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Fast Finite-Time Consensus for Multi-agent Systems with Diverse Topologies

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Abstract

Fast finite-time consensus problem of multi-agent systems under diverse topologies is investigated by a hybrid linear and fractional power protocol, where the linear item improves convergence performance when the state is far away from the equilibrium, and the fractional power one accelerates convergence process when the state is close to the equilibrium. Then a faster convergent rate is achieved in comparison with the individual asymptotic or finite-time consensus protocol. The leaderless multi-agent systems are firstly studied under undirected topology, and then it is extended to the leader-following case under the directed networks. Based on finite-time stability theory, the state consensus tracking errors are guaranteed to be zero within an upper bound of settling time. Finally, numerical simulations are presented to demonstrate the effectiveness and performance of the protocols.

Keywords Finite-time consensus · Multi-agent systems · Undirected · Directed topology

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1 Introduction

The consensus problems of multi-agent systems have received considerable attention in recent years, such as mobile sensor networks, unmanned air vehicles, autonomous underwater vehicles and multi-spacecraft alignments [10], and the software Evoplex has provided the extensible platform for simulations with the agent-based models and multi-agent systems on networks [2]. The main issues are focus on designing appropriate protocols for consensus behaviors based on the control theory and algebraic graph theory [6, 13, 18]. Compared with asymptotic algorithms, the finite-time cooperative controller possesses faster convergence speed and better disturbance rejection property, such that the states of agents could achieve an agreement in the guaranteed settling time [8, 21]. Based on Lyapunov function and homogeneity with dilation techniques, finite-time consensus results have been categorized into leaderless and leader–follower structure [14, 16].

Finite-time consensus of multi-agent systems can be achieved by a discontinuous or continuous protocol. Finite-time consensus of the first-order integrator dynamics with bounded disturbances rejection is achieved by the discontinuous interaction rule [3]. Some criteria for discontinuous finite-time consensus of the nonsmooth opinion dynamics have been applied to solve the distributed optimization problems over an unbalanced digraph, sufficient finite/fixed-time network modulus consensus criteria over signed digraphs are guaranteed with the sliding mode controller [11, 12]. Both finite-time and fixed-time consensus problems for multi-agent systems with discontinuous nonlinear inherent dynamics are studied in a leader-following framework based on Lipschitz continuous condition [9]. However, the discontinuous protocol can induce chattering both in numerical and practical implementation. General continuous but nonsmooth interaction rules with fractional power item are considered for both first-order and second-order multi-agent systems where undirected network topologies with a spanning tree are taken into account [4, 20]. The continuous finite time consensus protocols are investigated for the bidirectional and the unidirectional interaction cases, and finite-time stability has been proved for continuous static and time-varying weighted undirected graphs [17]. Due to the comparison principle, sufficient conditions are derived to guarantee finite-time consensus of nonlinear multi-agent networks with undirected switching topology [1]. For first order multi-agent systems with unknown nonlinear dynamics under undirected fixed and switching network topologies, the finite-time stability and finite-time parameter convergence are guaranteed by utilizing the local relative position state information, and linearly parameterized method [15]. The adaptive finite-time consensus control of nonlinear mechanical systems with parametric uncertainties are proposed for the multi-agent systems under an undirected graph [5]. Based on the continuous homogeneous finite-time consensus protocol for second-order multi-agent systems, the continuous integral sliding mode super-twisting protocols are developed to achieve accurate finite-time consensus [19]. The homogeneous functions and the finite-time observers have been applied to solve the finite-time consensus problem and tracking protocols for high-order linear multi-agent systems [7]. Multiple time delays and time-varying communication delay are investigated with finite-time consensus problems in [14]. Fast sliding mode control can guarantee faster finite-time convergence rate to reach the sliding surface. Combining with the fast sliding mode control, exponential finite time consensus is achieved for high-order and fractional-order multi-agent systems, respectively [8].

