

VANISHING OF LITTLEWOOD–RICHARDSON POLYNOMIALS IS IN P

ANSHUL ADVE, COLLEEN ROBICHAUX,
AND ALEXANDER YONG

Abstract. J. De Loera & T. McAllister and K. D. Mulmuley & H. Narayanan & M. Sohoni independently proved that determining the vanishing of Littlewood–Richardson coefficients has strongly polynomial time computational complexity. Viewing these as Schubert calculus numbers, we prove the generalization to the Littlewood–Richardson polynomials that control equivariant cohomology of Grassmannians. We construct a polytope using the edge-labeled tableau rule of H. Thomas, A. Yong. Our proof then combines a saturation theorem of D. Anderson, E. Richmond, A. Yong, a reading order independence property, and É. Tardos’ algorithm for combinatorial linear programming.

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1. Introduction

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0)$ be a partition with n non-negative parts. We identify it in the usual manner with its Ferrers/Young diagram, where the i -th row consists of λ_i boxes. Consider a grid with n rows and $m \geq n + \lambda_1 - 1$ columns. Place λ in the northwest corner; this is the **initial diagram** for λ .

For example, if $\lambda = (4, 1, 1, 0)$, the initial diagram is the first of the three below.

$$\begin{bmatrix} + & + & + & + & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} + & + & + & \cdot & \cdot & \cdot & \cdot \\ + & \cdot & \cdot & \cdot & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ + & \cdot & + & + & + & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

A **local move** is a mutation of any 2×2 subsquare of the form

$$\begin{pmatrix} + & \cdot & \cdot \\ \cdot & \cdot & + \end{pmatrix} \mapsto \begin{pmatrix} \cdot & \cdot \\ \cdot & + \end{pmatrix}.$$

from successive applications of the local move to the initial diagram for λ . Above, one sees two more of the many other plus diagrams for $\lambda = (4, 1, 1, 0)$.

Let $\text{Plus}(\lambda)$ be the set of plus diagrams for λ . Given $P \in \text{Plus}(\lambda)$, let $\text{wt}_x(P)$ be the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ where α_i is the number of $+$'s in row i of P . A finer statistic is

$$\text{wt}_{x,y}(P) = \prod_{(i,j)} x_i - y_j,$$

where the product is over all (i, j) such that there is a $+$ in row i and column j of P . For example, if P is the rightmost diagram above, then

$$\begin{aligned} \text{wt}_x(P) &= x_1 x_2^4 x_4 \quad \text{and} \\ \text{wt}_{x,y}(P) &= (x_1 - y_1)(x_2 - y_1)(x_2 - y_3)(x_2 - y_4)(x_2 - y_5)(x_4 - y_2). \end{aligned}$$

Let $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_{n+\lambda_1-1}\}$ be two collections of indeterminates. We consider two generating series, the second being a refinement of the first:

$$s_\lambda(X) = \sum_{P \in \text{Plus}(\lambda)} \text{wt}_x(P) \quad \text{and} \quad s_\lambda(X; Y) = \sum_{P \in \text{Plus}(\lambda)} \text{wt}_{x,y}(P).$$

These are the **Schur polynomial** and **factorial Schur polynomial**, respectively. A more standard description of these polynomials involves semistandard Young tableaux, see, e.g., [Macdonald](#)

(1992) and the references therein. The description above arises in, e.g., Knutson *et al.* (2009).

Let $\mathbf{Sym}[X]$ denote the ring of symmetric polynomials in X . The set of Schur polynomials $s_\lambda(X)$ over partitions λ with at most n , possibly empty, rows is a \mathbb{Z} -linear basis of $\mathbf{Sym}[X]$. Analogously, the factorial Schur polynomials form a $\mathbb{Z}[Y]$ -linear basis of $\mathbf{Sym}[X] \otimes_{\mathbb{Q}} \mathbb{Z}[Y]$.

