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Robust Static Output Feedback H_∞ Controller Design for Linear Parameter-Varying Time Delay Systems

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Abstract

This paper addresses the problem of synthesizing an H_∞ static output feedback controller for linear parameter-varying (LPV) time delay systems subject to time-varying delay. The motivation for this research stems from the challenges associated with designing controllers for such systems. In addition to considering the static output feedback controller design, we also explore the design of a dynamic output feedback controller within the context of H_∞ control. This is achieved by augmenting the system dynamics with the inclusion of the dynamic controller state and formulating the dynamic output feedback controller in an equivalent form of static output feedback control for the overall system. To tackle the synthesis problem, a quadratic Lyapunov–Krasovskii functional is selected, and a sufficient matrix inequality condition is developed by utilizing the well-known Jensen-type integral inequality. This approach enhances the robustness of the controller design and accounts for the time-varying delay characteristics of the system. An iterative algorithm is introduced to efficiently search for a potential feasible solution set of the synthesis problem. The effectiveness of the proposed controller design methods is demonstrated through several numerical examples. The application of the proposed approaches is showcased, and a comparative analysis is performed. Specifically, the results are compared with the existing approaches in the literature, considering both dynamic output feedback controllers and static output feedback controllers for H_∞ control. This comparison highlights the advantages and differences of the proposed methodology. Overall, this research contributes to the field by offering systematic methodologies for synthesizing H_∞ static output feedback controllers for LPV time delay systems subject to time-varying delay. The incorporation of dynamic output feedback controller design

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provides additional insights and flexibility. The numerical examples validate the applicability of the proposed methods and underscore their potential advantages in terms of robustness and performance.

Keywords Time delay systems · Uncertainty · Static output feedback · H_∞ control · Iterative approaches

1 Introduction

Time delay systems with time-varying delay and external disturbances pose significant challenges in control design due to their inherent complexity and uncertainty. The delay phenomena which inevitably exist in dynamical systems are usually the source of instability and poor performance of the system [6]. The synthesis of robust controllers that can ensure stability and performance in the presence of these challenges remains a critical research area. For several decades, the stabilization of systems has been studied in the literature, see [2–25] and the references therein. In this context, the design of H_∞ static output feedback controllers for linear parameter-varying (LPV) time delay systems has gained considerable attention. It appears that often output feedback based controller structures are reasonably preferred and the performance of the system under external disturbances is evaluated with the minimization of the H_∞ norm for the disturbance attenuation problem.

The motivation for synthesizing an H_∞ static output feedback controller for LPV systems with time-varying delays stems from the need to address control design challenges in practical applications. Many real-world systems exhibit time-varying dynamics due to changing operating conditions, environmental factors, or system configurations. Time delays are often present in these systems, and they can introduce performance degradation, stability issues, or even system failure if not adequately accounted for in the control design. Hence, developing effective control strategies for LPV systems with time-varying delays is of paramount importance.

The problem of synthesizing an H_∞ static output feedback controller for LPV systems with time-varying delays has broad practical applications. These applications span various domains such as aerospace, automotive, process control, robotics, and more. For example, in aerospace systems, time delays can arise from communication delays between aircraft components or sensors. In automotive applications, delays can occur in electronic control units (ECUs) or data transmission in vehicular networks. Industrial process control systems often face transport delays in pipelines or feedback loops. Robotics applications involve delays in control signals and sensory feedback. Addressing time-varying delays in these applications is crucial to achieving stability, robustness, and optimal performance.

Designing an H_∞ static output feedback controller for LPV systems with time-varying delays presents several challenges. First, time-varying delays introduce uncertainties that can degrade the system's performance and stability. The control design must account for these uncertainties and provide robustness against them. Second, the synthesis process needs to ensure that the control objectives, such as disturbance rejection, tracking performance, or stability margins, are met under varying operating conditions and parameter variations. Third, the control design should be computationally efficient, as practical implementation requires real-time or near-real-time computation capabilities. Finally, the control solution needs to strike a balance between complexity and performance, ensuring practical feasibility and ease of implementation.

We now provide a comprehensive review of the related literature, highlighting the advancements and limitations of existing control design techniques for LPV time delay systems. The simultaneous static output feedback low gain H_∞ control problem is investigated in Wu et al. [23] for a collection of linear systems subject to state and input delays. Kong et al. [12] study a non-fragile static output feedback control problem for linear uncertain time delay systems using a linear matrix inequality (LMI) optimization and a hybrid particle swarm optimization algorithm. Choi et al. [5] develop an output feedback H_∞ controller for active half-vehicle suspension systems with time-varying input delay, employing an auxiliary function-based integral inequality method and the reciprocally convex combination approach. Hu et al. [9] design a class of parameter-dependent state feedback controllers for LPV systems subject to time-varying delay and external disturbances within the context of uniformly finite-time boundedness. Cardeliquio et al. [1] introduce an output feedback controller for time delay systems that depends on the output at the present time and the maximum delay, as well as an arbitrary number of intermediate values in between, aiming to minimize the H_∞ norm and maximize the allowed delay. Shao et al. [19] design a robust output feedback H_∞ controller for active suspension systems with actuator faults and time delay, satisfying specific performance constraints. Nejem et al. [15] utilize the LMI dilation approach to design an H_∞ dynamic output feedback controller for linear parameter-varying delayed systems. Nazargah et al. [14] develop both state feedback and dynamic output feedback controllers for suppressing vibration in seismically excited building structures under the effect of time delay. Redondo et al. [17] propose an LMI-based H_∞ output feedback controller to compensate for input and output delays in a roll stability control system. Tlili [22] addresses the design of an H_∞ dynamic output feedback decentralized controller for nonlinear distributed time delay systems through an optimization problem subject to LMI constraints. Parlakci and Jafarov [16] develop a robust delay-dependent guaranteed optimal cost PID multivariable output feedback controller for linear uncertain time delay systems with nonlinear parameter perturbations using an LMI approach. Finally, Shahbazzadeh and Sadati [18] handle the H_∞ control problem for LPV time-delay systems subject to L_2-norm disturbances by employing a dynamic output feedback controller. They appropriately transform the design conditions into LMIs. While the existing literature has made significant contributions to the control design for LPV time delay systems, there are still limitations that need to be addressed. Some works focus on specific aspects such as input delays, actuator faults, or vibration suppression, while others consider either static or dynamic output feedback controllers. In this paper, we aim to overcome these limitations by proposing a comprehensive framework that addresses the synthesis of both H_∞ static and dynamic output feedback controllers for LPV time delay systems subject to time-varying delays and external disturbances. The proposed approach offers enhanced robustness, reduced computational complexity, and improved performance compared to existing techniques. By providing a thorough review of the literature and identifying the gaps and limitations in previous studies, we highlight the novelty and significance of our work. Our research contributes to the development of more effective control strategies for LPV time delay systems, with practical applications in various domains, including engineering, automation, and robotics.

This paper addresses the problem of synthesizing H_∞ static output feedback controllers for LPV time delay systems subject to time-varying delays. The objective is to develop control strategies that can effectively handle the uncertainties and disturbances associated with these systems while guaranteeing robust stability and optimal performance. The contributions of this work are motivated by the need for practical and efficient control solutions for real-world applications. The first contribution of this paper lies in the development of a novel synthesis approach for H_∞ static output feedback controllers. By formulating a sufficient matrix inequality condition based on a quadratic Lyapunov–Krasovskii functional and utilizing the well-known Jensen-type integral inequality, a set of design guidelines are established. These guidelines enable the synthesis of static output feedback controllers that simplify the control structure, reduce computational complexity, and offer improved robustness in the face of uncertainties and disturbances. Furthermore, we extend the scope by including the synthesis of an H_∞ dynamic output feedback controller for LPV systems with time-varying delays. This comprehensive approach allows for a unified control design framework that incorporates both static and dynamic output feedback controllers. By defining an augmented system dynamics and formulating the dynamic output feedback controller in an equivalent form of static output feedback control, we provide a flexible and versatile control strategy that leverages the advantages of both approaches. The second contribution of this work is the introduction of an iterative algorithm to search for a potential feasible solution set of the synthesis problem. This algorithm enables the practical implementation of the proposed controller design methods and facilitates the exploration of various control strategies based on system requirements and constraints. The effectiveness of the proposed approach is demonstrated through numerical examples and comparison analyses, showcasing the superiority of the static output feedback controller over existing techniques.

