluing problem into one of deformation theory of complex singular spaces. Floer, on the other hand, worked directly with the 4-manifolds and used some tools from the theory of elliptic operators on non-compact manifolds with cylindrical ends.

One of the motivations for the present work was the desire to extend the basic gluing theorem to handle the case where the (X_j, c_j) are conformally ASD orbifolds with isolated singular points. Such an extension opens up the possibility of obtaining new examples of conformally ASD spaces by desingularizing conformally ASD orbifolds within the ASD category. Indeed, the process of resolution of orbifold singularities amounts to taking a (generalized) connected sum with a suitable standard orbifold with precisely one singular point, and examples of such standard orbifolds are now known for many of the finite subgroups of SO_4 [Kr], [L1], [GL]. The simplest possible case of $\mathbb{Z}_2$-singularities was studied in [LS2], where the methods of [DF] were extended to give gluing theorems for these orbifolds. The method was further extended to cover cyclic singularities, by Jian Zhou [Z], but becomes increasingly complicated owing to the singularities developed by the corresponding twistor spaces.

On general grounds, however, one should expect that Floer's analytical approach might provide a simpler framework for such generalized gluing theorems. Indeed, in that approach the first step is to blow up X_j at the marked point 0_j at which the gluing takes place. This blow-up results in a manifold with an *infinite cylindrical end*, or more-or-less equivalently, a compact manifold with boundary, equipped with a b-metric [Me]. The cross-section of the cylinder is diffeomorphic to the added boundary component, both being the link in X_j of 0_j, and the singularity has disappeared completely. Having reached this point, it is quite reasonable to take 4-manifolds with *boundary*, equipped with conformally ASD b-metrics as the basic entities to glue, regarding compact manifolds and orbifolds as special cases. Our main result here is indeed a gluing theorem for pairs of conformally ASD b-manifolds (X_j, c_j), the connected sum being replaced by the 'join' $X = X_1 \cup_Y X_2$ of X_1 and X_2 across $Y \subset \partial X_j$, where Y is a common piece (union of connected components) of the boundaries of the X_j. The precise statement involves a considerable notational overhead in the definition of the obstruction spaces and is deferred to §6; suffice it to say that once the obstruction spaces have been correctly defined, the result is precisely analogous to the prototype mentioned before. It should perhaps be emphasised that this theorem genuinely operates in the b-category, in that ∂X can be non-empty, in which case the gluing theorem produces a conformally ASD b-metric on X or, in more traditional language, a complete conformally ASD metric on $X \backslash \partial X$ with cylindrical asymptotics.

Such a gluing theorem immediately poses questions about the existence

of conformally ASD b-metrics. The first observation is that near each component Y of the boundary, the metric must be asymptotic to a conformally ASD product metric on $Y \times \mathbb{R}$. Thus Y must have constant sectional curvature [Be], so the new possibilities (not arising from orbifolds) are $Y =$ the 3-torus T^3 or $Y =$ a hyperbolic 3-manifold. It seems that not much is known about the existence of conformally ASD b-metrics on manifolds with such boundary components, so we argue in §7 that Taubes' method [T1], [T2] can be adapted to yield conformally ASD b-metrics on $X \sharp N\overline{\mathbb{C}P^2}$ for large enough N, if (X, g) is any Riemannian b-manifold such that g is conformally flat near ∂X. In a companion paper [KoS], we also give a simple example of a conformally ASD (in fact hyperKähler) b-metric on a manifold with boundary equal to T^3.

In this paper we also study Hermitian-ASD conformal structures on complex surfaces. These are particularly interesting because of their relation to scalar-flat Kähler geometry [L2], [LS1], [KLP], [L3]. Such Kähler metrics with zero scalar curvature are of interest from the point of view of Calabi's extremal metric programme [Be], and even for complex surfaces there is no systematic existence theory. It is fortunate, then, that for surfaces, scalar-flat Kähler metrics can be approached through Hermitian-ASD conformal structures, and hence through gluing theorems. As for the full ASD equations, we consider the general gluing problem for conformally Hermitian-ASD b-metrics on compact complex surfaces with boundary, and obtain similar results. We also illustrate our general results with a simple application, showing that the blow-up of $\mathbb{C}^2$ at an arbitrary set of points p_j admits scalar-flat Kähler metrics that are asymptotic to the Euclidean metric at ∞. This generalizes LeBrun's explicit construction [L2, Theorem 1] of S^1-invariant scalar-flat Kähler metrics on this blow-up when the p_j lie on a complex line in $\mathbb{C}^2$.

It should perhaps be remarked that the ASD condition requires that an orientation be chosen on the underlying 4-manifold, and accordingly some care has to be taken with the construction of $X_1 \cup_Y X_2$ to ensure that this has an orientation that is compatible with the given orientations on the X_j. This is particularly true for gluing complex surfaces: a moment's reflection will convince the reader that the connected sum of 2 complex surfaces *never* has a complex structure compatible with the given complex structures on the summands. One will, however, be able to glue the asymptotic region of a suitable non-compact surface to the complement of a neighbourhood of a point in a compact surface. In this way one gets gluing theorems that

give sufficient conditions for the blow-up of a compact scalar-flat Kähler metric to admit a scalar-flat Kähler metric or more generally results about resolution of orbifold singularities of scalar-flat Kähler metrics within the scalar-flat Kähler category.

1.2 Strategy. Gluing theorems have been pursued vigorously in many different contexts over the last 10 or so years, and there are many approaches to the problem. For nonlinear problems like the ones studied in this paper, involving the construction of connections or metrics with prescribed curvature, the strategy is always the same: construct a family of 'approximate solutions' on the join of the two spaces, and then use some variant of the implicit function theorem to obtain a nearby genuine solution. In the notation of the rest of this paper, this family of solutions will depend on a large parameter ρ, essentially the length of the neck joining the two spaces. As $\rho \to \infty$, the approximate solution gets better and better. In particular, the implicit function theorem needs to be applied when ρ is very large, and for this some good control of the linearization of the problem is needed in this limit. For this reason a good deal of the work concerns the behaviour of linear operators on manifolds with long necks. These linear problems are of interest in their own right in the context of gluing formulae in index theory and for η-invariants: recent work in this direction, from a point of view that is close to that of this paper, can be found for example in [MM], [HMM]. For a more leisurely description of gluing theorems of this type, the reader is referred to [DK, Chapter 7], [F] or to [T2] for a recent survey of gluing problems for instantons and ASD conformal structures. Some other geometric (nonlinear) gluing problems are surveyed in [MP].

Having given these pointers to the literature, we shall concentrate in the rest of this paper on the technical details of our particular problems, without too much further motivation.

1.3 Contents. In more detail, the remaining sections of the paper are as follows:

§2: The b-category is introduced, Fredholm properties of b-differential operators are described, and gluing of b-metrics is explained in detail.

§3: Conformal geometry is recalled, with special reference to the ASD equations in 4 dimensions. The relevant PDE aspects of these equations are described.

§4: Linear aspects of our problem are discussed here, including a rather general account of the behaviour of the kernel and cokernel of elliptic oper-

ators on manifolds with long necks. The application of these general results to the ASD problem is also treated along with 'comparison theorems' which allow us to compare the linearization of the ASD equations on a compact orbifold with the linearization on the corresponding 'blown-up' b-manifold.

§5: Nonlinear aspects of the analysis appear here, centring around the application of the implicit function theorem to obtain a weak solution of the ASD equations, and elliptic regularity arguments to show that this solution is C^∞ (and has optimal behaviour at the boundary, if there is one).

§6: The main gluing theorems are summarized here, both for conformally ASD and conformally Hermitian-ASD metrics. The construction of scalar-flat Kähler metrics on an arbitrary blow-up of $\mathbb{C}^2$ also appears here.

§7: The b-version of Taubes' existence theorem is given here.

§8: A number of vanishing theorems for the obstruction space $H^2_c(X)$ are collected here.

1.4 Acknowledgements. We acknowledge useful conversations and encouragement from David Calderbank, Dominic Joyce, Claude LeBrun, Rafe Mazzeo, Richard Melrose and Mario Micallef. Both authors were supported by EPSRC while most of this work was carried out. Partially supported by The European Contract Human Potential Programme, Research Training Network HPRN-CT-2000-00101.

Finally our approach owes much to the work of Andreas Floer [F]. We hope to honour this exceptional mathematician with the present work, by bringing his insights about gluing ASD conformal structures to a wider audience.

2 Gluing b-manifolds

In this section we recall Richard Melrose's approach to spaces with cylindrical ends, and explain the relevance of these ideas in conformal geometry. This centres around the notion of conformal blow-up by means of which a compact Riemannian manifold (or orbifold) with a marked point is changed into a manifold with a conformally related b-metric. There is an analogous notion of conformal blow-down for weakly asymptotically locally Euclidean (WALE) spaces (§2.1.3).

We also give the elements of the Fredholm theory of b-elliptic operators. The latter involves us in a short account of the notion of 'polyhomogeneous' functions: such functions arise naturally and inevitably in the study of b-differential operators and should be thought of as an extension of the idea

of a function that is smooth up to the boundary.

Finally in this section, we give a careful description of the process of gluing a pair of Riemannian b-manifolds, and the construction of a suitable b-metric on their join. Thus all the material in this section is well known to the right people; it is necessary to summarize it here in the interests of making the present paper self-contained and fixing notation that will be used throughout.

2.1 The *b*-category. We begin with the basic definitions, and then provide some motivation. The reader can find a detailed account in [Me].

Let X be a compact n-manifold with smooth boundary ∂X; neither X nor ∂X are assumed connected. When working near ∂X it is convenient to fix a function $x \in C^\infty(X)$ with values in $[0,2]$, such that $x(p) = 0$ if and only if $p \in \partial X$ and $dx(p) \neq 0$ for all p with $x(p) \in [0,1]$. Such a function is often called a *boundary defining function*. Here and subsequently, $f \in C^\infty(X)$ means that f is smooth up to the boundary of X.

Near a boundary point p, one can introduce adapted coordinate systems $(x, y_1, \ldots, y_{n-1})$, where the y_j are local coordinates near p in ∂X. Using such coordinates we can define the b-tangent and cotangent bundles ${}^bTX$ and ${}^bT^*X$. These are smooth bundles of rank n over X; over the interior $X^o = X \backslash \partial X$ they are equal to TX^o, T^*X^o respectively, but at the boundary, near p, ${}^bTX$ is spanned by the elements

$$x\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y_1}, \ldots, \tfrac{\partial}{\partial y_{n-1}} \tag{2.1}$$

and dually ${}^bT^*X$ is spanned by the elements

$$\tfrac{dx}{x}, dy_1, \ldots, dy_{n-1}\,.$$

It is easy to see that ${}^bTX$ and ${}^bT^*X$ are smooth up to the boundary of X; the basic idea of the b-category is to use these bundles in place of TX and T^*X in the development of differential geometry and analysis for manifolds with boundary.

A basic example is the notion of a Riemannian b-metric. By definition, this is just a positive-definite inner product on ${}^bTX$, smooth up to the boundary. We shall not need the most general such metric, but only 'exact' b-metrics, which take the form

$${}^bg = \tfrac{dx^2}{x^2} + h(x,y)\,, \tag{2.2}$$

where $h(x,y)$ is a symmetric tensor such that ${}^bg$ is everywhere positive-definite. To begin with we assume that $h(x,y)$ is C^∞ up to ∂X, but we shall soon need to allow h to be merely polyhomogeneous (see §2.2.5).

We now give some examples to explain how b-metrics arise naturally in conformal geometry.

Example 2.1.1: conformal blow-up. Let $(\overline{X}, \overline{g})$ be a compact n-dimensional Riemannian manifold without boundary, let $0 \in \overline{X}$ be any point. In geodesic polar coordinates centred at 0, we have

$$\overline{g} = dr^2 + r^2\big(d\omega^2 + r^2\eta(r,\omega)\big)$$

where r is geodesic distance from 0, $d\omega^2$ is the standard round metric on S^{n-1} and $\eta(r,\omega)$ is a family of symmetric tensors on S^{n-1}, uniformly bounded as $r \to 0$. By multiplying g by a constant, we can assume that these coordinates are defined for $0 < r < 2$, say. The (oriented) blow-up X of $\overline{X}$ at 0 is a smooth manifold with boundary ∂X canonically identified with the unit sphere-bundle of 0 in $\overline{X}$. The pull-back x of r is then a boundary defining function and the pull-back of $g = r^{-2}\overline{g}$ is the b-metric

$$g = \tfrac{dx^2}{x^2} + d\omega^2 + x^2\eta$$

where η is smooth up to ∂X. We call (X, g) the *conformal blow-up* of $\overline{X}$ at 0.

2.1.2 Generalization: conformal blow-up of orbifolds. Recall [B] that an n-dimensional orbifold $\overline{X}$ is defined analogously to a manifold, but a neighbourhood U_p of $p \in \overline{X}$ is homeomorphic to $\mathbb{R}^n/a_p$, where $a_p : \Gamma_p \times \mathbb{R}^n \to \mathbb{R}^n$ is an effective action of the finite group Γ_p (the 'local isotropy group') on $\mathbb{R}^n$. The covering map $\mathbb{R}^n \to U_p$ is called a *local uniformizing chart* centred at p.

If $\Gamma_p = \{1\}$ then p is a *smooth point* of $\overline{X}$, otherwise p is a *singular* point. The set of all singular points of $\overline{X}$ is denoted $\overline{X}_{\text{sing}}$.

In this paper we shall use the term 'orbifold' to mean'orbifold with isolated singular points'. Then for each point p, $a_p(\gamma)$ only fixes 0 for each $\gamma \neq 1$ in Γ_p.

A Riemannian metric $\overline{g}$ on $\overline{X}\backslash\overline{X}_{\text{sing}}$ is called a *smooth orbifold metric*, and $(\overline{X}, \overline{g})$ is called a Riemannian orbifold, if the pull-back of $\overline{g}$ to a local uniformizing chart extends smoothly to $0 \in \mathbb{R}^n$ and a_p acts by isometries. In particular a_p gives a representation of Γ_p in O_n, the orthogonal group of $T_0\mathbb{R}^n$.

By working in a local uniformizing chart one sees that the conformal blow-up (X, g) of a singular point p of $\overline{X}$ can be defined as in §2.1.1. The only difference is that ∂X is canonically identified with the spherical space-form S^{n-1}/a_p.

Example 2.1.3: conformal blow-down of asymptotically Euclidean spaces. Let (M, g) be a Riemannian manifold that is *weakly asymptotically locally Euclidean* (WALE). By this we mean that there exists a compact subset $K \subset M$ and a diffeomorphism $\phi : (\mathbb{R}^n \backslash B)/a_\infty \to M \backslash K$ (where B is some closed ball in $\mathbb{R}^n$ and $a_\infty : \Gamma \times \mathbb{R}^n \to \mathbb{R}^n$ is an action of the finite group Γ) such that

$$|g - g_0| = O(r^{-k}), \quad |\nabla_0^j g| = O(r^{-j-k}),$$

where g_0 is the Euclidean metric, r is the distance from the origin of $\mathbb{R}^n$ and ∇_0 is the covariant derivative of the Euclidean metric. This definition is mainly of interest when $k \geq 2$. The term 'asymptotically locally Euclidean' (ALE) has become standard for the very strong decay with $k = 4$. Writing the metric in polar coordinates again, we have

$$g = dr^2 + r^2(d\omega^2 + r^{-k}\eta)$$

where $|\nabla_0^j \eta| = O(r^{-j-k})$ for $j = 0, 1, 2, \ldots$ and η is again a family of symmetric 2-tensors on S^{n-1}/a_∞. Set $x = r^{-1}$ and we obtain

$$x^2 g = \tfrac{dx^2}{x^2} + d\omega^2 + x^k \eta,$$

a b-metric on the radial compactification of M. We refer to this process as conformal blow-down of a WALE space.

Remark 2.1.4. Observe that from this 'b' point of view, the conformal blow-up of a point and the conformal blow-down of the infinity of a WALE space both look exactly the same; a manifold with boundary a spherical space-form, equipped with an exact b-metric which is the standard metric of constant curvature at the boundary.

Remark 2.1.5. By the change of variables $t = \pm \log x$, $dt = \pm dx/x$, the interior of a b-manifold with b-metric becomes a manifold with a cylindrical end diffeomorphic to $\partial X \times (t_0, \infty)$ or $\partial X \times (-\infty, t_0)$ (t_0 some constant), depending on the sign chosen. In these coordinates, a b-metric becomes a metric which approaches a Riemannian product metric on $\partial X \times \mathbb{R}$ at an exponential rate in t, with similar estimates on all derivatives of the metric. We shall use this change of variables to describe the process of gluing arbitrary b-manifolds in §2.3. We shall also often confuse a manifold with cylindrical ends with a b-manifold (even though this runs counter to the idea of the b-category, which is to replace non-compactness by degeneracy at the boundary!).

In this section we have tried to show that there is a precise sense in which the b-category unifies orbifolds and WALE spaces. Equally important

is that there is a good Fredholm theory for partial differential operators naturally associated to b-metrics. This will now be outlined.

2.2 On b-differential operators. Throughout this section, X is a C^∞ manifold with boundary ∂X, $x \geq 0$ is a boundary defining function, $t = -\log x$, and $X^o = X \backslash \partial X$.

Definition 2.2.1. A C^∞ b-differential operator $P : C^\infty(X, E) \to C^\infty(X, F)$, where E and F are two vector bundles over X, is a differential operator which can be written locally in the form $P = p(x, y; x\partial_x, \partial_y)$ where p is smooth in (x, y) and polynomial in $x\partial_x$ and ∂_y (cf. (2.1)).

Any operator 'naturally associated to' a b-metric – for example the Laplacian or Dirac operator – will automatically be a b-differential operator. Fredholm theory for such operators has been developed by Lockhart and McOwen [LoM], and in much more detail by Melrose and Mendoza [Me] and is the main technical tool needed to prove our gluing theorems.

Given a b-differential operator P, there is a canonically associated *indicial operator* $I(P)$ which is given locally by $I(P) = p(0, y; \partial_t, \partial_y)$, regarded as a t-invariant differential operator on the cylinder $\partial X \times \mathbb{R}$. In terms of the cylindrical model of X^o, all coefficients of $P - I(P)$ decay exponentially as $t \to \infty$.