The effectiveness of the finite-time consensus protocols are evaluated by the convergence rate. According to the fractional power item, the above finite-time convergent process is slower when the system is far away from the equilibrium. Motivated by the above investigation, a fast finite-time consensus strategy is studied in this paper, the main contributions are listed as follows: (1) With aid of an additional linear control item, a hybrid finite-time consensus protocol is proposed for single-integrator agents to ensure the consensus and tracking errors converging to zero faster in the whole process, the fast finite-time consensus under undirected topology is proved that the settling time is upper bounded for any initial condition. (2) Different from the reported finite-time consensus in undirected topology, the proposed fast finite-time consensus protocol is extended to the leader–follower case under directed topology, and with the semi-positive definite Lyapunov function, the state consensus tracking errors are guaranteed to be zero within an upper bound of settling time under directed information flow.

The rest of this paper is organized as follows. Section 2 introduces some preliminaries and algebraic graph theory. The main results of the fast finite-time consensus problems under undirected and directed topology are presented in Sect. 3. Section 4 gives a numerical example to verify the correctness. Conclusion is summarized in the last section.

2 Preliminaries and Problem Formulation

In this section, some basic concepts on information consensus and results on agents in a network are introduced about fast finite-time stability and algebraic graph theory.

2.1 Algebraic Graph Theory

The communication topology among N agents is represented by a weighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{V} = \{1, 2, \ldots, N\}$ denotes the set of nodes, $\mathcal{E} \in (\mathcal{V} \times \mathcal{V})$ denotes the set of ordered pairs of the nodes, called edges. Assume that there is no self-edge, i.e., $(i, i) \notin \mathcal{E}$ for any $i \in \mathcal{V}$, and the set of neighbors of agent j is $N_j = \{i \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. An edge $(i, j) \in \mathcal{E}$ in graph $\mathcal{G}$ means that agent j can receive information from agent i, but not necessarily conversely. a_{ij} is the coupling strength of the directed edge (i, j) satisfying $a_{ij} > 0$ if (i, j) is an edge of $\mathcal{G}$ and $a_{ij} = 0$ otherwise. For any pair of vertices (i, j), if $a_{ij} = a_{ji}$, the graph is called an undirected graph. An undirected graph is regarded as connected if a path exists between any two distinct vertices (i, j). The corresponding adjacency matrix $A = \left(a_{ij}\right)_{N\times N}$ is symmetric, i.e. $A^T = A$, and $\lambda_{\min}(A)$ is the minimum eigenvalue of a symmetric matrix A. The Laplacian matrix $\boldsymbol{L} = (l_{ij})$ are defined as follows:

Lemma 1 [21] Let $L = \left[l_{ij}\right] \in R^{N \times N}$ denote a graph Laplacian, which is defined by

$$l_{ij} = \begin{cases} \sum\limits_{k=1,k\neq i}^{N} a_{ik}, & i = j \\ -a_{ij}, & i \neq j \end{cases} \tag{1}$$

then L have the following properties in undirected graph as follows:

1) 0 is an eigenvalue of L and 1_N is the associated eigenvector.

2) For any $\varepsilon = [\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n]^T \in R^n$, $\varepsilon^T L \varepsilon = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\left(\varepsilon_j - \varepsilon_i\right)^2$, which implies that all eigenvalues of L are non-negative real numbers.

3) If the graph is undirected and connected, then the second smallest eigenvalue of L, which is denoted by $\lambda_2(L)$, is larger than zero and $\lambda_2(L) = min_{\varepsilon \neq 0, 1^T\varepsilon = 0} \frac{\varepsilon^T L \varepsilon}{\varepsilon^T \varepsilon}$. Therefore, if $1^T\varepsilon = 0$, then $\varepsilon^T L \varepsilon \geq \lambda_2(L)\varepsilon^T\varepsilon$.