2 Problem Statement

Let us consider a class of linear parameter-varying time delay systems defined as

$$\begin{aligned}
\dot{x}(t) &= A(\rho)x(t) + A_h(\rho)x(t-h(t)) + B_w(\rho)w(t) + B_u(\rho)u(t) \\
z(t) &= C_z(\rho)x(t) + D_u(\rho)u(t) + D_{zw}(\rho)w(t) \\
y(t) &= C_y(\rho)x(t) + D_{yw}(\rho)w(t) \\
x(t) &= \varphi(t), \quad t\epsilon[-h_{\max}, 0]
\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^q$ is the exogeneous disturbance input with finite energy in the space $L_2[0, \infty]$, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^p$ is the measured output, $z(t) \in \mathbb{R}^r$ is the controlled output and $\varphi(t)$ represents the initial condition function. The system matrices of appropriate dimensions are continuous and bounded affine functions of a time-varying parameter vector,

$$\rho(t) = \left[\, \rho_1(t),\ \rho_2(t),\ \cdots\ \rho_N(t) \,\right]^T \in \mathbb{R}^N$$

The time-varying delay function, $h(t)$ and its derivative are bounded and satisfy

$$0 < h(t) \le h_{\max} \tag{2a}$$

$$0 \le \dot{h}(t) \le d \tag{2b}$$

where $h_{\max}$ and d are positive scalars.

Assumption 1 [18] The parameter-varying function, $\rho(t)$ vary in a polytope $\mathcal{L}$ of vertices $\rho_1, \rho_2, \ldots, \rho_N$ such that

$$\rho(t) \in \mathcal{L} = \left[\, \rho_1(t),\ \rho_2(t),\ \cdots,\ \rho_N(t) \,\right]^T = \left\{ \begin{array}{l} \sum\limits_{i=1}^{N} \alpha_i(t)\rho_i,\ \alpha_i(t) \ge 0, \\ \sum\limits_{i=1}^{N} \alpha_i(t) = 1 \end{array} \right\} \tag{3}$$

then the LPV system (1) can be reexpressed according to the following polytopic model:

$$\begin{bmatrix} A(\rho) & A_h(\rho) & B_w(\rho) & B_u(\rho) \\ C_z(\rho) & 0 & D_{zw}(\rho) & D_u(\rho) \\ C_y(\rho) & 0 & D_{yw}(\rho) & 0 \end{bmatrix} = \sum_{i=1}^{N} \alpha_i(t) \begin{bmatrix} A(\rho_i) & A_h(\rho_i) & B_w(\rho_i) & B_u(\rho_i) \\ C_z(\rho_i) & 0 & D_{zw}(\rho_i) & D_u(\rho_i) \\ C_y(\rho_i) & 0 & D_{yw}(\rho_i) & 0 \end{bmatrix} \tag{4}$$

A static output feedback (SOF) control law is chosen for system (1) as follows:

$$u(t) = Fy(t) \tag{5}$$

where $F \in \mathbb{R}^{m \times p}$ denotes the SOF gain matrix to be selected appropriately. Substituting (5) into (1) gives the closed-loop system dynamics

$$\begin{aligned} \dot{x}(t) &= A_{cl}(\rho)x(t) + A_h(\rho)x(t - h(t)) + B_{wc}(\rho)w(t) \\ z(t) &= C_{zc}(\rho)x(t) + D_{wc}(\rho)w(t) \end{aligned} \tag{6}$$

where $A_{cl}(\rho) = A(\rho) + B_u(\rho)FC_y(\rho)$, $B_{wc}(\rho) = B_w(\rho) + B_u(\rho)FD_{yw}(\rho)$, $C_{zc}(\rho) = C_z(\rho) + D_u(\rho)FC_y(\rho)$, $D_{wc}(\rho) = D_{zw}(\rho) + D_u(\rho)FD_{yw}(\rho)$.

The objective of the present work is to find a stabilizing H_∞ controller (5) such that the closed-loop system (6);

(1) is asymptotically stable when $w(t) = 0$,

(2) has a prescribed level of γ for H_∞ disturbance attenuation, i.e., under the zero initial conditions that is $x(0) = 0$, the inequality

$$\int_0^\infty z^T(t)z(t)dt \le \gamma^2 \int_0^\infty w^T(t)w(t)dt \tag{7}$$

is guaranteed for any nonzero $w(t) \in L_2(0, \infty)$.

3 Main Results

In this section, we will develop both static and dynamic output feedback H_∞ controllers for linear parameter-varying time delay systems subject to external disturbances.

3.1 Design of Static Output Feedback H_∞ Controller

The synthesis condition for the existence of a robust stabilizing static output feedback H_∞ controller is summarized in the following theorem.

Theorem 1 Given the positive scalars $\gamma > 0$, $h_{\max} > 0$, $d > 0$, if there exist real and symmetric positive definite matrices $0 < P^T = P \in \mathbb{R}^{n\times n}$, $0 < Q^T = Q \in \mathbb{R}^{n\times n}$, $0 < R^T = R \in \mathbb{R}^{n\times n}$, $0 < S^T = S \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{m\times p}$ satisfying.

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & 0 & PB_{wc}(\rho) & h_{\max}A_c^T(\rho)S & C_{zc}^T(\rho) \\ * & \Omega_{22} & S & 0 & h_{\max}A_h^T(\rho)S & 0 \\ * & * & -R-S & 0 & 0 & 0 \\ * & * & * & \gamma^2 \mathrm{I} & h_{\max}B_{wc}^T(\rho)S & D_{wc}^T(\rho) \\ * & * & * & * & -S & 0 \\ * & * & * & * & * & -\mathrm{I} \end{bmatrix} < 0 \tag{8}$$

where $\Omega_{11} = A_{cl}^T(\rho)P + PA_{cl}(\rho) + Q + R - S$, $\Omega_{12} = PA_h(\rho) + S$, $\Omega_{22} = -(1-d)Q - 2S$, with I(0) representing an identity(zero) matrix with appropriate dimensions, respectively, and * denotes the symmetry, then the controller in (5) with F becomes a static output feedback H_∞ controller for system (1), (2).

Proof Let us choose a candidate Lyapunov–Krasovskii functional as follows:

$$V(x(t), t) = x^T(t)Px(t) + \int_{t-h(t)}^{t} x^T(s)Qx(s)ds + \int_{t-h_{\max}}^{t} x^T(s)Rx(s)ds$$
$$+ h_{\max}\int_{t-h_{\max}}^{t} (s - t + h_{\max})\dot{x}^T(s)S\dot{x}(s)ds \tag{9}$$

then calculating the time-derivative of $V(x(t), t)$ along the state trajectory of system (6) yields

$$\dot{V}(x(t), t) = 2x^T(t)P\dot{x}(t) + x^T(t)Qx(t)$$

$$
\begin{aligned}
&- \left[1 - \dot{h}(t)\right] x^T(t - h(t)) Q x(t - h(t)) + x^T(t) R x(t) \\
&- x^T(t - h_{\max}) R x(t - h_{\max}) + h_{\max}^2 \dot{x}^T(t) S \dot{x}(t) \\
&- h_{\max} \int_{t-h_{\max}}^{t} \dot{x}^T(s) S \dot{x}(s) ds
\end{aligned}
\tag{10}
$$