The structure constants with respect to these bases are defined by

$$s_\lambda(X)s_\mu(X) = \sum_{\nu} c_{\lambda,\mu}^{\nu} s_{\nu}(X) \quad \text{and}$$

$$s_\lambda(X; Y)s_\mu(X; Y) = \sum_{\nu} C_{\lambda,\mu}^{\nu} s_{\nu}(X; Y).$$

Here, $c_{\lambda,\mu}^{\nu}$ is the **Littlewood–Richardson coefficient**; this is known to be a nonnegative integer. Following the terminology of Molev (2009), the **Littlewood–Richardson polynomial** is $C_{\lambda,\mu}^{\nu} \in \mathbb{Z}[Y]$. These latter coefficients generalize the former, i.e.,

$$c_{\lambda,\mu}^{\nu} = C_{\lambda,\mu}^{\nu} |_{y_1=0, y_2=0, \dots, y_{n+\lambda_1-1}=0}.$$

In general, $c_{\lambda,\mu}^{\nu} = 0$ unless $|\lambda| + |\mu| = |\nu|$, whereas $C_{\lambda,\mu}^{\nu} = 0$ unless $|\lambda| + |\mu| \geq |\nu|$, where here $|\lambda| = \sum_i \lambda_i$. It is a theorem of Graham (2001) that $C_{\lambda,\mu}^{\nu}$ is uniquely expressible as a polynomial, with nonnegative integer coefficients in the variables $\{\beta_i := y_{i+1} - y_i : i \geq 1\}$.

For example, the interested reader may verify that

$$s_{(1,0)}(x_1, x_2; Y)^2 = s_{(2,0)}(x_1, x_2; Y) + s_{(1,1)}(x_1, x_2; Y) \\ + (y_3 - y_2)s_{(1,0)}(x_1, x_2; Y).$$

De Loera & McAllister (2006) and Mulmuley *et al.* (2012) independently proved the vanishing problem for $c_{\lambda,\mu}^{\nu}$ has strongly polynomial time complexity. The following result completes the parallel above:

THEOREM 1.1. *The vanishing of $C_{\lambda,\mu}^{\nu}$ can be decided in strongly polynomial time.*

In contrast, Narayanan (2006) has shown that computation of $c_{\lambda,\mu}^\nu$ is a $\#\text{P}$ -complete problem in L. Valiant's complexity theory for counting problems Valiant (1979). Now, $c_{\lambda,\mu}^\nu$ is a special case of $C_{\lambda,\mu}^\nu$ of when $|\nu| = |\lambda| + |\mu|$. In this case, $C_{\lambda,\mu}^\nu|_{\beta_i=1} = C_{\lambda,\mu}^\nu$. Hence, it follows that determining the value of $C_{\lambda,\mu}^\nu|_{\beta_i=1} \in \mathbb{Z}_{\geq 0}$ is $\#\text{P}$ -hard. In particular, no polynomial time algorithm for either counting problem can exist unless $\text{P} = \text{NP}$.

Overview of proof of Theorem 1.1. Our argument is a modification of that used in De Loera & McAllister (2006); Mulmuley *et al.* (2012). In Section 3, we construct, with explicit inequalities, a polytope $\mathcal{P}_{\lambda,\mu}^\nu$ with the property that $\mathcal{P}_{\lambda,\mu}^\nu$ has a lattice point if and only if $C_{\lambda,\mu}^\nu \neq 0$. Now, if $\mathcal{P}_{\lambda,\mu}^\nu$ is nonempty, it has a rational vertex. In that case, some dilation $N\mathcal{P}_{\lambda,\mu}^\nu$ contains an integer lattice point. Moreover, by our construction, $N\mathcal{P}_{\lambda,\mu}^\nu = \mathcal{P}_{N\lambda,N\mu}^{N\nu}$, which means $C_{N\lambda,N\mu}^{N\nu} \neq 0$. Thus, by a saturation theorem of Anderson *et al.* (2013), we conclude

$$C_{\lambda,\mu}^\nu \neq 0 \iff C_{N\lambda,N\mu}^{N\nu} \neq 0 \iff \mathcal{P}_{\lambda,\mu}^\nu \neq \emptyset.$$

To determine if $\mathcal{P}_{\lambda,\mu}^\nu \neq \emptyset$, one needs to ascertain feasibility of any linear programming problem involving $\mathcal{P}_{\lambda,\mu}^\nu$. The Klee–Minty cube shows that the practically efficient simplex method has exponential worst-case complexity. Instead, one can appeal to ellipsoid/interior point methods for polynomiality. Better yet, our inequalities are of the form $A\mathbf{x} \leq \mathbf{b}$ where the entries of A are from $\{-1, 0, 1\}$ and the vector \mathbf{b} is integral. Hence, our polytope is *combinatorial* and so one can achieve strongly polynomial time complexity using É. Tardos' algorithm; see Grotschel *et al.* (1993); Tardos (1986). \square