A directed path from agent i_1 to agent i_s is a sequence of edges of the form $(i_k, i_{k+1}), k = 1, 2, \ldots, s-1$. A digraph has a spanning tree if there is an agent called root, such that there is a directed path from the root to each other agent in the graph. Then the topology $\mathcal{G}$ is strongly connected, and directed topological graphs are not symmetric. For simplicity, denote $L_B = L + B$, and B is a nonnegative diagonal matrix defined by $B = \text{diag}(b_1, b_2, \cdots, b_N)$, where $b_i > 0$ means that the leader is accessible by the i th agent, and $b_i = 0$ otherwise. Some preliminary assumption and lemmas about directed network graph are introduced briefly in the following.

Lemma 2 [19] A network $\mathcal{G}$ is strongly connected if and only if its corresponding Laplacian matrix L is irreducible.

Lemma 3 [6] The Laplacian matrix L has a simple eigenvalue zero, and all the other eigenvalues have positive real parts if and only if the directed network has a directed spanning tree.

Lemma 4 [6] Suppose that L is irreducible. Then, $L1_N = 0$, and there is a positive vector $\xi = [\xi_1, \xi_2, \cdots, \xi_N]^T$ such that $\xi^T L = 0$. Denote $\Xi = \text{diag}(\xi_1, \xi_2, \cdots, \xi_N)$. Then, $\Xi > 0$ and $\Xi^{-1} > 0$.

Assumption 1 Suppose that the underlying topology of leader and followers contains a directed spanning tree and the subgraph describing the communication topology among followers is strongly connected.

Lemma 5 [14] Suppose that Assumption 1 holds. Then L_B is invertible and $\Xi L_B + L_B^T \Xi$ is positive definite.

2.2 Fast Finite-Time Stability

Finite-time stability means the system state can converge to the equilibrium in finite time and stay there afterwards, and the corresponding Lyapunov stability theorem is defined as follows:

Lemma 6 [17] Supposing that function $V(t) : [0, \infty) \to [0, \infty)$ is differentiable and satisfies the condition

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \le -K_1 V(t) - K_2 V(t)^{\alpha} \tag{2}$$

where $K_1, K_2 > 0$ and $0 < \alpha < 1$, then $V(t)$ reaches zero at t^* and $V(t) = 0$, for all $t \ge t^*$.

$$t^* = \frac{1}{K_1(1-\alpha)} \ln \frac{K_1 V(0)^{1-\alpha} + K_2}{K_2} \tag{3}$$

Remark 1 For the above differential inequality (2), if $K_2 = 0$, $V(t)$ approaches zero asymptotically, whereas reaches zero in finite time at a lower rate when $K_1 = 0$.

Furthermore, define $s^{[\alpha]} = \mathrm{sign}(s)|s|^{\alpha}$, where $s \in R$, $\mathrm{sign}(s)$ is the sign function and $\alpha > 0$ is a constant, and the following Lemma will be used in the stability analysis.

Lemma 7 [20] If $\xi_1, \xi_2, \cdots, \xi_N \ge 0$ and $0 < \alpha \le 1$ then

$$\left(\sum_{i=1}^{N} \xi_i\right)^{\alpha} \le \sum_{i=1}^{N} \xi_i^{\alpha} \le N^{1-\alpha}\left(\sum_{i=1}^{N} \xi_i\right)^{\alpha} \tag{4}$$

2.3 Problem Formulation

The multi-agent systems consist of N dynamic agents, labeled 1 through N. Let $x_i(t) \in R$ denote the state of agent i, and $x(t) = [x_1(t), x_2(t), \cdots, x_N(t)]^T$. The dynamical model of each agent is described by

$$\dot{x}_i(t) = u_i(t) \tag{5}$$

where $u_i(t)$ is a local state feedback, called the protocol. If $\forall x_i(0)$ and $\left|x_i(t) - x_j(t)\right| \to 0$ as $t \to \infty$, the closed-loop system with the protocol u_i can reach or achieve consensus asymptotically. It is said to achieve finite-time consensus, if for $\forall x_i(0)$, there is a settling time $T \in [0, \infty)$ such that

$$\begin{cases} \lim\limits_{t \to T} \left|x_i(t) - x_j(t)\right| \to 0 \\ x_i(t) = x_j(t), \forall t \ge T \end{cases}$$

3 Main Results

The general framework of the fast finite-time consensus protocols is developed with a group of agents to reach agreement with undirected information flow, then extend the results to the directed topology.