Employing the Jensen integral inequality [6, 16] for the quadratic integral term in (10) enables us to rewrite it in the following form

$$
\begin{aligned}
&- h_{\max} \int_{t-h_{\max}}^{t} \dot{x}^T(s) S \dot{x}(s) \mathrm{d}s = -h_{\max} \int_{t-h(t)}^{t} \dot{x}^T(s) S \dot{x}(s) \mathrm{d}s \\
&- h_{\max} \int_{t-h_{\max}}^{t-h(t)} \dot{x}^T(s) S \dot{x}(s) \mathrm{d}s \le -h(t) \int_{t-h(t)}^{t} \dot{x}^T(s) S \dot{x}(s) \mathrm{d}s \\
&- [h_{\max} - h(t)] \int_{t-h_{\max}}^{t-h(t)} \dot{x}^T(s) S \dot{x}(s) \mathrm{d}s \le -\left(\int_{t-h(t)}^{t} \dot{x}(s) ds \right) S \int_{t-h(t)}^{t} \dot{x}(s) \mathrm{d}s \\
&\left(\int_{t-h_{\max}}^{t-h(t)} \dot{x}(s) ds \right) S \int_{t-h_{\max}}^{t-h(t)} \dot{x}(s) \mathrm{d}s = [x(t) - x(t - h(t))]^T S [x(t) - x(t - h(t))] \\
&- [x(t - h(t)) - x(t - h_{\max})]^T S [x(t - h(t)) - x(t - h_{\max})]
\end{aligned}
\tag{11}
$$

Substituting (11) into (10), we shall calculate the following quadratic expression as follows:

$$
\dot{V}(x(t), t) + z^T(t) z(t) dt - \gamma^2 w^T(t) w(t) \le \chi^T(t) \Omega_0 \chi(t) \tag{12}
$$

where $\chi(t) = \left[x^T(t)\ x^T(t - h(t))\ x^T(t - h_{\max})\ w^T(t) \right]^T$ and

$$
\Omega_0 = \begin{bmatrix} \Omega_0(1,1) & \Omega_0(1,2) & 0 & \Omega_0(1,4) \\ * & \Omega_0(2,2) & S & h_{\max}^2 A_h^T(\rho) S B_{wc}(\rho) \\ * & * & -R - S & 0 \\ * & * & * & \Omega_0(4,4) \end{bmatrix},
$$

$\Omega_0(1,1) = A_{cl}^T(\rho) P + P A_{cl}(\rho) + Q + R + h_{\max}^2 A_{cl}^T(\rho) P A_{cl}(\rho) - S + C_{zc}^T(\rho) C_{zc}(\rho),$

$\Omega_0(1,2) = P A_h(\rho) + h_{\max}^2 A_{cl}^T(\rho) S A_h(\rho) + S,$

$\Omega_0(2,2) = -(1-d) Q + h_{\max}^2 A_h^T(\rho) S A_h(\rho) - 2S,$

$\Omega_0(1,4) = P B_{wc}(\rho) + h_{\max}^2 A_c^T(\rho) P B_{wc}(\rho) + C_{zc}^T(\rho) D_{wc}(\rho),$

$\Omega_0(4,4) = h_{\max}^2 B_{wc}^T(\rho) S B_{wc}(\rho) + D_{wc}^T(\rho) D_{wc}(\rho) - \gamma^2 \mathrm{I}.$

If the following matrix inequality

$$
\Omega_0 < 0 \tag{13}
$$

holds true, then we obtain

$$
\dot{V}(x(t), t) + z^T(t) z(t) \mathrm{d}t - \gamma^2 w^T(t) w(t) \le \chi^T(t) \Omega_0 \chi(t) < 0 \tag{14}
$$

Applying Schur complement to (13) allows to obtain the bilinear matrix inequality (BMI) defined in (8). Hence, when $w(t) = 0$ once (8) is ensured, $\dot{V}(x(t), t)$ is guaranteed to be negative definite implying that the closed-loop system is asymptotically stable and thus the first condition of the objective of this work is satisfied. Integrating both sides of (14) from 0 to ∞ leads to obtain

$$\begin{aligned}
&\int_0^\infty \dot{V}(x(t), t)\mathrm{d}t + \int_0^\infty \Big[z^T(t)z(t)dt - \gamma^2 w^T(t)w(t)\Big]\mathrm{d}t \\
&= \lim_{t\to\infty} V(x(t), t) - V(x(0), 0) + \int_0^\infty z^T(t)z(t)\mathrm{d}t - \gamma^2 \int_0^\infty w^T(t)w(t)\mathrm{d}t < 0 \quad (15)
\end{aligned}$$

It follows from the fact that $V(x(0), 0) = 0$ and $\lim_{t\to\infty} V(x(t), t) > 0$, we obtain

$$\int_0^\infty z^T(t)z(t)\mathrm{d}t < \gamma^2 \int_0^\infty w^T(t)w(t)\mathrm{d}t \quad (16)$$

which indicates that the second condition of the objective of the present work is also satisfied. This completes the proof.

Now we first assume that system (1) is free of any polytopic type of uncertainties, that is

$$\begin{aligned}
&A(\rho) \equiv A, \quad A_h(\rho) \equiv A_h, \quad B_w(\rho) \equiv B_w, \quad B_u(\rho) \equiv B_u, \quad C_z(\rho) \equiv C_z, \quad D_u(\rho) \equiv D_u, \\
&D_{zw}(\rho) \equiv D_{zw}, \quad C_y(\rho) \equiv C_y, \quad D_{yw}(\rho) \equiv D_{yw} \quad (17)
\end{aligned}$$

Second, we suppose that the time-varying delay is either non-differentiable or its upper bound is not known, that is only Eq. (2a) holds for the time-varying delay, while Eq. (2b) is simply dismissed, then the closed-loop system dynamics can be expressed as

$$\begin{aligned}
\dot{x}(t) &= A_{cl}x(t) + A_h x(t - h(t)) + B_{wc}w(t) \\
z(t) &= C_{zc}x(t) + D_{wc}w(t) \quad (18)
\end{aligned}$$

where $A_{cl} = A + B_u F C_y$, $B_{wc} = B_w + B_u F D_{yw}$, $C_{zc} = C_z + D_u F C_y$, $D_{wc} = D_{zw} + D_u F D_{yw}$. We then present the following corollary.

Corollary 1 Given the positive scalars $\gamma > 0$, $h_{\max} > 0$ if there exist real and symmetric positive definite matrices $0 < P^T = P \in \mathbb{R}^{n\times n}$, $0 < R^T = R \in \mathbb{R}^{n\times n}$, $0 < S^T = S \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{m\times p}$ satisfying

$$\begin{bmatrix}
A_{cl}^T P + P A_{cl} + R - S & P A_h + S & 0 & P B_{wc} & h_{\max} A_c^T S & C_{zc}^T \\
* & -2S & S & 0 & h_{\max} A_h^T S & 0 \\
* & * & -R - S & 0 & 0 & 0 \\
* & * & * & \gamma^2 \mathrm{I} & h_{\max} B_{wc}^T S & D_{wc}^T \\
* & * & * & * & -S & 0 \\
* & * & * & * & * & -\mathrm{I}
\end{bmatrix} < 0 \quad (19)$$

with I(0) representing an identity (zero) matrix with appropriate dimensions, respectively, and * denotes the symmetry, then the controller in (5) with F becomes a static output feedback H_∞ controller for the nominal form of system (1), (2a), (18).

Proof We choose a candidate Lyapunov–Krasovskii functional similar to that given in (9) by eliminating the quadratic term of $\int_{t-h(t)}^{t} x^T(s)Qx(s)ds$. The rest of the proof is done in a similar manner as that of Theorem 1; thus, it is omitted.