Definition 2.2.2. Let $P : C^\infty(X, E) \to C^\infty(X, F)$ be a b-differential operator of order m. Then

$$\mathrm{spec}_b(P) = \left\{\lambda \in \mathbb{C} : \text{ there exists } u(y) \neq 0 \text{ such that } I(P)(e^{i\lambda t}u(y)) = 0\right\}.$$

In other words $\mathrm{spec}_b(P)$ is the set of complex numbers λ for which $I(P)$ has a non-trivial exponential solution with exponent $i\lambda$. Notice that here E and F have implicitly been trivialized in the t direction along the infinite cylinder $\partial X \times \mathbb{R}$: to be precise, E and F have been identified with bundles pulled back from ∂X by the projection $\partial X \times \mathbb{R} \to \partial X$.

2.2.3 It is a basic fact that if P is elliptic over X^o, then $\mathrm{spec}_b(P)$ is a discrete set which meets every horizontal strip $\{a < \mathrm{Im}\,(\lambda) < b\}$ in a finite number of points. For the translation-invariant operator $I(P)$, real elements of spec_b are obstructions to its invertibility in L^p spaces:

Proposition 2.3. *The elliptic operator $I(P)$ extends to a bounded map*

$$L^p_k(\partial X \times \mathbb{R}, E) \to L^p_{k-m}(\partial X \times \mathbb{R}, F)$$

for each p and k. This map is invertible if and only if $\mathrm{spec}_b I(P) \cap \mathbb{R} = \emptyset$.

Here and below $L^p_k(X)$ is the Sobolev space of functions u such that $Pu \in L^p$ for every b-differential operator of order $\leq k$ and the measure used

to define L^p is that defined by any C^∞ b-metric on X. For the cylinder $\partial X \times \mathbb{R}$ an example of such a metric would be a (t-invariant) Riemannian product metric.

2.2.4 Fully elliptic *b*-differential operators. The b-differential operator P is said to be *fully elliptic* if P is elliptic over X^o and $\mathrm{spec}_b(P) \cap \mathbb{R} = \emptyset$. In view of Proposition 2.3, the latter is saying that P is 'invertible at ∞ (or at ∂X)'. It turns out that an elliptic b-differential operator is Fredholm in L^p if and only if it is fully elliptic. Before stating the theorem which summarizes the mapping properties of fully elliptic b-differential operators, we need to introduce polyhomogeneous functions.

2.2.5 Polyhomogeneity. If X is the conformal blow-up at a point of $\overline{X}$, then the pull-back u to X of a C^∞ function on $\overline{X}$ is smooth and in particular has an asymptotic expansion $u \sim \sum_{j=0}^\infty u_j(y)x^j$ near the boundary, where $u_j \in C^\infty(\partial X)$ for each j.

A polyhomogeneous (phg) function is C^∞ in X^o and has a similar asymptotic expansion near ∂X, but more general powers of x can appear, as well as polynomials in $\log x$. The set of powers that can occur is called an *index set* $\mathcal{I}$ and must be a discrete subset of $\mathbb{C}$ having the additional property that if $z_j \in \mathcal{I}$ and $|z_j| \to \infty$, then $\mathrm{Re}\,(z_j) \to +\infty$. Given an index set $\mathcal{I}$, $u \in C^\infty(X^o)$ is called polyhomogeneous (with respect to $\mathcal{I}$) if

$$u \sim \sum_{z \in \mathcal{I}} x^z u_z(y, \log x) \tag{2.4}$$

where for each z, $u_z(y,t)$ is C^∞ in y and polynomial in t. The symbol $\sim$ in (2.4) is meant in the following strong sense: if

$$u_N = \sum_{z \in \mathcal{I}, \mathrm{Re}\,(z) \le N} x^z u_z(y, \log x)\,,$$

then $u - u_N \in C^N(X)$ and all derivatives of order $\le N$ of $u - u_N$ vanish at ∂X. For more details, see [Me, §5.10]; there, however, a more refined notion of index set is used, designed to keep track of the degrees of the polynomials $u_z(y, \cdot)$. We have chosen to ignore this refinement here.

We are now ready to summarize the essential properties of fully elliptic operators on X.

Theorem 2.5. *Let $P : C^\infty(X,E) \to C^\infty(X,F)$ be a fully elliptic b-differential operator of order m. Then*

(i) P extends to a Fredholm map $L^p_k(X,E) \to L^p_{k-m}(X,F)$ for every p and k, with index independent of p and k.

(ii) *If $u \in \ker(P)$ then u is C^∞ in X^o and is phg relative to the index set $\mathcal{I} = i\mathrm{spec}_b(P) \cap \{Re\, z > 0\}$.*

(iii) *The cokernel of P can be identified with the L^p kernel of P^*, where P^* is the L^2-adjoint of P with respect to a b-metric on X. In particular $PL^p_k(X, E)$ can be complemented by phg sections of F.*

For a proof see [Me] or [LoM].

2.2.6 Conjugation and weights. Let δ be a real number and let $P^{(\delta)} = x^\delta P x^{-\delta} = e^{-\delta t} P e^{\delta t}$. Then $\mathrm{spec}_b(P^{(\delta)}) = \mathrm{spec}_b(P) - i\delta$ and so if P is elliptic, then $P^{(\delta)}$ will be fully elliptic for all $\delta \notin \mathrm{Im}\,\mathrm{spec}_b(P)$. By the remark at the beginning of §2.2.3, $P^{(\delta)}$ is fully elliptic for all but a discrete set of real values δ. The index of $P^{(\delta)}$ is locally constant in δ and jumps as δ passes through a point in $\mathrm{Im}\,\mathrm{spec}_b(P)$.

An equivalent formulation of this observation is that an elliptic b-operator P defines a Fredholm operator between *weighted Sobolev spaces*

$$e^{\delta t} L^p_k(X, E) \to e^{\delta t} L^p_{k-m}(X, F)$$

for all $\delta \notin \mathrm{Im}\,\mathrm{spec}_b(P)$. Note carefully, however, that the weighted Fredholm alternative (analogue of Theorem 2.5 (iii)) identifies the cokernel of this map with the kernel of the map

$$e^{-\delta t} L^p_k(X, F) \to e^{-\delta t} L^p_{k-m}(X, E)\,.$$

We shall need to make use of these ideas, for $0 \in \mathrm{spec}_b$ for the linearization of the conformal ASD equations.

Remark 2.2.7. If $\partial X = Y_1 \cup \ldots \cup Y_n$ where the Y_j are connected then $\mathrm{spec}_b = \mathrm{spec}_b^{(1)} \cup \ldots \cup \mathrm{spec}_b^{(n)}$ where $\mathrm{spec}_b^{(j)}$ is the contribution from Y_j. Then one can shift these pieces independently of each other by conjugating by a function which is equal to $e^{\delta_j t_j}$ near Y_j. We shall not develop a systematic notation for this situation.

2.2.8 Polyhomogeneous *b*-differential operators. Since polyhomogeneous functions are at least as natural on a manifold with boundary as functions that are C^∞ up to the boundary, it is natural to widen the class of operators we consider by allowing their coefficients to be polyhomogeneous relative to some index set $\mathcal{J}$ and continuous up to the boundary. Such operators arise naturally as operators canonically associated to a *phg b-metric*, i.e. a metric ${}^b g$ of the form (2.2) where $h(x, y)$ is phg and continuous up to the boundary. The results of this section, in particular Theorem 2.5, go through in this case, the only difference being in part (ii) where u will now have a phg expansion relative to the index set $\mathcal{I} \cup \mathcal{J}$, $\mathcal{I}$ being as before.

2.3 On gluing *b*-manifolds. We shall now explain how to glue b-manifolds across (a part of) their boundaries. The construction is complicated slightly by the need to keep track of orientations. The main points of the discussion are contained in §2.3.2, §2.3.5 and §2.3.6.

2.3.1 Data. For $j = 1, 2$ let X_j be a smooth, oriented, compact n-manifold with boundary and suppose that an oriented boundaryless manifold Y occurs in ∂X_j with each orientation:

$$Y \subset \partial X_1\,, \quad -Y \subset \partial X_2\,.$$

(In what follows, we shall not indicate the orientation of Y unless orientation issues are being discussed.) Let x_j be defining functions for $Y \subset \partial X_j$ and assume that $x_j = 1$ near $\partial X_j \backslash Y$. Let $t_1 = -\log x_1$, $t_2 = \log x_2$; then there exist open sets $U_j \subset X_j^o$ diffeomorphic to cylinders

$$U_1 = Y \times \{0 < t_1 < \infty\}\,, \quad U_1 = Y \times \{-\infty < t_2 < 0\}\,,$$

such that a sequence of points tending towards Y corresponds to $|t_j| \to \infty$. It is important to notice that U_j inherits from X_j orientation $dt_j \wedge \mathrm{or}_Y$ where or_Y is a given orientation of Y.

Definition 2.3.2. The manifold X_ρ is defined by truncating the U_j at $\pm t_j = \rho$ and identifying the boundaries $Y \times \{t_1 = \rho\}$ and $Y \times \{t_2 = -\rho\}$.

After the remarks of the previous paragraph, it is clear that X_ρ is oriented, this orientation agreeing with the given orientations on the X_j. Furthermore X_ρ contains a *neck* of length 2ρ given by $\{0 < t_1 \leq \rho\} \cup \{-\rho \leq t_2 \leq 0\}$.

2.3.3 Notation. On X_ρ the function t is defined so that $t = t_1 - \rho$ for $0 \leq t_1 \leq \rho$ and $t = t_2 + \rho$ for $-\rho \leq t_2 \leq 0$, and extended smoothly to $-\rho$ on the rest of X_1 and to equal ρ on the rest of X_2. The central slice of the neck then corresponds to $t = 0$. See Figure 1.

Example 2.3.4. Suppose that X_1 is the conformal blow-up at a point 0, say, of a compact orbifold $\overline{X}$. In particular a neighbourhood of 0 in $\overline{X}$ is homeomorphic to the quotient $\mathbb{R}^4/a$, where $a : \Gamma \times \mathbb{R}^4 \to \mathbb{R}^4$ is an action of the finite group Γ on $\mathbb{R}^4$. Suppose further that X_2 is the conformal blow-down of the asymptotic region of a WALE space $\widehat{X}$, where a neighbourhood of ∞ is homeomorphic to a neighbourhood of ∞ in $\mathbb{R}^4/a$. Then $\partial X_1 = Y = S^3/a$ and $\partial X_2 = -Y$ and the construction of §2.3.1 can be applied. The result is a generalized blow-up of $0 \in \overline{X}$, where a small ball centred at 0 is replaced by the complement of a neighbourhood of infinity in $\widehat{X}$.

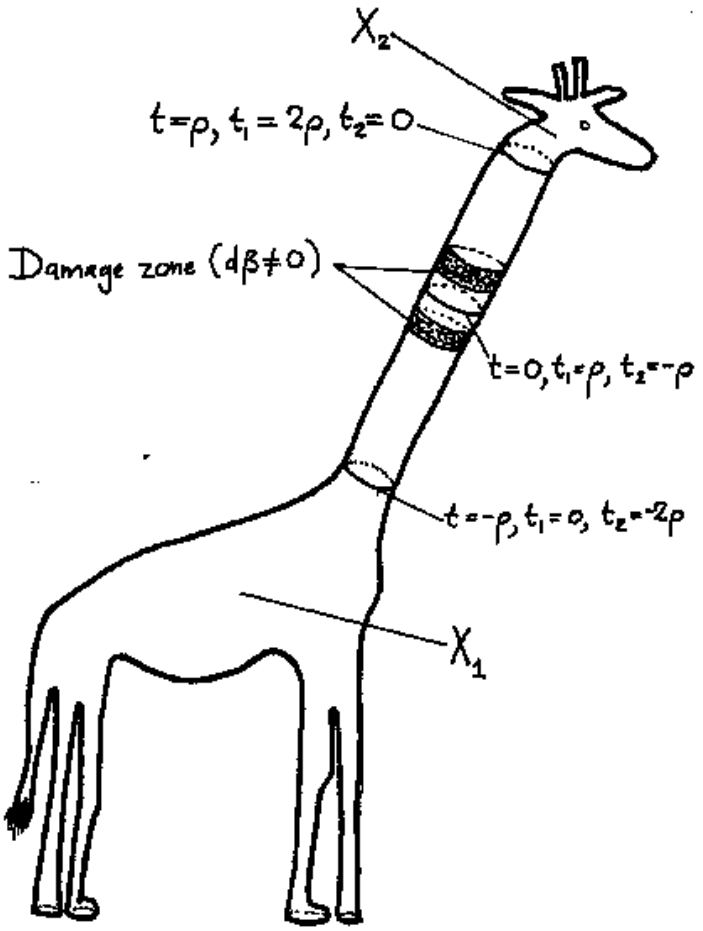

Figure 1: The manifold X_ρ with a long neck constructed by gluing the b-manifolds X_1 and X_2. (Adapted from a drawing by F.N. Singer.)

2.3.5 Gluing Riemannian *b*-metrics. Let X be a b-manifold. A C^∞ Riemannian metric on X^o is called a polyhomogeneous b-metric on X if and only if it has the form (2.2) where $h(x,y)$ is continuous up to ∂X and has a polyhomogeneous expansion relative to some index set. Assume now that the X_j in §2.3.1 are equipped with such phg b-metrics g_j and that near Y, $h_1(0,y) = h_2(0,y) =: h(y)$, say. In terms of the cylindrical parameters t_j, the g_j both approach the metric

$$g_0 = dt^2 + h(y) \text{ on } X_0 := \mathbb{R} \times Y$$

as $|t_j| \to \infty$.

We construct a Riemannian metric g_ρ on X_ρ by picking once and for all a standard non-increasing cut-off function $\beta : \mathbb{R} \to [0,1]$, such that $\beta(t) = 1$ for $t \le -1/2$ but $\beta(t) = 0$ for $t \ge 1/2$ and using it to define new metrics

$$\widetilde{g}_{1,\rho} = \beta(t_1 - \rho + 1)g_1 + \big(1 - \beta(t_1 - \rho + 1)\big)g_0$$

on X_1 and

$$\widetilde{g}_{2,\rho} = \beta(\rho - 1 - t_2)g_2 + \big(1 - \beta(\rho - 1 - t_2)\big)g_0$$

on X_2. These formulae cut off the exponentially decreasing terms in the asymptotic expansions of the g_j leaving the standard t-independent metric g_0 for $|t_j| \ge \rho$. In particular the identification used in the construction of X_ρ is now an isometry and we define g_ρ on X_ρ to be equal to $\widetilde{g}_{1,\rho}$ for $t \le 0$ and to be equal to $\widetilde{g}_{2,\rho}$ for $t \ge 0$.

2.3.6 First properties of g_ρ. It is clear from the construction that g_ρ is equal to g_1 or g_2 for $|t| \ge 2$ and is equal to g_0 for $|t| < 1/2$. Even in the *damage zone* $\{-1/2 \le |t-1| \le 1/2\}$, g_ρ is *exponentially close* to g_0. By this we mean that there exists $\eta > 0$ such that

$$\sup_K |g_\rho - g_0| = O(e^{-\eta\rho}) \text{ as } \rho \to \infty\,, \tag{2.6}$$

where $K = \{-1/2 \le |t-1| \le 1/2\}$ and the pointwise norm is that induced by g_0. In fact, (2.6) holds for $K = \{|t| \le T\}$, for any fixed $T > 0$, and similar estimates hold for all derivatives of $g - g_0$; such estimates follow from the assumed polyhomogeneous expansions of the g_j near Y.

In particular if P_j and P_ρ are differential operators canonically associated to the metrics g_j and g_ρ then P_ρ is exponentially close to the P_j and hence also to P_0 on $\{|t| < T\}$ for any fixed T, as $\rho \to \infty$.

3 The Anti-self-duality Equations

In this section we first review Riemannian and conformal geometry, passing in §3.2 to the special case of 4 dimensions where we stay for most of

the rest of the paper. In particular, the relevant analytical aspects of the ASD equations are given here, the main facts being summarized in Proposition 3.14. In §3.3 we study the Hermitian version of the ASD equations from the same point of view, listing the main points in Proposition 3.19. Thus experts in 4-dimensional geometry may well be able to move straight to these propositions, referring back if necessary to the earlier parts of this section.

3.1 Preliminaries on metrics and conformal structures.

3.1.1 Metrics and curvature. Let (X, g) be a Riemannian n-manifold and let h be a g-symmetric endomorphism of TX. Then the bilinear form

$$g_h(\xi, \eta) := g(\xi, \eta) + g(h\xi, \eta) \quad (\xi, \eta \in C^\infty(X, TX)) \tag{3.1}$$

is symmetric and defines a Riemannian metric if $|h| := (\mathrm{tr}\ (h^2))^{1/2} < 1$ at each point of X. Let ∇ and ∇^h denote respectively the metric connections of g and g_h. Then we have

$$\nabla_h \xi = \nabla \xi + Q\big((1+h)^{-1}, \nabla h\big)\xi \tag{3.2}$$

for any section ξ of TX, where $Q(u, v)$ is an endomorphism-valued 1-form, bilinear in (u, v).

Hence the curvature R_h of g_h has the form

$$R_h = R_0 + d^\nabla Q\big((1+h)^{-1}, \nabla h\big) + Q\big((1+h)^{-1}, \nabla h\big) \wedge Q\big((1+h)^{-1}, \nabla h\big)\,. \tag{3.3}$$

where R_0 is the curvature of g and d^∇ is the covariant exterior derivative defined by ∇. Separating the linear and nonlinear terms in (3.3), we obtain

$$R_h = R_0 + R_0'[h] + \varepsilon_1(1+h, \nabla h \otimes \nabla h) + \varepsilon_2(1+h, h \otimes \nabla\nabla h) \tag{3.4}$$

where R_0' is a linear differential operator and each ε_j is real-analytic in the first variable and linear in the second variable, with coefficients depending only upon g and its derivatives.

3.1.2 Convention. From now on we shall use $\varepsilon(u, v)$, $\varepsilon_1(u, v)$, $\varepsilon_2(u, v)$, etc., generically for 'error terms' with the properties just mentioned. That is, they are real-analytic in the 0-jet of u near $u = 1$ and linear in the 0-jet of v. For example, the ε_j in (3.13) below are not identical to those in (3.4). The point is that in order to estimate these non-linear terms later on, all we shall need to know is their qualitative dependence upon h and its derivatives.

3.1.3 Conformal structures. We adopt a 'modern' approach to conformal structures for which we claim no originality. For more details of this approach, the reader could consult, for example, [CP].