3.1 Fast Finite-Time Consensus Under Undirected Topology

In this subsection, a multi-agent system (1) is investigated under undirected topology. The protocol utilized to solve the fast finite-time consensus problem is

$$u_i = -k_1\left(\sum\nolimits_{j\in N_j} a_{ij}\left(x_i - x_j\right)\right) - k_2\left(\sum\nolimits_{j\in N_j} a_{ij}\left(x_i - x_j\right)\right)^{[\alpha]} \tag{6}$$

where $k_1 > 0$, $k_2 > 0$, $0 < \alpha < 1$.

Theorem 1 Supposing that communication topology $\mathcal{G}(A)$ of the multi-agent systems (1) is undirected and connected, then the protocol (6) solves the fast finite-time consensus problem.

Proof Given the undirected and connected topology, $a_{ij} = a_{ji}$ for all $i, j \in I_N$, $I_N = \{1, 2, \cdots, N\}$ then we obtain.

$$\sum\nolimits_{i=1}^{N} \dot{x}_i = \sum\nolimits_{i=1}^{N} u_i = 0 \tag{7}$$

The Lyapunov function is taken as

$$V = \frac{1}{2}x^T Lx = \frac{1}{4}\sum\nolimits_{i=1}^{N}\sum\nolimits_{j=1}^{N} a_{ij}\left(x_j(t) - x_i(t)\right)^2 \tag{8}$$

According to Lemma 1, $V(x(t)) = 0$ if and only if $x(t) \in \mathrm{span}\{1_N\}$, the symmetry of the adjacent matrix gives that

$$\frac{\partial V(x)}{\partial x_i} = -\sum\nolimits_{j=1}^{N} a_{ij}\left(x_j - x_i\right) \tag{9}$$

The derivative of $V(x)$ versus time is

$$\begin{aligned}\dot{V}(x) = \sum_{i=1}^{N}\frac{\partial V(x)}{\partial x_i}\dot{x}_i &= -\sum_{i=1}^{N}\left(\sum_{j\in N_j} a_{ij}\left(x_j(t) - x_i(t)\right)\right)\\ &\times\left[k_1\left(\sum_{j\in N_j} a_{ij}\left(x_j - x_i\right)\right) + k_2\left(\sum_{j\in N_j} a_{ij}\left(x_j - x_i\right)\right)^{[\alpha]}\right]\end{aligned}$$

$$= -k_1 \sum_{i=1}^{N} \left(\sum_{j \in N_j} a_{ij} \left(x_j(t) - x_i(t) \right) \right)^2 - k_2 \sum_{i=1}^{N} \left(\sum_{j \in N_j} a_{ij} \left(x_j(t) - x_i(t) \right) \right)^{[\alpha+1]}$$

$$= -k_1 \sum_{i=1}^{N} \left(\sum_{j \in N_j} a_{ij} \left(x_j(t) - x_i(t) \right) \right)^2 - k_2 \sum_{i=1}^{N} \left(\left(\sum_{j \in N_j} a_{ij} \left(x_j(t) - x_i(t) \right) \right)^2 \right)^{\frac{[\alpha+1]}{2}} \tag{10}$$