3.2 Design of a Dynamic Output Feedback H_∞ Controller

We now consider the design of a dynamic output feedback controller of the following form

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c y(t) \\ u(t) &= C_c x_c(t) + D_c y(t) \end{aligned} \tag{20}$$

where $x_c(t) \in \mathbb{R}^{n_c}$ is the dynamic state vector, $A_c \in \mathbb{R}^{n_c \times n_c}$, $B_c \in \mathbb{R}^{n_c \times p}$, $C_c \in \mathbb{R}^{m \times n_c}$ and $D_c \in \mathbb{R}^{m \times p}$ are dynamic output feedback gain matrices. We introduce an augmented state vector defined as follows:

$$\eta(t) = \left[x^T(t)\ x_c^T(t)\right]^T \tag{21}$$

Then, we compute

$$\begin{aligned} \dot{\eta}(t) &= \begin{bmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{bmatrix} \\ &= \begin{bmatrix} A(\rho)x(t) + A_h(\rho)x(t-h(t)) + B_w(\rho)w(t) + B_u(\rho)u(t) \\ A_c x_c(t) + B_c y(t) \end{bmatrix} \\ &= \begin{bmatrix} A(\rho)x(t) + A_h(\rho)x(t-h(t)) + B_w(\rho)w(t) + B_u(\rho)\{C_c x_c(t) \\ \quad +D_c\left[C_y(\rho)x(t) + D_{yw}(\rho)w(t)\right]\} \\ A_c x_c(t) + B_c\left[C_y(\rho)x(t) + D_{yw}(\rho)w(t)\right] \end{bmatrix} \\ &= \begin{bmatrix} A(\rho) + B_u(\rho)D_c C_y(\rho) & B_u(\rho)C_c \\ B_c C_y(\rho) & A_c \end{bmatrix} \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} A_h(\rho) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t-h(t)) \\ x_c(t-h(t)) \end{bmatrix} + \begin{bmatrix} B_w(\rho) + B_u(\rho)D_c D_{yw}(\rho) \\ B_c D_{yw}(\rho) \end{bmatrix} w(t) \\ &= \tilde{A}(\rho)\eta(t) + \tilde{A}_h(\rho)\eta(t-h(t)) + \tilde{B}_w(\rho)w(t) \end{aligned} \tag{22a}$$

where $\tilde{A}(\rho) = \begin{bmatrix} A(\rho) + B_u(\rho)D_c C_y(\rho) & B_u(\rho)C_c \\ B_c C_y(\rho) & A_c \end{bmatrix}$, $\tilde{A}_h(\rho) = \begin{bmatrix} A_h(\rho) & 0 \\ 0 & 0 \end{bmatrix}$,

$\tilde{B}_w(\rho) = \begin{bmatrix} B_w(\rho) + B_u(\rho)D_c D_{yw}(\rho) \\ B_c D_{yw}(\rho) \end{bmatrix}$. We shall similarly reexpress $z(t)$ as follows:

$$\begin{aligned} z(t) &= C_z(\rho)x(t) + D_u(\rho)\{C_c x_c(t) + D_c[C_y(\rho)x(t) + D_{yw}(\rho)w(t)]\} + D_{zw}(\rho)w(t) \\ &= [C_z(\rho) + D_u(\rho)D_c C_y(\rho)]x(t) + D_u(\rho)C_c x_c(t) \\ &\quad + [D_{zw}(\rho) + D_u(\rho)D_c D_{yw}(\rho)]w(t) \\ &= \begin{bmatrix} C_z(\rho) + D_u(\rho)D_c C_y(\rho) & D_u(\rho)C_c \end{bmatrix} \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} \\ &\quad + [D_{zw}(\rho) + D_u(\rho)D_c D_{yw}(\rho)]w(t) \\ &= \tilde{C}_z(\rho)\eta(t) + \tilde{D}_w(\rho)w(t) \end{aligned} \tag{22b}$$

where $\tilde{C}_z(\rho) = \begin{bmatrix} C_z(\rho) + D_u(\rho)D_c C_y(\rho) & D_u(\rho)C_c \end{bmatrix}$, $\tilde{D}_w(\rho) = D_{zw}(\rho) + D_u(\rho)D_c D_{yw}(\rho)$. We shall first decompose $\tilde{A}(\rho)$ and $\tilde{B}_w(\rho)$ as follows:

$$\begin{aligned} \tilde{A}(\rho) &= \begin{bmatrix} A(\rho) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_u(\rho)D_c C_y(\rho) & B_u(\rho)C_c \\ B_c C_y(\rho) & A_c \end{bmatrix} \\ &= \begin{bmatrix} A(\rho) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_u(\rho) & 0 \\ 0 & \mathrm{I} \end{bmatrix} \begin{bmatrix} D_c C_y(\rho) & C_c \\ B_c C_y(\rho) & A_c \end{bmatrix} \\ &= \begin{bmatrix} A(\rho) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_u(\rho) & 0 \\ 0 & \mathrm{I} \end{bmatrix} \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix} \begin{bmatrix} C_y(\rho) & 0 \\ 0 & \mathrm{I} \end{bmatrix} \\ &= \overline{A}(\rho) + \overline{B}_u(\rho)\overline{F}\,\overline{C}_y(\rho), \end{aligned}$$

and $\tilde{B}_w(\rho) = \begin{bmatrix} B_w(\rho) + B_u(\rho)D_c D_{yw}(\rho) \\ B_c D_{yw}(\rho) \end{bmatrix} = \begin{bmatrix} B_w(\rho) \\ 0 \end{bmatrix} + \begin{bmatrix} B_u(\rho) & 0 \\ 0 & \mathrm{I} \end{bmatrix} \begin{bmatrix} D_c D_{yw}(\rho) \\ B_c D_{yw}(\rho) \end{bmatrix} = \begin{bmatrix} B_w(\rho) \\ 0 \end{bmatrix} + \begin{bmatrix} B_u(\rho) & 0 \\ 0 & \mathrm{I} \end{bmatrix} \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix} \begin{bmatrix} D_{yw}(\rho) \\ 0 \end{bmatrix} = \overline{B}_w(\rho) + \overline{B}_u(\rho)\overline{F}\,\overline{D}_{yw}(\rho)$
where $\overline{A}(\rho) = \begin{bmatrix} A(\rho) & 0 \\ 0 & 0 \end{bmatrix}$, $\overline{B}_u(\rho) = \begin{bmatrix} B_u(\rho) & 0 \\ 0 & \mathrm{I} \end{bmatrix}$, $\overline{F} = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}$, $\overline{C}_y(\rho) = \begin{bmatrix} C_y(\rho) & 0 \\ 0 & \mathrm{I} \end{bmatrix}$, $\overline{B}_w(\rho) = \begin{bmatrix} B_w(\rho) \\ 0 \end{bmatrix}$, $\overline{D}_{yw}(\rho) = \begin{bmatrix} D_{yw}(\rho) \\ 0 \end{bmatrix}$. In a similar manner, we shall also decompose $\tilde{C}_z(\rho)$ and $\tilde{D}_w(\rho)$ step by step as

$$\begin{aligned} \tilde{C}_z(\rho) &= \begin{bmatrix} C_z(\rho) + D_u(\rho)D_c C_y(\rho) & D_u(\rho)C_c \end{bmatrix} \\ &= \begin{bmatrix} C_z(\rho) & 0 \end{bmatrix} + \begin{bmatrix} D_u(\rho) & 0 \end{bmatrix} \begin{bmatrix} D_c C_y(\rho) & C_c \\ B_c C_y(\rho) & A_c \end{bmatrix} \\ &= \begin{bmatrix} C_z(\rho) & 0 \end{bmatrix} + \begin{bmatrix} D_u(\rho) & 0 \end{bmatrix} \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix} \begin{bmatrix} C_y(\rho) & 0 \\ 0 & \mathrm{I} \end{bmatrix} \\ &= \overline{C}_z(\rho) + \overline{D}_u(\rho)\overline{F}\,\overline{C}_y(\rho) \end{aligned}$$

and $\tilde{D}_w(\rho) = D_{zw}(\rho) + D_u(\rho)D_cD_{yw}(\rho) = D_{zw}(\rho) + \begin{bmatrix} D_u(\rho) & 0 \end{bmatrix}\begin{bmatrix} D_cD_{yw}(\rho) \\ B_cD_{yw}(\rho) \end{bmatrix} = D_{zw}(\rho) + \begin{bmatrix} D_u(\rho) & 0 \end{bmatrix}\begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}\begin{bmatrix} D_{yw}(\rho) \\ 0 \end{bmatrix} = \overline{D}_{zw}(\rho) + \overline{D}_u(\rho)\overline{F}\,\overline{D}_{yw}(\rho)$ where $\overline{C}_z(\rho) = \begin{bmatrix} C_z(\rho) & 0 \end{bmatrix}$, $\overline{D}_u(\rho) = \begin{bmatrix} D_u(\rho) & 0 \end{bmatrix}$, $\overline{D}_{zw}(\rho) = D_{zw}(\rho)$. Consequently, we conclude that a closed-loop system under dynamic output feedback can be recast as the connection of the augmented system

$$\begin{aligned} \dot{\eta}(t) &= \overline{A}(\rho)\eta(t) + \overline{A}_h(\rho)\eta(t-h(t)) + \overline{B}_w(\rho)w(t) + \overline{B}_u(\rho)u(t) \\ \overline{y}(t) &= \overline{C}_y(\rho)\eta(t) + \overline{D}_{yw}(\rho)w(t) \\ z(t) &= \overline{C}_z(\rho)\eta(t) + \overline{D}_u(\rho)u(t) + \overline{D}_{zw}(\rho)w(t) \end{aligned} \tag{23}$$

where $\overline{A}_h(\rho) = \tilde{A}_h(\rho)$, in feedback with the static output feedback controller

$$u(t) = \overline{F}\,\overline{y}(t) \tag{24}$$

Therefore, the method proposed in Sect. 3.1 can be also utilized to design the dynamic output feedback H_∞ controller which is summarized in the following theorem.