We point out some aspects of our modification. In De Loera & McAllister (2006); Mulmuley *et al.* (2012), the authors use the original saturation theorem of Knutson & Tao (2003). In addition, the polytope used has precisely $c_{\lambda,\mu}^\nu$ many lattice points. Our polytope does not have any such exact counting feature. To construct it, we need to deduce a new result about the edge-labeled tableau rule of Thomas & Yong (2018). The remainder of our argument is Proposition 3.2.

In recent years, there has been significant work on the complexity of computing Kronecker coefficients; see, e.g., Bürgisser & Ikenmeyer (2008); Ikenmeyer *et al.* (2017); Pak & Panova (2017) and

the references therein. In the context of the representation theory of the symmetric group, these are an extension of the Littlewood–Richardson coefficients. This paper initiates a study of the analogous complexity issues in Schubert calculus, by interpreting the Littlewood–Richardson coefficients as triple intersections of Schubert varieties in the Grassmannian; see [Section 4](#) for further discussion and open problems.

2. A factorial Littlewood–Richardson rule

[Molev & Sagan \(1999\)](#) gave the first combinatorial rule for $C_{\lambda, \mu}^{\nu}$. The first rule that exhibited the positivity of [Graham \(2001\)](#) was found by [Knutson & Tao \(2003\)](#) in terms of *puzzles*. Later, [Kreiman \(2010\)](#) and [Molev \(2009\)](#) independently gave essentially equivalent tableaux rules with the same positivity property. We also mention [Zinn-Justin \(2009\)](#) which gives a quantum integrability proof of the puzzle rule.

Actually, we will use yet another rule, due to [Thomas & Yong \(2018\)](#). This is also the rule utilized in the proof of the saturation theorem of [Anderson *et al.* \(2013\)](#) that we need. Indeed, we will observe a new property of the rule that may be of some independent interest.

2.1. The edge-labeled rule. We now recall the rule for $C_{\lambda, \mu}^{\nu}$ of [Thomas & Yong \(2018\)](#).

Suppose $\lambda \subseteq \nu$. An **edge-labeled tableau** T of skew shape ν/λ and content μ is an assignment of μ_i many labels i to the boxes of ν/λ and the horizontal edges weakly south of the “southern border” of λ (thought of as a lattice path, in the usual way). Each box contains exactly one label. Each edge contains a (possibly empty) set of labels. Moreover:

- (i) the *box* labels weakly increase along rows;
- (ii) the labels strictly increase along columns; and
- (iii) no edge label k is **too high**, i.e., every edge label k must be weakly below the southern edges of row k .

We will refer to (i) and (ii) as **semistandardness** conditions.

A tableau is **lattice** if for each label k and column j , the number of k 's in column j and to the right is weakly greater than the number of $(k + 1)$'s that appear in the same region. This can be stated in terms of a column reading word $w_c(T)$, obtained by reading the columns top to bottom, right to left. When reading a set-valued edge, read entries in increasing order.

We will also need the row reading word $w_r(T)$. This is obtained by reading the rows right to left and top to bottom, and reading set-valued edges in increasing order.

We say a word is a **lattice** if for every t and label k , in reading the first t letters, there are weakly more k 's than $(k + 1)$'s.

EXAMPLE 2.1. Consider the following tableaux with $\nu = (4, 2, 2)$ and $\lambda = (2, 2, 0)$: ◇

$$T_1 = \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline \frac{1}{2} & \frac{12}{3} & & \\ \hline \end{array} \quad T_2 = \begin{array}{|c|c|c|c|} \hline & & 1 & 2 \\ \hline \frac{1}{2} & \frac{12}{3} & & \\ \hline \end{array} \quad T_3 = \begin{array}{|c|c|c|c|} \hline & & \frac{1}{23} & 1 \\ \hline \frac{1}{2} & \frac{12}{3} & & \\ \hline \end{array}.$$

