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COMPLEXITY OF TROPICAL AND MIN-PLUS LINEAR PREVARIETIES

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Abstract. A tropical (or min-plus) semiring is a set $\mathbb{Z}$ (or $\mathbb{Z} \cup \{\infty\}$) endowed with two operations: $\oplus$, which is just usual minimum, and $\odot$, which is usual addition. In tropical algebra, a vector x is a solution to a polynomial $g_1(x) \oplus g_2(x) \oplus \cdots \oplus g_k(x)$, where the $g_i(x)$s are tropical monomials, if the minimum in $\min_i(g_i(x))$ is attained at least twice. In min-plus algebra solutions of systems of equations of the form $g_1(x) \oplus \cdots \oplus g_k(x) = h_1(x) \oplus \cdots \oplus h_l(x)$ are studied.
In this paper, we consider computational problems related to tropical linear system. We show that the solvability problem (both over $\mathbb{Z}$ and $\mathbb{Z} \cup \{\infty\}$) and the problem of deciding the equivalence of two linear systems (both over $\mathbb{Z}$ and $\mathbb{Z} \cup \{\infty\}$) are equivalent under polynomial-time reductions to mean payoff games and are also equivalent to analogous problems in min-plus algebra. In particular, all these problems belong to $\mathsf{NP} \cap \mathsf{coNP}$. Thus, we provide a tight connection of computational aspects of tropical linear algebra with mean payoff games and min-plus linear algebra. On the other hand, we show that computing the dimension of the solution space of a tropical linear system and of a min-plus linear system is NP-complete.

Keywords. Tropical linear systems, min-plus linear systems, tropical prevarieties, min-plus prevarieties, mean payoff games.

Subject classification. 15A80, 68Q17, 91A43.

1. Introduction

A *min-plus* or *tropical semiring* is defined by the set K endowed with two operations $\oplus$ and $\odot$. For K, we can take $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Z} \cup \{+\infty\}$, $\mathbb{R} \cup \{+\infty\}$ and so on. In this paper, we mainly consider the cases of $\mathbb{Z}$ and $\mathbb{Z}_\infty = \mathbb{Z} \cup \{+\infty\}$. Our results also extend to the cases of $\mathbb{Q}$ and $\mathbb{Q}_\infty = \mathbb{Q} \cup \{\infty\}$. The operations *tropical addition* $\oplus$ and *tropical multiplication* $\odot$ are defined in the following way:

$$x \oplus y = \min\{x, y\}, \quad x \odot y = x + y.$$

The *tropical linear system* associated with a matrix $A \in K^{m \times n}$ is the system of expressions

$$\min_{1 \le j \le n} \{a_{ij} + x_j\},\ 1 \le i \le m, \tag{1.1}$$

or, to state it the other way, the vector $A \odot x$ for $x = (x_1, \ldots, x_n)$. We say that $x \neq (\infty, \ldots, \infty)$ is a solution to the system (1.1) if for every row $1 \le i \le m$, there are two columns $1 \le k < l \le n$ such that

$$a_{ik} + x_k = a_{il} + x_l = \min_{1 \le j \le n} \{a_{ij} + x_j\}.$$

For example, consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Clearly $x = (0, -1, 0)$ is a solution to the corresponding tropical linear systems. Also if we add some positive number to the first coordinate of this solution, we will still have a solution. That is, $x = (t, -1, 0)$ is a solution for any $t \ge 0$. Finally, note that if some vector (x_1, x_2, x_3) is a solution, then the vector $(x_1 + t, x_2 + t, x_3 + t)$ is also a solution. It is not hard to see that there are no other solutions to this system.

Following the notation of Richter-Gebert *et al.* (2003), we call the set of solutions of a tropical linear system a *tropical linear prevariety*. It follows from the analysis of Richter-Gebert *et al.* (2003) that this set is a union of polyhedra of possibly different dimensions (this is one of the reasons for using pre- in "prevariety"). The *dimension* of a tropical prevariety is the largest dimension of a polyhedron contained in it.

The *(two-sided) min-plus linear system* associated with a pair of matrices $A, B \in K^{m \times n}$ is the system

$$\min_{1 \le j \le n} \{a_{ij} + x_j\} = \min_{1 \le j \le n} \{b_{ij} + x_j\},\ 1 \le i \le m. \tag{1.2}$$

We note that for all systems, we consider it is not essential which of the functions min or max we use. The whole theory remains the same.

The two branches of algebra related to $(\min, +)$ structure—tropical algebra and min-plus algebra—have different origins. Tropical algebra had arisen in algebraic geometry (see the surveys Itenberg *et al.* 2007; Sturmfels 2002) and min-plus algebra had arisen in combinatorial optimization and scheduling theory (see recent monograph Butkovič 2010). Thus, these two branches developed mostly independently in parallel. In this paper, we are interested in computational aspects of these algebras. Naturally, the most basic and important computational problems concern linear algebra. In the case of classical algebra, the Gauss algorithm solves linear systems in polynomial time. In the case of tropical semirings, things turn out to be more complicated and no polynomial-time algorithm is known either for tropical linear systems or for min-plus linear systems. For the tropical case, it is known, however, that the solvability problem is in $\mathsf{NP} \cap \mathsf{coNP}$. Pseudopolynomial algorithms (Akian *et al.* 2012; Grigoriev 2013) are known, that is the algorithms with complexity polynomial in the size of a system and in the absolute values of its coefficients. Also it is known that the problem reduces to the well-known and long-standing problem *mean payoff games* (Akian *et al.* 2012) (see Section 2 for the definition). Concerning the algorithms, Grigoriev (2013) has constructed an algorithm which is pseudopolynomial and polynomial for constant size matrices, that is at the same time, its running time is bounded by $\mathrm{poly}(m, n)M \log M$ and $\mathrm{poly}(2^{nm}, \log M)$, where n is the number of columns, m is the number of rows, and M is the largest absolute value of matrix entries. Concerning the dependence on n and m in the second upper bound, the best refinement for Grigoriev's algorithm is the roughly $\binom{m+n}{n}$ upper bound which was proven by Davydow (2012). It was also shown in Davydow (2012) that this upper bound is tight for Grigoriev's algorithm. Later Davydow (2013) constructed another algorithm which works in time $\binom{m}{n}\mathrm{poly}(n, m, \log M)$.

We note that the hardness of the solvability problem for tropical linear systems lies in the case when the number of rows m

is substantially greater than the number of columns n. Indeed, if $m < n$, then the system always has a solution. If $m \leq n + c$ for constant c, then the solvability can be checked in polynomial time, which follows for example from the upper bound of Davydow (2013), mentioned above. In the same paper, it was also shown that for $m \leq n + c$, we not only can check the solvability, but also can find a solution in polynomial time, if it exists.

More is known about the solvability problem for min-plus linear systems. In addition to containment in $\mathsf{NP} \cap \mathsf{coNP}$ and pseudopolynomial algorithms, as for tropical systems, it was proven (Akian *et al.* 2012; Bezem *et al.* 2010) that the problem is polynomial-time *equivalent* to mean payoff games. Note that the latter paper deals with systems of min-plus inequalities, but they are equivalent to systems of min-plus equations, see the preliminaries section.

The first result of our paper is that the solvability problem for tropical linear systems is also equivalent to mean payoff games. Thus, on the one hand, we characterize the complexity of the solvability problem of tropical linear systems and on the other hand give a new reformulation of mean payoff games.