Given that $(\alpha + 1)/2 \in (0, 1)$, with Lemma 7, we have

$$\dot{V}(x) \le -k_1 \sum\nolimits_{i=1}^{N} \left(\sum\nolimits_{j \in N_j} a_{ij} \left(x_j(t) - x_i(t) \right)^2 \right) - k_2 N^{\frac{[1-\alpha]}{2}} \left(\sum\nolimits_{i=1}^{N} \left(\sum\nolimits_{j \in N_j} a_{ij} \left(x_j(t) - x_i(t) \right) \right)^2 \right)^{\frac{[\alpha+1]}{2}} \tag{11}$$

The semi-positive property of L ensures $L = Q^T Q$, $Q \in R_{N*N}$ is a semi-positive matrix. For $V(x) \neq 0$, then

$$\frac{\sum_{i=1}^{N} \left(\sum_{j \in N_j} a_{ij} \left(x_j - x_i \right) \right)^2}{V(x)} = \frac{x^T L^T L x}{\frac{1}{2} x^T L x} = \frac{2 x^T Q^T Q Q^T Q x}{x^T Q^T Q x} \ge 2\lambda_2 \left(Q Q^T \right) = 2\lambda_2(L) \tag{12}$$

where $\lambda_2(L) > 0$, then

$$\dot{V}(x) \le -k_1 (2\lambda_2(L) V(x)) - k_2 N^{\frac{[1-\alpha]}{2}} (2\lambda_2(L))^{\frac{[1+\alpha]}{2}} V(x)^{\frac{[1+\alpha]}{2}} \tag{13}$$

If $V \neq 0$, let $z = (2\lambda_2(L) V(x))^{\frac{1}{2}}$, then

$$\dot{z} = -\lambda_2 k_1 z - k_2 N^{\frac{[1-\alpha]}{2}} \lambda_2 z^{[\alpha]} \tag{14}$$

$\frac{1}{z^{[\alpha]}} \frac{\mathrm{d}z}{\mathrm{d}t} = -\left(k_2 N^{\frac{[1-\alpha]}{2}} \lambda_2 + \lambda_2 k_1 z^{[1-\alpha]} \right)$, it follows

$$\frac{1}{k_2 N^{\frac{[1-\alpha]}{2}} \lambda_2 + \lambda_2 k_1 z^{[1-\alpha]}} dz^{[1-\alpha]} = -[1-\alpha] \mathrm{d}t$$

Let $\varphi\left(z^{[1-\alpha]}\right) = \int_0^{z^{[1-\alpha]}} \frac{1}{k_1 N^{\frac{[1-\alpha]}{2}} \lambda_2 + \lambda_2 k_2 z^{[1-\alpha]}} \mathrm{d}z^{[1-\alpha]}$. The derivative of $\varphi\left(z^{[1-\alpha]}\right)$ is $\varphi'\left(z^{[1-\alpha]}\right) = \frac{1}{k_1 N^{\frac{[1-\alpha]}{2}} \lambda_2 + \lambda_2 k_2 z^{[1-\alpha]}} > 0$, then the function $\varphi\left(z^{[1-\alpha]}\right)$ is monotonically increasing.

Integrating both sides of the equation yields

$$\varphi\left(z^{[1-\alpha]}(t)\right) = \varphi\left(z^{[1-\alpha]}(0)\right) - [1-\alpha]t$$

Since $\varphi\left(z^{[1-\alpha]}\right) = 0$ if and only if $z^{[1-\alpha]} = 0$, which means $V = 0$. Then $\lim_{t\to T(z_0)} V(x) = 0$, the settling time function is given by

$$T(z_0) = \frac{1}{1-\alpha}\varphi\left(z^{[1-\alpha]}(0)\right) = \frac{1}{1-\alpha}\varphi\left((2\lambda_2(L)V(x))^{\frac{[1-\alpha]}{2}}(0)\right)$$