Then,

$$w_c(T_1) = 1\ 1\ 1\ 2\ 3\ 1\ 2\ 3 \quad w_r(T_1) = 1\ 1\ 1\ 2\ 1\ 3\ 2\ 3$$

$$w_c(T_2) = 2\ 1\ 1\ 2\ 3\ 1\ 2\ 3 \quad w_r(T_2) = 2\ 1\ 1\ 2\ 1\ 3\ 2\ 3.$$

T_1 and T_2 are both edge-labeled tableaux; however, only T_1 is lattice. Further, notice that both $w_c(T_1)$ and $w_r(T_1)$ are lattice, whereas both $w_c(T_2)$ and $w_r(T_2)$ are not lattice. This is the point of Theorem 2.3 below. In T_3 , the edge labels on the southern border of the first row are too high. Therefore, while T_3 is lattice, it is not an edge-labeled tableau.

Let $\text{EdgeTab}_{\lambda, \mu}^\nu$ be the set of edge-labeled tableaux T such that $w_c(T)$ is lattice. The main theorem of Thomas & Yong (2018) is that there is a weight $\text{apwt}(T)$ such that

$$C_{\lambda, \mu}^\nu = \sum_{T \in \text{EdgeTab}_{\lambda, \mu}^\nu} \text{apwt}(T).$$

We do not actually need $\text{apwt}(T)$ in this paper, so we suppress this detail. Instead, to discuss nonvanishing, we only need the following immediate consequence:

PROPOSITION 2.2 (Anderson *et al.* 2013, Corollary 3.3). $C_{\lambda,\mu}^\nu = 0$ if and only if $\text{EdgeTab}_{\lambda,\mu}^\nu = \emptyset$.

2.2. Reading order independence. It is well known to experts in the theory of Young tableaux that “any reasonable reading order works.” An instantiation of this imprecise statement is that a (classical, i.e., non-edge-labeled) semistandard tableaux is lattice for the column reading word (top to bottom, right to left) if and only if it is lattice for the row reading word (right to left, top to bottom).

The original formulation of the rule from Thomas & Yong (2018) uses column reading order. However, since the saturation property concerns stretching rows, we will need the following:

THEOREM 2.3. Let T be an edge-labeled tableau. Then, $w_c(T)$ is lattice if and only if $w_r(T)$ is lattice.

PROOF (Theorem 2.3). Let T be an edge-labeled tableau. Let $T_{i,j}$ = the label of the box in row i column j in matrix notation. Similarly,

$T_{i+\frac{1}{2},j}$ = the (set) filling of the southern edge of (i, j) .

Accordingly, we let (x, y) denote either a box or edge position of the tableau, i.e., $(x, y) = (i, j)$ or $(x, y) = (i + \frac{1}{2}, j)$. Let

$w_r |_{(x,y)}(T)$ = the row reading word of T ending at (x, y) , and

$w_c |_{(x,y)}(T)$ = the column reading word of T ending at (x, y) .

EXAMPLE 2.4. Let $T = T_1$ from Example 2.1. Then,

$$\begin{aligned} w_c |_{(2,1)}(T) &= 1\ 1\ 1\ 2\ 3 & w_r |_{(2,1)}(T) &= 1\ 1 \\ w_c |_{(2+\frac{1}{2},1)}(T) &= 1\ 1\ 1\ 2\ 3\ 1 & w_r |_{(2+\frac{1}{2},1)}(T) &= 1\ 1\ 1\ 2\ 1. \end{aligned}$$

◇

CLAIM 2.5.

(I) All labels weakly northeast of (x, y) are read by $w_r \upharpoonright_{(x,y)}(T)$ and $w_c \upharpoonright_{(x,y)}(T)$.

(II) If ℓ is read by $w_c \upharpoonright_{(x,y)}(T)$ but not $w_r \upharpoonright_{(x,y)}(T)$ then $\ell > T_{x,y}$.

(III) If ℓ is read by $w_r \upharpoonright_{(x,y)}(T)$ but not $w_c \upharpoonright_{(x,y)}(T)$ then $\ell < T_{x,y}$.