Let X be a C^∞ manifold and let $\Omega = |\Lambda^n T^*X|$ be the bundle of densities on X ([Hö1, p. 148]). Since Ω is an $\mathbb{R}_+$-bundle, Ω^w has a canonical meaning for any real w. By a *Riemannian conformal structure* c on X we shall mean a suitably normalized positive-definite C^∞ section of the bundle $S^2T^*X \otimes \Omega^{-2/n}$. Since the top exterior power of $\Omega^{1/n}TX$ is canonically trivial a possible normalization is the condition $\det c = 1$, but we shall make a different choice in §3.1.5. (Here we have written $\Omega^{1/n}TX$ for $\Omega^{1/n} \otimes TX$. We continue to omit such tensor product signs below.) Note that c can also be viewed as a normalized metric on the *weightless tangent bundle* $\Omega^{1/n}TX$.

A positive trivialization or *choice of length-scale* μ of $\Omega^{-1/n}$ determines a *compatible Riemannian metric* $g_\mu = \mu^{-2}c$ on TX; any two such are *conformally related* in the sense that

$$g_{\mu'} = (\mu'/\mu)^2 g_\mu \tag{3.5}$$

where μ'/μ is a positive C^∞ function, so that any two compatible metrics are related by *conformal rescaling*. In particular, the present approach is equivalent to the more traditional one in which a conformal structure is taken as a conformal equivalence class of Riemannian metrics. We shall occasionally write $g \in c$ to mean that g is compatible with c, i.e. that g arises from c by a choice of scale μ or say that g belongs to the conformal class of c.

The length-scale μ also defines a metric $|\cdot|_\mu$ on the tensor bundle $\Omega^{-w/n}TX^{\otimes j} \otimes T^*X^{\otimes k}$ (and any sub- or quotient bundle) by the formula

$$|s|_\mu := |\mu^{-w-j+k}s|_c \tag{3.6}$$

(generalizing (3.5)). This bundle is therefore said to have *conformal weight* $w + j - k$, because of the formula

$$|s|_{\mu'} = (\mu'/\mu)^{w+j-k}|s|_\mu\,. \tag{3.7}$$

A conformal structure c determines an identification $\Omega^{1/n}TX \to \Omega^{-1/n}T^*X$, the conformal version of index-lowering. Similarly indices are raised with c^{-1}. Note that these operations *preserve conformal weight*.

Definition 3.1.4. A bundle E of conformal weight 0 is called a *weightless* bundle.

If E is a weightless bundle then a conformal structure c defines a genuine metric on E. It is clear that any bundle associated to the tangent bundle of

X can be written uniquely in the form $\Omega^{-a/n}E$, where E is weightless. We shall frequently write bundles in this way in situations where it is necessary to keep track of conformal weights.

3.1.5 Parameterization of conformal structures. Let (X, c) be a conformal n-manifold with $\det c = 1$. If h is a c-symmetric endomorphism of TX, we define

$$c_h(\xi, \eta) := c(\xi, \eta) + c(h\xi, \eta) \quad \big(\xi, \eta \in C^\infty(X, TX)\big)\,. \tag{3.8}$$

For suitable h (in particular if $|h| < 1$ at each point) then c_h will be positive-definite. We choose to normalize c_h by the requirement that h be trace-free, rather than $\det c_h = 1$. To first order in h, these conditions agree.

Denote by E^1 the bundle of c-symmetric trace-free endomorphisms of TX; then E^1 is weightless and using c can be identified with $\Omega^{-2/n}S_0^2T^*X$, where S_0^2 = trace-free part of the symmetric square.

In terms of E^1, we can summarize this paragraph as follows: there is an open neighbourhood B of the zero-section in $C^\infty(X, E^1)$, containing all h with $\sup_X |h| < 1$, such that $h \mapsto c_h$ is a diffeomorphism between B and the set of all conformal structures on X.

3.1.6 Conformal invariance. It is well known in conformal geometry that certain differential operators, which *a priori* depend upon a Riemannian metric, are unchanged by conformal rescaling. If such a *conformally invariant* operator $P : C^\infty(\Omega^{-a/n}E) \to C^\infty(\Omega^{-b/n}F)$ is of order k, say, where E and F are weightless bundles, then its symbol provides a map

$$\Omega^{-a/n}S^kT^*X \otimes E = \Omega^{-a/n+k/n}S^k(\Omega^{-1/n}T^*) \otimes E \to \Omega^{-b/n}F$$

which must be weightless. Accordingly we must have $b = a - k$.

The formal adjoint of a conformally invariant operator defines a conformally invariant operator but the conformal weights may change. Indeed it is a familiar fact in the analysis literature (e.g. [Hö2, p. 93] that a differential operator $P : C^\infty(\Omega^{1/2}E) \to C^\infty(\Omega^{1/2}F)$ has a formal adjoint $P^* : C^\infty(\Omega^{1/2}F^*) \to C^\infty(\Omega^{1/2}E^*)$ independent of any metric. If we replace $\Omega^{1/2}E$ and $\Omega^{1/2}F$ by $\Omega^{-a/n}E$ and $\Omega^{k/n-a/n}F$ where E and F are now assumed weightless, then using c to identify E with E^* and F with F^*, we obtain

$$P^* : C^\infty(\Omega^{(1+(a-k)/n)}F) \to C^\infty(\Omega^{(1+a/n)}E)$$

which is conformally invariant if P is conformally invariant.

3.2 Background to 4-dimensional geometry. 4-dimensional Riemannian geometry is enriched by the existence of the special isomorphism

$\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. The geometric counterpart of this algebraic fact is the decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^- \tag{3.9}$$

for 2-forms on a 4-dimensional vector space equipped with a metric and orientation. As far as the present work is concerned, the main consequence of this is the presence of the anti-self-duality equations for curvatures on an oriented 4-manifold. We start, however, with some basic algebra.

3.2.1 Algebraic preliminaries. Consider Euclidean space $\mathbb{R}^4$ with its standard metric and orientation $dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$. The purpose of this paragraph is to explain the invariant isomorphism $S^2_0\mathbb{R}^4 = \Lambda^+ \otimes \Lambda^-$, which will often be used below.

Consider the standard bases

$$e_1 = dx_0 \wedge dx_1 + dx_2 \wedge dx_3\,, \quad e_2 = dx_0 \wedge dx_2 + dx_3 \wedge dx_1\,,$$
$$e_3 = dx_0 \wedge dx_3 + dx_1 \wedge dx_2\,,$$

of Λ^+ and

$$\overline{e}_1 = dx_0 \wedge dx_1 - dx_2 \wedge dx_3\,, \quad \overline{e}_2 = dx_0 \wedge dx_2 - dx_3 \wedge dx_1\,,$$
$$\overline{e}_3 = dx_0 \wedge dx_3 - dx_1 \wedge dx_2\,,$$

of Λ^-. Using the metric, e_r and $\overline{e}_r$ operate on $\mathbb{R}^4$ as orthogonal complex structures I_r and $\overline{I}_r$, respectively, corresponding to the left and right action of the quaternions i, j and k. In particular

$$I_1^2 = I_2^2 = I_3^2 = I_1 I_2 I_3 = -1$$

but

$$I_r \overline{I}_s = \overline{I}_s I_r \text{ for all } r \text{ and } s\,.$$

Now let A be an endomorphism of $\mathbb{R}^4$. The irreducible components of A consist of

- $\mathrm{tr}\,(A) \in \mathbb{R}$,
- $\sum \mathrm{tr}\,(I_r A) e_r \in \Lambda^+$,
- $\sum \mathrm{tr}\,(\overline{I}_r A) \overline{e}_r \in \Lambda^-$,
- $\sum \mathrm{tr}\,(I_r \overline{I}_s A) e_r \otimes \overline{e}_s \in \Lambda^+ \otimes \Lambda^-$,

and the maps implied by these formulae are equivariant. It is easy to see, using standard properties of the trace, that the last of these is indeed an isomorphism of the space of symmetric trace-free endomorphisms of $\mathbb{R}^4$ with $\Lambda^+ \otimes \Lambda^-$.

For the rest of this section, (X, c) will be an oriented conformal 4-manifold, and g will be a Riemannian metric in the conformal class c.

3.2.2 The anti-self-duality equation for conformal structures. The curvature R of g, viewed as a symmetric endomorphism of Λ^2 decomposes as

$$R = \begin{pmatrix} W^+ + s/12 & \Phi \\ \Phi^t & W^- + s/12 \end{pmatrix} : \begin{pmatrix} \Lambda^+ \\ \Lambda^- \end{pmatrix} \to \begin{pmatrix} \Lambda^+ \\ \Lambda^- \end{pmatrix} \tag{3.10}$$

where Φ is the trace-free part of the Ricci tensor of g, viewed as a section of $\Lambda^- \otimes \Lambda^+$ as in §3.2.1, s is the scalar curvature and W^+ and W^- are respectively the self-dual and anti-self-dual parts of the *Weyl curvature* W. The metric g is said to be *conformally ASD* if $W^+(g) = 0$. Since $W = W^+ + W^-$ and (3.9) are conformally invariant, we also write $W^+(c) = 0$ and call c an ASD conformal structure.

3.2.3 The deformation complex. If $W^+(c) = 0$ then to first order in h the condition $W^+(c_h) = 0$ is equivalent to a conformally invariant differential equation which will be denoted $D_c h = 0$. This operator is part of the *deformation complex* for the conformal ASD equations,

$$C^\infty(X, \Omega^{-1/4}E^0) \xrightarrow{L_c} C^\infty(X, E^1) \xrightarrow{D_c} C^\infty(X, \Omega^{1/2}E^2)\,, \tag{3.11}$$

of conformally invariant operators, where

$$\Omega^{-1/4}E^0 = TX\,, \quad E^1 = \Omega^{-1/2}S_0^2T^*X\,, \quad E^2 = \Omega^{-1}S_0^2\Lambda^+T^*X\,. \tag{3.12}$$

In particular each of the E^j is weightless. As we have said, D_c is the linearization of the map $h \mapsto W^+(c_h)$ and is a second-order operator; L_c gives the action of infinitesimal diffeomorphisms on conformal structures, so that $L_c\xi$ is equal to the trace-free part of the Lie derivative of c along ξ.

The operator D_c is easier to understand after use of the isomorphism $S_0^2(\Omega^{-1/4}T^*X) = \Lambda^-(\Omega^{-1/4}T^*X) \otimes \Lambda^+(\Omega^{-1/4}T^*X)$ described in §3.2.1. Suppressing powers of Ω for the moment, recall the existence of a natural second-order elliptic operator $d^+d^* : C^\infty(\Lambda^-) \to C^\infty(\Lambda^+)$. This can be coupled to any vector bundle V with connection A to yield an operator $d_A^+d_A^* : C^\infty(\Lambda^- \otimes V) \to C^\infty(\Lambda^+ \otimes V)$. Then the second-order part of D_c is the composite of this map (with $V = \Lambda^+$) and projection $\Lambda^+ \otimes \Lambda^+ \to S_0^2\Lambda^+$. D_c also has a zeroth-order term given by multiplication by the trace-free part of the Ricci tensor.

The composite D_cL_c vanishes if c is ASD and then the *deformation cohomology groups* H_c^* are defined and have a standard role [KiK] in describing the local properties of the moduli space of ASD conformal structures near c. (3.11) is an elliptic complex, so the $H_c^*(X)$ are finite-dimensional vector spaces if X is a compact manifold (or orbifold) without boundary.

3.2.4 The full ASD equations. The self-dual Weyl tensor $W_h^+ := W^+(c_h)$ is obtained by orthogonal projection of R_h onto $\Omega^{1/2}E_h^2 = \Omega^{-1/2}S_0^2\Lambda_h^+$, where Λ_h^+ is the bundle of self-dual 2-forms for c_h. Since this orthogonal projection depends real-analytically upon the 0-jet of $(1+h)$, we have a formula analogous to (3.4):

$$W_h^+ = W_0^+ + D_c h + \varepsilon_1(1+h, h\otimes\nabla\nabla h) + \varepsilon_2(1+h, \nabla h\otimes\nabla h) + \varepsilon_3(1+h, h\otimes h) \quad (3.13)$$

(Recall the convention regarding the ε_j explained in §3.1.2.)

It is convenient to remove the dependence of the target space upon h by projecting orthogonally from E_h^2 onto E^2 (recall that both E^2 and E_h^2 are subbundles of $\Omega^{-1}S^2\Lambda^2$). In this way we obtain a nonlinear map $F(h) : B \to C^\infty(X, \Omega^{1/2}E^2)$, such that $F(h) = 0$ if and only if c_h is conformally ASD. Summing up,

Proposition 3.14. *Let (X, c) be an oriented conformal 4-manifold and let the E^j be as in (3.12). Then there exists a C^∞ map $F : B \to C^\infty(X, \Omega^{1/2}E^2)$, where $B := \{h \in C^\infty(X, E^1) : \sup_X |h| < 1\}$, with an expansion of the form*

$$F(h) = W_0^+ + D_c h + \varepsilon_1(1+h, h\otimes\nabla\nabla h) + \varepsilon_2(1+h, \nabla h\otimes\nabla h) + \varepsilon_3(1+h, h\otimes h) \quad (3.15)$$

such that $F(h) = 0$ if and only if c_h is ASD. In (3.15), W_0^+ is the self-dual Weyl tensor of c, D_c is the linearization of $h \mapsto W^+(c_h)$, and the ε_j are nonlinear terms which conform to Convention 3.1.2.

3.3 Hermitian-ASD conformal structures. Let (X, J) be a complex surface and let c be a J-Hermitian conformal structure. We shall repeat our discussion of deformations of ASD conformal structures now preserving the J-Hermitian condition.

3.3.1 Hermitian deformations of *c*. In the parameterization of deformations of c by endomorphisms h (§3.1.5) it is easily checked that c_h is J-Hermitian if and only if $[h, J] = 0$. Working equivalently with $\Lambda^-\otimes\Lambda^+$ (and forgetting about conformal weights for now) the J-Hermitian deformations correspond to elements of the form $u\otimes\omega$ where $u \in \Lambda^-$ and ω is the fundamental 2-form defined by c and J.

3.3.2 The conformal weight of *J*. For reasons that will emerge in a moment, it is convenient to regard ω not as weightless but as a section of the bundle $\Omega^{-1/4}E_J^2 := \Omega^{-3/4}\Lambda^+$. Then J defines not an endomorphism of TX but a map $\Omega^{-w/4}TX \to \Omega^{-(w+1)/4}TX$. (In the presence of c, ω

and J are interchangeable.) Given c and $\omega \in \Lambda^+$ such that ω is nowhere 0 in X, there is a unique choice of gauge such that $J_\mu := \mu^{-1}\omega$ satisfies $J_\mu^2 = -1$, so there is no loss in giving ω this strange conformal weight. Having done so, we are forced to take $u \in C^\infty(\Omega^{1/4}E_J^1 := \Omega^{-1/4}\Lambda^-)$ so that $h = u \otimes \omega$ is weightless. We now explain why this choice of conformal weight is advantageous.

3.3.3 Scalar-flat Kähler metrics. It is well known [Bo], [V] that a Kähler metric g on a complex surface (X, J) is conformally ASD if and only if its scalar curvature is 0; we refer to such g as *scalar-flat Kähler*. For any Kähler metric g, we have $\nabla J = 0$; the complex structure is parallel for the metric connection. An alternative characterization is that the associated Kähler form ω should satisfy the *twistor equation*

$$T\omega = 0, \quad T\omega = (\nabla\omega)_0 \tag{3.16}$$

where the subscript 0 denotes trace-free part, that is, the image by the projection

$$\Lambda^1 \otimes \Lambda^+ \to (\Lambda^1 \otimes \Lambda^+)_0$$

(whose kernel is Λ^1). Indeed, a short calculation shows that if $T\omega = 0$ and $|\omega| = 1$, then $\nabla\omega = 0$.

On the other hand, T is a conformally invariant operator

$$T : C^\infty(\Omega^{-1/4}E_J^2) \to C^\infty\big((\Omega^{-3/4}\Lambda^1 \otimes \Lambda^+)_0\big)$$

so that we can regard a pair (c, ω) where c is a conformal structure and ω is a nowhere-vanishing solution of the twistor equation (3.16), as being a *conformally invariant description* of a Kähler metric on a 4-manifold. Furthermore, c is ASD if and only if the Kähler metric is scalar-flat Kähler.

As in §3.3.2, the Kähler representative of c is fixed by the unique scale which gives ω constant length equal to 1 at each point of X.

Finally we note that any Hermitian-ASD conformal structure on a compact complex surface X of Kähler type (equivalently $b_1(X)$ even) is conformally equivalent to a scalar-flat Kähler metric [Bo], [V]. Thus looking for Hermitian-ASD conformal structures is, under suitable topological conditions, *equivalent* to looking for scalar-flat Kähler metrics. We shall apply this observation to construct scalar-flat Kähler metrics on multiple blow-ups of $\mathbb{C}^2$ in Theorem F.

3.3.4 Deformation theory of Hermitian-ASD conformal structures. Since Hermitian deformations are parameterized by a bundle of rank 3, it might be expected that the ASD equations are now overdetermined. This is not the case: it is shown in [Bo] that if c is Hermitian, then

$W^+(c)$ lies in the image of the map

$$\Omega^{3/4}E^2_J \xrightarrow{\omega\otimes} \Omega^{1/2}E^2_J \otimes E^2_J \to \Omega^{1/2}E^2$$

(recall that E^2_J is our shorthand for the weightless version of Λ^+). In other words, 2 of the 5 components of W^+ are automatically zero in the Hermitian case. Accordingly the linearization of the ASD equations is given by a second-order operator

$$D_{c,J} : C^\infty(\Omega^{1/4}E^1_J) \to C^\infty(\Omega^{3/4}E^2_J)$$

where $(\omega \otimes D_{c,J}u)_0 = D_c(\omega \otimes u)$. If ω satisfies the twistor equation, then $D_{c,J}$ simplifies to a conformally invariant operator which we shall denote by S

$$S : C^\infty(X, \Omega^{1/4}E^1_J) \to C^\infty(X, \Omega^{3/4}E^2_J) \tag{3.17}$$

which *depends only upon* c and not upon ω. This S-operator was studied in detail in [LS1]; it has the form $d^+d^* + \Phi$ where $\Phi : \Omega^{1/4}E^1_J \to \Omega^{3/4}E^2_J$ is defined by regarding the trace-free part of the Ricci tensor as a section of $\mathrm{Hom}(\Omega^{1/4}E^1_J, \Omega^{3/4}E^2_J)$. The operator $D_{c,J}$ or S has the same role, in the study of Hermitian-ASD conformal structures, as the deformation complex (3.11) for ASD conformal structures. In particular if we put

$$H^1_{c,J}(X) = \ker(S)\,, \quad H^2_{c,J}(X) = \operatorname{coker} S = \ker S^* \tag{3.18}$$

then $H^1_{c,J}$ is formally the tangent space to the moduli space of Hermitian-ASD deformations of c, while $H^2_{c,J}$ is an obstruction space.