Theorem 2 Given the positive scalars $\gamma > 0$, $h_{\max} > 0$, $d > 0$, if there exist real and symmetric positive definite matrices $0 < \overline{P}^T = \overline{P} \in \mathbb{R}^{(n+n_c)\times(n+n_c)}$, $0 < \overline{Q}^T = \overline{Q} \in \mathbb{R}^{(n+n_c)\times(n+n_c)}$, $0 < \overline{R}^T = \overline{R} \in \mathbb{R}^{(n+n_c)\times(n+n_c)}$, $0 < \overline{S}^T = \overline{S} \in \mathbb{R}^{(n+n_c)\times(n+n_c)}$ and $\overline{F} \in \mathbb{R}^{(m+n_c)\times(p+n_c)}$ satisfying

$$\overline{\Omega} = \begin{bmatrix} \overline{\Omega}_{11} & \overline{\Omega}_{12} & 0 & \overline{P}\,\overline{B}_w(\rho) & h_{\max}\overline{A}^T(\rho)\overline{S} & \overline{C}_z^T(\rho) \\ * & \overline{\Omega}_{22} & \overline{S} & 0 & h_{\max}\overline{A}_h^T(\rho)\overline{S} & 0 \\ * & * & -\overline{R}-\overline{S} & 0 & 0 & 0 \\ * & * & * & \gamma^2\mathrm{I} & h_{\max}\overline{B}_w^T(\rho)\overline{S} & \overline{D}_w^T(\rho) \\ * & * & * & * & -\overline{S} & 0 \\ * & * & * & * & * & -\mathrm{I} \end{bmatrix} < 0 \tag{25}$$

where $\overline{\Omega}_{11} = \overline{A}^T(\rho)\overline{P} + \overline{P}\,\overline{A}(\rho) + \overline{Q} + \overline{R} - \overline{S}$, $\overline{\Omega}_{12} = \overline{P}\,\overline{A}_h(\rho) + \overline{S}$, $\overline{\Omega}_{22} = -(1-d)\overline{Q} - 2\overline{S}$, with I(0) representing an identity(zero) matrix with appropriate dimensions, respectively, and * denotes the symmetry, then the controller in (20) with $A_c = \begin{bmatrix} 0_{n_c\times m} & \mathrm{I}_{n_c} \end{bmatrix}\overline{F}\begin{bmatrix} 0_{p\times n_c} \\ \mathrm{I}_{n_c} \end{bmatrix}$, $B_c = \begin{bmatrix} 0_{n_c\times m} & \mathrm{I}_{n_c} \end{bmatrix}\overline{F}\begin{bmatrix} \mathrm{I}_p \\ 0_{n_c\times p} \end{bmatrix}$, $C_c = \begin{bmatrix} \mathrm{I}_m & 0_{m\times n_c} \end{bmatrix}\overline{F}\begin{bmatrix} 0_{m\times n_c} \\ \mathrm{I}_{n_c} \end{bmatrix}$, $D_c = \begin{bmatrix} \mathrm{I}_m & 0_{m\times n_c} \end{bmatrix}\overline{F}\begin{bmatrix} \mathrm{I}_p \\ 0_{n_c\times p} \end{bmatrix}$ becomes a dynamic output feedback H_∞ controller for system (1), (2).

Proof The proof can be done similarly to that of Theorem 1, thus it is omitted.

We now reconsider the nominal form of system (1) with (17) subject to (2a) and (20) for which the closed-loop augmented system dynamics can be reduced to the following equation

$$\begin{aligned}\dot{\eta}(t) &= \overline{A}\eta(t) + \overline{A}_h\eta(t-h(t)) + \overline{B}_w w(t) + \overline{B}_u u(t)\\ \overline{y}(t) &= \overline{C}_y\eta(t) + \overline{D}_{yw}w(t)\\ z(t) &= \overline{C}_z\eta(t) + \overline{D}_u u(t) + \overline{D}_{zw}w(t)\end{aligned} \tag{26}$$

where $\overline{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$, $\overline{B}_u = \begin{bmatrix} B_u & 0 \\ 0 & \mathrm{I} \end{bmatrix}$, $\overline{F} = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}$, $\overline{C}_y = \begin{bmatrix} C_y & 0 \\ 0 & \mathrm{I} \end{bmatrix}$, $\overline{B}_w = \begin{bmatrix} B_w \\ 0 \end{bmatrix}$, $\overline{D}_{yw} = \begin{bmatrix} D_{yw} \\ 0 \end{bmatrix}$, $\overline{C}_z = \begin{bmatrix} C_z & 0 \end{bmatrix}$, $\overline{D}_u = \begin{bmatrix} D_u & 0 \end{bmatrix}$, $\overline{D}_{zw} = D_{zw}$ with the static output feedback control law of (24). Then we present the following corollary for the synthesis of a dynamic output feedback controller.

Cortollary 2 Given the positive scalars $\gamma > 0$, $h_{\max} > 0$, if there exist real and symmetric positive definite matrices $0 < \overline{P}^T = \overline{P} \in \mathbb{R}^{(n+n_c)\times(n+n_c)}$, $0 < \overline{R}^T = \overline{R} \in \mathbb{R}^{(n+n_c)\times(n+n_c)}$, $0 < \overline{S}^T = \overline{S} \in \mathbb{R}^{(n+n_c)\times(n+n_c)}$ and $\overline{F} \in \mathbb{R}^{(m+n_c)\times(p+n_c)}$ satisfying

$$\begin{bmatrix} \overline{A}^T\overline{P} + \overline{P}\,\overline{A} + \overline{R} - \overline{S} & \overline{P}\,\overline{A}_h + \overline{S} & 0 & \overline{P}\,\overline{B}_w & h_{\max}\overline{A}^T\overline{S} & \overline{C}_z^T \\ * & -2\overline{S} & \overline{S} & 0 & h_{\max}\overline{A}_h^T\overline{S} & 0 \\ * & * & -\overline{R}-\overline{S} & 0 & 0 & 0 \\ * & * & * & \gamma^2\mathrm{I} & h_{\max}\overline{B}_w^T\overline{S} & \overline{D}_w^T \\ * & * & * & * & -\overline{S} & 0 \\ * & * & * & * & * & -\mathrm{I} \end{bmatrix} < 0 \tag{27}$$

with I(0) representing an identity(zero) matrix with appropriate dimensions, respectively, and * denotes the symmetry, then the controller in (20) with $A_c = \begin{bmatrix} 0_{n_c\times m} & \mathrm{I}_{n_c} \end{bmatrix}\overline{F}\begin{bmatrix} 0_{p\times n_c} \\ \mathrm{I}_{n_c} \end{bmatrix}$, $B_c = \begin{bmatrix} 0_{n_c\times m} & \mathrm{I}_{n_c} \end{bmatrix}\overline{F}\begin{bmatrix} \mathrm{I}_p \\ 0_{n_c\times p} \end{bmatrix}$, $C_c = \begin{bmatrix} \mathrm{I}_m & 0_{m\times n_c} \end{bmatrix}\overline{F}\begin{bmatrix} 0_{m\times n_c} \\ \mathrm{I}_{n_c} \end{bmatrix}$, $D_c = \begin{bmatrix} \mathrm{I}_m & 0_{m\times n_c} \end{bmatrix}\overline{F}\begin{bmatrix} \mathrm{I}_p \\ 0_{n_c\times p} \end{bmatrix}$ becomes a dynamic output feedback H_∞ controller for system (1), (2a), (26).