In particular, our result means that the solvability problem for tropical linear systems is equivalent to the solvability problem for min-plus linear systems. Thus, we establish a tight connection between two branches of algebra over the operations min and $+$. Also from our reduction, the translation of Grigoriev's algorithm to mean payoff games follows. We are not aware of a "natural"[1] algorithm for mean payoff games with similar complexity bounds as in Grigoriev (2013), see above. This indicates that this translated algorithm might be essentially different from known algorithms for mean payoff games.

[1] We note that if we have two algorithms for some problem with time upper bounds $t_1(n)$ and $t_2(n)$, where n is the size of the input, then we can easily construct an algorithm with the time bound $\min(O(t_1(n)), O(t_2(n)))$ simply running both algorithms in parallel and stopping when one of them stops. Thus, we can construct an algorithm with the same time upper bounds as Grigoriev's algorithm from two different known algorithms for mean payoff games. However, this algorithm will hardly be natural and most likely will essentially differ from Grigoriev's algorithm.

Next, we study other problems related to tropical linear systems: the problem of equivalence of two given tropical linear systems and the problem of computing the dimension of a tropical prevariety. The former problem turns out to be also equivalent to mean payoff games. The analogous statement for min-plus linear systems is also true and follows from the known results (see Lemma 3.13 below).

Interestingly, the dimension problem of the tropical prevariety turns out to be NP-complete. More precisely we prove NP-completeness of the following problem: given an $m \times n$ matrix A and a number k, decide whether the dimension of the tropical prevariety of the tropical linear system corresponding to A is at least k. We also prove the analogous result for the case of min-plus linear systems.

All results above we prove for both $\mathbb{Z}$ and $\mathbb{Z}_\infty$ domains (there is no obvious translation between these two cases).

The techniques of our proofs are mostly combinatorial. For equivalence of the solvability problem for tropical linear systems to mean payoff games, we use the result of Möhring *et al.* (2004) in which the equivalence of mean payoff games to the *max-atom problem* (MAP for short) was shown (see Section 2 for definitions). This result was already used in Bezem *et al.* (2010) to show that the solvability problem of min-plus linear systems is equivalent to mean payoff games (for the case of $\mathbb{Z}$)—it was shown there that the solvability problem for min-plus linear systems is equivalent to MAP. For our result, we show that the solvability problem for tropical linear systems is equivalent to MAP. The main difficulty with the tropical case is that MAP is easier to use for studying min-plus structures than for the tropical ones. Indeed, a MAP instance is very similar to the system of min-plus linear inequalities, and the reductions between them are rather direct and simple. The systems of tropical linear equations, however, look different, and it seems that there is not such a simple connection. Another approach to the solvability problem of min-plus linear systems was taken in Akian *et al.* (2012), where a very nice and clear direct connection between min-plus linear systems and mean payoff games was constructed. The authors of that paper also considered a similar analysis for tropical linear systems, but only proved a reduction in one direction: from the solvability problem of tropical linear systems to mean payoff games. Once again, since the inner structure of the tropical equations is different, it seems that there is not such a simple direct connection to mean payoff games.

For the dimension problem of tropical linear systems, we give a reduction from the vertex cover problem. The main technical ingredient here is a combinatorial characterization of the dimension of the tropical prevariety of a given tropical linear system.

In this paper, we deduce some equivalences between purely tropical computational problems through the connection to mean payoff games. We note that in the technical report Grigoriev & Podolskii (2012) preceding this paper we also gave direct (not referring to mean payoff games) combinatorial proofs of reductions between some of these problems.

We also mention other complexity results in min-plus algebra. First of all, it was shown by Theobald (2006) that if we do not restrict ourselves to linear systems and consider systems of tropical polynomial equations of arbitrary degree (and even of degree 2), then the solvability problem for such systems is NP-complete. Also Kim and Roush have shown that computing both tropical rank (Kim & Roush 2005) of a matrix and Kapranov rank (Kim & Roush 2006) of a matrix is NP-complete.

The rest of the paper is organized as follows. In Section 2, we give definitions and state the facts we need on the tropical linear systems. In Section 3, we prove the result on the equivalence of the solvability problem for tropical linear systems and of mean payoff games. We also deduce the result on the equivalence problem there. In Section 4, we discuss a relation between the dimension of the solution space of a tropical linear system and the known notions of matrix rank in min-plus algebra. In Sections 5 and 6, we prove NP-completeness of the dimension of a tropical prevariety: In the former, we give a combinatorial characterization of the dimension, and in the latter, we use it to prove NP-completeness.

2. Preliminaries

Throughout the paper for an integer n, we denote by $[n]$ the set $\{1, 2, \ldots, n\}$. By $\leq_p$, we denote Karp reductions (polynomial time many to one reductions). We also consider Cook reductions (polynomial-time Turing reductions). See Arora & Barak (2009) for the definitions. When we do not specify the type of reduction, we mean Karp reduction.

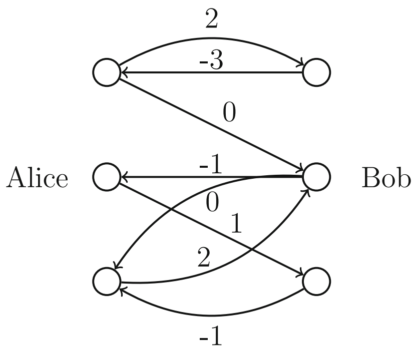
2.1. Mean payoff games. In an instance of a mean payoff game, we are given a directed graph $G = (V, E)$, whose vertices are divided into two disjoint sets $V = V_1 \sqcup V_2$, some fixed initial node $v \in V_1$ and a function $w\colon E \to \mathbb{Z}$ assigning weights to the edges of G (see Figure). In the beginning of the game, a token is placed on the initial vertex v. At each turn, one of the two players moves the token to some other node of the graph. Each turn of the game is organized as follows. If the token is currently in some node $u \in V_1$, then the first player (Alice) can move it to any node w such that $(u, w) \in E$. If, on the other hand, $u \in V_2$, then the second player (Bob) can move the token to any node w such that $(u, w) \in E$. The game is infinite, and the process of the game can be described by a sequence of nodes $v_0, v_1, v_2, \ldots$ which the token visits. Note that $v_0 = v$. The first player wins the game if

$$\liminf_{n\to\infty} \frac{1}{t} \sum_{i=1}^{t} w(v_{i-1}, v_i) > 0. \tag{2.1}$$

The corresponding mean payoff game problem is to decide whether the first player has a winning strategy.

In the game on the figure, if the game starts in the upper left vertex, then for Alice, it is not reasonable to move to the right upper vertex since Bob will go back and win one point overall. If instead Alice moves to the middle vertex on the right, then it is easy to see that she will win whatever Bob does. So in this game, Alice wins.

For more information on mean payoff games, see the survey Klauck (2002). It is known that both of the players have an optimal positional strategy, that is, strategies depending only on the current position of the token and not on the history. From this, in particular, it follows that the optimal value of the game (the largest left-hand side of (2.1) that the first player can achieve) is a rational number with the denominator polynomial in the number of vertices of G.



Mean payoff game, an example.

Also it is clear that the negated mean payoff game problem, that is the problem whether the second player has a winning strategy, is Karp reducible to mean payoff games. Indeed, just change the roles of the players and add the new initial vertex v' with no ingoing edges and one outgoing edge (v', v) to pass the move to the second player. The problem that the value of the game might be zero can be handled by changing all weights to small rational numbers (after that the value of the game is always nonzero) and multiplying them by the denominator to make them integer.