3.3.5 Nonlinear terms. The analysis of nonlinear terms in the Hermitian-ASD equations is the same as for the full ASD equations and leads to the following, which is the analogue of Proposition 3.14:

Proposition 3.19. *Let (X, c, J) be a conformal Hermitian surface, let ω be the (conformal) fundamental 2-form, and let*

$$E^1_J = \Omega^{-1/2}\Lambda^-\,, \quad E^2_J = \Omega^{-1/2}\Lambda^+. \tag{3.20}$$

Then there exists a C^∞ map $F_J : B_J \to C^\infty(X, \Omega^{3/4}E^2_J)$, where B_J is an open neighbourhood of the zero-section in $C^\infty(X, \Omega^{1/4}E^1)$, with an expansion of the form

$$\begin{aligned} F_J(u) = W^+_0 + D_{c,J}u + \varepsilon_1(1 + u, u \otimes \nabla\nabla u) \\ + \varepsilon_2(1 + u, \nabla u \otimes \nabla u) + \varepsilon_3(1 + u, u \otimes u)\,. \end{aligned} \tag{3.21}$$

such that $F_J(u) = 0$ if and only if $c_{\omega\otimes u}$ is Hermitian-ASD. In (3.21), W^+_0 is the self-dual Weyl tensor of c, $D_{c,J}$ is the linear operator mentioned above and the ε_j are nonlinear terms which conform to Convention 3.1.2. Finally if ω satisfies the twistor equation, so that (c, ω) is conformal to scalar-flat Kähler, then $D_{c,J}$ can be replaced by the operator S of (3.17).

This completes our study of the PDE aspects of the two problems of interest in this paper. We have seen that when regarded as 'perturbation problems', both have elliptic linearizations (in the case of the full ASD equations it is necessary to divide by the action of the diffeomorphism group). We are going to apply Propositions 3.14 and 3.19 on the manifold X_ρ to expand the ASD equations about the approximate solution g_ρ constructed in §2.3. In the next section the behaviour of the linearizations D_c and $D_{c,J}$ will be studied, leading to an application of the implicit function theorem to solve the equations in §5.

4 Linear Theory on X_ρ

This section is devoted to a discussion of the linear aspects of our gluing problems. The first part includes the 'main estimate' (Proposition 4.2) which is one of the key technical results needed for all our gluing theorems. This proposition states, roughly speaking, that whenever a fully elliptic operator P_ρ on X_ρ is obtained by gluing fully elliptic operators P_j on X_j then $\ker P_\rho$ and $\operatorname{coker} P_\rho$ are well approximated by $\ker P_1 \oplus \ker P_2$ and $\operatorname{coker} P_1 \oplus \operatorname{coker} P_2$, and that P_ρ induces a uniformly bounded isomorphism between suitable complementary subspaces.

In §4.2 we discuss the linearized operator that arises in the deformation complex and note that it is never fully elliptic on a b-manifold. This entails a discussion of the linearization over cylinders which is taken further than is strictly needed for most of the gluing theorems.

In §4.3 we consider b-manifolds that are conformal blow-ups and blow-downs of compact or WALE spaces and compare the 'Hodge' version of the deformation complex for a b-manifold with the analogous definitions in terms of the compact or WALE models. Here conformal invariance is an essential tool.

Finally in §4.4 we give comparison results for the operator S that arises in the Hermitian-ASD problem.

4.1 Stretching the neck. In this section we consider the behaviour of linear elliptic operators on X_ρ as $\rho \to \infty$, under the assumption that the corresponding operators over X_0, X_1 and X_2 are *fully elliptic*. The main points of the argument are based closely on Floer's paper, but with several simplifications and generalizations. Because the analysis presented here is also needed in other geometric applications [Ko], [KoS], we work here in some generality.

4.1.1 Notation. The geometric set-up and notation used will be as in §2.3. In particular X_1 and X_2 will be b-manifolds and Y a piece of ∂X_j at which the gluing is taking place.

Now suppose that for $j = 1, 2$,

$$P_j : C^\infty(X_j, E_j) \to C^\infty(X_j, F_j)$$

are fully elliptic phg b-differential operators of order m. Suppose that over the cylindrical subset U_j of §2.3.1 we have identifications $E_j = \pi^* E_0$, $F_j = \pi^* F_0$, where $\pi : U_j \to Y$ is the obvious projection. Suppose further that with these identifications we have $P_j = p_j(x_j, y; x_j\partial_{x_j}, \partial_y)$ and the indicial operators agree in the sense

$$P_0 := p_1(0, y; \partial_t, \partial_y) = p_2(0, y; -\partial_t, \partial_y).$$

(The sign in p_2 is to take care of the sign convention in the definition of the t_j.) Then we can glue the P_j across $\pm t_j = \rho$ to obtain bundles E_ρ, F_ρ and a fully elliptic b-operator

$$P_\rho : C^\infty(X_\rho, E_\rho) \to C^\infty(X_\rho, F_\rho)$$

in exactly the same way that the metric g_ρ was constructed from the g_j in §2.3.5. We have analogous estimates to those of §2.3.6:

$$\sup_{|t|\le T} |P_j - P_0| = O(e^{-\eta\rho}), \quad \sup_{|t|\le T} |P_\rho - P_0| = O(e^{-\eta\rho}) \text{ as } \rho \to \infty, \tag{4.1}$$

for any fixed T, where $\eta > 0$ is some constant. Here we have written $|P_j - P_0|$ for the sum of the moduli of the coefficients of $P_j - P_0$. Note that in our application, the P_j arise as operators canonically associated to some geometric data (a metric or conformal structure). Then by cutting off and gluing these data and *then* constructing the corresponding P-operator one will get a slightly different operator from P_ρ, constructed by gluing the P_j directly. Since in either case we shall have estimates like those of (4.1), this difference is not important, and will be ignored in the sequel.

4.1.2 Definition of asymptotic kernels and cokernels. According to Proposition 2.3,

$$P_0 : L^p_m(X_0, E_0) \to L^p(X_0, F_0)$$

is an isomorphism and by Theorem 2.5, the P_j are Fredholm in L^p for every p, with index independent of p. Let the L^p null-space of P_j be denoted by N_j and the L^p null-space of the L^2 adjoint P_j^* be denoted by M_j. By part (ii) of Theorem 2.5, N_j and M_j consist of phg sections. In particular, these sections are exponentially decreasing as $t_1 \to \infty$ or $t_2 \to -\infty$, and the same is true of all derivatives of these sections. Because of this,

$$N_{1,\rho} = \beta(t_1 - \rho/2)N_1, \quad M_{1,\rho} = \beta(t_1 - \rho/2)M_1$$

and

$$N_{2,\rho} = \beta(-t_2 + \rho/2)N_2\,, \quad M_{2,\rho} = \beta(-t_2 + \rho/2)M_2$$

obtained by cutting off at distance $\rho/2$ will be very good approximations if ρ is large. Moreover, we can regard $N_{j,\rho}$ as a subspace of $U_\rho = L^p_m(X_\rho, E_\rho)$ and $M_{j,\rho}$ as a subspace of $V_\rho = L^p(X_\rho, E_\rho)$. Define $U'_\rho = N_{1,\rho} \oplus N_{2,\rho}$ to be the *asymptotic kernel* of P_ρ and $V'_\rho = M_{1,\rho} \oplus M_{2,\rho}$ to be its *asymptotic cokernel*. Finally denote by U''_ρ (resp. V''_ρ) the L^2-orthogonal complement of U'_ρ (resp. V'_ρ) in U_ρ (resp. V_ρ).

4.1.3 The main estimate.

PROPOSITION 4.2. *There exists $\rho_* > 0$ such that for all $\rho > \rho_*$, the induced map $P''_\rho : U''_\rho \to V''_\rho$ is an isomorphism and the operator norm of $G_\rho = [P''_\rho]^{-1}$ is bounded independent of ρ.*

Proof. We shall prove that there exists ρ_* and a constant $\varepsilon > 0$ such that if $\rho > \rho_*$, then

$$\|P''_\rho u\| \geq \varepsilon \|u\| \text{ for all } u \in U''_\rho\,. \tag{4.3}$$

Such an estimate shows that P_ρ is injective, with a uniform estimate for a left-inverse $V''_\rho \to U''_\rho$. Our set-up is symmetric under adjoints, however, so by the analogous estimate with P_ρ replaced by P^*_ρ, we see that P''_ρ is surjective, and the left-inverse is a true inverse, with norm bounded independent of ρ.

The proof of (4.3) goes by contradiction. If it fails, there exists a sequence $\rho_n \to \infty$ and $u_n \in U''_n$ such that

$$\|u_n\| = 1 \tag{4.4}$$

but

$$\|P''_n u_n\| \to 0 \tag{4.5}$$

as $n \to \infty$. (Here we begin to use obvious notational simplifications, replacing ρ_n by n wherever this is unambiguous.) From the definition of P''_ρ, there exist further sequences $v_n^{(j)} \in V'_{j,n}$ such that

$$\|P_n u_n - v_n^{(1)} - v_n^{(2)}\| \to 0 \tag{4.6}$$

as $n \to \infty$. The main step in the proof is contained in the following:

LEMMA 4.7. *There exists a subsequence of u_n (which by abuse of notation we continue to denote by u_n), such that*

$$\|u_n\|_{L^p_m(|t|\leq 2)} \to 0 \text{ as } n \to \infty\,.$$

(We shall see from the proof that 2 could be replaced by any larger positive real number.)

Proof of Lemma 4.7. Multiply u_n by a bump-function so that it is cut off to zero at $t = \pm\rho_n/2 - 1$ Denote by $u_n^{(0)}$ the resulting section over X_0. Clearly the L^p_m-norm of $u_n^{(0)}$ is uniformly bounded as $n \to \infty$, so by passing to a subsequence we may assume that

$$u_n^{(0)} \to u^{(0)} \text{ weakly in } L^p_m\,.$$

Over any fixed compact $K \subset X_0$, we have by (4.6)

$$\|P_0 u_n^{(0)}\|_{L^p(K)} \le \big\|(P_0 - P_n)u_n\big\|_{L^p(K)} + \|P_n u_n\|_{L^p(K)} \to 0 \text{ as } n \to \infty$$

since the support of $v_n^{(1)} + v_n^{(2)}$ does not meet K. Now the seminorm $u \mapsto \|P_0 u\|_{L^p}$ is continuous with respect to the L^p_m-norm, so by lower-semicontinuity of weak limits, $\|P_0 u^{(0)}\|_{L^p(K)} = 0$. Since this is true for every compact set K, $P_0 u^{(0)} = 0$ over X_0 and hence $u^{(0)} = 0$ because $P^{(0)}$ is an isomorphism in L^p.

To complete the proof of the lemma, recall that weak convergence in $L^p_m(K)$ implies strong convergence in $L^p(K)$ if K is compact. The lemma now follows by taking $K = \{|t| \le 2\}$ and applying the elliptic estimate

$$\|u_n\|_{L^p_m(K)} \le C\big(\|P_n u_n\|_{L^p(K)} + \|u_n\|_{L^p(K)}\big)\,.$$

Proof of Proposition 4.2. Replace u_n by the subsequence in Lemma 4.7 and construct sequences $u_n^{(j)}$ over X_j by cutting off u_n in the intervals $\rho_n - 2 < t_1 < \rho_n - 1$ and $1 - \rho_n < t_2 < 2 - \rho_n$. More precisely, we have $u_n^{(j)} = \beta_n^{(j)} u_n$, say, where $\beta_n^{(j)}$ is a suitable translation of a standard cut-off function. We have

$$P_j u_n^{(j)} = \beta_n^{(j)} P_n u_n + [P_j, \beta_n^{(j)}] u_n$$

since $P_n = P_j$ where $\beta_n^{(j)} \ne 0$. Therefore

$$\|P_j u_n^{(j)} - v_n^{(j)}\| \le \|P_n u_n - v_n^{(1)} - v_n^{(2)}\| + \big\|[P_j, \beta_n^{(j)}] u_n\big\|$$

and the first term here tends to zero by (4.6), the second by Lemma 4.7, for $[P_j, \beta_n^{(j)}]$ is a differential operator of order $\le m$ which vanishes outside of $\{|t| \le 2\}$. Since $P_j u_n^{(j)}$ and $v_n^{(j)}$ lie in complementary subspaces, it follows that

$$\|P_j u_n^{(j)}\| \to 0 \text{ and } \|v_n^{(j)}\| \to 0\,.$$

On the other hand $u_n^{(j)}$ lies in a subspace on which P_j is injective, so $u_n^{(j)} \to 0$ in L^p_m as $n \to \infty$.

Finally we combine these estimates with (4.4) to obtain

$$1 = \|u_n\| \le \|\beta_n^{(1)} u_n\| + \|\beta_n^{(2)} u_n\| + \big\|(1 - \beta_n^{(1)} - \beta_n^{(2)}) u_n\big\| \to 0\,.$$

This contradiction completes the proof of Proposition 4.2.

4.2 Application to ASD problem. Let X be a compact oriented 4-manifold with a b-metric g. Let

$$\mathcal{D} = (D_g, L_g^*) : C^\infty(X, E^1) \to C^\infty(X, E^2) \oplus C^\infty(X, E^0) \tag{4.8}$$

so that if $\partial X = \emptyset$ and g were conformally ASD we should have

$$H_c^1 = \ker \mathcal{D}\,, \quad H_c^2 \oplus H_c^0 = \operatorname{coker} \mathcal{D}$$

the direct-sum decomposition of the cokernel corresponding to the direct-sum decomposition in (4.8). We wish to apply the main estimate to $\mathcal{D}_\rho$ obtained by gluing $\mathcal{D}_j$. Therefore we need to check whether $\mathcal{D}_g$ is fully elliptic when g is a b-metric. For this we must examine the indicial operator $I(\mathcal{D})$ or equivalently the operator $\mathcal{D}_0$ corresponding to the cylinder $X_0 = Y \times \mathbb{R}$, where $g_0 = h(y) + dt^2$ is a product metric. We assume that g_0 is conformally ASD, hence conformally flat (consider the reflection isometry $t \mapsto -t$). Since $dt^2 + h(y)$ is conformally flat if and only if h is a metric of constant curvature we assume this from now on. Then the section $(dt \otimes dt)_0$ of E^1 is parallel so $\mathcal{D}_0(dt \otimes dt)_0 = (\Phi_0 \cdot (dt \otimes dt)_0$ where Φ_0 is the trace-free part of the Ricci tensor of g_0 (which is itself a multiple of $(dt \otimes dt)_0$) and $(\cdots)_0$ denotes the projection into $S_0^2\Lambda^+$. Hence this term is 0 and it follows that $\mathcal{D}_g$ is *never* fully elliptic.

We can however apply the conjugation trick of §2.2.6 to obtain nearby operators that *are* fully elliptic. To this end, denote by $\mathcal{D}_g^\pm$ the operator

$$\mathcal{D}_g : x^{\pm\delta} L_2^2(X, E^1) \to x^{\pm\delta}\big(L^2(X, E^2) \oplus L_1^2(X, E^0)\big) \tag{4.9}$$

and set

$$\ker \mathcal{D}_g^\pm = H_c^{1,\pm}\,, \quad \operatorname{coker} \mathcal{D}_g^\pm = H_c^{2,\pm} \oplus H_c^{0,\pm}\,,$$

the direct-sum decomposition of the cokernel corresponding, as before, to that of the target of $\mathcal{D}$. Here we assume

$$\delta > 0\,, quad\big\{0 < |\operatorname{Im}\lambda| \le \delta\big\} \cap \operatorname{spec}_b(\mathcal{D}_g) = \emptyset\,. \tag{4.10}$$

It follows that $\mathcal{D}^\pm$ are Fredholm and that the $H^{j,\pm}$ are independent of the choice of Sobolev spaces in (4.9) and of the choice of δ satisfying (4.10).

From the definitions,

$$H_g^{1,+} \subset H_g^{1,-}\,, \quad H_g^{2,-} \subset H_g^{2,+},\ H_g^{0,-} \subset H_g^{0,+}\,.$$

We are now in a situation in which Proposition 4.2 can be applied. We take this up in §5 and the reader who wants to see at once how this leads to gluing theorems for ASD conformal structures can now start reading there.

The remainder of this section is however taken up with a more detailed discussion of the indicial operator $\mathcal{D}_0$.

4.2.1 $\mathcal{D}$ on a cylinder. With (X_0, g_0) the cylinder $Y \times \mathbb{R}$ with product metric $g_0 = h(y) + dt^2$, where h is a metric of constant curvature, we wish to study the non-vanishing $H_c^{j,\pm}(Y \times \mathbb{R})$, viz. $H_c^{0,+}(Y \times \mathbb{R})$, $H_c^{1,-}(Y \times \mathbb{R})$, $H_c^{2,+}(Y \times \mathbb{R})$. By definition, these are respectively the parts of the null-spaces of L, $\mathcal{D}$, D^*, that consist of t-invariant or t-periodic sections. An index theorem gives

$$\dim H_c^{0,+}(Y \times \mathbb{R}) - \dim H_c^{1,-}(Y \times \mathbb{R}) + \dim H_c^{2,+}(Y \times \mathbb{R}) = 0$$

for these spaces can be identified with cohomology groups for an associated elliptic complex over Y. Furthermore, it is not hard to see that $H_c^{0,+}(Y \times \mathbb{R}) = \mathbb{R} \oplus \mathrm{Isom}(Y)$, where the summand $\mathbb{R}$ corresponds to translations of the cylinder.

4.2.2 Geometric interpretation of $H_c^{1,-}(Y \times \mathbb{R})$. If $H_c^{1,-}(Y \times \mathbb{R}) \neq 0$, then there exists a t-periodic or t-invariant solution of $\mathcal{D}_0 u = 0$. In particular, we can consider u to be a solution on $Y \times S^1$, where the length of the circle is determined by the period of the solution. Thus u gives an infinitesimal conformally flat deformation of the product conformally flat structure on $Y \times S^1$. Now there is an obvious space of deformations of this structure: constant-curvature deformations of the constant-curvature metric on Y plus isometries of Y used to change the way in which the two copies of Y are identified when making $Y \times S^1$ from $Y \times I$ (I is an interval). Such deformations correspond to t-invariant solutions u, and in the cases which are understood, when the curvature of Y is non-negative, all solutions are of this form.