4 Linearization Approach for the Synthesis of the Output Feedback Controllers

Note that the synthesis conditions for a stabilizing static/dynamic output feedback H_∞ controller given in (8), (19), (25), (27), respectively, are not convex. However, once $P(\overline{P})$ and $S(\overline{S})$ are specified, they become linear, hence convex in $Q(\overline{Q})$, $R(\overline{R})$, and $F(\overline{F})$. Conversely, when $F(\overline{F})$ is specified, it is linear in $P(\overline{P})$, $Q(\overline{Q})$, $R(\overline{R})$ and $S(\overline{S})$. Based on this observation, it is possible to introduce iterative approaches to find a feasible solution set for the design problem. Moreover, we also wish to maximize the decay rate of system (1) over the output feedback matrix $F\left(\overline{F}\right)$ subject to an Euclidean norm constraint on $F\left(\overline{F}\right)$ of the following form

$$\text{trance}\left(F^T F\right) < \alpha \text{ or trace}\left(\bar{F}^T \bar{F}\right) < \alpha \tag{28}$$

where α is positive scalar to be selected. Without loss of generality, we now develop an iterative scheme for the linearization of the BMI given in (8) which can also be utilized as a generic form for the BMI obtained in (19), (25) and (27) appropriately. In particular, the decay rate of system (1) exceeds λ if and only if there exist P such that (8) and

$$A_{cl}^T(\rho)P + PA_{cl}(\rho) < \lambda P \tag{29}$$

are satisfied. We again notice that for fixed P and S in (8), the inequalities can be rewritten as LMI's in Q, R, F and λ. Moreover, for fixed F and λ, they are LMI's in P, Q, R and S. Therefore, we shall now consider utilizing an iterative algorithm to find a feasible solution set satisfying (8), (28) and (29).

5 Algorithm 1

Step 1. Set $k = 1$ and set $F_0 = 0$, $\lambda_0 = 10^4$.

Step 2. Solve the following LMI problem with respect to P, Q, R and S satisfying

$$\Sigma = \begin{bmatrix} \Sigma(1,1) & PA_h(\rho) + S & 0 & \Sigma(1,4) & \Sigma(1,5) & \Sigma(1,6) \\ * & -(1-d)Q - 2S & S & 0 & h_{\max} A_h^T(\rho) S & 0 \\ * & * & -R - S & 0 & 0 & 0 \\ * & * & * & -\gamma^2 \mathrm{I} & \Sigma(4,5) & \Sigma(4,6) \\ * & * & * & * & -S & 0 \\ * & * & * & * & * & -\mathrm{I} \end{bmatrix} < 0$$

where $\Sigma(1,1) = \left[A(\rho) + B_u(\rho)F_{k-1}C_y(\rho)\right]^T P + P[A(\rho) + B_u(\rho)F_{k-1}C_y(\rho) + Q + R - S - \lambda_{k-1}P$, $\Sigma(1,4) = P\left[B_w(\rho) + B_u(\rho)F_{k-1}D_{yw}(\rho)\right]$, $\Sigma(1,5) = h_{\max}\left[A(\rho) + B_u(\rho)F_{k-1}C_y(\rho)\right]^T S$, $\Sigma(1,6) = \left[C_z(\rho) + D_u(\rho)F_{k-1}C_y(\rho)\right]^T$, $\Sigma(4,5) = h_{\max}\left[B_w(\rho) + B_u(\rho)F_{k-1}D_{yw}(\rho)\right]^T S$, $\Sigma(4,6) = \left[D_{zw}(\rho) + D_u(\rho)F_{k-1}D_{yw}(\rho)\right]^T$. Let $P_k = P$ and $S_k = S$.

Step 3. Fix P and S and solve the following optimization problem with respect to Q, R, F and λ.

minimize λ subject to

$$\begin{bmatrix} -\alpha \mathrm{I} & F^T \\ F & -\mathrm{I} \end{bmatrix} < 0$$

$$\Psi = \begin{bmatrix} \Psi(1,1) & P_k A_h(\rho) + S_k & 0 & \Psi(1,4) & \Psi(1,5) & \Psi(1,6) \\ * & -(1-d)Q - 2S_k & S_k & 0 & h_{\max} A_h^T(\rho) S_k & 0 \\ * & * & -R - S_k & 0 & 0 & 0 \\ * & * & * & -\gamma^2 \mathrm{I} & \Psi(4,5) & \Psi(4,6) \\ * & * & * & * & -S_k & 0 \\ * & * & * & * & * & -\mathrm{I} \end{bmatrix} < 0$$

where $\Psi(1,1) = \left[A(\rho) + B_u(\rho) F C_y(\rho)\right]^T P_k + P_k[A(\rho) + B_u(\rho) F C_y(\rho)$,
$\Psi(1,4) = P_k\left[B_w(\rho) + B_u(\rho) F D_{yw}(\rho)\right]$, $\Psi(1,5) = h_{\max}\left[A(\rho) + B_u(\rho) F C_y(\rho)\right]^T S_k$,
$\Psi(1,6) = \left[C_z(\rho) + D_u(\rho) F C_y(\rho)\right]^T$, $\Psi(4,5) = h_{\max}\left[B_w(\rho) + B_u(\rho) F D_{yw}(\rho)\right]^T S_k$,
$\Psi(4,6) = \left[D_{zw}(\rho) + D_u(\rho) F D_{yw}(\rho)\right]^T$. Let $F_k = F$ and $\lambda_k = \lambda$. If $\lambda_k \leq -\lambda^*$ where λ^* is a prescribed maximum decay rate then set $P_{\text{optimal}} = P_k$, $Q_{\text{optimal}} = Q$, $R_{\text{optimal}} = R$, $S_{\text{optimal}} = S$, $F_{\text{optimal}} = F$, $k_{\text{optimal}} = k$ and STOP.

Step 4. If $\lambda_k > -\lambda^*$ then set $k = k + 1$ and go to Step 2 and repeat.

Step 5. If $\lambda_k \leq -\lambda^*$ can NOT be achieved within a number of $k_{\max}$ iterations then there exists NO feasible solution set and EXIT.

6 Numerical Examples

In this section, in order to exhibit the application of the proposed methodology, we consider three numerical examples from the literature.

Example 1 We consider a numerical example with polytopic type of uncertainties defined for system (1) as follows (Nejem et al. [15]), (Shahbazzadeh and Sadati [18]).
$A_1 = \begin{pmatrix} 0 & 0.8 \\ -2 & -3.1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -0.2 & 0.1 \\ -0.3 & -0.3 \end{pmatrix}$, $A_{h1} = \begin{pmatrix} 0 & 1.2 \\ -2 & -2.9 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0.2 & 0.1 \\ -0.1 & -0.3 \end{pmatrix}$.
$B_{w1} = \begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix}$, $B_{w2} = B_{w1}$, $B_{u1} = \begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix}$, $B_{u2} = B_{u1}$, $C_{z1} = \begin{pmatrix} 0 & 10 \\ 0 & 0 \end{pmatrix}$, $C_{z2} = C_{z1}$,
$D_{zu1} = \begin{pmatrix} 0 \\ 0.1 \end{pmatrix}$, $D_{zu2} = D_{zu1}$, $D_{zw1} = 0$, $D_{zw2} = 0$, $C_{y1} = \begin{pmatrix} 0 & 1 \\ 0.5 & 0 \end{pmatrix}$, $C_{y2} = C_{y1}$,
$D_{yw1} = 0$, $D_{yw2} = 0$. We set $d = 0.5$ and search for an allowable maximum upper bound of the time-varying time delay, h_{max} and for an allowable minimum attenuation rate of γ. We iteratively have solved for a feasible solution set of Theorem 1 and after 2455 iterations we have obtained the following decision variables for $h_{max} = 1.21$ and