Proof (Claim 2.5): (I) is by definition. (II) and (III) follow since T is semistandard. \square

(\Rightarrow) Suppose $w_c(T)$ is lattice, but $w_r(T)$ is not. Hence, there exists a label k and position (x, y) such that $w_r \upharpoonright_{(x,y)}$ contains more $(k + 1)$'s than k 's. We may assume without loss of generality that (x, y) contains $k + 1$. Then by (II) and (III) of the claim, the excess of $(k + 1)$'s must be blamed on the region weakly northeast of (x, y) . However, (I) implies $w_c(T)$ is not lattice, a contradiction.

(\Leftarrow) Conversely, suppose $w_r(T)$ is lattice and $w_c(T)$ is not. Take a label k and position (x, y) such that $w_c \upharpoonright_{(x,y)}(T)$ contains more $(k + 1)$'s than k 's. We may assume that if (x, y) is a box position then $T_{x,y} = k + 1$, and if (x, y) is an edge position then $k + 1 \in T_{x,y}$. Further, we may assume (x, y) is the first (topmost and rightmost) position of such a failure.

Case 1: $[(x, y) = (i, j)$ is a box] By (II), among the labels read by $w_c \upharpoonright_{(i,j)}(T)$ but not $w_r \upharpoonright_{(i,j)}(T)$, no k or $k + 1$ appears. Therefore, since $w_c \upharpoonright_{(i,j)}(T)$ is not lattice, in the region read by both, there are more $(k + 1)$'s than k 's. Since $w_r(T)$ is lattice, in the region only read by $w_r \upharpoonright_{(i,j)}(T)$, there must exist at least one k . Where can such an additional k appear? By semistandardness, it must be in row $i - 1$, strictly to the left of column j , as either a box or edge label. Moreover, again by semistandardness, any such extra k in column $j' < j$ must have a "paired" $k + 1$ in the box (i, j') below it. Hence, it follows that $w_r(T)$ is also not lattice, a contradiction.

Case 2: $[(x, y) = (i + \frac{1}{2}, j)]$ As in **Case 1**, there must exist an extra k in the region R weakly north of row i and strictly west of column j . Now, if there is a box label k in R , then by semistandardness, we conclude $T_{i,j} = k$. This implies (x, y) is not the first violation of latticeness for $w_c(T)$.

3. Proof of the main theorem

Suppose $T \in \text{EdgeTab}_{\lambda, \mu}^\nu$. Let $r_k^i = r_k^i(T)$ denote the number of k 's in the i th row of T and $r_k^{i+\frac{1}{2}} = r_k^{i+\frac{1}{2}}(T)$ the number of k 's in the southern edges of the i th row of T , where

$$k \in \{1, 2, \dots, l(\mu)\} \quad \text{and} \quad i \in \{1, 2, \dots, l(\nu)\}.$$

Recall, $l(\mu)$ is the number of nonzero parts of μ , etc. By convention, let

$$r_{l(\mu)+1}^i = r_k^{l(\nu)+1} = 0.$$

EXAMPLE 3.1. For instance, consider

$$T = \begin{array}{|c|c|c|c|} \hline & & & \\ \hline & & \frac{1}{2} & 1 \\ \hline & \frac{1}{2} & \frac{2}{3} & \\ \hline & \frac{2}{3} & & \\ \hline \end{array}.$$

Then,

$$r_1^2 = 2, r_1^{2+\frac{1}{2}} = 1, r_2^{2+\frac{1}{2}} = 1, r_2^3 = 1, r_3^3 = 1, r_3^{3+\frac{1}{2}} = 1,$$

and all other values are zero. ◇

Next, examine the following conditions (which modify those of a preprint version of [Mulmuley et al. \(2012\)](#)):

(A) Nonnegativity: For all i, k ,

$$r_k^i \geq 0, r_k^{i+\frac{1}{2}} \geq 0.$$

(B) Shape constraints: For all i ,

$$\lambda_i + \sum_k r_k^i = \nu_i.$$

(C) Content constraints: For all k ,

$$\sum_i r_k^i + r_k^{i+\frac{1}{2}} = \mu_k.$$

(D) Gap constraints: For all i, k ,

$$r_k^{i+\frac{1}{2}} \leq \left(\lambda_i + \sum_{k' < k} r_{k'}^i \right) - \left(\lambda_{i+1} + \sum_{k' \leq k} r_{k'}^{i+1} \right).$$