4.2.3 Positive curvature: $Y = S^3/\Gamma$. According to the previous paragraph, the t-invariant part of $H^{1,-}$ contains $\mathbb{R} \oplus \mathfrak{so}_4^\Gamma$. The $\mathbb{R}$ summand corresponds to homotheties of $h(y)$ (or to $(dt \otimes dt)_0$) while $\mathfrak{so}_4^\Gamma$ is the space of infinitesimal isometries of S^3/Γ. Recall that homotheties are the only constant-curvature deformations of the round metric on S^3. Floer has shown, moreover, that this is the whole of $H_0^{1,-}(S^3 \times \mathbb{R})$ [F, §5]. Summing up,

$$H_c^{0,+}(S^3/\Gamma \times \mathbb{R}) = H_c^{1,-}(S^3/\Gamma \times \mathbb{R}) = \mathbb{R} \oplus \mathfrak{so}_4^\Gamma\,, \quad H_0^{2,+}(S^3/\Gamma \times \mathbb{R}) = 0$$

by the index formula in §4.2.1.

4.2.4 The case $Y = T^3$. The metric on $Y \times \mathbb{R}$ is in this case flat and it follows from $\mathcal{D}u = 0$ that $\Delta^2 u = 0$. If u is also bounded, then u must be parallel. In particular, every solution is t-invariant and $H_c^{1,-}(T^3 \times \mathbb{R}) = \mathbb{R}^9$. (This also agrees with the geometric interpretation of §4.2.2.) Similarly we

have

$$H_c^{0,+}(T^3 \times \mathbb{R}) = \mathbb{R}^4\,, \quad H_c^{2,+}(T^3 \times \mathbb{R}) = \mathbb{R}^5\,.$$

4.2.5 The case Y is hyperbolic. In this case, §4.2.2 would predict that $H_0^{1,-}(Y \times \mathbb{R}) = \mathbb{R}$ because of the rigidity of hyperbolic structures. In fact, by following Floer's calculations through one can verify that if Y is a hyperbolic rational homology 3-sphere, then the t-invariant part of $H_0^{1,-}(Y \times \mathbb{R})$ is indeed 1-dimensional. We have not been able to eliminate the possibility of periodic solutions in this case, but make the following

Conjecture 4.11. *If Y is a hyperbolic rational homology 3-sphere, then $H_0^{1,-}(Y \times \mathbb{R}) = \mathbb{R}$.*

In any case, we have $H^{0,+}(Y \times \mathbb{R}) = \mathbb{R}$. If the conjecture is true, then we should also have $H_0^{2,+}(Y \times \mathbb{R}) = 0$.

This concludes our digression on the properties of $\mathcal{D}_0$.

4.3 Comparison theory. We now turn to b-manifolds that arise by conformal blow-up or blow-down (cf. §2.1.1, §2.1.3). The main results of this section are the following 'comparison theorems':

Theorem 4.12. *Let (X, g) be the conformal blow-up at the point 0 of the compact conformally ASD orbifold $(\overline{X}, \overline{g})$. Then the framed cohomology groups $H_{\overline{c}}^*(\overline{X}, 0)$ of the framed deformation complex*

$$C_{(0)}^\infty(\overline{X}, \Omega^{-1/4}E^0) \to C^\infty(\overline{X}, E^1) \to C^\infty(\overline{X}, \Omega^{1/2}E^2)\,,$$

where

$$C_{(0)}^\infty(\overline{X}, V) = \left\{v \in C^\infty(\overline{X}, V) : v(0) = 0\right\}\,,$$

agree with the kernel and cokernel of $\mathcal{D}_g^+$:

$$H_{\overline{c}}^j(\overline{X}, 0) = H_g^{j,+}(X), \text{ for } j = 0, 1, 2\,.$$

Moreover,

$$H_c^{2,+}(X) = H_{\overline{c}}^2(\overline{X}) = \ker\left(\overline{D}^* : C^\infty(\overline{X}, \Omega^{1/2}E^2) \to C^\infty(\overline{X}, \Omega E^1)\right)\,.$$

This result was stated and partly proved by Floer [F, §3]. The following is the analogue for comparison with a WALE space:

Theorem 4.13. *Let (X, g) be the conformal blow-down at ∞ of the conformally ASD, WALE space $(\widehat{X}, \widehat{g})$. Assume that $\widehat{g}$ is such that g is a smooth b metric. Then the cohomology groups $H_{\widehat{c}}^*(\widehat{X})$ of the complex*

$$C^\infty(\widehat{X}, \Omega^{-\frac{1}{4}}E^0) \cap O(1) \to C^\infty(\widehat{X}, E^1) \cap O(R^{-1}) \to C^\infty(\widehat{X}, \Omega^{\frac{1}{2}}E^2) \cap O(R^{-3})\,,$$

where

$$C^\infty(\widehat{X}, V) \cap O(R^{-a}) = \left\{v \in C^\infty(\widehat{X}, V) : |\nabla^j v| = O(R^{-a-j}), \text{ as } R \to \infty, \text{ for all } j\right\}$$

agree with the kernel and cokernel of $\mathcal{D}_g^-$*:*

$$H^j_{\widehat{c}}(\widehat{X}) = H^{j,-}_g(X), \text{ for } j = 0, 1, 2\,.$$

Moreover,

$$H^{2,-}_c(X) = H^2_{\widehat{c}}(\widehat{X}) = \ker\big(\widehat{D}^* : C^\infty(\widehat{X}, \Omega^{1/2}E^2) \cap O(R^{-4}) \to C^\infty(\widehat{X}, \Omega E^1)\big).$$

Note that the decay conditions used to define the $H^j_{\widehat{c}}$ are the natural ones on a WALE space. In particular, a differential operator of order m canonically associated with $\widehat{g}$ will automatically map $O(R^{-a})$-sections to $O(R^{-a-m})$-sections.

Proof. Recall first the relation between the cylindrical metric, and the Euclidean metrics g_0 and g_∞ (near 0 and ∞, respectively):

$$dt^2 + d\omega^2 = r^{-2}(dr^2 + r^2 d\omega^2) = R^{-2}(dR^2 + R^2 d\omega^2) \tag{4.14}$$

where

$$t = e^{-r} = e^{-x}\,, \quad t = e^R\,, \quad R = r^{-1}\,. \tag{4.15}$$

Here we are assuming that 0 is a smooth point, or if not, we pass to a uniformizing chart centred at 0. It will be clear that Γ-equivariance is preserved throughout, so we can afford to ignore singularities from now on. Denoting the pointwise norms that correspond to the different choices of conformal gauge in (4.14) by $|\cdot|_b$, $|\cdot|_0$ and $|\cdot|_\infty$ respectively, we have by (3.5) and (3.6),

(i) If $\xi \in \Omega^{-1/4}E^0$, then $|\xi|_b = r^{-1}|\xi|_0 = R^{-1}|\xi|_\infty$;
(ii) If $h \in E^1$, then $|h|_b = |h|_0 = |h|_\infty$;
(iii) If $\psi \in \Omega^{1/2}E^2$, then $|\psi|_b = r^2|\psi|_0 = R^2|\psi|_\infty$.

We compare first H^0 and H^2. By the weighted Fredholm alternative, we have

$$H^{0,\pm}_c(X) = \ker\big(x^{\mp\delta}L^p_k(X, \Omega^{1/4}E^0) \xrightarrow{L} x^{\mp\delta}L^p_{k-1}(X, E^1)\big)$$

and

$$H^{2,\pm}_c(X) = \ker\big(x^{\mp\delta}L^p_k(X, \Omega^{1/2}E^2) \xrightarrow{D^*} x^{\mp\delta}L^p_{k-1}(X, \Omega E^1)\big)\,,$$

the formal adjoint being taken with respect to the b-metric g. Therefore by conformal invariance of these differential operators and the rescaling formulae (i) and (iii) above,

$$H^{0,\pm} = \big\{\overline{\xi} \in C^\infty(\overline{X}\backslash\{0\}, \Omega^{-1/4}E^0) : \overline{L\xi} = 0,\ |\xi|_0 = O(r^{1\mp\delta}) \text{ as } r \to 0\big\}\,,$$

and

$$H^{2,\pm} = \big\{\overline{\psi} \in C^\infty(\overline{X}\backslash\{0\}, \Omega^{1/2}E^2) : \overline{D}^*\overline{\psi} = 0,\ |\psi|_0 = O(r^{-2\mp\delta}) \text{ as } r \to 0\big\}\,.$$

We argue next that the singularity at 0 is removable in all cases except $H^{2,+}$. First note that in all cases, $\overline{\xi}$ and $\overline{\psi}$ admit extensions as distributional

sections to the whole of $\overline{X}$ [Hö1, §3.2]. Denoting these extensions by the same symbols, we obtain equations of the form $\overline{L\xi} = \alpha$, $\overline{D}^*\overline{\psi} = \beta$, where α and β are linear combinations of derivatives of the Dirac distribution δ_0. Thus α and β are sums of terms homogeneous of degree $-4-m$ for $m = 0, 1, \ldots$, and this is not compatible with the orders of $\overline{L}$ and $\overline{D}^*$ and the growth conditions on $\overline{\xi}$ and $\overline{\psi}$ that appear in $H^{0,\pm}$ and $H^{2,-}$. So in these three cases, $\alpha = 0$ and $\beta = 0$, and since $\overline{D}^*$ and $\overline{L}$ are overdetermined elliptic the solutions are actually smooth. Hence we obtain

$$H^{0,+} = \{\overline{\xi} \in C^\infty(\overline{X}, \Omega^{-1/4}E^0) : \overline{L\xi} = 0,\ \overline{\xi}(0) = 0\} \tag{4.16}$$

$$H^{0,-} = \{\overline{\xi} \in C^\infty(\overline{X}, \Omega^{-1/4}E^0) : \overline{L\xi} = 0,\ \overline{\xi}(0) = 0, \overline{\nabla\xi}(0) = 0\} \tag{4.17}$$

$$= \{\widehat{\xi} \in C^\infty(\widehat{X}, \Omega^{-1/4}E^0) : \widehat{L}\widehat{\xi} = 0\} \cap O(1)\,, \tag{4.18}$$

the last following from the formula (i) relating the lengths of an element of $\Omega^{1/4}E^0$ in the compact and WALE models. We also have

$$H^{2,-} = \{\overline{\psi} \in C^\infty(\overline{X}, \Omega^{1/2}E^2) : \overline{D}^*\overline{\psi} = 0\} \tag{4.19}$$

$$= \{\widehat{\psi} \in C^\infty(\widehat{X}, \Omega^{1/2}E^2) : \widehat{D}^*\widehat{\psi} = 0\} \cap O(R^{-4})\,. \tag{4.20}$$

In order to overcome the problem encountered with $H^{2,+}$ we note a result of Biquard [Bi] which provides exact comparisons of b-Sobolev spaces of X with ordinary Sobolev spaces on $\overline{X}$ for a good choice of weight δ and exponent p. The version we need states that the obvious map on compactly supported functions extends to an isomorphism

$$\lambda : x^\delta L^p_2(X, E^1) \simeq L^p_2(X, 0; E^1) := \{h \in L^p_2(\overline{X}, E^1) : h(0) = 0\}$$

provided that

$$0 < \delta = 2 - 4/p < 1\,. \tag{4.21}$$

Note that the vanishing condition makes sense because $L^p_2 \subset C^0$ for $2 - 4/p > 0$ in 4 dimensions and that Biquard's result for functions applies here because E^1 is weightless (and so behaves conformally like the trivial bundle). By direct calculation (in which the factor $\Omega^{1/2}$ is crucial) we see also that there is an isomorphism

$$\mu : x^\delta L^p(X, \Omega^{1/2}E^2) \simeq L^p(\overline{X}; \Omega^{1/2}E^2)$$

if δ and p are related as in (4.21). Using conformal invariance of D we have $D^+ = \mu^{-1}\overline{D}\lambda$ and in particular

$$H^{2,+} = \operatorname{coker}\big(\overline{D} : L^p_2(\overline{X}, 0; E^1) \to L^p(\overline{X}, \Omega^{1/2}E^2)\big)$$

It follows that $H^2_{\overline{c}}(\overline{X}) = H^{2,+}_g$ if

$$\overline{D}\big(L^p_2(\overline{X}, E^1)\big) = \overline{D}\{\overline{h} \in L^p_2(\overline{X}, E^1) : \overline{h}(0) = 0\}\,.$$

To prove this, it is enough to show that given $h \in L^p_2(\overline{X}, E^1)$, there exists $\xi \in L^p_3(\overline{X}, E^0)$ such that $h(0) - \overline{L}\xi(0) = 0$. Working in normal coordinates x^a near 0 ($a = 1, \ldots, 4$), we have the formula

$$(\overline{L}\xi)_{ab} = \partial_a \xi_b + \partial_b \xi_a - [\partial_c \xi^c] g_{ab}/2 \text{ at } 0$$

just as in $\mathbb{R}^4$ with the Euclidean metric. Then $\overline{L}(\beta\, h_{ab} x^a)(0) = 2h_{ab}$ if h_{ab} is constant and $\beta = 1$ in a neighbourhood of 0. This completes the proofs of all statements pertaining to H^0 and H^2 in Theorems 4.12 and 4.13.

4.3.1 Comparison of H^1. Fix p and δ as in (4.21). Then by Biquard's result and conformal invariance, there is a natural map $H^{1,+} \to H^1_{\overline{c}}(\overline{X}, 0)$ given by mapping $h \in H^{1,+}_c$ to the cohomology class $[\overline{h}]$ of the image of h in $L^p_2(\overline{X}, 0; E^1)$. We claim first that this map is injective. Indeed, if $\overline{h} = \overline{L\xi}$ for some $\overline{\xi}$ with $\overline{\xi}(0) = 0$, then transferring back to X we obtain ξ, such that $|\xi|_b = O(1)$ as $t \to \infty$ and satisfying $L\xi = h$. But $h \in H^{1,+}_c$ implies that $L^* L\xi = 0$ and since h is in L^2, we may integrate by parts, getting $h = L\xi = 0$. This establishes the injectivity.

To prove surjectivity, we need to show that if $\overline{h}$ represents a class in $H^1_{\overline{c}}$, then we can find $\overline{\xi} \in C^\infty(E^0)$, such that $\overline{\xi}(0) = 0$ and

$$L^*(L\xi + h) = 0\,.$$

By a previous argument we may assume $\overline{h}(0) = 0$ so that $L^* h \in x^\delta L^p_2(X, E^1)$. It is straightforward to show

$$L^* L\big(x^\delta L^p_k(X, E^0)\big) = L^*(x^\delta L^p_{k+1})$$

by adapting standard Hodge-theory arguments. In particular, we can solve the equation and transfer to $\overline{X}$, getting a section $\overline{\xi}$ which vanishes at 0. This completes the comparison of H^1 for conformal blow-ups, and so the proof of Theorem 4.12. The comparison of H^1 in Theorem 4.13 follows very similar lines, using an analogue of Biquard's theorem to compare weighted Sobolev spaces on X with Sobolev spaces on $\widehat{X}$. The details are omitted. □

4.4 Linear theory for the Hermitian-ASD problem. In §3.3, we saw that the operator

$$S : C^\infty(X, \Omega^{1/4} E^1_J) \to C^\infty(X, \Omega^{3/4} E^2_J)\,,$$

where E^1_J and E^2_J are the weightless versions of Λ^- and Λ^+ respectively, controls the deformation theory of conformally scalar-flat Kähler metrics. In contrast to $\mathcal{D}$, it turns out that S is fully elliptic on a b-manifold X if each component of ∂X is a spherical space-form. This makes for substantial simplifications, particularly for the comparison theorems for this problem.

PROPOSITION 4.22. *Let $X_0 = S^3 \times \mathbb{R}$, $g_0 = h(y) + dt^2$, where h is the round metric on S^3, S_0 the S-operator associated to g_0. Then if for some $u(y)$, we have $S_0(u(y)e^{i\lambda t}) = 0$, it follows that $i\lambda \in \mathbb{Z}\backslash 0$.*

Proof. Use the conformal isometry $S^3 \times \mathbb{R} \to \mathbb{R}^4\backslash 0$ given by $r = e^t$ and the conformal invariance of S. Because of the conformal weights, we have if u is a section of $\Omega^{1/4}E^1_J$

$$|u|_{\text{cylinder}} = r|u|_{\text{Euclidean}}$$

so that a solution $\overline{S}\overline{u} = 0$ in $\mathbb{R}^4\backslash 0$, homogeneous of degree λ, translates into an exponential solution with factor $e^{(\lambda+1)t}$ on the cylinder. In particular the constant solution in $\mathbb{R}^4$ gives rise to a solution that goes like e^t along the cylinder.

We use a 'removable singularities' argument like the one in the proof of Theorem 4.12. If $Su = 0$ in $\mathbb{R}^4\backslash 0$ and u has homogeneity λ in r, then Su is a distribution supported at $\{0\}$ and homogeneous of degree $\lambda - 2$. If this distribution vanishes at 0, then by elliptic regularity, u is smooth near 0 and hence λ is a non-negative integer. If the distribution is non-vanishing, then Su must be a multiple of some derivative of δ_0, hence $\lambda - 2 = -4 - m$ for some integer $m \geq 0$. Hence $\lambda = -2 - m$ and so $\lambda = -1$ cannot occur. □

REMARK 4.4.1. On $T^3 \times \mathbb{R}$, however, the existence of non-trivial parallel sections obstructs the full ellipticity of S on a b-manifold some of whose boundary components are tori.

4.4.2 Comparison theory for S. Consider now the situation of §4.3. We have:

Theorem 4.23. *Let (X, g) be a conformally ASD manifold with smooth b metric, and suppose that (X, g) is the conformal blow-up at 0 in $(\overline{X}, \overline{g})$ and the conformal blow-down of $\infty \in (\widehat{X}, \widehat{g})$. Let $H^1_{c,J}(X)$, $H^2_{c,J}(X)$ be respectively the kernel and cokernel of*

$$S_g : L^p_k(X, \Omega^{1/4}E^1_J) \to L^p_{k-2}(X, \Omega^{1/4}E^2_J)\,,$$

let $H^1_{\overline{c},J}(X)$, $H^2_{\overline{c},J}(X)$ be respectively the kernel and cokernel of $\overline{S} = S_{\overline{g}}$, and let

$$H^1_{\widehat{c},J}(\widehat{X}) = \ker(\widehat{S}) \cap O(R^{-2})\,, \quad H^2_{\widehat{c},J}(\widehat{X}) = \ker(\widehat{S}^*) \cap O(R^{-2})\,,$$

(where the adjoint is taken relative to the WALE metric $\widehat{g}$). Then

$$H^1_{c,J} = H^1_{\overline{c},J} = H^1_{\widehat{c},J} \quad \text{and} \quad H^2_{c,J} = H^2_{\overline{c},J} = H^2_{\widehat{c},J}\,.$$

Proof. If $Su = 0$, $u \in L^p_k(X, E^1_J)$, then from Proposition 4.22, u has a phg expansion where the index set is just the positive integers. Translating to

$\overline{X}$ and remembering the conformal weight, we get $\overline{u}$ which satisfies $\overline{S}\overline{u} = 0$ and $|\overline{u}|_{\overline{g}} = O(1)$ near 0. Hence by elliptic regularity, the L^p-null space of S on X agrees with the standard null-space of $\overline{S}$ on $\overline{X}$. The argument for H^2 is the same, for $H^2_{c,J}$ can be identified with the L^p null-space of S^* and this is conformally invariant as an operator between bundles with the same conformal weights as for S (cf. §3.1.6).