During our reductions, sometimes we will be in the situation that we reduce some problem to solution of several instances of another problem equivalent to mean payoff games, that is the input to the original problem will be a "yes" instance if all inputs constructed during the reduction are "yes" inputs of the problem equivalent to mean payoff games. In this case, we can actually substitute several inputs by one since we can do this for mean payoff games. Indeed, we can just consider the graph consisting of pairwise unconnected copies of all graphs corresponding to the several inputs and we have added the node belonging to the second player from which he can reach all starting nodes of all connected components and add one more node to pass the first move to the first player.

2.2. Tropical and min-plus linear systems. Consider an arbitrary tropical linear system (1.1). Note that its tropical prevariety S is closed under tropical scalar multiplication, or, to state it the other way, $S = S + \mathbb{Z}\vec{1}$, where by $\vec{1}$, we denote the vector of all ones. Thus, we can consider the set of solutions of (1.1) as a set in the projective space $\mathbb{TP}^{n-1} = \mathbb{R}^n/\langle\vec{1}\rangle_{\mathbb{R}}$. In this paper, we will alternatively consider the solution prevariety in the spaces $\mathbb{R}^n$ and $\mathbb{TR}^{n-1}$ depending on which one is more convenient in the current argument. Note that in the definition of the dimension of the tropical prevariety, we consider the prevariety in the space $\mathbb{R}^n$. It is convenient also to introduce the *projective dimension* of the tropical prevariety to be just the dimension of the prevariety in the space $\mathbb{TR}^{n-1}$. Clearly, the projective dimension is always smaller by one than the usual dimension.

Consider some matrix $A \in \mathbb{Z}^{m\times n}$. Note that adding some number to all entries of some row of A does not change the tropical prevariety of system (1.1). Thus, in the course of the proofs, we can freely add and subtract some number from some row of the matrix under consideration.

Let us add the same vector $\vec{v} \in \mathbb{Z}^n$ to all rows of A and denote the resulting matrix by $A_{\vec{v}}$. Then the tropical prevariety of $A_{\vec{v}}$ is a linear translation of the tropical prevariety of A. Since many important properties survive after translations, we will apply this kind of transformations to matrices.

Finally, let us multiply all entries of the matrix by the same constant $c \in \mathbb{N}$. Note that all vectors in the tropical prevariety are also multiplied by the same constant. Sometimes we will perform this operation also. In particular, this observation implies that all our results are also true for the domains $\mathbb{Q}$ and $\mathbb{Q} \cup \{\infty\}$.

All observations above in this subsection are also true for min-plus systems of equations.

Besides the systems of min-plus linear equations, we can consider the systems of min-plus linear inequalities. The definition is very similar: the *(two-sided) min-plus linear system of inequalities* associated with a pair of matrices $A, B \in K^{m\times n}$ is the system

$$\min_{1\leq j\leq n}\{a_{ij} + x_j\} \leq \min_{1\leq j\leq n}\{b_{ij} + x_j\},\ 1 \leq i \leq m. \qquad (2.2)$$

It turns out that min-plus linear systems of equations and min-plus linear systems of inequalities are essentially the same. More precisely, given a min-plus system of linear equations, it is easy to construct an equivalent system of min-plus linear inequalities and vice versa. Indeed, each min-plus linear equation $L_1(x) = L_2(x)$ is equivalent to the pair of min-plus inequalities $L_1(x) \geq L_2(x)$ and $L_1(x) \leq L_2(x)$. On the other hand, the min-plus linear inequality $L_1(x) \leq L_2(x)$ is equivalent to the min-plus equation $L_1(x) = \min(L_1(x), L_2(x))$. It is not hard to see that the last equation can be transformed to the form of a min-plus linear equation. These simple observations immediately give polynomial-time reductions between min-plus linear systems of equations and inequalities for all problems we consider. Thus, in this paper, we will switch freely between min-plus linear equations and min-plus linear inequalities and trivially all complexity results are true for both equations and inequalities.

Consider a tropical linear system with the matrix $A \in \mathbb{Z}^{m \times n}$ and assume that $a_{ij} \geq 0$ for all $i \in [m]$, $j \in [n]$ (we can reduce any matrix to this form adding vectors $c \cdot \vec{1}$ to the rows). Assume that the entries of the matrix are bounded by some value M, that is $a_{ij} \leq M$.

We will use the following lemma proven in Grigoriev (2013) and bounding the size of the smallest solution of the tropical linear system.

Lemma 2.3 (Grigoriev 2013). *If the system has a solution $(x_1, \ldots, x_n)$, then it has a solution $(x'_1, \ldots, x'_n)$ satisfying $0 \leq x'_j \leq M$ for all $1 \leq j \leq n$.*

In this paper, we consider the following problems.

- TropSolv. In this problem we are given an integer matrix $A \in \mathbb{Z}^{m \times n}$. The problem is to decide whether the corresponding tropical system (1.1) is solvable.

- TropEquiv. In this problem we are given two integer matrices $A \in \mathbb{Z}^{m \times n}$ and $B \in \mathbb{Z}^{k \times n}$. The problem is to decide whether the corresponding tropical systems (1.1) over the same set of variables are equivalent.

- TropImpl. In this problem we are given an integer matrix $A \in \mathbb{Z}^{m\times n}$ and a vector $l \in \mathbb{Z}^n$. The problem is to decide whether the tropical system (1.1) corresponding to A implies the tropical equation corresponding to l.
- TropDim. In this problem we are given an integer matrix $A \in \mathbb{Z}^{m\times n}$ and a number $k \in \mathbb{N}$. The problem is to decide whether the dimension of the tropical prevariety corresponding to the tropical system (1.1) is at least k.

For all problems above, there are also variants of them over $\mathbb{Z}_\infty$. We denote them by the subscript ∞, for example, in the problem TropSolv$_\infty$, we are given a matrix $A \in \mathbb{Z}_\infty^{m\times n}$ and the problem is to decide whether the corresponding tropical system over $\mathbb{Z}_\infty$ is solvable. For local dimension of tropical prevariety (that is the dimension of the neighborhood of some point) over $\mathbb{Z}_\infty$ in a point with some infinite coordinates, we consider just the dimension over finite coordinates only.

Recall that when we consider systems over $\mathbb{Z}_\infty$, we do not allow solutions consisting only of infinities.

Next, we show some simple relations between the $\mathbb{Z}$ and $\mathbb{Z}_\infty$ cases.

Lemma 2.4. *(i)* TropSolv $\leq_p$ TropSolv$_\infty$*;*

(ii) TropImpl $\leq_p$ TropImpl$_\infty$*;*

(iii) TropDim $\leq_p$ TropDim$_\infty$*.*

Proof. For the first reduction, if we are given a tropical linear system with coefficients in $\mathbb{Z}$, then it is solvable over $\mathbb{Z}$ iff it is solvable over $\mathbb{Z}_\infty$. For the nontrivial direction of this statement, if there is a solution over $\mathbb{Z}_\infty$ in which some coordinates are infinite, we can just substitute them by large enough finite numbers.

For the second reduction, if we are given a tropical linear system and a tropical linear equation over $\mathbb{Z}$, consider them over $\mathbb{Z}_\infty$. If there was no implication over $\mathbb{Z}$, that is there is a solution over $\mathbb{Z}$ of the system, which is not a solution of the equation, then clearly the same is true over $\mathbb{Z}_\infty$, and there is also no implication. If there is no implication over $\mathbb{Z}_\infty$, then there is a solution over $\mathbb{Z}_\infty$ of the system, which is not a solution of the equation. Substituting infinities in the solution by large enough constants, we get that there is also no implication over $\mathbb{Z}$.