(E) Too high: For all $i < k$,

$$r_k^{i+\frac{1}{2}} = 0.$$

(F) Reverse lattice word constraints: For all i, k ,

$$\sum_{i' < i} r_k^{i'} + r_k^{i'+\frac{1}{2}} \geq r_{k+1}^i + \sum_{i' < i} r_{k+1}^{i'} + r_{k+1}^{i'+\frac{1}{2}}.$$

Define a polytope

$$\mathcal{P}_{\lambda, \mu}^\nu = \{ (r_k^i, r_k^{i+\frac{1}{2}}) : \text{(A)-(F)} \} \subseteq \mathbb{R}^{2l(\nu)l(\mu)}.$$

PROPOSITION 3.2. $\text{EdgeTab}_{\lambda, \mu}^\nu \neq \emptyset \iff \mathcal{P}_{\lambda, \mu}^\nu \cap \mathbb{Z}^{2l(\nu)l(\mu)} \neq \emptyset.$

PROOF (Proposition 3.2). (\implies) Let $T \in \text{EdgeTab}_{\lambda, \mu}^\nu$. Clearly, r_k^i and $r_k^{i+\frac{1}{2}}$ satisfy (A), (B), (C), and (E) above. The tableau constraint (D) asks that there be enough edges in row $i + \frac{1}{2}$, between the rightmost k in row $i + 1$ and the leftmost k in row i , to accommodate $r_k^{i+\frac{1}{2}}$ many k 's; this holds by semistandardness of T .

Finally, (F) merely asks that the row word will be lattice after reading all the $(k + 1)$'s in row i ; this is certainly true of T as it is row lattice.

(\impliedby) Let $(r_k^i, r_k^{i+\frac{1}{2}}) \in \mathcal{P}_{\lambda, \mu}^\nu \cap \mathbb{Z}^{2l(\nu)l(\mu)}$. Construct a tableau T^* of shape ν/λ and content μ as follows. First, for all i, k , (uniquely) place r_k^i many k 's in row i , such that the k 's are weakly increasing along each row. At this point, the tableau has no edge labels but,

by (B) has the correct skew shape ν/λ . Moreover, (A) and (D) combined implies that for all i, k ,

$$\lambda_{i+1} + \sum_{k' \leq k} r_{k'}^{i+1} \leq \lambda_i + \sum_{k' < k} r_{k'}^i.$$

This precisely asserts that the partially built T^* is column strict.

Next, place $r_k^{i+\frac{1}{2}}$ many k 's as far to the right as possible in row $i + \frac{1}{2}$ without breaking the semistandardness of T . To be precise, the last k will be in column $\lambda_i + \sum_{k' < k} r_{k'}^i$ and the remaining k 's will be in adjacent columns to the left, namely columns:

$$(3.3) \quad \left(\lambda_i + \sum_{k' < k} r_{k'}^i \right) - r_k^{i+\frac{1}{2}} + 1, \left(\lambda_i + \sum_{k' < k} r_{k'}^i \right) - r_k^{i+\frac{1}{2}} + 2, \\ \dots, \left(\lambda_i + \sum_{k' < k} r_{k'}^i \right) - 1, \lambda_i + \sum_{k' < k} r_{k'}^i.$$

This completes T^* .

(D) asserts that column strictness is preserved in the final step where we have added the edges: We are placing the k 's in row $i + \frac{1}{2}$ to the right of the box labels $\leq k$ in row $i + 1$. Also, in row i , the columns (3.3) contain box labels $< k$. Now, (E) says no edge label is too high.

However, (F) does not *a priori* show $w_r(T^*)$ is lattice (see Example 3.6 below). Thus, we need:

CLAIM 3.4. $w_r(T^*)$ is lattice.

Proof of claim: Consider row $i = 1$. In this case, (F) asserts $r_{k+1}^1 \leq 0$ for all $k \geq 1$. In view of (A), this means that there are no labels greater than 1 in the first row of T^* . Moreover, if we know row latticeness has not failed before reading row $i > 1$, then (F) immediately says no violation can occur in row i either.