The argument for comparison with $\widehat{X}$ is also closely analogous. □

5 Nonlinear Theory

We come now to the problem of finding an exactly conformally ASD b-metric on X_ρ as a perturbation of the metric g_ρ constructed in §2.3.5. More precisely, assume that (X_j, g_j) in §2.3 are conformally ASD b-manifolds so that g_ρ is conformally ASD except in the damage zone $\{-1/2 \leq |t-1| \leq 1/2\}$ near the middle of the neck. Using the conformal class c_ρ of g_ρ as the reference point c in Proposition 3.14 we need to find a small ($\sup_{X_\rho} |h| < 1$), smooth h satisfying (3.15),

$$0 = F_\rho(h) = W^+_\rho + D_\rho h + \varepsilon_1(1+h, h \otimes \nabla\nabla h) + \varepsilon_2(1+h, \nabla h \otimes \nabla h) + \varepsilon_3(1+h, h \otimes h) \quad (5.1)$$

where $W^+_\rho := W^+(c_\rho)$, $D_\rho := D_{c_\rho}$.

The strategy is to solve this equation in a suitable Banach space by use of a version of the implicit function theorem (IFT). Then the solution will be proved to be C^∞ and polyhomogeneous near ∂X_ρ. For both parts of the argument it is useful to supplement (5.1) with the gauge-fixing condition

$$L^*_\rho h = 0 \quad (5.2)$$

(the $*$ denoting L^2 adjoint with respect to g_ρ). Then the linearization of the map

$$h \mapsto (F_\rho(h), L^*_\rho h) \quad (5.3)$$

is equal to the elliptic operator $\mathcal{D}_\rho$.

5.1 Weak solution. In the next few paragraphs we explain how to arrange matters so that this strategy can be successfully pursued. We shall need to choose Banach spaces so that F_ρ extends to a C^∞ map with uniform behaviour as $\rho \to \infty$. This involves estimating certain Sobolev norms of the nonlinear terms in F_ρ. But we begin by adjusting F_ρ so that the neck-stretching analysis of §4.1 can be applied to its linearization.

5.1.1 Introduction of weights. The main estimate, Proposition 4.2 does not apply directly to $\mathcal{D}_\rho$ because, as we saw at the start of §4.2, this operator does not arise by gluing a pair of fully elliptic operators. Therefore we replace (5.3) by

$$u \mapsto \left(F_\rho^w(u), w^{-1}L_\rho^*(wu)\right) \text{ where } F_\rho^w(u) := w^{-1}F_\rho(wu)\,, \tag{5.4}$$

where $w := w_\rho$ is a suitable weight-function on X_ρ.

5.1.2 Definition of w_ρ. For $j = 1, 2$, let w_j be equal to a generically chosen positive power of a boundary defining function for $\partial X_j \backslash Y$. We assume $0 \le w_j \le 1$ on X_j, with $x_j = 0$ only at $\partial X_j \backslash Y$ and $w_j = 1$ near Y. Extend the neck parameter t smoothly to X_ρ (and denote the extension also by t) so that the range of t is $[-\rho - 1, \rho + 1]$ and $t = -\rho - 1$ near $\partial X_1 \backslash Y$ and $t = \rho + 1$ near $\partial X_2 \backslash Y$. Now for δ satisfying the conditions of (4.10), we put

$$w_\rho = w_1 w_2 e^{-\delta(t+\rho+1)}.$$

If ρ is fixed, we have $0 \le w_\rho \le 1$, with $w_\rho = 0$ only at ∂X_ρ; w_ρ is a power of a boundary defining function near ∂X_ρ and decreases exponentially along the neck.

The linearization of (5.4) is $P_\rho := w_\rho^{-1}\mathcal{D}_\rho w_\rho$ which is obtained by gluing the fully elliptic operators

$$P_1 = (w_1 x_1^\delta)^{-1}\mathcal{D}_1(w_1 x_1^\delta)$$

and

$$P_2 = (w_2 x_2^{-\delta})^{-1}\mathcal{D}_1(w_2 x_2^{-\delta})$$

as in §4.1.1. According to the main estimate we can now invert P_ρ in a controlled way in Sobolev spaces over X_ρ. The next task is to choose Sobolev spaces such that (5.4) extends to a smooth map between them.

5.1.3 Choice of Sobolev space. Because D_ρ is a second-order operator, it is natural to take $u \in L_2^p(X_\rho, E^1)$ for some p. If $p > 2$ we have the estimate

$$\sup_{X_\rho} |u| \le C\|u\|_{L_2^p(X_\rho)} \tag{5.5}$$

and we can assume C is independent of ρ. This uniformity of C (and also (5.6)) follows from standard Sobolev embedding theorems for the complete Riemannian manifolds X_0, X_1 and X_2, by a partition of unity argument [Au, §2.23]. We chose $w < 1$ so that (5.5) implies that if the L_2^p-norm of u is sufficiently small, then the pointwise norm of $h = wu$ is everywhere < 1 and $c_\rho(1 + h)$ defines a genuine (C^0) conformal structure.

So L_2^p, for any $p > 2$, will take care of the linear term in (5.1). The nonlinearities in F^w are much easier to control, however, if we take $p > 4$ so that (5.5) can be strengthened to

$$\sup_{X_\rho} \big(|u| + |\nabla u|\big) \le C\|u\|_{L_2^p(X_\rho)} \tag{5.6}$$

for some other constant C that is independent of ρ. Therefore we fix $p > 4$ and show next that $u \mapsto F^w(u)$ extends to a smooth map from a neighbourhood of 0 in $L_2^p(X_\rho, E^1)$ to $L^p(X_\rho, E^2)$.

5.1.4 Estimation of the nonlinearities. From (5.1) and the properties of the ε_j (§3.1.2) we obtain

$$\big(F_\rho^w(u), w^{-1}L^*(wu)\big) = w^{-1}W_\rho^+ + P_\rho u + Q(u) \tag{5.7}$$

where

$$\begin{aligned} Q(u) := {} & w\varepsilon_1(1 + wu, u \otimes \nabla^w\nabla^w u) \\ & + w\varepsilon_2(1 + wu, \nabla^w u \otimes \nabla^w u) + w\varepsilon_3(1 + wu, u \otimes u) \end{aligned}$$

and $\nabla^w u = w^{-1}\nabla(wu)$. (In the interests of legibility we have dropped the notational dependence of the weight w upon ρ. There seems little danger of confusion.) We can now state the main result:

Proposition 5.8. *For any fixed $p > 4$, the map*

$$f_\rho : u \mapsto \big(F_\rho^w(u), w^{-1}L_\rho^*(wu)\big)$$

extends to a smooth map from the ball B_ρ of radius r in $U_\rho = L_2^p(X_\rho, E^1)$ to V_ρ where

$$V_\rho = L^p(X_\rho, E^2) \oplus L_1^p(X_\rho, E^0)\,.$$

Moreover for a suitable choice of the parameters in the definition of w, f_ρ has the following properties:

(i) $\|f_\rho\| \to 0$ *as* $\rho \to \infty$*;*

(ii) There exist decompositions $U_\rho = U_\rho' \oplus U_\rho''$, $V_\rho = V_\rho' \oplus V_\rho''$, where U_ρ' and V_ρ' are finite-dimensional, U_ρ'' and V_ρ'' are closed and the map $P_\rho'' : U_\rho'' \to V_\rho''$ induced by P_ρ has a uniformly bounded inverse $G_\rho : V_\rho'' \to U_\rho''$.

(iii) The nonlinearity $Q(u)$ satisfies

$$\big\|Q(u) - Q(v)\big\| \le C\big(\|u\| + \|v\|\big)\|u - v\| \tag{5.9}$$

for every u, v in B_ρ, where C is bounded independent of ρ.

Proof. (i) Note first by the discussion in §2.3.6 that the L^p-norm of W_ρ^+ is $O(e^{-\eta\rho})$ for some $\eta > 0$. Taking $0 < \delta < \eta$, $\|f_\rho(0)\| = O(e^{-(\eta-\delta)\rho}) \to 0$ as $\rho \to \infty$. To check that the map is smooth, it evidently suffices to show

that the nonlinear terms define a smooth map. Note first by the properties of w that $\|\nabla^w u\|_{L^p_k} \leq C\|u\|_{L^p_{k+1}}$ where C is independent of ρ. Hence from (5.6) we have

$$\|u\nabla^w\nabla^w u\|_{L^p} \leq (\sup|u|)\|\nabla^w\nabla^w u\|_{L^p} \leq C\|u\|^2_{L^p_2}\,.$$

Similarly

$$\begin{aligned}\|\nabla^w u \otimes \nabla^w u\|_{L^p} &\leq C\sup(|u| + |\nabla u|)\|u\|_{L^p_1}\\ &\leq C\sup(|u| + |\nabla u|)\|u\|_{L^p_1} \leq C\|u\|^2_{L^p_2}\end{aligned}$$

and

$$\|u \otimes u\|_{L^p} \leq C\|u\|^2_{L^p_2}.$$

(Here C is a generic constant bounded independent of ρ but possibly varying from line to line.) This is not quite enough because the ε_j also have a real-analytic dependence on the 0-jet of u. However, this is convergent for all u with $\sup|u| < 1$ and by multiplication properties of elements of L^p_2 with $p > 4$, these extend to define smooth maps from a fixed ball $B \subset U_\rho$ into V_ρ. Combining these observations with the previous estimates for the terms in the derivatives of u, we obtain part (i).

(ii) This follows from the main estimate (Proposition 4.2) and the fact that P_ρ is obtained by gluing fully elliptic operators.

(iii) This is deduced by a simple modification of the arguments used to prove f_ρ smooth. The details are omitted. □

5.1.5 Implicit function theorem. Relative to the decompositions of Proposition 5.8, write $u = (u_1, u_2)$, $f(0) = (f_1(0), f_2(0))$, $P = (P_{ij})$ and $Q = (Q_1, Q_2)$. The equation to be solved becomes the pair

$$f_1(0) + P_{11}u_1 + P_{12}u_2 + Q_1(u_1, u_2) = 0\,, \tag{5.10}$$

$$f_2(0) + P_{21}u_1 + P_{22}u_2 + Q_2(u_1, u_2) = 0\,. \tag{5.11}$$

By the proposition, P_{22} is invertible, while from construction of the asymptotic kernels and cokernels in §4.1, the operator norms of the other P_{ij} tend to zero as $\rho \to \infty$. Thus for each fixed u_1, (5.11) can be reformulated as a fixed-point problem

$$u_2 = T_{u_1}(u_2) := -P_{22}^{-1}f_2(0) - P_{22}^{-1}P_{21}u_1 - P_{22}^{-1}Q_2(u_1, u_2)\,.$$

From Proposition 5.8 it is easy to show that if ρ is sufficiently large and $\|u_1\|$ is sufficiently small, then T_{u_1} is a contraction mapping on a sufficiently small neighbourhood of 0 in U''_ρ. To be more precise there exist $r_1 > 0$, $r_2 > 0$ independent of ρ assumed large, and a function

$$\varphi_\rho : \left\{u_1 \in U'_\rho : \|u_1\| \leq r_1\right\} \to \left\{u_2 \in U''_\rho : \|u_2\| \leq r_2\right\}$$

such that every solution (u_1, u_2) of (5.11) with $\|u_j\| \leq r_j$ is of the form $(u_1, \varphi_\rho(u_1))$.

Thus we have solved an 'infinite-dimensional component' of the conformal ASD equations (5.11); these are reduced to finding zeros of the nonlinear map between finite dimensional spaces got by substituting $u_2 = \varphi(u_1)$ into (5.10): put

$$\psi_\rho(u_1) = f_1(0) + P_{11}u_1 + P_{12}\varphi_\rho(u_1) + Q_1\big(u_1, \varphi(u_1)\big)$$

and let $\sigma_\rho(u_1)$ be the component of this in $V'_\rho \cap L^p(E^2)$. To summarize:

Proposition 5.12. *For sufficiently large ρ, there exists a nonlinear map σ_ρ from a ball in U'_ρ to V'_ρ whose zeros correspond to L^p_2 conformally ASD metrics near c_ρ. In particular if $V'_\rho = 0$ such conformally ASD metrics on X_ρ always exist.*

5.2 Considerations of regularity. Our final task is to establish that the weak (L^p_2) solution found in the previous section is actually smooth. In fact we prove both interior regularity and that the resulting metric has optimal boundary regularity – in other words that it is polyhomogeneous (relative to an index set that we do not specify). In this section ρ is large but fixed and we drop it from the notation.

Proposition 5.13. *Let $u \in L^p_2(X)$ satisfy*

$$F^w(u) = a,\ w^{-1}L^*(wu) = b \tag{5.14}$$

where a and b are C^∞, polyhomogeneous and vanish at the boundary. Then if $\sup_X |u|$ is sufficiently small, $u \in C^\infty(X_\rho)$ and is polyhomogeneous at ∂X_ρ.

Proof. Combine the equations in the form

$$P_u u = wQ(u, \nabla^w u) + \theta \tag{5.15}$$

where θ is a polyhomogeneous section which is a linear combination of W_0^+, a and b,

$$Q(u, \nabla^w u) = \varepsilon_2(1 + wu, \nabla^w u \otimes \nabla^w u) + \varepsilon_3(1 + wu, u \otimes u)$$

and

$$P_u v = Pv - wu\varepsilon_1(1 + wu, \nabla^w \nabla^w v)\,.$$

(i) *Interior regularity.* That u is C^∞ in X^o now follows from standard regularity results, which are applicable because u is already $C^{1,\alpha}$, where $\alpha = 1 - 4/p$ [ADN].

(ii) *Boundary regularity.* The method we use is closely analogous to that used by Mazzeo in [M]; we are indebted to him for useful discussions on this point.

Near the boundary, $w = x^\alpha$, where $\alpha > 0$ and $\theta = x^\alpha \theta'$, say, where θ' has a phg asymptotic expansion. Then (5.15) takes the form

$$P_u u = Pu - x^\alpha \varepsilon_1(1 + x^\alpha u, \nabla\nabla u) = x^\alpha Q(u, \nabla u) + x^\alpha \theta' \tag{5.16}$$

and we already know that $\sup(|u| + |\nabla u|)$ is uniformly bounded as $x \to 0$. Now if we had the indicial operator $I(P)$ in place of P_u on the LHS of (5.16), we could use the fact that $I(P)$ has an inverse which behaves well on the b-Sobolev spaces to conclude that $u \in x^\alpha L^p_k$ for every k and some fixed p. It would follow that $|(x\partial_x)^j \partial_y^\beta u|$ is continuous at ∂X for all j and all multi-indices β. Continuing with the argument under the simplifying assumption that $I(P)$ not P_u is on the LHS of (5.16), we can now use the fact that $I(P)$ and its inverse also preserve spaces of polyhomogeneous functions. To do so, assume by induction that u has a phg expansion up to some order N, say. Then because of the factor x^α on the RHS of (5.16), $I(P)u$ has an expansion to order $N+\alpha$, and so u also has such an expansion. Hence u has a complete phg expansion at the boundary.

The result with P_u replacing $I(P)$ comes from a suitable approximation argument. The details of this are straightforward but lengthy, and are omitted. □

6 Main Theorems

6.1 Gluing ASD *b*-manifolds. We now summarize our work so far by giving statements of the main theorems. For the reader's convenience we gather first the relevant notation.

6.1.1 Notation. For $j = 1, 2$, X_j is a 4-manifold with boundary and g_j is a conformally ASD polyhomogeneous b-metric. A piece (union of compact connected components) Y of ∂X_j is given, such that the g_j approach isometric cylindrical metrics near Y. Weights w_j are chosen as in §5.1.2 and a real number δ is chosen as in (4.10) and finite-dimensional 'cohomology spaces' are defined by the exactness of the sequence

$$0 \to H_j^{1,\pm} \to w_j x_j^{\pm\delta} L^p_k(X_j, E^1) \to$$
$$w_j x_j^{\pm\delta}\big[L^p_{k-2}(X_j, E^2) \oplus L^p_{k-1}(X_j, E^0)\big] \to H_j^{2,\pm} \oplus H_j^{0,\pm} \to 0\,.$$

For large real ρ, X_ρ is constructed by gluing the X_j across Y and the approximately conformally ASD b-metric g_ρ is constructed on X_ρ as in §2.3.

Theorem A. *Let the notation be as in §6.1.1. Then for all sufficiently large ρ, there exists a map σ_ρ from a neighbourhood of* 0 *in* $H_1^{1,+} \oplus H_2^{1,-}$

into $H_1^{2,+} \oplus H_2^{2,-}$ *such that* $\sigma_\rho^{-1}(0)$ *parameterizes the set of conformally ASD metrics* $\tilde{g}_\rho$ *sufficiently close to* g_ρ*. Here 'sufficiently close' means in particular that*

$$\sup_{X_\rho} |\tilde{g}_\rho - g_\rho| \to 0 \text{ as } \rho \to \infty$$

and that $\tilde{g}_\rho$ *has a polyhomogeneous expansion near* ∂X_ρ*, such that* $\tilde{g}_\rho$ *coincides with* g_ρ *at* ∂X_ρ*.*

Remark 6.1.2. The term 'parameterizes' is used in a loose sense here. The family of conformally ASD metrics given by $\sigma_\rho^{-1}(0)$ is *complete* in the sense that every diffeomorphism class of conformally ASD metrics sufficiently close to g_ρ appears in the family. However the gauge action of the diffeomorphism group has only been fixed up to a finite dimensional residual gauge freedom and correspondingly the true moduli-space will in general be got by dividing $\sigma_\rho^{-1}(0)$ by a suitable compact Lie group. Further details of this are omitted.