For the last reduction, again if we have a tropical linear system with coefficients in $\mathbb{Z}$ and we have some solution with infinite coordinates then if we substitute infinities by large enough finite numbers, the local dimension at this point does not decrease. □

2.3. Max-atom problem. For the proof of our first result, we need an intermediate *max-atom problem* or MAP. In this problem, we are given a system of m inequalities in variables $x_1, x_2, \ldots, x_n$. The system is given by integers $a_1, \ldots, a_m$, and each inequality has the form

$$\max\{x_{j_1}, x_{j_2}\} + a_j \geq x_{j_3}, \tag{2.5}$$

where $j_1, j_2, j_3 \in [n]$. The problem is to decide whether there is a solution to the system (over integers). It is known that this problem is equivalent to mean payoff games Möhring *et al.* (2004).

3. Solving tropical systems is equivalent to mean payoff games

In this section, we prove that the solvability problem for tropical linear systems is equivalent to mean payoff games. For this, we show that TropSolv is equivalent to MAP. First, we prove the following simple lemma.

Lemma 3.1. TropSolv *reduces in polynomial time to the solvability problem for a system of min-plus equations. Moreover, for a given tropical linear system, we can effectively construct a system of min-plus equations over the same set of variables and with the same set of solutions. The same is true for the domain $\mathbb{Z}_\infty$.*

Remark 3.2. *This lemma is standard, and the arguments of this type appears frequently in this area. We provide a proof for the sake of completeness.*

PROOF. Let A be some tropical linear system. For each of its equations, we construct a system of min-plus equations over the same set of variables which is *equivalent* to this tropical equation.

For this, let

$$\min\{x_1 + a_1, x_2 + a_2, \ldots, x_n + a_n\} \tag{3.3}$$

be one of the rows of the system A. For notational simplicity, we denote $y_i = x_i + a_i$ for $i = 1, \ldots, n$. Then we can rewrite (3.3) as $\min\{y_1, \ldots, y_n\}$.

It is easy to see that the fact that the minimum in the expression above is attained at least twice is equivalent to the fact that for any $i = 1, \ldots, n$ it is true that

$$\min\{y_1, \ldots, y_n\} = \min\{y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n\}. \tag{3.4}$$

And each of these equations is in turn equivalent to the equation

$$\begin{aligned}&\min\{y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_n\}\\ &\quad = \min\{y_1, \ldots, y_{i-1}, y_i + 1, y_{i+1}, \ldots, y_n\}.\end{aligned}$$

The last equation is already in min-plus form, and thus, we have that any tropical equation is equivalent to a system of min-plus equations. To get a min-plus system equivalent to the tropical system, we just unite the min-plus systems for all equations of A.

Note that exactly the same analysis works for the case $\mathbb{Z}_\infty$. □

REMARK 3.5. *It was proven in Akian et al. (2012, Corollary 3.7) that the solvability problem for systems of min-plus inequalities (and thus the same problem for the systems of equations) over both $\mathbb{Z}$ and $\mathbb{Z}_\infty$ is equivalent to mean payoff games. It was also proven there that* TROPSOLV *and* TROPSOLV$_\infty$ *reduces to mean payoff games. The lemma above shows, in particular, that the latter result follows easily from the former.*

As a corollary of Lemma 3.1, we have a reduction from TROPSOLV to MAP.

COROLLARY 3.6. TROPSOLV $\leq_p$ MAP.

Proof. First, by Lemma 3.1, we can reduce TropSolv to the solvability problem for min-plus linear systems of equations. Multiplying all equations of the system by -1 and substituting $-x_i$ by x_i for all i, we can easily reduce min-plus linear systems of equations to max-plus linear systems of equations. Finally, it was shown in Bezem *et al.* (2010, Theorem 1) that the solvability problem for max-plus systems of linear equations is polynomial-time equivalent to MAP. □

Now we proceed to the reduction in the reverse direction. For this, we will need the following technical lemma.

Lemma 3.7. *Let $k \leq n$ and consider an arbitrary vector $\vec{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$. Then for any $C \in \mathbb{Z}$, there is a tropical linear system $A \in \mathbb{Z}^{m \times n}$, where $m = n - k + 1$, such that*

- *for any $i \in [m]$ and any $j \in [k]$ we have $a_{ij} = a_j$;*
- *for any $i \in [m]$ and any $j \in [n] \setminus [k]$ we have $a_{ij} \geq C$;*
- *for any solution of A and for any row the minimum is attained at least twice in the $\vec{a}$-part of the row.*

Proof. To prove the lemma, we will introduce several tropical equations and the system A will be the union of them. First consider the row corresponding to the following vector

$$l_0 = (\vec{a}, C+1, \ldots, C+1),$$

where $l_0 \in \mathbb{Z}^n$. Next, for each $i = k+1, \ldots, n$ let

$$l_i = l_0 - e_i = (\vec{a}, C+1, \ldots, C, \ldots, C+1),$$

where $e_i \in \mathbb{Z}^n$ is a vector with 1 in the i-th coordinate and 0 in all other coordinates. We let A be the system consisting of equations $l_0, l_{k+1}, \ldots, l_n$.

To make following the argument below easier, we provide an example of the system A with n=6, k=3, $\vec{a}$=$(1, 2, 3)$ and C=9:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 10 & 10 & 10 \\ 1 & 2 & 3 & 9 & 10 & 10 \\ 1 & 2 & 3 & 10 & 9 & 10 \\ 1 & 2 & 3 & 10 & 10 & 9 \end{array}\right).$$

Suppose, by way of contradiction, that A has a solution such that in some row l_i, there is at most one minimum in the $\vec{a}$-part. This means that in this row, there is a minimum in a column j such that $k+1 \leq j \leq n$. If $j-k \neq i$, consider the row l_{j-k}. It is easy to see that this row contains exactly one minimum (in the column j) and this is the contradiction. Thus, the minimum in the row l_i outside of the $\vec{a}$-part can be situated only in the column $i+k$ (this corresponds to one of the 9-entries in the example) and, in particular, $i \neq 0$. But since the minimum is attained at least twice, there is at least one minimum in the $\vec{a}$-part of l_i. Thus, there is exactly one minimum in the $\vec{a}$-part of the row l_i and one more minimum in the column $i+k$. Now consider the row l_0. Clearly, there is exactly one minimums in this row, and this gives a contradiction. □

To prove the desired reduction, we will make use of the following lemma bounding the size of the minimal solution of MAP which was proven in Bezem *et al.* (2010).

LEMMA 3.8 (Bezem *et al.* 2010). *Let M be a* MAP *system over variables $x_1, \dots, x_n$ and let C be the sum of absolute values of all constants in M. If M is solvable, then it has a solution $\vec{x}$ such that $\max_{i\in[n]}\{x_i\} - \min_{i\in[n]}\{x_i\} \leq C$.*

Now, we are ready to prove the reduction in the backwards direction.

THEOREM 3.9. MAP $\leq_p$ TROPSOLV.

PROOF. Suppose we are given a system A of inequalities of the form $\max\{x, y\} + k \geq z$. First multiply all inequalities by (-1) and make a transformation of variables $x \mapsto (-x)$. Then we have a system B of inequalities of the form $\min\{x, y\} + k \leq z$ which is solvable if and only if the initial system is solvable. We denote by C the sum of absolute values of all constants in B.