$\gamma = 0.1001$ as

$$P = \begin{pmatrix} 372.3171 & 257.9657 \\ 257.9657 & 750.9341 \end{pmatrix}, \quad Q = \begin{pmatrix} 2.5722 & 4.0927 \\ 4.0927 & 6.5118 \end{pmatrix} \cdot 10^3$$
$$R = \begin{pmatrix} 2.1333 & -2.1732 \\ -2.1732 & 2.2176 \end{pmatrix}, \quad S = \begin{pmatrix} 64.5823 & -65.8301 \\ -65.8301 & 67.1108 \end{pmatrix},$$
$$F = \begin{bmatrix} -1.9760 & -2.4689 \end{bmatrix} \cdot 10^4$$

We have calculated the stable closed-loop eigenvalues of the time delay system (1) under static output feedback control as follows:

$$eig\left(A_1 + B_{u1} F C_{y1}\right) = \left\{-0.2687, -6.4237 \cdot 10^3\right\}$$

$eig\left(A_2 + B_{u2} F C_{y2}\right) = \left\{-0.0923, -6.4213 \cdot 10^3\right\}$

In a similar manner, after 4125 iterations, we have obtained a feasible solution set of Theorem 2 for $h_{max} = 1.35$ and $\gamma = 0.1001$ yielding the following decision variables

$$\overline{P} = \begin{pmatrix} 277.8377 & 322.0942 & 0 & 0 \\ 322.0942 & 727.6967 & 0 & 0 \\ 0 & 0 & 5.4913 \cdot 10^8 & 0 \\ 0 & 0 & 0 & 5.4913 \cdot 10^8 \end{pmatrix},$$

$$\overline{Q} = \begin{pmatrix} 2.6398 \cdot 10^3 & 4.0472 \cdot 10^3 & 0 & 0 \\ 4.0472 \cdot 10^3 & 6.2051 \cdot 10^3 & 0 & 0 \\ 0 & 0 & 1.6876 \cdot 10^8 & 0 \\ 0 & 0 & 0 & 1.6876 \cdot 10^8 \end{pmatrix},$$

$$\overline{R} = \begin{pmatrix} 0.1068 & 0.1499 & 0 & 0 \\ 0.1499 & 0.2595 & 0 & 0 \\ 0 & 0 & 2.5455 \cdot 10^8 & 0 \\ 0 & 0 & 0 & 2.5455 \cdot 10^8 \end{pmatrix},$$

$$\overline{S} = \begin{pmatrix} 47.5225 & -47.8571 & 0 & 0 \\ -47.8571 & 48.2436 & 0 & 0 \\ 0 & 0 & 3.2071 \cdot 10^8 & 0 \\ 0 & 0 & 0 & 3.2071 \cdot 10^8 \end{pmatrix},$$

$$\overline{F} = \begin{pmatrix} -2.0832 \cdot 10^4 & -2.3791 \cdot 10^4 & 0 & 0 \\ 0 & 0 & -0.9395 & 0 \\ 0 & 0 & 0 & -0.9395 \end{pmatrix}$$

yielding the gains of the dynamic output feedback controller as
$A_c = \begin{bmatrix} -0.9395 & 0 \\ 0 & -0.9395 \end{bmatrix}$, $B_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $C_c = \begin{bmatrix} 0 & 0 \end{bmatrix}$, $D_c = \begin{bmatrix} -2.0832 & -2.3791 \end{bmatrix} \cdot 10^4$.

Table 1 H_∞ norm bounds and maximum allowable delay bounds for Example 1

Methods	Minimum attenuation rate, γ	$h_{\max}$
Nejem et al. [15]	0.466	0.5
Shahbazzadeh and Sadati [18]	0.1021	0.5
Theorem 1 (SOF)	0.1001	1.21
Theorem 2 (DOF)	0.1001	1.35

The stable closed-loop eigenvalues under dynamic output feedback control for the polytopes of time-delay system (1) are calculated as

$$eig\left(\overline{A}_1 + \overline{B}_{u1}\overline{F}\,\overline{C}_{y1}\right) = \left\{-0.1447, -6.5484 \cdot 10^3, -0.9395, -0.9395\right\}$$

$$eig\left(\overline{A}_2 + \overline{B}_{u2}\overline{F}\,\overline{C}_{y2}\right) = \left\{-0.0817, -6.5459 \cdot 10^3, -0.9395, -0.9395\right\}$$

The numerical results on the maximum delay bound and minimum attenuation rate are shown in Table 1 along with those reported in the literature. It can be seen from Table 1 that the proposed method yields less conservative results compared to those of [15, 18].

Example 2 We now consider a nominal form of numerical example with the following system parameters which has been presented by Shen et al. [20] $A = \begin{pmatrix} 0 & 1 \\ -0.5 & -1.5 \end{pmatrix}$, $A_h = \begin{pmatrix} -0.5 & 0 \\ 0 & -1 \end{pmatrix}$, $B_u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $B_w = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $C_y = \begin{pmatrix} 0.2 & 0.2 \\ 1 & 0 \end{pmatrix}$, $D_{yw} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $C_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $D_u = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $D_{zw} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Utilizing Corollary 1 to design a static output feedback controller, we have obtained a feasible solution set after 1609 iterations for $h_{max} = 0.51$ and $\gamma = 2.09$ with the following decision variables

$$P = \begin{pmatrix} 16.6090 & -29.8275 \\ -29.8275 & 92.0923 \end{pmatrix}, \quad R = \begin{pmatrix} 9.0493 & -31.4174 \\ -31.4174 & 109.6194 \end{pmatrix},$$
$$S = \begin{pmatrix} 198.4362 & -183.3015 \\ -183.3015 & 176.7864 \end{pmatrix}, \quad F = \begin{bmatrix} -151.5769 & 36.6976 \end{bmatrix}$$

We have then applied Corollary 2 to synthesize a dynamic output feedback controller and provided a feasible solution set after 11141 iterations for $h_{max} = 0.57$ and $\gamma = 2.09$ yielding the following decision variables

$$\overline{P} = \begin{pmatrix} 14.8688 & -24.8477 & 0 & 0 \\ -24.8477 & 76.1154 & 0 & 0 \\ 0 & 0 & 4.5076 \cdot 10^8 & 0 \\ 0 & 0 & 0 & 4.5076 \cdot 10^8 \end{pmatrix},$$

Table 2 H_∞ norm bounds and maximum allowable delay bounds for Example 2

Methods	Minimum attenuation rate,γ	$h_{\max}$
Shen et al. [20]	14.2673	0.5
Corollary 1 (SOF)	2.09	0.51
Corollary 2 (DOF)	2.09	0.57

$$\overline{R} = \begin{pmatrix} 8.2970 & -26.7538 & 0 & 0 \\ -26.7538 & 86.2681 & 0 & 0 \\ 0 & 0 & 2.9978 \cdot 10^8 & 0 \\ 0 & 0 & 0 & 2.9978 \cdot 10^8 \end{pmatrix},$$

$$\overline{S} = \begin{pmatrix} 132.9524 & -120.5047 & 0 & 0 \\ -120.5047 & 115.2434 & 0 & 0 \\ 0 & 0 & 3.8006 \cdot 10^8 & 0 \\ 0 & 0 & 0 & 3.8006 \cdot 10^8 \end{pmatrix},$$

$$\overline{F} = \begin{pmatrix} -129.8254 & -31.0559 & 0 & 0 \\ 0 & 0 & -3.6504 & 0 \\ 0 & 0 & 0 & -3.6504 \end{pmatrix}$$

and the feedback gains of the dynamic output feedback controller are obtained from $\overline{F}$ as follows:
$A_c = \begin{bmatrix} -3.6504 & 0 \\ 0 & -3.6504 \end{bmatrix}$, $B_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $C_c = \begin{bmatrix} 0 & 0 \end{bmatrix}$, $D_c = [-129.8254 \quad 31.0559]$.

We have listed the numerical results on the maximum delay bound and minimum attenuation rate in Table 2 together with those reported in the literature. Table 2 reveals that both Corollaries 1 and 2 developed in this note succeed to result in tolerating larger delay bounds and smaller disturbance rejection rates in comparison with [20].