Proof. Combine Propositions 5.12 and 5.13. The estimate on $\tilde{g}_\rho - g_\rho$ follows from the fact that the L_2^p-norm of u_2 in §5.1.5 is of the same order of magnitude as $\|W_\rho^+\| = O(e^{-(\eta-\delta)\rho})$ and the estimate (5.5). The completeness of the family of metrics constructed comes directly from the implicit function theorem. □

When the boundary components are spherical space-forms, this theorem can be combined with our comparison results to give the following 'ASD desingularization theorem':

Theorem B. *Let* (M, g) *be a compact conformally ASD orbifold, and let* $0 \in M$ *be a point with a neighbourhood modelled on* $\mathbb{R}^4/a$*, where* a *is an action of the finite group* Γ *with isolated fixed-point set. If* (N, h) *is a conformally ASD, WALE space whose asymptotic region is also modelled by* $\mathbb{R}^4/a$*, then there exists a smooth map* σ *from a neighbourhood of* 0 *in* $H_c^1(M) \oplus H_c^1(N)$ *into* $H_c^2(M) \oplus H_c^2(N)$ *whose zeroes give conformally ASD metrics on the connected sum* $M \sharp N$ *obtained by joining the asymptotic region of* N *to a neighbourhood of* 0 *in* M*. Here the deformation cohomology groups are as in Theorems 4.12 and 4.13.*

Proof. Combine Theorem A with Theorems 4.12 and 4.13. □

Another variant covers the case of a connected sum of compact conformally ASD orbifolds.

Theorem C. *For* $j = 1, 2$*, let* (M_j, g_j) *be a compact conformally ASD orbifold, and let* $0_j \in M_j$ *be a point with a neighbourhood modelled on the*

origin in $\mathbb{R}^4/a_j$. *Suppose further that* a_1 *and* a_2 *define* complementary singularities *in the sense that there is an orientation-reversing linear isometry* ϕ *of* $\mathbb{R}^4$ *which intertwines* a_1 *and* a_2. *Then there is a smooth map* σ *from a neighbourhood of* 0 *in* $H^1_c(M_1)\oplus\mathbb{R}\oplus(\mathfrak{so}_4)^a\oplus H^1_c(M_2)$ *into* $H^2_c(M_1)\oplus H^2_c(M_2)$ *whose zeroes give conformally ASD metrics on the connected sum* $M_1\sharp M_2$ *obtained by joining at* 0_1 *and* 0_2.

Proof. Take X_1 to be the conformal blow up at $\{0_1, 0_2\}$ of $M_1 \sqcup M_2$. Take X_2 to be the b-manifold obtained by gluing $(-\infty, 0] \times (S^3/a_1)$ to $(-\infty, 0] \times (S^3/a_2)$ by identifying $0 \times S^3/a_1$ with $0 \times S^3/a_2$ by ϕ. Then X_2, viewed as a b-manifold, has boundary equal to $-Y \sqcup Y$, while the boundary of X_1 is $Y \sqcup -Y$. Applying Theorem A to X_1 and X_2 now gives the conclusion, in view of the calculations of the deformation cohomology groups for a cylinder given in §4.2.3 and Theorem 4.12. (We have denoted by a the action induced by a_1 or a_2 on $S^3 \times \mathbb{R}$.) □

Remark 6.1.3. The domain of σ_ρ in this case has a natural interpretation, the three summands corresponding to deformations of the conformally ASD structures of the M_j together with deformations of the 'gluing map' ϕ.

Remark 6.1.4. Clearly Theorems A, B and C give existence theorems for conformally ASD metrics if the indicated obstruction spaces vanish. On the other hand, it is sometimes possible to calculate the leading term in the map σ in terms of the given data, as in [DF], and so obtain existence results even in the presence of obstructions.

6.2 Gluing Hermitian-ASD *b*-manifolds. The results are very similar to those for ASD conformal structures, and will follow in the same way (from the methods of §5) once it has been explained how to glue Hermitian-ASD manifolds. This will be done in the next few paragraphs. For simplicity we deal only with the case that the metrics being glued are scalar-flat Kähler.

6.2.1 Notation and assumptions. For $j = 1, 2$, let (X_j, J_j) be complex b-manifolds of (real) dimension 4, and let $Y \subset \partial X_j$ be a piece of the boundary. We make the assumption that the cylindrical neighbourhoods U_j of Y in X_j (cf. §2.3.1) are *biholomorphic*. This will be the case, for example, if X_1 is the conformal blow-up of a point in a compact surface and X_2 is obtained by conformal blow-down of ∞ in an asymptotically Euclidean space.

Now introduce b-metrics g_j on X_j and assume as in §2.3 that in U_j, g_j approaches a standard cylindrical metric g_0. Assume that the isometry

also preserves the complex structures, so that g_0, g_1, g_2 are all Hermitian with respect to the given complex structure.

Then we can construct (X_ρ, g_ρ, J_ρ) by gluing just as before. Note that J_ρ is a genuine integrable complex structure on X_ρ and that g_ρ is J_ρ-Hermitian.

The Riemannian product metric on the cylinder is not necessarily Kähler, but if we assume that (X_j, c_j, ω_j) is conformally Kähler as in §3.3.3, so that ω_j is a solution of the twistor equation (3.16), then when we glue, we get g_ρ as before and ω_ρ which still defines an integrable complex structure when rescaled to unit length. However ω_ρ is only an approximate solution of the twistor equation defined by the conformal class of g_ρ. We can now introduce weights and repeat the work of §5 for the nonlinear map F_J of Proposition 3.19. This yields the Hermitian analogue of Theorem A:

Theorem D. *For $j = 1, 2$, let (X_j, g_j, ω_j) be conformally Hermitian-ASD b manifolds, where ω_j satisfies the twistor equation defined by g_j. Choose weights w_j as in §5.1.2 and define finite-dimensional vector spaces $H^{r,\pm}_{j,J}$ by the exactness of the sequence*

$$0 \to H^{1,\pm}_{j,J} \to w_j x_j^{\pm\delta} L^p_k(X_j, \Omega^{1/4} E^1_J) \\ \to w_j x_j^{\pm\delta} L^p_{k-2}(X_j, \Omega^{3/4} E^2_J) \to H^{2,\pm}_{j,J} \to 0\,.$$

Suppose further that there is a piece Y of ∂X_j such that cylindrical neighbourhoods U_j of Y in X_j are biholomorphic. Then for all sufficiently large ρ, there is a map σ_ρ from a neighbourhood of 0 in $H^{1,+}_{1,J} \oplus H^{1,-}_{2,J}$ into $H^{2,+}_{1,J} \oplus H^{2,-}_{2,J}$ whose zeroes parameterize the conformally Hermitian-ASD metrics near g_ρ. Moreover these metrics have the same boundary behaviour as the metrics constructed in Theorem A.

Restricting to orbifolds and using the comparison theorem (4.23), we obtain a result about desingularization (or just blow-up) of compact Hermitian-ASD orbifolds.

Theorem E. *Let (M, g) be a compact scalar-flat Kähler metric and let $0 \in M$ be a point with a neighbourhood biholomorphic to a neighbourhood of 0 in $\mathbb{C}^2/a$, where a is an action of the finite group Γ with isolated fixed-point set. If (N, h) is a scalar-flat Kähler, WALE space whose asymptotic region is biholomorphic to a neighbourhood of ∞ in $\mathbb{C}^2/a$, then there exists a smooth map σ from a neighbourhood of 0 in $H^1_{c,J}(M) \oplus H^1_{c,J}(N)$ into $H^2_{c,J}(M) \oplus H^2_{c,J}(N)$ whose zeroes give scalar-flat Kähler metrics on the connected sum $M \sharp N$ obtained by joining the asymptotic region of N to a neighbourhood of 0 in M. Here the deformation cohomology groups are as in Theorem 4.23.*

Proof. It is clear that Theorem D combined with Theorem 4.23 gives a Hermitian-ASD conformal structure c on $M\sharp N$. However it is obvious that $M\sharp N$ is Kälerian, so by [Bo], there is a Kähler representative of the conformal class c and this must necessarily be scalar-flat. □

6.3 Weakly asymptotically Euclidean scalar-flat Kähler metrics on the blow-up of $\mathbb{C}^2$. Finally we give a simple application of Theorem D which is not quite included in Theorem E.

Theorem F. *Let $p_1, \ldots, p_n$ be a collection of n distinct points in $\mathbb{C}^2$. Then the (complex) blow-up M of $\mathbb{C}^2$ at the p_j admits weakly asymptotically Euclidean scalar-flat Kähler metrics.*

Proof. Recall first the Burns metric (B, g_B) [L2], a weakly asymptotically Euclidean scalar-flat Kähler metric on the blow-up B of $\mathbb{C}^2$ at the origin. The complement of the exceptional divisor in B is biholomorphic to $\mathbb{C}^2\backslash 0$, so we can construct the multiple blow-up M, with its standard complex structure, by gluing a copy of B at each of the p_j. More precisely, let $(\check{B}, \check{g}_B)$ denote the conformal blow-down of B. Then $H^2_{c,J}(\check{B}) = 0$ by Theorem 8.4 and Theorem 4.23. So let X_1 stand for the b-manifold obtained from $\mathbb{C}^2$ by conformal blow-up of each of the p_j and conformal blow-down of ∞. By Theorem 4.23,

$$H^2_{c,J}(X_1) = H^2_{c,J}(\mathbb{C}^2) \cap O(R^{-2})$$

and this is zero, because if $Su = 0$ in Euclidean space then every component of u satisfies $\Delta^2 u = 0$ and so $u = 0$ by the growth condition at ∞. We take X_2 to be n copies of $\check{B}$ and apply Theorem D, taking all weights equal to 1 since all boundary components are spherical (Proposition 4.22). The conclusion is that M admits a Hermitian-ASD conformal structure with good asymptotic behaviour at the boundary. It remains to verify that there is a weakly asymptotically Euclidean Kähler representative of this conformal class.

The fundamental 2-form ω of the metric constructed by Theorem D has the form $dt \wedge e_1 + e_2 \wedge e_3 + O(e^{-t})$, where the e_j form a standard basis of left-invariant 1-forms on S^3. Following the method of [Bo], introduce a 1-form β which measures the failure of ω to be Kähler:

$$d\omega + \beta \wedge \omega = 0\,.$$

From the asymptotic formula for ω, $\beta = 2dt + \beta'$ where β' is a 1-form whose length decays exponentially as $t \to \infty$. It is a local calculation that on a Hermitian-ASD manifold $d\beta$ is ASD hence β is harmonic. Because $d\beta = d\beta'$ and β' is in L^2, we conclude by the standard integration-by-parts argument

that $d\beta = 0$. Hence β' is closed and decaying and so by [Me, Ch. 6] represents an element of the relative cohomology group $H^1(M, \partial M) = 0$. By other results in [Me, Ch. 6], $\beta' = du$ where du is also exponentially decaying. It follows that $d(e^{u+2t}\omega) = 0$ and this is the conformal factor that yields the Kähler representative. □

7 Existence of ASD Conformal Structures for Manifolds with Boundary.

Let (M, g_M) be an oriented Riemannian 4-manifold without boundary. Taubes [T1] has shown that there exists $N > 0$ such that $M_N := M \sharp N \overline{\mathbb{C}P^2}$ admits a conformally ASD metric. In this section we shall sketch the adaptations of his argument that are needed to prove the analogous result in the b-category:

Theorem G. *Let X be a compact oriented 4-manifold with boundary, g_0 an exact b-metric that is conformally flat near each component of ∂X. Then there exists $N > 0$ such that $X \sharp N \overline{\mathbb{C}P^2}$ admits a conformally ASD b-metric g, such that $|g - g_0| \to 0$ at ∂X.*

The strategy of the proof of this result for manifolds without boundary is summarized in [T2, Ch. 7]. The proof can be divided into three steps, each of which is substantial:

Step 1 Show that for $N > 0$ there is a way to construct a Riemannian metric g_N on M_N with the property that $\|W^+(g_N)\| \to 0$ as $N \to \infty$.

Step 2 Apply the implicit function theorem (IFT) to find a small perturbation $g' = g_N 1 + h(g_N)$ that is conformally ASD modulo the vanishing of a set of constraint functions (essentially the map σ of Proposition 5.12). Interpret the constraint functions as a finite number of nonlinear conditions upon g_N.

Step 3 Show that by replacing M_N by M_{N+n}, a metric g_{N+n} can be constructed with small $\|W^+(g_{N+n})\|$ and vanishing constraint functions. Apply the IFT in Step 2 to obtain a perturbation of g_{N+n} that is conformally ASD.

Another way of interpreting Step 3 is to say that one can reduce to an unobstructed deformation problem by forming the connected sum with sufficiently many copies of $\overline{\mathbb{C}P^2}$, this number depending only upon the original data (M, g_M). An important technical point is that the norm used in Step 1 is not a standard Sobolev norm; it is scale-invariant (like L^2) but

a little stronger. This means that the IFT used in Step 2 is to be applied in non-standard Banach spaces; this in turn requires the development of other non-standard estimates for linear elliptic operators.

7.1 Sketch Proof I. Now let us turn to the b-manifold (X, g_0) of Theorem G. We start by applying Taubes' theorem to the double (M, g_M) of X. That is, regarding X as a non-compact manifold with a cylindrical end, we cut off the cylinder and glue the resulting manifold to another copy of itself (with opposite orientation). The closed manifold M then contains a neck $Y \times [-\rho, \rho]_t$ on which the metric is a conformally flat Riemannian product metric. Here the double is used for definiteness only. Any closed, oriented 4-manifold containing the subset $\{t \leq \rho\}$ of X would do just as well.

An examination of Step 1 above reveals that $\|W^+\|$ can be decreased by gluing copies of $\overline{\mathbb{C}P^2}$ onto $\mathrm{Supp}(W^+(g_M))$. In particular for our double M, we can construct (M_N, g_N) such that $\|W^+(g_N)\| \to 0$ as $N \to \infty$, but leaving the cylindrical neck in M untouched.

Now we take Steps 2 and 3 to obtain a conformally ASD metric g', say, on M_{N+n}. We claim that near the middle ($t = 0$) of the neck, g' will be a very small perturbation of the product metric. Now return to a b-manifold X' by cutting the middle of the neck and gluing on a semi-infinite cylinder $Y \times [0, \infty)$ to produce a manifold with an end $Y \times (-\rho, \infty)$. Glue g' to the product metric by means of a cut-off function to obtain a metric g'' on X' with the property that $W^+(g'') \neq 0$ only in a small neighbourhood of $t = 0$. We can moreover assume that a suitable weighted Sobolev norm of $W^+(g'')$ is as small as we please.

Thus we are now in the same framework as for the deformation theory in the rest of this paper. So we invoke once more the IFT to find a conformally ASD b-metric g''' as a small perturbation of g''. We claim that arguments analogous to those of Step 3 allow us to overcome the obstructions that could arise here.

The inelegant double use of the IFT in this argument is intended to avoid the need to adapt to b-manifolds the non-standard norms mentioned above. This completes our sketch of a proof of Theorem G.

7.2 Sketch Proof II. It is also possible to argue slightly differently: apply Step 1 as outlined above, and then pass back to a b-manifold X' by cutting at $t = 0$ and gluing in $Y \times [0, \infty)$. Now adapt Steps 2 and 3 to apply to b-manifolds with small $\|W^+\|$. This direct approach is attractive conceptually and several of the main steps go through without major changes. However, it is technically subtle as we have already indicated because of

the non-standard norms used throughout Taubes' argument. In particular, Taubes' analysis makes heavy use of the spectral theory of (bundle-valued) Laplacians $\nabla^*\nabla$: for some estimates it is necessary to expand sections as a linear combination of eigensections of $\nabla^*\nabla$, and spectral projection is used to define finite-dimensional subspaces corresponding to 'small eigenvalues'. Because $\nabla^*\nabla$ has continuous spectrum on a b-manifold, it is not simple to extend such arguments to b-manifolds. However we claim that one can satisfactorily glue Taubes' estimates over the compact piece $t \leq 0$ of X' onto standard weighted Sobolev space estimates for a product metric on the cylindrical end $Y \times (0, \infty)$. This gives another approach to the proof of Theorem G.

8 Vanishing Theorems

This section is summarizes some vanishing theorems for the obstruction-spaces that arise in the main gluing theorems. They apply to conformally ASD b-manifolds that are either obtained by conformal blow-up or blow-down of a conformally ASD manifold whose metric has additional geometric properties, for example, Einstein or scalar-flat Kähler.

The analysis of the Einstein cases rests on the use of Weitzenböck formulae for D^*, while the analysis of the scalar-flat Kähler story was done in [LS1] in the compact case and is a modification of this in the ALE case.

8.1 D^* and Dirac operators. In §3.2.3 we have described D in terms of a coupled version of the operator d^+d^*. There is a useful alternative description using spinors, which we now explain. Spinors cannot be introduced globally on a 4-manifold unless the second Stiefel–Whitney class vanishes, but we shall only ever need tensor products of an even number of spin-bundles, and these always exist globally. Indeed the link between our two accounts of the operator D_g is obtained precisely by carrying through the necessary identifications of tensor products of spin-bundles with certain bundles of tensors (associated to the tangent bundle) [PR].

On an oriented Riemannian 4-manifold, the spin-bundles $V^\pm$ are naturally $SU(2)$-bundles and that the Dirac operators interchange $+$ and $-$:

$$\partial\!\!\!/^+ : C^\infty(V^-) \to C^\infty(V^+) \quad \partial\!\!\!/^- : C^\infty(V^+) \to C^\infty(V^-)\,.$$

Again there are coupled versions of these, $\partial\!\!\!/^\pm_E : C^\infty(V^\mp \otimes E) \to C^\infty(V^\pm \otimes E)$, for any vector bundle E equipped with a unitary connection. In particular, there is a second-order operator $C^\infty(S^2V^-) \to S^2(V^+)$ given by composing the Dirac operators $C^\infty(V^- \otimes V^-) \to C^\infty(V^+ \otimes V^-)$ and

$C^\infty(V^+ \otimes V^-) \to C^\infty(V^+ \otimes V^+)$. Up to a constant factor, this can be identified with $d^+\delta$, using the natural isomorphisms $S^2V^\pm = \Lambda^\pm$. Now ignoring conformal weights,

$$E^0 = V^+ \otimes V^- \,, \quad E^1 = S^2V^+ \otimes S^2V^- \,, \quad E^2 = S^4V^+ \,.$$

(Here the canonical symplectic forms on $V^\pm$ have been used to eliminate all appearances of dual spin spaces.)