Now, we are ready to construct a tropical linear system T. Let us denote the variables of B by $x_1, \dots, x_n$. Our tropical linear system for each variable x_i of B will have two corresponding variables x_i and x_i'. We would like these variables to be equal in any solution of T. This can be easily achieved by means of Lemma 3.7. For this, let in this lemma $k = 2$, $\vec{a} = (0, 0)$, $C = C$ and apply it to the variables x_i, x'_i. As a result, we get the system T_i which guarantees that in each solution the variables x_i and x'_i are equal. We include systems T_i for all i into the system T.

Next, we have to guarantee that for any inequality $\min\{x, y\} + k \leq z$ of B, where x, y, z are some variables among $x_1, \dots, x_n$, the same inequality is true for the solutions of T. Since we already know that the variables x_i and x'_i are equal for each solution of T, it suffices to say that

$$\min\{x, x', y, y', z - k, z' - k + 1\}$$

is attained at least twice. However, we have to add other variables to this tropical polynomial. This can be done again by Lemma 3.7. For this let in this lemma $k = 6$, $\vec{a} = (0, 0, 0, 0, -k, -k+1)$, $C = C$, apply the lemma to the variables x, x', y, y', z, z' and include the resulting system into the system T.

Now the construction of T is finished, and we have to show that it is solvable if and only if B is solvable. Assume first that T has a solution. Then it follows from the construction of T that for each $i = 1, \dots, n$ the variables x_i and x'_i are equal. And from this and again from the construction of T, it follows that each inequality of B is true.

On the other hand, suppose that B is satisfiable. Then, by Lemma 3.8, there is a solution $\vec{x}$ such that

$$\max_{i \in [n]}\{x_i\} - \min_{i \in [n]}\{x_i\} \leq C.$$

Since we can add any constant to all coordinates of $\vec{x}$, we can assume that $\min_{i \in [n]}\{x_i\} = 0$ and thus for all i we have $0 \leq x_i \leq C$. For the solution of T, let x_i be the same as in the solution of B and let $x'_i = x_i$ for all i. It is left to check that this vector is a solution of T. We can check it for all rows separately. If a row is in T_i for some i, then clearly the minimum is attained on x_i and x'_i due to the choice of the constant C in application of Lemma 3.7. And if a row came from some inequality $\min\{x, y\} + k \leq z$ of B, then clearly the minimum is attained either on x and x', or on y and y'. □

From Theorem 3.9 and Corollary 3.6 we conclude the following.

COROLLARY 3.10. *The problem* TROPSOLV *is polynomially equivalent to mean payoff games.*

Moreover, we can also conclude the same for the problem TROPSOLV_∞.

COROLLARY 3.11. *The problem* TROPSOLV_∞ *is polynomially equivalent to mean payoff games.*

PROOF. It was proven in Akian *et al.* (2012) that TROPSOLV_∞ is Karp reducible to mean payoff games (see also the remark after Lemma 3.1). Theorem 3.9 gives us that mean payoff games can be reduced to TROPSOLV. Finally, TROPSOLV reduces to TROPSOLV_∞ by Lemma 2.4 and thus all three problems are equivalent. □

In particular, it follows that the problems TROPSOLV and TROPSOLV_∞ are polynomial-time equivalent. But the given proof of equivalence of these two purely tropical problems rather unnaturally goes through mean payoff games. In the preliminary version of this paper Grigoriev & Podolskii (2012), we give a direct combinatorial proof of this equivalence (and also of analogous equivalences for min-plus systems).

One more corollary of our analysis concerns the equivalence and implication problems for tropical linear systems.

COROLLARY 3.12. *The problems* TROPEQUIV, TROPEQUIV_∞ *are equivalent to mean payoff games under Karp reductions. The problems* TROPIMPL *and* TROPIMPL_∞ *are equivalent to mean payoff games under Cook reductions.*

PROOF. It is easy to see that the problem TROPEQUIV is equivalent to the problem TROPIMPL (under a Cook reduction). Indeed, suppose we are given a tropical system A and a tropical equation l. Deciding whether l follows from A is equivalent to deciding whether systems A and $A \cup \{l\}$ are equivalent. On the other hand, if we need to check whether two systems A and B are equivalent, it is enough to check whether each equation of the second system follows from the first system and vise versa. Thus, we have that TropEquiv is equivalent to TropImpl. The same argument gives us also that TropEquiv_∞ is equivalent to TropImpl_∞. Note that the same argument works also for min-plus systems.

Next, it is easy to construct the reduction from TropSolv to TropEquiv. Indeed, to check whether some system is solvable, it is enough to check whether it is equivalent to some fixed nonsolvable system.

The reduction of TropImpl to TropImpl_∞ is in Lemma 2.4.

Thus, it is only left to show that TropEquiv_∞ reduces to mean payoff games. Assume that we are given two tropical systems A_1 and A_2 and we have to check whether they are equivalent. First by Lemma 3.1 for each of the systems, we construct the system of min-plus equations with the same solution sets. By the argument in Preliminaries, each system of min-plus equations is equivalent to a system of min-plus inequalities. Then we reduce the equivalence problem for the systems of inequalities to the implication problem for inequalities by the same argument as above. And finally we can apply the result of Allamigeon *et al.* (2011) stating that the implication problem for min-plus inequalities over $\mathbb{Z}_\infty$ is equivalent to mean payoff games.

Keeping in mind the discussion in the preliminaries, it is easy to see that these reductions can be transformed into Karp reductions for the case of the problems TropEquiv and TropEquiv_∞. □

It is not hard to see that analogous results for min-plus linear systems follow along the same lines from known results.

Lemma 3.13. *The equivalence and implication problems for min-plus systems of linear equations over both $\mathbb{Z}$ and $\mathbb{Z}_\infty$ are equivalent to mean payoff games.*

Proof. The same proof as for Corollary 3.12 works, only we do not need Lemma 3.1 here.

The result on the implication problem for min-plus systems of linear inequalities over $\mathbb{Z}_\infty$ was already proven in Allamigeon *et al.* (2011). □

In the preprint version of this paper Grigoriev & Podolskii (2012), we give direct combinatorial proofs of equivalence between solvability and equivalence problems for both min-plus and tropical linear systems (both over $\mathbb{Z}$ and $\mathbb{Z}_\infty$).

4. Dimension and the tropical rank

In the case of classical linear systems, the dimension of the solution space is closely related to the rank of the matrix. The natural idea is that maybe the dimension of the tropical prevariety is also related to some "rank" of the tropical matrix and NP-completeness can be derived from the completeness for this "rank."

There are three widely considered notions of the "rank" in tropical algebra studied in the literature: *Barvinok rank*, *Kapranov rank* and *tropical rank*. We refer the reader to the paper Develin *et al.* (2005) for the definitions and further information on these notions. For us, only the tropical rank is relevant and it will be convenient to use the following definition. The *tropical rank* of the matrix A of size $n \times m$ is the largest integer r such that there is a subset of r columns of A such that the tropical linear system generated by them is unsolvable.

Also it is important for us that there is a relation

$$\textit{tropical rank}(A) \leq \textit{Kapranov rank}(A) \leq \textit{Barvinok rank}(A), \tag{4.1}$$

for any matrix A. All inequalities in (4.1) can be strict (Develin *et al.* 2005). It is known that the problem of computing the tropical rank is NP-hard (Kim & Roush 2005) and the problem of computing the Kapranov rank is also NP-hard (Kim & Roush 2006).

We will show the following result.