Therefore, it remains to check that while reading the edge labels in a row $i + \frac{1}{2}$, one always remains lattice. Suppose not. (F) combined with (A) implies

$$(3.5) \quad \sum_{i' < i+1} r_k^{i'} + r_k^{i'+\frac{1}{2}} \geq \sum_{i' < i+1} r_{k+1}^{i'} + r_{k+1}^{i'+\frac{1}{2}}.$$

That is, after reading the *entirety* of row $i + \frac{1}{2}$, the row reading word has at least as many k 's as $(k + 1)$'s.

Say edge $(i + \frac{1}{2}, j)$ contains a violating label $k + 1$ that breaks the latticeness of the row word. We may assume this $k + 1$ is first (rightmost) among all such labels. By (3.5), and/or the sentence immediately after it, there must be an “extra” edge label k in row $i + \frac{1}{2}$ and in column j or to its left.

Case 1: $[T_{i,j} < k]$ The rightmostness of the placement of the k 's (see (3.3)) implies that $k \in T_{i+\frac{1}{2},j}$. Hence, the row word has more $(k + 1)$'s than k 's before reading edge $(i + \frac{1}{2}, j)$, a contradiction of the rightmostness of $k + 1$. That is, this case cannot occur.

Case 2: $[T_{i,j} = k]$ By semistandardness, for every $k + 1$ in an edge $(i + \frac{1}{2}, j')$ with $j' \geq j$, there is $k \in T_{i,j'}$. It is then straightforward to conclude there are more $(k + 1)$'s than k 's before reading the edge $(i + \frac{1}{2}, j)$, and in particular, before it even reads the rightmost k in row i , a contradiction. \square

This completes the proof of the proposition.

EXAMPLE 3.6. To illustrate the proof of (\Leftrightarrow) above, take $\lambda = (2, 2, 1, 1, 0)$, $\mu = (2, 2, 2, 1, 1)$, and $\nu = (2, 2, 2, 2, 2)$. Now, $r_1^3 = r_2^4 = r_1^{4+\frac{1}{2}} = r_2^{4+\frac{1}{2}} = r_4^{5+\frac{1}{2}} = r_5^{5+\frac{1}{2}} = 1$ and $r_3^5 = 2$ (all other components are zero) defines a lattice point in $\mathcal{P}_{\lambda,\mu}^\nu$. There are four edge-labeled tableaux that have these statistics, namely

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The first is not lattice, but the other three are. The rightmost of them is T^* . \diamond

Conclusion of proof (Theorem 1.1): Notice that since (A)–(F) is linear and homogeneous in the components of λ , μ and ν , it follows that $\mathcal{P}_{N\lambda, N\mu}^{N\nu} = N\mathcal{P}_{\lambda,\mu}^\nu$. Thus, in view of Proposition 3.2, we have

constructed the desired polytope $\mathcal{P}_{\lambda,\mu}^\nu$ alluded to in the introduction. Since this is the only missing component of the argument given there, the theorem follows.

Using indicator variables, one easily modifies the above argument to construct a polytope $\mathcal{Q}_{\lambda,\mu}^\nu$ whose lattice points are in bijection with the tableaux in $\text{EdgeTab}_{\lambda,\mu}^\nu$. However, this polytope does not satisfy $N\mathcal{Q}_{\lambda,\mu}^\nu = \mathcal{Q}_{N\lambda,N\mu}^{N\nu}$. Counting lattice points of this polytope is not equivalent to computing $C_{\lambda,\mu}^\nu$ since one needs a weighted count based on `apwt`.

4. Schubert calculus and complexity theory

The Littlewood–Richardson polynomials appear in the topic of equivariant Schubert calculus of Grassmannians. The usage of “equivariant” refers to “equivariant cohomology,” a type of enriched cohomology theory. We refer the reader to [Knutson & Tao \(2003\)](#) for background.

Another enriched cohomology theory is K -theory (i.e., the Grothendieck ring of algebraic vector bundles). There is a lattice rule [Buch \(2002\)](#) for the corresponding K -theoretic Littlewood–Richardson coefficients $k_{\lambda,\mu}^\nu$ (another lattice rule uses *genomic tableaux* [Pechenik & Yong \(2017\)](#)).