where the settling time is bound by

$$\begin{aligned}\lim_{z_0\to\infty} T(z_0) &= \lim_{z_0\to\infty} \frac{1}{1-\alpha}\varphi\left(z^{[1-\alpha]}(0)\right)\\ &= \lim_{z_0\to\infty} \frac{1}{1-\alpha}\left(\int_0^{z^{[1-\alpha]}} \frac{1}{k_1 N^{\frac{[1-\alpha]}{2}}\lambda_2 + \lambda_2 k_2 z^{1-\alpha}} \mathrm{d}z^{[1-\alpha]}\right)\\ &= \frac{1}{k_1\lambda_2(1-\alpha)} \ln \frac{k_2 N^{\frac{[1-\alpha]}{2}}\lambda_2 + k_1\lambda_2(2\lambda_2 V(0))^{\frac{1-\alpha}{2}}}{k_2 N^{\frac{[1-\alpha]}{2}}\lambda_2}\end{aligned}$$

If $V(x) = 0$, then $x_i = x_j$. Therefore, the proposed protocol guarantees the system's stability and solves the fast finite-time consensus problem.

Remark 2 The hybrid protocol consists of non-linear and linear terms, which solves the fast finite-time consensus problem. The upper bound of convergence time offered by Theorem 1 only relates to the parameters of protocol (6), the order N of the multi-agent system, and the algebraic connectivity of $\mathcal{G}(A)$. When $\alpha = 1$, the protocol is typical linear consensus protocol, whereas is typical finite-time consensus protocol when $k_1 = 0$.

3.2 Fast Finite-Time Leader-Following Consensus Under Directed Topology

In this part, the tracking consensus protocol under the directed networks is considered. The dynamics of the leader has the following form:

$$\dot{x}_0 = u_0 = 0$$

Remark 3 Here the leader's control input is assumed as $u_0 = 0$ for clear expression. As a fact, if $u_0 \neq 0$, we can design a distributed observer for each follower to estimate it in finite time.

The protocol guarantees the system's stability and solves the fast finite-time leader–follower consensus problem

$$u_i(t) = -k_1\left[\sum\nolimits_{j=1}^{N} a_{ij}\left(x_i - x_j\right) + b_i(x_i - x_0)\right] \\ - k_2\left[\sum\nolimits_{j=1}^{N} a_{ij}\left(x_i - x_j\right) + b_i(x_i - x_0)\right]^{[\alpha]} \tag{15}$$

where $k_1 > 0, k_2 > 0, 0 < \alpha < 1$.

Theorem 2 Supposing that Assumption 1 holds. Then, the fast finite-time leader-following consensus tracking problem for multi-agent systems (1) with the leader $\dot{x}_0 = u_0$ can be solved by the protocol (15).

Proof Given that the topology is directed and connected. Set $e_i = x_i - x_0$, for all $i, j \in I_n$.

$$\dot{e}_i = \dot{x}_i - \dot{x}_0 = -k_1\left[\sum_{j=1}^{N} a_{ij}\left(x_i - x_0 - \left(x_j - x_0\right)\right) + b_i(x_i - x_0)\right] \\ - k_2\left[\sum_{j=1}^{N} a_{ij}\left(x_i - x_0 - \left(x_j - x_0\right)\right) + b_i(x_i - x_0)\right]^{[\alpha]}$$

Then the error system is

$$\dot{e}_i = -k_1\left[\sum\nolimits_{j=1}^{N} a_{ij}\left(e_i - e_j\right) + b_i e_i\right] - k_2\left[\sum\nolimits_{j=1}^{N} a_{ij}\left(e_i - e_j\right) + b_i e_i\right]^{[\alpha]} \tag{16}$$

Based on Lemmas 2 and 3, the dynamics of agents can be written in a compact vector form

$$\dot{e} = -k_1(L_B e) - k_2(L_B e)^{[\alpha]} \tag{17}$$

Denote $y = [y_1, \cdots, y_N]^T = L_B e$, then

$$\dot{y} = -k_1 L_B y - k_2 L_B (y)^{[]} \tag{18}$$

Choose a Lyapunov candidate

$$V = \frac{k_1}{2}\sum_{i=1}^{N} \xi_i |y_i|^2 + \frac{k_2}{\alpha+1}\sum_{i=1}^{N} \xi_i |y_i|^{\alpha+1} \tag{19}$$