Lemma 4.2. *For any matrix $A \in \mathbb{R}^{m\times n}$ we have*

$$n - \textit{tropical dimension}(A) \leq \textit{tropical rank}(A),$$

and the inequality can be both tight and strict. Here by the tropical dimension, we mean the affine variant of dimension.

This lemma together with (4.1) shows that there is a relation between the tropical dimension and rank of the tropical matrix, but this relation is not enough for computational needs.

Proof. To prove the inequality, let the tropical rank of the matrix A be equal to r and consider a maximal set C of columns of A such that the tropical linear system generated by them is unsolvable. The size of this set of columns is equal to r. Add one of the remaining $n-r$ columns to C and denote the resulting $m \times (r+1)$ matrix by C'. Due to the maximality property of C, there is a solution to the tropical linear system with the columns C'. This solution can be extended to a solution of the whole system by fixing all coordinates x_i with $i \in [n] \setminus C'$ to be large enough numbers. Note that these coordinates of the resulting solution of A can be changed locally (if the numbers are chosen large enough). Thus, we have that the solution space contains a subspace of dimension $n-(r+1)$. But note that currently we have projective dimension: some of the coordinates never change in this subspace. So, we can add the vector $(1, \dots, 1)$ to our subspace and get the desired subspace of dimension $n-r$.

To show that the inequality can be tight consider for example the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

It is easy to see that the solution space of the corresponding tropical system consists of points (c, c, c) for any c and thus has dimension 1. The tropical rank of this matrix is 2. To see this, consider the submatrix defined by the first two columns.

To show that on the other hand, the inequality can be strict consider the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

The tropical rank of this matrix is 4. To see this, consider the submatrix defined by the first four columns. On the other hand, the dimension of the solution space is also 4 since it contains the subspace generated by $(1,0,0,0,0)$, $(0,1,0,0,0)$, $(0,0,1,0,0)$, $(1,1,1,1,1)$.

Both of the examples above can be easily generalized to arbitrary matrix size. □

5. Combinatorial characterization of the dimension of a tropical prevariety

In our analysis, it will be convenient to use the following definition.

DEFINITION 5.1. *Let A be a matrix of size $m \times n$. We associate with it the table A^* of the same size $m \times n$ in which we put the star $*$ to the entry (i,j) iff $a_{ij} = \min_k a_{ik}$ and we leave all other entries empty.*

The table A^* captures properties of the tropical system A essential to us. For example, the vector $x = (x_1, \dots, x_n)$ is a solution to the system A if there are at least two stars in every row of the table $(\{a_{ij} + x_j\}_{ij})^*$.

Next, we give a combinatorial characterization of local dimension (at a given point) of a tropical prevariety in terms of the table A^*. For this, we will use the following block triangular form of the matrix (which exists if the table corresponding to the matrix contains at least two stars in each row).

DEFINITION 5.2. *The block triangular form of size d of the matrix A is a partition of the set of rows of A into sets $R_1, R_2, \dots, R_d$ (some of the sets R_i might be empty) and a partition of the set of columns of A into nonempty sets $C_1, \dots, C_d$ with the following properties (see figure):*

(i) for every i each row in R_i has at least two stars in columns C_i in A^*;

(ii) if $1 \le i < j \le d$, then the rows in R_i have no stars in columns C_j in A^*.

$$
A^* = \begin{array}{c}
\\ R_1 \\ \\ \\ R_2 \\ \\ \\ \vdots \\ \\ \\ R_d \\ \\
\end{array}
\overset{\begin{array}{cccc} \quad C_1 \quad & \quad C_2 \quad & \quad \cdots \quad & \quad C_d \quad \end{array}}{\left(\begin{array}{ccc|ccc|ccc|ccc}
* & * & & & & & & & & & & \\
* & & * & & \emptyset & & & \emptyset & & & \emptyset & \\
& * & * & & & & & & & & & \\
\hline
* & & & * & * & & & & & & & \\
* & * & & & * & * & & \emptyset & & & \emptyset & \\
& & & * & & * & & & & & & \\
\hline
* & & & & & & * & & * & & & \\
* & & & * & * & * & * & * & & & \emptyset & \\
& * & & & & & & * & * & & & \\
\hline
& & & & * & & & & & & * & * \\
& & * & & & * & & * & & * & * & \\
* & * & & & & & * & & & * & * &
\end{array}\right)}
$$

For example, for the table

$$
\left(\begin{array}{c|c|c|c|c}
* & & & * & \\
* & & * & & \\
& * & & * & *
\end{array}\right)
$$

there is a block triangular form of size 2 with $C_1 = \{1, 3, 4\}, C_2 = \{2, 5\}, R_1 = \{1, 2\}, R_2 = \{3\}$. There is one more block triangular form of size 2 with $C_1 = \{1, 2, 3, 4\}, C_2 = \{5\}, R_1 = \{1, 2, 3\}, R_2 = \emptyset$. It is not hard to see that there is no block triangular form of size 3.

We are looking for a block triangular form of the matrix with the largest possible d. We next make several observations on the structure of the block triangular form of maximal size:

- Without loss of generality the pairs (C_i, R_i) with empty R_i can be moved to the beginning of the list permuting correspondingly the lists of C_i's and of R_i's.

- We can assume that the pairs with empty R_i have $|C_i| = 1$. Indeed, if $|C_i| > 1$ we can break it into several sets of size 1 without violating the properties of the block triangular form and the size will only increase.

Now we are ready to give a combinatorial characterization of the dimension of the prevariety.

Theorem 5.3. *Assume that the zero vector is a solution of the tropical linear system A over $\mathbb{Z}$. Then the local projective dimension of the system A in the zero solution is equal to the maximal d such that there is a block triangular form of A of size $d+1$.*

Clearly the case of arbitrary solutions can be reduced to the zero solution.

To prove the theorem, we will use the following technical lemma.

Lemma 5.4. *In any cone of dimension k in the space $\mathbb{R}^n$, there is a vector with at least k distinct coordinates.*

Proof. Consider some set of linearly independent vectors $f^1, f^2, \ldots, f^k$ in the cone.

For each coordinate i of the basis vectors $f^1, \ldots, f^k$, consider the tuple $f_i = (f_i^1, \ldots, f_i^k)$ of all ith coordinates of vectors in the basis. Note that $a = |\{f_1, f_2, \ldots, f_n\}| \geq k$, that is, the number of different vectors among f_i is at least k. Indeed, if the number of different tuples among f_i is less than k, then we can consider the vector equation $\sum_i c_i f^i = \vec{0}$ for $c_i \in \mathbb{R}$ as a linear system on $c_1, \ldots, c_k$. This system has fewer than k equations and thus has a nonzero solution. This means that $f^1, \ldots, f^k$ are linearly dependent and we have a contradiction.

Next, multiplying all vectors f^j by a positive number, we can assume that if $|f_i^j - f_{i'}^{j'}|$ is nonzero, then it is greater or equal to 2. Denote $C = \max_{i,j,i',j'} |f_i^j - f_{i'}^{j'}|$.

Now we can consider the vector

$$f' = \sum_{j=1}^{k} C^{2j-2} f^j.$$

We claim that the number of different coordinates of f' is at least k. For this, we show that if $f_i \neq f_{i'}$, then $f'_i \neq f'_{i'}$. Indeed, suppose this is not the case. Then

$$\sum_{j=1}^{k} C^{2j-2}(f_i^j - f_{i'}^j) = 0. \tag{5.5}$$

Consider the largest j such that $f_i^j - f_{i'}^j \neq 0$. Then the corresponding term in (5.5) is greater than C^{2j-2} in absolute value. On the other hand, each previous term l is at most C^{2l-1}. Since

$$C^{2j-2} > \sum_{l=1}^{2j-3} C^l$$

for $C \geq 2$, the sum in (5.5) cannot be zero and we have a contradiction. □

Proof (of the theorem). We can assume that the minimum in each row of A is 0.