QUESTION 4.1. *Is the decision problem $k_{\lambda,\mu}^\nu = 0$ in P?*

One cannot use the same general method of this paper to resolve Question 1. To be precise, in ([Buch 2002](#), Section 7), it is noted that (up to a sign convention) $k_{(1,0),(1,0)}^{(2,1)} = 1$ but $k_{(2,0),(2,0)}^{(4,2)} = 0$. That is, the saturation statement

$$k_{\lambda,\mu}^\nu \neq 0 \implies k_{N\lambda,N\mu}^{N\nu} \neq 0$$

is false. (The truth of the converse is not known.) Therefore, there cannot exist a polytope $\mathcal{K}_{\lambda,\mu}^\nu$ with the dilation property $N\mathcal{K}_{\lambda,\mu}^\nu = \mathcal{K}_{N\lambda,N\mu}^{N\nu}$ crucial for the argument we use.

Quantum cohomology of Grassmannians is another deformation of significant interest. Work of [Buch et al. \(2016\)](#) established a combinatorial rule for the coefficients $d_{\lambda,\mu}^\nu$ of this case. They proved the two-step case of a *puzzle* conjecture of A. Knutson; see

Buch *et al.* (2003). Also, Belkale (2008) has established a quantum saturation property for these quantum Littlewood–Richardson coefficients. However, even from these results a solution to the following is not clear to us:

QUESTION 4.2. *Is the decision problem $d_{\lambda,\mu}^\nu = 0$ in \mathbf{P} ?*

The problems of computing $k_{\lambda,\mu}^\nu$ and $d_{\lambda,\mu}^\nu$ are $\#\mathbf{P}$ -hard problems. This is because they contain as special cases the Littlewood–Richardson coefficients, which are $\#\mathbf{P}$ -complete by Narayanan (2006). To show both problems are actually $\#\mathbf{P}$ -complete, one needs an argument to establish both problems are actually in $\#\mathbf{P}$. This should be possible using one of the combinatorial rules for each of the numbers; the technicalities of such an argument (including recalling the rules) might appear elsewhere.

Finally, the Schur polynomials are special cases of *Schubert polynomials* $\mathfrak{S}_w(X)$, defined for any permutation $w \in S_n$. It is known that $\mathfrak{S}_{w'} = \mathfrak{S}_w$ if w' is the image of w under the natural embedding of $S_n \hookrightarrow S_{n+1}$. Therefore, it is unambiguous to refer to \mathfrak{S}_w for $w \in S_\infty$. These polynomials form a \mathbb{Z} -linear basis of the ring of polynomials $\mathbb{Q}[x_1, x_2, \dots]$. The structure constants $c_{u,v}^w$ relative to this basis are known to be nonnegative for Schubert calculus reasons. We refer to the book Manivel (2001) for background and references.

It is a longstanding open problem to find a combinatorial rule for $c_{u,v}^w$. In particular, it is not known if the problem is in $\#\mathbf{P}$. Since the Schubert structure constants also contain the Littlewood–Richardson coefficients in a specific way, the aforementioned theorem of H. Narayanan implies the problem is $\#\mathbf{P}$ -hard. I. Pak–A. Morales have informed us that they have a proof that the problem is in \mathbf{GapP} .

QUESTION 4.3. *Is the decision problem $c_{u,v}^w = 0$ NP-hard?*

Recently, it was shown that vanishing of Kronecker coefficients is NP-hard Ikenmeyer *et al.* (2017). This establishes a formal difference in difficulty between the Kronecker coefficients and the Littlewood–Richardson coefficients. Inspired by their results, Question 3 asks if one can similarly establish a formal complexity dif-

ference in Schubert calculus. On the other hand, if either decision problem in Questions 1 or 2 is NP-hard, then such a formal difference does not preclude existence of a general combinatorial rule, as rules exist in both of these research directions.

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ANSHUL ADVE
UCLA,
Los Angeles, CA 90095, USA.
anshul.adve@gmail.com

COLLEEN ROBICHAUX
Department of Mathematics,
U. Illinois at Urbana-Champaign,
Urbana, IL 61801, USA.
cer2@illinois.edu

ALEXANDER YONG
Department of Mathematics,
U. Illinois at Urbana-Champaign,
Urbana, IL 61801, USA.
ayong@uiuc.edu