Take the derivative of V along the trajectory, based on Lemma 4, it follows that:

$$\dot{V} = k_1 \sum_{i=1}^{N} \xi_i |y_i| \mathrm{sign}(y_i)\dot{y}_i + k_2 \sum_{i=1}^{N} \xi_i |y_i|^{\alpha} \mathrm{sign}(y_i)\dot{y}_i$$

$$
\begin{aligned}
&= k_1 y^T \Xi \dot{y} + k_2 \left(y^{[\alpha]}\right)^T \Xi \dot{y} = \left(k_1 y + k_2 y^{[\alpha]}\right)^T \Xi \dot{y} \\
&= -\left(k_1 y + k_2 y^{[\alpha]}\right)^T \Xi L_B \left(k_1 y + k_2 y^{[\alpha]}\right) \\
&= -\frac{1}{2}\left(k_1 y + k_2 y^{[\alpha]}\right)^T \left(\Xi L_B + L_B^T \Xi\right)\left(k_1 y + k_2 y^{[\alpha]}\right) \\
&\le -\frac{1}{2}\bar{\lambda}\left(k_1 y + k_2 y^{[\alpha]}\right)^T \left(k_1 y + k_2 y^{[\alpha]}\right) \\
&= -\frac{1}{2}\bar{\lambda}\sum_{i=1}^{n}\left(k_1 y_i + k_2 y_i^{[\alpha]}\right)^T \left(k_1 y_i + k_2 y_i^{[\alpha]}\right) \\
&= -\frac{1}{2}\bar{\lambda}\sum_{i=1}^{n}\left(k_1^2 |y_i|^2 + 2k_1 k_2 |y_i|^{1+\alpha} + k_2^2 |y_i|^{2\alpha}\right)
\end{aligned}
\tag{20}
$$

According to Lemma 5, $\overline{\lambda} = \lambda_{\min}\left(\Xi L_B + L_B^T \Xi\right) > 0$. Then

$$
V = \frac{k_1}{2}\sum_{i=1}^{N} \xi_i |y_i|^2 + \frac{k_2}{\alpha + 1}\sum_{i=1}^{N} \xi_i |y_i|^{\alpha+1} \tag{21}
$$

$$
V \le \frac{\xi_{\max} k_{\max}}{\alpha + 1}\left(\sum_{i=1}^{N} |y_i|^2 + \sum_{i=1}^{N} |y_i|^{\alpha+1}\right) \tag{22}
$$

where $\xi_{\max} = \max\{\xi_i\}$ is the largest element of the eigenvector in Lemma 4, $i \in I_n$, $k_{\max} = \max\{k_1, k_2\}$ is the larger gain of the linear one and fractional power one.

Consider $\overline{V} = \sum_{i=1}^{N} |y_i|^2 + \sum_{i=1}^{N} |y_i|^{\alpha+1}$, $(1+\alpha)/2 < 1$, according to Lemma 7, it follows that

$$
\begin{aligned}
\left(\sum_{i=1}^{N} |y_i|^2 + \sum_{i=1}^{N} |y_i|^{\alpha+1}\right)^{\frac{1+\alpha}{2}} &\le \left(\sum_{i=1}^{N} |y_i|^2\right)^{\frac{1+\alpha}{2}} + \left(\sum_{i=1}^{N} |y_i|^{\alpha+1}\right)^{\frac{1+\alpha}{2}} \\
&= \sum_{i=1}^{N} |y_i|^{1+\alpha} + \sum_{i=1}^{N} |y_i|^{\frac{(1+\alpha)^2}{2}}
\end{aligned}
\tag{23}
$$

where Lemma 7 is inserted in view of $(\alpha + 1)^2/2 - 2 < 0$ and $(\alpha + 1)^2/2 - 2\alpha > 0$.