Denote by d the size of the largest block triangular form minus 1 and denote the local projective dimension of the tropical prevariety in the zero solution by $\dim A$.

It is not hard to see that $\dim A \geq d$. Indeed, consider the block triangular form of size $d+1$ and take as the basis for the subspace of tropical prevariety the negated characteristic vectors of $\bigcup_{j \geq i} C_j$ for all $i = 2, \ldots, d+1$. It is clear that any point in the cone generated by these vectors close enough to the zero vector is a solution to the tropical system.

It remains to prove that $\dim A \leq d$. Consider the polytope (over $\mathbb{R}^n$) of the largest dimension $k = \dim A + 1$ in the tropical prevariety containing the zero point. We can restrict ourselves to a cone of the same dimension whose vertex is the zero point and such that the neighborhood of the vertex intersected with the cone lies in the polytope. By Lemma 5.4, we can find a vector f' in this cone with $a \geq k$ different coordinates. Dividing the vector by a large enough number, we can ensure it to be in the prevariety and all its coordinates to be at most 1. Since our prevariety is closed under the translation by a vector $c \cdot \vec{1} = (c, \ldots, c)$, we can assume that the coordinates of f' are nonpositive. Let us enumerate the coordinates of f' in increasing order: $b_1 < \cdots < b_a \leq 0$.

We denote by B_j for $j \in [a]$ the set of coordinates of f' with the values at most b_j, that is $B_j = \{l \in [n] \mid f'_l \leq b_j\}$. Note that $B_1 \subset B_2 \subset \cdots \subset B_a$.

Claim 5.6. *For every j and for every row l columns in B_j contain in l either no stars, or at least two stars in the table A^*.*

Proof (of the claim). The proof goes by induction on j.

For the base of induction, consider the set B_1. Note that the columns in B_1 are precisely the columns with the smallest coordinates of f'. Suppose that there is a star in row l and in the columns of B_1. Let us add to each column i of the matrix A the i-th coordinate of f' (which is non-positive). Since f' is a solution, the resulting matrix should have at least two stars in the row l. But the star among the columns of B_1 has the value b_1 which is the smallest possible value of coordinates in the row l. Thus, there should be one more coordinate with the same value, which can appear only in columns of B_1 and it can only be a star of A. Thus, the columns of B_1 have at least two stars in the row l.

For the induction step assume that we have proved the claim for B_{j-1} and consider the set B_j. If the row l contains two stars in B_{j-1}, it also contains two stars in B_j. Thus, we can assume that the row l contains no stars in B_{j-1}. Assume that there is a star in B_j. Again add coordinates of f' to the corresponding columns of A. Since there are no stars in B_{j-1}, all corresponding coordinates of the row l in these columns are positive (recall that the coordinates of f' are less than 1). The star in B_j has coordinate b_j, and this is the smallest possible value of this coordinate. Since f' is a solution to the system, there should be one more coordinate with the same value and this can be only the coordinate in B_j and also the initial star of A. Thus, there are at least two stars in B_j in the row l. □

Now, we are ready to describe the sets of rows and columns corresponding to the desired triangular form. The size of this form will be a. For the set C_{a-i+1}, we take $B_i \setminus B_{i-1}$, which is nonempty. The choice of R_i is straightforward: We take all rows that have at least two stars in the set C_i and no stars in $C_{i+1}, \ldots, C_k$.

The properties of the triangular form follow from the construction. We only have to check that every row is in some R_i. Consider an arbitrary row l and let i be the smallest number such that B_i contains a star of row l. By the claim, this star cannot be unique, and since by the choice of B_i, there are no stars in row l and in the columns of B_{i-1}, we have $l \in R_{a-i+1}$. □

Clearly, the same argument works for $\mathbb{R}$.

It is also easy to see that the same argument works for the tropical linear systems over $\mathbb{Z}_\infty$: We can ignore infinite coordinates of the solution we consider, and infinite entries in the matrix do not affect the proof. That is, given a solution $x \in \mathbb{Z}_\infty$, we remove from the matrix A_x (see Preliminaries for the definition) all columns for which the corresponding coordinate of x is infinite and denotes the resulting matrix by $\widetilde{A}_x$. Consider the corresponding table $\widetilde{A}_x^*$. It is not hard to see that the rows consisting of infinities do not affect the maximal size of the block triangular form. Note that infinities in other rows of the matrix cannot become stars in $\widetilde{A}_x^*$ in the neighborhood of x and thus if we substitute them by large enough numbers, neither the local dimension nor the block triangular forms of maximal size change.

Almost the same argument works for min-plus linear systems $A \odot x = B \odot x$, where A, B are in $\mathbb{Z}^{m\times n}$ or $\mathbb{Z}_\infty^{m\times n}$. Here, we consider the joint matrix $D = \left(\begin{array}{c|c} A & B \end{array} \right)$ and also consider the table D^*. The block triangular form of size d is now the row partition $R_1, R_2, \ldots, R_d$, where some of the sets R_i might be empty, and the partition $C_1, \ldots, C_d$ of $\{1, \ldots, n\}$, where all C_i are nonempty. For a given set C_i, we associate the columns in the A-part of D with the corresponding numbers and the columns in the B-part of D with the same numbers. The partitions should satisfy the following properties:

1. for every i each row in R_i has at least one star in columns with numbers C_i in the A-part of D and at least one star in columns with numbers C_i in the B-part of D;

2. if $1 \le i < j \le d$, then the rows in R_i have no stars in columns with numbers C_j in both parts of D^*.

The analog of Theorem 5.3 can be proven by a straightforward adaptation of the proof above.

6. Computing the dimension of tropical and min-plus linear prevarieties is NP-complete

Before proving the completeness result, we prove the following technical lemma.

LEMMA 6.1. *If we are given a tropical linear system A over n variables the entries of which are nonnegative and of value at most M, then the maximal dimension of the tropical prevariety is achieved at some point with all finite coordinates at most $(M+1)n$.*

PROOF. We have seen in Theorem 5.3 that the dimension of the tropical prevariety in a given point depends only on the star-table in this point. Given a star-table, we consider a graph whose nodes are stars in the table and two stars are connected by an edge if they are in the same column or in the same row. We call this graph the star-graph. We say that two columns of the table are connected if there are two stars in these columns which are connected by a path in the star-graph. Note that if there is a path, there is always a path of length at most $2n$ (n row-steps and n column-steps). If all columns are connected, then for each pair of solution coordinates there is a path in a star-graph of length at most $2n$ connecting these two columns. It is not hard to see that for each consecutive solution coordinates in this path, their difference is at most M.

If not all columns are connected, then there are several connected components. We take one of them and reduce all coordinates in this component of the solution by the same number until a new star appears in this set of columns. It is easy to see that this star connects two different components. After that we increase all the coordinates we have just reduced by 1. Then in the place of a new star, we have an entry which is larger by 1 than the star-entries in the same row. Instead of the star, we put the symbol $\circ$ in this entry. And from now on, consider a star-circle-graph. Thus, reducing components one by one and introducing new $\circ$-entries, we get a connected graph. Applying the argument for connected graphs, we obtain the desired $(M+1)n$ upper bound. □

LEMMA 6.2. TROPDIM $\in$ NP *and* TROPDIM$_\infty$ $\in$ NP.