Then D_g^* can be written in terms of coupled Dirac operators D_1 and D_2 where

$$D_1 : C^\infty(V^- \otimes S^3V^+) \to C^\infty(V^- \otimes V^- \otimes S^2V^+)$$

and

$$D_2 : C^\infty(S^4V^+) \to C^\infty(V^- \otimes S^3V^+)\,;$$

namely $D_g^* = S(D_1D_2) + \Phi$. The operator S is the algebraic operation of symmetrization: $V^- \otimes V^- \otimes S^2V^+ \to S^2V^- \otimes S^2V^+$. There is a similar formula for D_g, which we do not write down.

The above formula for D_g^* has an important simplification if g is ASD. This is that the symmetrization is *unnecessary*. The reason is that the skew part of D_1D_2 is an algebraic operator $S^4V^+ \to \Lambda^2V^- \otimes S^2V^+ = S^2V^+$, given by partial contraction with some component of the curvature tensor. In general, the only component that could provide such a map is $W^+ \in C^\infty(S^4V^+)$, which we have assumed is zero. The conclusion is as follows: if g is ASD, then D_g^* can be identified with the operator

$$D_1D_2 + \Phi : C^\infty(S^4V^+) \to C^\infty(S^2V^- \otimes S^2V^+)\,.$$

8.2 Vanishing theorems when X is the conformal blow-up of a compact ASD-Einstein orbifold. To simplify notation, let us drop the bars which have been used to distinguish a compact manifold from its conformal blow-up; this should cause no confusion since the latter will not be used in this section. Our first result is the following:

PROPOSITION 8.1. *Let (X, g) be a compact 4-orbifold such that g is ASD and Einstein with positive scalar curvature s. Then $H_c^2(X) = 0$. If instead (X, g) is ASD and Ricci-flat, then $H_c^2(X)$ consists of parallel sections of E^2.*

This is a folklore theorem, but we reproduce the short proof.

Proof. If X is Einstein, we can identify D_g^* with the composite D_1D_2 of Dirac operators as above. To prove the proposition, it is enough to note the Weitzenböck formulae

$$D_1^*D_1 = \nabla^*\nabla + \tfrac{5}{12}s\,, \quad D_2^*D_2 = \nabla^*\nabla + \tfrac{1}{2}s\,,$$

which hold whenever $\Phi = 0$ and $W^+ = 0$. (The verification of these is left to the reader.) Suppose $D_g^*\psi = 0$; set $\chi = D_2\psi$, so that $D_1\chi = 0$. If

$s > 0$ then $D_1^* D_1$ is invertible, so $\chi = 0$, i.e. $D_2\psi = 0$. Similarly $D_2^* D_2$ is invertible, so $\psi = 0$. This completes the proof of the first part.

Suppose now that $s = 0$. With χ and ψ as before, we deduce first that χ is a *parallel* section. In particular, $D_2^*\chi = 0$. But χ also lies in the image of D_2 (by definition), so χ must be zero, by the Fredholm alternative. Thus $D_2\psi = 0$ and, applying the Weitzenböck formula, ψ is parallel. Thus we have identified the kernel of D_g^* with the space of parallel sections of S^4V^+ in the Ricci-flat case. □

Of course compact ASD-Einstein manifolds with $s \geq 0$ are rather rare. A well-known result of Hitchin states that the only examples with $s > 0$ are the complex projective plane (with the opposite orientation) and the 4-sphere. And when $s = 0$, one has only (quotients of) the K3-surface or the flat 4-torus. Since the former is simply connected and the latter has trivial holonomy, we have $\dim H_c^2 = 5$ in each of these cases.

When the class of spaces is widened to compact, ASD–Einstein *orbifolds* with $s > 0$, many more examples appear. These include the weighted projective spaces of Galicki–Lawson [GL].

The next class of examples consists of the compact scalar-flat Kähler surfaces. Here there is a fundamental dichotomy according as the Ricci tensor does or does not vanish. Since the Ricci-flat case was already analyzed, we may as well suppose that the surface is not Ricci-flat. Then we have the vanishing theorem of [KLP] to the effect that $H_c^2(X) = 0$ whenever the scalar-flat Kähler surface is *non-minimal*; that is to say, whenever it contains at least one rational curve of self-intersection -1. We shall not repeat the argument (though the reader will see many of the details in our discussion below of the case of non-compact but ALE scalar-flat Kähler surfaces).

Remark. Although we do not give a formal statement, it also often possible to compute H_c^2 when c is conformally flat. Indeed, in this case the ASD deformation theory is essentially the same as the conformally flat deformation theory; in particular if the latter is unobstructed, then so is the former. On the other hand, the conformally flat deformation theory is given by a flat complex (de Rham complex with twisted coefficients). The cohomology groups can sometimes be computed by topological methods. For example, for generic conformally flat structures, $H_c^2(S^1 \times S^3 \sharp \cdots \sharp S^1 \times S^3)=0$ by a Meyer–Vietoris argument.

8.3 Vanishing theorems for WALE spaces. In this subsection we consider the case that the cylindrical-end model has a conformal blow-up

that is WALE and either Ricci-flat or scalar-flat Kähler. Once again, we shall not have any use for the b-manifold here and therefore drop the use of hats.

Recall from previous discussion that we are interested in the part of the kernel of D_g^* that is $O(R^{-2})$ near ∞.

PROPOSITION 8.2. *Suppose (X, g) an ALE space that is ASD and Ricci-flat. Then $H_g^2(X) = 0$.*

Proof. Recall the formula $D^* = D_1 D_2$ from the previous proposition. Let $\psi \in H_g^2$ and let $\chi = D_2\psi$. Then $|\chi| = O(R^{-3})$ as $R \to \infty$, and so

$$0 = \int_{R \le R_0} (\chi, D_1^* D_1 \chi) = \int_{R \le R_0} (\chi, \nabla^* \nabla \chi) = \int_{R \le R_0} |\nabla \chi|^2 + O(R_0^{-4})$$

and letting $R_0 \to \infty$ we conclude as before that χ is parallel. Since $|\chi| \to 0$ at ∞, moreover, $\chi = 0$. Hence $D_2\psi = 0$. We make the same argument as above using the Weitzenböck formula. This time the boundary term is $O(R_0^{-2})$ but we still conclude that ψ is parallel and hence 0, since 0 at ∞. □

Now let us take up the Kähler story. By definition a non-compact Kähler surface (X, g, J) is said to be ALE if g is ALE and Kähler with respect to J *and if* the chart at infinity ϕ can be chosen to be a biholomorphic map $X - K \to (\mathbb{C}^2 - B)/\Gamma$. We shall now show that in this case $H_g^2(X) = 0$. Our method is to analyze D_g^* in terms of holomorphic data on X, decaying at ∞. This is helpful because of the following lemma:

LEMMA 8.3. *Suppose that X is an ALE Kähler surface and suppose that T is a holomorphic tensor field on X, $|T| \to 0$ at ∞. Then $T = 0$.*

Proof. Transfer T to $(\mathbb{C}^2 - B)/\Gamma$ and pull back to $\mathbb{C}^2 - B$. Then T becomes a holomorphic section of a trivial vector bundle and so each component of T is a holomorphic function that decays at ∞. But by the removable singularity theorem of Hartogs, each component extends uniquely to a holomorphic function on $\mathbb{C}^2$ and by the maximum principle must therefore be identically zero. This argument shows that T is identically zero on $X - K$. But then by uniqueness of analytic continuation, T is identically zero on X. □

Now the main theorem of this subsection can be given:

Theorem 8.4. *Suppose that (X, g, J) is an ALE scalar-flat Kähler surface. Then $H_g^2(X) = H_{g,J}^2(X) = 0$.*

Proof. Recall first that

$$\Lambda^+ \otimes \mathbb{C} = K \oplus 1 \oplus K^{-1}, \quad \Lambda^- \otimes \mathbb{C} = \Lambda_0^{(1,1)}.$$

Here K is the canonical-bundle of X, and the trivial bundle 1 is embedded in Λ^+ by multiplication by ω. Let us for the moment write $K \oplus K^{-1} = E$ so that $\Lambda^+ = 1 \oplus E$. This decomposition is preserved by the connection since J is parallel, so that D^* decomposes as a pair of operators

$$S^* = d^-\delta + \Phi : C^\infty(\Lambda^+) \to C^\infty(\Lambda^-) \tag{8.5}$$

(taking the component of ω) and

$$d^-_E\delta_E + \Phi : \Omega^+(E) \to \Omega^-(E)\,. \tag{8.6}$$

Now we shall decompose the other factor of Λ^+. If ψ is a section of E^2, write its components as follows

$$\begin{pmatrix} \psi_0 & \psi_1 \\ \psi_1 & \psi_{11} \end{pmatrix} \in C^\infty \begin{pmatrix} 1 & 1 \otimes E \\ E \otimes 1 & E \otimes E \end{pmatrix}.$$

Now the condition $D_g^*\psi = 0$ may be written as the two equations

$$d_- d^c\psi_0 + \rho\psi_0 + d_-\delta\psi_1 = 0 \tag{8.7}$$

and

$$d^E_-(d^E)^c\psi_1 + \rho\psi_1 + d^E_-\delta^E\psi_{11} = 0 \tag{8.8}$$

The vanishing theorem will be proved according to the following scheme:

(i) $H^0(X,\Theta) \cap O(R^{-2}) = 0$ implies $\psi_0 = 0$;
(ii) $H^0(X,\mathcal{O}(K)) \cap O(R^{-2}) = 0$ implies $\psi_1 = 0$;
(iii) $H^0(X,\Omega^1 \otimes K) \cap O(R^{-2}) = 0$ and $H^0(X,\mathcal{O}(K^2)) \cap O(R^{-2}) = 0$ implies $\psi_{11} = 0$.

In other words, relative to $S^2_0\Lambda^+ = K^{-2} \oplus K^{-1} \oplus 1 \oplus K \oplus K^2$, we eliminate first the component in the trivial bundle, next the components in $K^{\pm 1}$, finally those in $K^{\pm 2}$. Remark that complex conjugation carries K^r into K^{-r} so it is enough to deal with the components in 1, K and K^2.

Proof of (i). As in [LS1], the real function ψ_0 satisfies Lichnerowicz's differential equation; on a compact manifold it follows that $\nabla^{1,0}\psi_0$ is a holomorphic vector field. The argument requires integration by parts but in our situation we have sufficient decay at ∞ so that the conclusion holds. In fact, $\nabla^{1,0}\psi_0$ is holomorphic and decays at ∞, so by Lemma 8.3 ψ_0 is constant. Finally ψ_0 is $O(R^{-2})$ so $\psi_0 = 0$; the proof of (i) is complete.

Proof of (ii). Referring to (8.7), ψ_1 satisfies the equation $d^-\delta\psi_1 = 0$. As in the compact case, this implies $d\delta\psi_1 = 0$ (cf. the proof of the existence of the conformal factor). In particular ψ_1 is harmonic. Because the Hodge and $\overline{\partial}$-Laplacians agree (up to a factor of 2) we infer that $\overline{\partial}^*\overline{\partial}\gamma = 0$ where $\psi_1 = \gamma + \overline{\gamma}$ is the decomposition of according to components in K and K^{-1}.

Integration-by-parts is applicable now to show that γ is holomorphic. The proof is completed by applying Lemma 8.3 to γ.

Finally we consider implication (iii). Since $\psi_0 = 0$ and $\psi_1 = 0$, $\psi_{11} = \alpha + \overline{\alpha}$, say, where α is a section of K^2, and (8.8) gives $d_K^- \delta_K \alpha = 0$. These operators are the usual Hodge-de Rham operators, coupled to the holomorphic line bundle K and α lies in the space $\Omega^{2,0}(K)$. For reasons of degree, and using the Kähler identity $\partial^* = i[\Lambda, \overline{\partial}]$,

$$d_K d_K^* \alpha = i d_K \Lambda \overline{\partial}_K \alpha = \omega\lambda + \mu$$

where λ is a section of K and μ is a section of K^2. More precisely,

$$i\overline{\partial}_K \Lambda \overline{\partial}_K \alpha = \omega\lambda\,. \tag{8.9}$$

Since $\overline{\partial}_K^2 = 0$ (the curvature is of type $(1,1)$), we obtain $\omega \wedge \overline{\partial}_K \lambda = 0$ and hence λ is a holomorphic section of K. Since λ decays at ∞, we have $\lambda = 0$. Two more steps, each involving an application of Lemma 8.3 complete the proof. For with $\lambda = 0$, equation (8.9) says that $\Lambda\overline{\partial}_K \alpha$ is a decaying holomorphic section of $\Lambda^{1,0}(K)$, hence zero. Since $\Lambda : \Omega^{2,1} \to \Omega^{1,0}$ is an isomorphism, it follows that α is a decaying, holomorphic section of K^2. This completes the proof of (iii) and hence the vanishing theorem. □

References

[ADN] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for the solutions of elliptic partial differential equations satisfying general boundary conditions, II, Comm. Pure Appl. Math. 17 (1964), 35–92.

[AtHS] M.F. Atiyah, N. Hitchin, I.M. Singer, Self-duality in four-dimensional Riemannian geometry. Proc. R. Soc. London A 362 (1978), 425–461.

[Au] T. Aubin, Nonlinear Analysis on Manifolds. Monge-Ampère Equations, Grundlehren 252, Springer-Verlag, 1982.

[B] W.L. Baily, The decomposition theorem for V-manifolds, Am. J. Math. 78 (1956), 862–888.

[Be] A. Besse, Einstein Manifolds, Springer-Verlag, 1987.

[Bi] O. Biquard, Fibrés paraboliques stables et connexions singulières plates. Bull. Soc. Math. France 119 (1991), 231–257.

[Bo] C.P. Boyer, Conformal duality and compact complex surfaces, Math. Ann. 274 (1986), 517–526.

[CP] D.M.J. Calderbank, H. Pedersen, Einstein–Weyl geometry, in "Essays on Einstein Manifolds" (C.R. LeBrun, M.Wang, eds.), Surveys in Differential Geometry V (Suppl. J. Diff. Geom.), International Press (1999), 387–423.

[DF] S.K. Donaldson, R.D. Friedman, Connected sums of self-dual manifolds and deformations of singular spaces, Nonlinearity 2 (1989), 197–239.

[DK] S.K. Donaldson, P.B. Kronheimer, Geometry of Four-manifolds, Oxford University Press, 1991.

[F] A. Floer, Self-dual conformal structures on $l\,\mathbb{C}P^2$. J. Diff. Geom. 33 (1991), 551–573.

[GL] K. Galicki, H.B. Lawson, Jr., Quaternionic reduction and quaternionic orbifolds, Math. Ann. 282 (1988), 1–21.

[HMM] A. Hassell, R.R. Mazzeo, R.B. Melrose, Analytic surgery and the accumulation of eigenvalues, Comm. Anal. Geom. 3 (1995), 115–222.

[Hö1] L. Hörmander, The Analysis of Linear Partial Differential Operators I (second edition) Springer-Verlag, 1990.

[Hö2] L. Hörmander, The Analysis of Linear Partial Differential Operators III, Grundlehren 274, Springer-Verlag, 1985.

[KLP] J. Kim, C. LeBrun, M. Pontecorvo, Scalar-flat Kähler surfaces of all genera, J. Reine Angew. Math. 486 (1997), 69–95.

[KiK] A.D. King, D. Kotschick, The deformation theory of anti-self-dual conformal structures, Math. Ann. 294 (1992), 591–609.

[Ko] A.G. Kovalev, Twisted connected sums and special Riemannian holonomy, preprint, University of Edinburgh (2000).

[KoS] A.G. Kovalev, M.A. Singer, On the Kummer construction of complete hyperKähler 4-manifolds, in preparation.

[Kr] P.B. Kronheimer, The construction of ALE spaces as hyper-Kähler quotients, J. Diff. Geom 29 (1989), 665–683.

[L1] C. LeBrun, Counter-examples to the generalized positive action conjecture, Comm. Math. Phys. 118 (1988), 591–596.

[L2] C. LeBrun, Explicit self-dual metrics on $\mathbb{C}P_2\sharp\cdots\sharp\mathbb{C}P_2$, J. Diff. Geom. 34 (1991), 223–253.

[L3] C.R. LeBrun, Anti-self-dual metrics and Kähler geometry, Proceedings of the International Congress of Mathematicians (Zürich, 1994), Birkhäuser (1995), 498–507.

[LS1] C. LeBrun, M. Singer, Existence and deformation theory for scalar-flat Kähler metrics on compact complex surfaces, Invent. Math. 112 (1993), 273–313.

[LS2] C. LeBrun, M. Singer, A Kummer-type construction of self-dual 4-manifolds, Math. Ann. 300 (1994), 165–180.

[LoM] R.B. Lockhart, R.C. McOwen, Elliptic differential operators on noncompact manifolds, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 12 (1985), 409–447.

[M] R.R. Mazzeo, Regularity for the singular Yamabe problem, Indiana Univ. Math. J. 9 (1991), 303–344.

[MM] R.R. Mazzeo, R.B. Melrose, Analytic surgery and the eta invariant,

Geom. Funct. Anal. 5 (1995), 14–75.

[MP] R.R. Mazzeo, D. Pollack, Gluing and moduli for noncompact geometric problems, Geometric theory of singular phenomena in partial differential equations (Cortona, 1995). Sympos. Math. 38, Cambridge University Press, 1998.

[Me] R.B. Melrose, The Atiyah–Patodi–Singer Index Theorem, A K Peters Ltd., Wellesley, MA, 1993.

[P] R. Penrose, Nonlinear gravitons and curved twistor theory, Gen. Relativ. Grav. 1 (1976), 31–52.

[PR] R. Penrose, W. Rindler, Spinors and Space-Time 1, Cambridge University Press, 1984.

[T1] C.H. Taubes, The existence of anti-self-dual conformal structures, J. Diff. Geom. 36 (1992), 163–253.

[T2] C.H. Taubes, Metrics, Connections and Gluing Theorems, Reg. Conf. Ser. in Math. 89, AMS, 1996.

[V] I. Vaisman, On locally conformal almost Kähler manifolds, Israel J. Math. 24 (1976), 338–351.

[Z] Jian Zhou, PhD thesis, SUNY at Stony Brook.

Alexei Kovalev, Department of Pure Mathematics and Mathematical Statistics, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, UK
A.G.Kovalev@dpmms.cam.ac.uk

Michael Singer, Department of Mathematics and Statistics, University of Edinburgh, King's Buildings, Edinburgh EH9 3JZ, UK michael@maths.ed.ac.uk

Submitted: October 2000
Revision: May 2001