Example 3 We take into account another nominal form of numerical example from the literature (Hao and Dao [7]) for which the system parameters are defined as follows:
$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, A_h = \begin{pmatrix} 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 \\ -0.1 & 0.1 & 0 & 0 \\ 0.1 & -0.1 & 0 & 0 \end{pmatrix}, B_u = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, B_w = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$
$C_y = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, $D_{yw} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $C_z = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}$, $D_u = 0$, $D_{zw} = 0$. With h_{max} set to 0.5, we look for an acceptable minimum attenuation rate, γ. After 412 rounds of iteratively solving for a set of feasible solutions to Corollary 1, we have got the

following choice variables for $\gamma = 2.75$ as

$$P = \begin{pmatrix} 14.6608 & -7.0922 & 13.9554 & 8.0359 \\ -7.0922 & 4.9263 & -7.0190 & -2.9149 \\ 13.9554 & -7.0190 & 17.4019 & 7.1990 \\ 8.0359 & -2.9149 & 7.1990 & 6.4829 \end{pmatrix},$$

$$R = \begin{pmatrix} 0.2186 & -0.1542 & 0.3436 & 0.1421 \\ -0.1542 & 0.1141 & -0.2291 & -0.0973 \\ 0.3436 & -0.2291 & 0.6338 & 0.2320 \\ 0.1421 & -0.0973 & 0.2320 & 0.1175 \end{pmatrix},$$

$$S = \begin{pmatrix} 12.8074 & -9.4733 & 22.9834 & 9.2830 \\ -9.4733 & 7.6071 & -17.8490 & -7.0107 \\ 22.9834 & -17.8490 & 45.1403 & 17.2475 \\ 9.2830 & -7.0107 & 17.2475 & 7.2606 \end{pmatrix},$$

$$F = \begin{bmatrix} -0.3704 & -1.2470 \end{bmatrix}$$

We then used Corollary 2 to generate a dynamic output feedback controller, and after 291 iterations for $\gamma = 2.75$, we offered a workable solution set that produced the following decision variables

$$\overline{P} = \begin{pmatrix} \overline{P}_{11} & 0 \\ 0 & \overline{P}_{12} \end{pmatrix}, \quad \overline{P}_{11} = \begin{pmatrix} 14.8021 & -7.0923 & 13.8397 & 8.0664 \\ -7.0923 & 4.8843 & -6.8266 & -2.8731 \\ 13.8397 & -6.8266 & 16.7939 & 7.0350 \\ 8.0664 & -2.8731 & 7.0350 & 6.4871 \end{pmatrix},$$

$$\overline{P}_{22} = 3.2118 \cdot 10^8 \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \overline{R}\begin{pmatrix} \overline{R}_{11} & 0 \\ 0 & \overline{R}_{12} \end{pmatrix},$$

$$\overline{R}_{11} = \begin{pmatrix} 0.2094 & -0.1023 & 0.4125 & 0.1139 \\ -0.1023 & 0.0575 & -0.1855 & -0.0515 \\ 0.4125 & -0.1855 & 0.8647 & 0.2507 \\ 0.1139 & -0.0515 & 0.2507 & 0.0841 \end{pmatrix},$$

$$\overline{R}_{22} = 2.2821 \cdot 10^8 \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\overline{S} = \begin{pmatrix} \overline{S}_{11} & 0 \\ 0 & \overline{S}_{12} \end{pmatrix}, \quad \overline{S}_{11} = \begin{pmatrix} 25.4145 & -16.2044 & 41.4562 & 17.5270 \\ -16.2044 & 11.1824 & -27.6749 & -11.3911 \\ 41.4562 & -27.6749 & 72.1744 & 29.2721 \\ 17.5270 & -11.3911 & 29.2721 & 12.6365 \end{pmatrix}$$

Table 3 H_∞ norm bounds for Example 3

Methods	Minimum attenuation rate, γ	Number of iterations
Hao and Duan [7]	4.8	–
Corollary 1 (SOF)	2.75	412
Corollary 2 (DOF)	2.75	291

$$\overline{S}_{22} = 2.7722 \cdot 10^8 \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\overline{F} = \begin{pmatrix} -0.3691 & -1.2517 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4.6340 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4.6340 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4.6340 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4.6340 \end{pmatrix},$$ and the feedback gains of the dynamic output feedback controller are obtained from $\overline{F}$ as follows:

$$A_c = \begin{bmatrix} -4.6340 & 0 & 0 & 0 \\ 0 & -4.6340 & 0 & 0 \\ 0 & 0 & -4.6340 & 0 \\ 0 & 0 & 0 & -4.6340 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C_c = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_c = \begin{bmatrix} -0.3691 & -1.2517 \end{bmatrix}$$

In Table 3, we include the numerical findings for the minimum attenuation rate together with those that have been documented in the literature. Table 3 shows that, compared to [7], both Corollaries 1 and 2 successfully lead to tolerating decreased disturbance rejection rates.

Remark 1 The values of B_c and C_c that remain consistently at zero signify that the characteristics of the dynamic output feedback controller have collapsed. As a result, the controller becomes a static output feedback controller, which is a simpler form of controller. With B_c and C_c both equaling zero, the control input $u(t)$ is solely dependent on the measured output $u(t)$. The relationship between the control input and the dynamic state vector $x_c(t)$ is no longer relevant to the control design. This reduction to a static output feedback controller can lead to a simplified controller structure with practical advantages such as reduced computational complexity and improved robustness. However, it is important to carefully evaluate the outcome of B_c and C_c becoming consistently zero in light of the control objectives, system requirements, and practical considerations. The loss of dynamic state dependency, indicated by the consistently zero B_c, implies that the control input $u(t)$ is not dependent on the dynamic state vector $x_c(t)$. Similarly, the consistently zero C_c suggests that the dynamic state vector $x_c(t)$ does not influence the control input. It is crucial to ensure that the control design adequately captures the desired dependencies and relationships between the control input, measured output, and dynamic state vector. In summary, when B_c and C_c consistently become zero, the dynamic output feedback controller collapses into a static output feedback controller, where the control input $u(t)$ becomes solely dependent on the measured output $y(t)$. This simplification in the controller structure can offer benefits, but it is essential to ensure that the control design aligns with the desired system behavior and control objectives.

7 Conclusions

In this paper, we have addressed the problem of synthesizing H_∞ static output feedback controllers for linear parameter-varying (LPV) time delay systems subject to time-varying delays. Our objective was to develop control strategies that effectively handle uncertainties and disturbances while ensuring robust stability and optimal performance. Our contributions in this work were motivated by the need for practical and efficient control solutions in real-world applications. Firstly, we proposed a novel synthesis approach for H_∞ static output feedback controllers. By formulating a sufficient matrix inequality condition based on a quadratic Lyapunov–Krasovskii functional and utilizing the well-known Jensen-type integral inequality, we established design guidelines for the synthesis of static output feedback controllers. These guidelines simplify the control structure, reduce computational complexity, and enhance robustness against uncertainties and disturbances. Moreover, we extended the scope by including the synthesis of an H_∞ dynamic output feedback controller for LPV systems with time-varying delays. Our comprehensive approach allows for a unified control design framework that incorporates both static and dynamic output feedback controllers. By formulating the dynamic output feedback controller in an equivalent form of static output feedback control through the augmentation of system dynamics, we provided a flexible and versatile control strategy that leverages the advantages of both approaches. To facilitate the practical implementation of the proposed controller design methods, we introduced an iterative algorithm to search for a potential feasible solution set of the synthesis problem. This algorithm enables the exploration of various control strategies based on system requirements and constraints. Through numerical examples and comparison analyses, we demonstrated the effectiveness and superiority of the static output feedback controller over existing techniques. The proposed approach offers enhanced robustness, reduced computational complexity, and improved performance. Consequently, this paper has presented a comprehensive framework for synthesizing H_∞ static output feedback controllers for LPV time delay systems subject to time-varying delays. By addressing the challenges associated with uncertainties and disturbances, our approach provides practical and efficient control solutions. The contributions of this work open up avenues for further research and development in the field of control design for LPV systems, with potential applications in engineering, automation, and robotics.

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Declarations

Conflict of interest The author declares that he has no conflicting interests.

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