PROOF. As a certificate of an inequality $\dim A \geq k$, one can take a solution x at which the local dimension is at least k, together with a block triangular form of $A_x = \{a_{ij} + x_j\}_{i,j}$ of size at least $k+1$ (see Theorem 5.3). By Lemma 6.1, there is a solution as needed with small enough coordinates. It is easy to check in polynomial time that the given vector is a solution and that the given row and column partitions indeed give a block triangular form of needed size.

The same proof works for TROPDIM$_\infty$. □

To prove NP-completeness, we give a reduction of the VERTEXCOVER problem to our problem.

DEFINITION 6.3. VERTEXCOVER: *given an undirected graph G and a natural number k, decide whether there is a vertex cover of size at most k in G, that is whether there is a subset K of vertices of G of size at most k such that each edge of G has at least one end in K.*

Let n be the number of vertices in G and m be the number of edges in G. We make the following additional assumptions on G and k:

1. G is connected;
2. $k \leq 2n/3$.

With these additional assumptions, vertex cover problem is still NP-complete (this follows from the standard proof of its completeness in Garey & Johnson 1979).

THEOREM 6.4. TROPDIM *is* NP-*complete.*

PROOF. Given a fixed graph G, we will construct a matrix A of a tropical linear system. The matrix A will have $(n+1)$ columns, m rows and all its entries will be 0 or 1, that is $A \in \{0,1\}^{m\times(n+1)}$. The zero vector will be a solution of the tropical system A and the global dimension will be attained at this solution.

Now, we construct the matrix A. The first column of A consists of zeros (and thus the first column of A^* consists of stars). All other columns are labeled by the vertices of G and rows are labeled by the edges of G. The entry (v, e) is set to 0 if and only if v is one of the endpoints of e (see Figure). In particular, this means that every row of A contains exactly 3 zeros and one of them is in the first column.

$$
\begin{array}{c}
\\ \\ \\ \\ e \\ \\ \\
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} & & v & & u & \end{array} \\
\left(\begin{array}{c|c|c|c|c|c}
0 & & & & & \\
0 & & & & & \\
\vdots & & & & & \\
0 & 1\ \ldots 1 & 0 & 1\ \ldots 1 & 0 & 1\ \ldots 1 \\
\vdots & & & & & \\
0 & & & & &
\end{array}\right)
\end{array}
$$

The matrix A

Now, let us consider the zero solution to the tropical system A. We are going to prove that the local dimension of the solution space in this solution is at least $n - k$ if and only if G has a vertex cover of size k. Here, we consider the projective dimension.

First consider a vertex cover $V_1 \subseteq V$ of the graph G with $|V_1| = k$. Consider the set of columns V_1 in A and add the first column to it. It is not hard to see that this set of columns contains at least two zeros in any row: one in the first column and the other one in V_1, since V_1 is a vertex cover. Thus, we can increase all other columns and the codimension is at least $n - k$.

Now suppose that the dimension of the tropical prevariety is $n - d$. Thus, there is a block triangular form of A of size $n - d + 1$ (see Theorem 5.3). Consider the set C_i containing the first column of A. We claim that for all $j \neq i$, the sets R_j are empty. To see this suppose that the set R_j is nonempty for some $j \neq i$. First of all note that $j > i$, otherwise we will have a star over the diagonal blocks in block triangular form. Consider the largest j with nonempty R_j. It is not hard to see that C_j contains all columns except the first one. Indeed, if C_j corresponds to a proper subset of the set of vertices of the graph G, then due to the connectedness of G, there are vertices v, u such that $v \in C_j$, $u \notin C_j$ and there is an edge $\{v, u\}$ in G. This edge is clearly not in R_j, but it also cannot lie in any R_k with

$k < j$, otherwise once more we get a star over the diagonal blocks in the block triangular form. Thus, C_j contains all columns except the first one. But then the size of this block triangular form is 2, but we know that there is a larger block triangular form (recall that the size of a minimal vertex cover is at most $2n/3$). Thus, we have proved that for all $j \neq i$ the sets R_j are empty and thus we can assume that $i = n - d + 1$.

Thus, we have that the block triangular form has the following structure. $R_1, \ldots, R_{n-d}$ are empty, $|C_1| = \ldots = |C_{n-d}| = 1$ and thus $|C_{n-d+1}| = d + 1$ and $R_{n-d+1} = \{1, \ldots, m\}$. Also, the first column is in C_{n-d+1}. It is easy to see that the set of all other columns in C_{n-d+1} forms a vertex cover and thus $k \leq d$.

Now it is only left to show that the zero solution of the system A achieves the maximal dimension in the prevariety. Consider any solution x of the system (1.1). Since we are in the projective tropical space, we can assume that $x_1 = 0$. This means that the first column of the matrix

$$B = \{a_{ij} + x_j\}_{i,j} \tag{6.5}$$

is the same as in A.

Claim 6.6. *For all $j = 1, \ldots, n$ we can assume that $x_j \geq 0$.*

Proof (of the claim). Assume on the contrary that $\alpha = \min_j x_j < 0$. Let $C_1 = \{j \mid x_j = \alpha\}$. The set of columns C_1 corresponds to some set $V_1 \subseteq V$ of vertices of the graph G (note that $1 \notin C_1$). There are two cases.

Case 1. $V_1 \neq V$. Since G is connected, there is an edge e with one end in V_1 and the other end in $V \setminus V_1$. Consider the row of the matrix (6.5) corresponding to e. It is clear that in one entry in this row, we have α and in all others we have numbers greater than α. Thus, this row in the table B^* contains only one star and we have a contradiction.

Case 2. $V_1 = V$. Then to obtain B, we have decreased all columns of A by the same integer. Thus, there are exactly two stars in each row of B^*. And since the graph is connected, the maximal triangular form in this case has size two: the first column with empty set of rows and all other columns with all rows. Thus, the dimension in this point of the prevariety is only 1 which is less than for the zero solution. □

Now consider some column j such that $x_j > 0$. It is not hard to see that all entries of the matrix B in this column are greater than zero. And since the first column consists of zeros, in the column j in B^* there are no stars. Thus, it is easy to describe how the table B^* differs from A^*: we just remove all stars in A^* from the columns j such that $x_j > 0$. It is only left to show that the size of the largest triangular form for A is at least the size of the largest triangular form for B. For this consider the largest triangular form for B. Note that each column j such that $x_j > 0$ should constitute a separate set C_i with an empty set R_i and note that we can assume that all these sets are in the beginning of the list of C_i's. Consider the same system of C_i's and R_i's for the matrix A. It is easy to see that this system is a triangular form for this matrix as well. Thus, the maximal size of a triangular form for A can only be greater than that for B and thus the dimension of the prevariety attains its maximum at the zero solution. □

As a corollary we have the following result.

Corollary 6.7. TropDim$_\infty$ *is* NP*-complete.*

Proof. The containment in NP was already proven in Lemma 6.2. The completeness follows since there is a simple reduction from TropDim to TropDim$_\infty$ given by Lemma 2.4. □

The results of this section can be easily extended to the case of min-plus linear systems.

Theorem 6.8. *Given a min-plus linear system and a natural number k, the problem of deciding whether the solution space of the system has dimension at least k is* NP*-complete.*

Proof. Indeed, the analogs of Lemmas 6.1 and 6.2 can be proven in the same way. For completeness note that Lemma 3.1 gives the reduction from TropDim. □

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