$$
\sum_{i=1}^{N} |y_i|^{\frac{(1+\alpha)^2}{2}} \le \sum_{i=1}^{N} |y_i|^2 + \sum_{i=1}^{N} |y_i|^{2\alpha}
$$

Combining the above formula results in

$$
\overline{V}^{\frac{1+\alpha}{2}} \le \sum_{i=1}^{N} |y_i|^{1+\alpha} + \sum_{i=1}^{N} |y_i|^2 + \sum_{i=1}^{N} |y_i|^{2\alpha} \le \sum_{i=1}^{N}\left(|y_i|^{1+\alpha} + |y_i|^2 + |y_i|^{2\alpha}\right) \tag{24}
$$

Similarly, $|y_i|^{\alpha+1} \le |y_i|^{2\alpha} + |y_i|^{\alpha+1}$,

$$
\sum_{i=1}^{N} |y_i|^2 + \sum_{i=1}^{N} |y_i|^{\alpha+1} \le \sum_{i=1}^{N} |y_i|^2 + \sum_{i=1}^{N} |y_i|^{2\alpha} + \sum_{i=1}^{N} |y_i|^{\alpha+1}
$$

It follows that:

$$\overline{V} \leq \sum_{i=1}^{N} |y_i|^2 + \sum_{i=1}^{N} |y_i|^{2\alpha} + \sum_{i=1}^{N} |y_i|^{\alpha+1} \leq \sum_{i=1}^{N} \left(|y_i|^2 + |y_i|^{2\alpha} + 2|y_i|^{\alpha+1} \right)$$

Then,

$$\dot{V} \leq -\frac{1}{4} k_{\max} \overline{\lambda} \left[\overline{V} + \overline{V}^{\frac{\alpha+1}{2}} \right] = -K_1 \overline{V} - K_2 \overline{V}^{\frac{\alpha+1}{2}} \tag{25}$$

where $K_1 = \frac{1}{4} \frac{k_{\max}}{\xi_{\max}} \overline{\lambda}(\alpha + 1)$ and $K_2 = \frac{1}{4} k_{\max} \overline{\lambda} \left(\frac{\alpha+1}{\xi_{\max}} \right)^{\frac{\alpha+1}{2}}$. According to Lemma 6, y reaches to zero, i.e., consensus tracking is achieved, in finite time T_0.

$$T_0 \leq \frac{2}{K_1(1-\alpha)} \ln \frac{K_1 V(0)^{\frac{1-\alpha}{2}} + K_2}{K_2} \tag{26}$$

Remark 4 A transformation and some inequality approaches are utilized in Theorem 2. It is shown that the upper bound of the convergence time is related to the topology, the designed parameters and the initial states.

4 Simulation

In this section, three examples are provided to illustrate the effectiveness of the fast finite-time strategy by comparing with the typical finite-time ones in [16, 17, 21]. Without loss of generality, it is assumed $N = 6$ for all the examples.

Example 1 [17]: Consider the case of the fast finite-time consensus protocol (7) under undirected topology. Set control gain $k_1 = k_2 = 1$, the weight of all edges is 1 in this study. The initial state is $[-1, -0.5, -0.2, 0.2, 0.5, 1]^T$. The communication topology is shown in Figs. 1, 2 shows the fast finite-time state trajectories of agents when $\alpha = 0.5, 0.8$. Comparisons with the traditional finite-time (dotted line)$u_i = -k_2 \left(\sum_{j \in N_j} a_{ij} \left(x_j - x_i \right) \right)^{[\alpha]}$ using the same design parameters show that the convergence time of protocol (7) is faster.

Example 2 [21]: Set nonzero weights $a_{ij} = 2$, two initial scenarios: (a) $[-5, -3, 3, 8, 4, 5]^T$, (b) $[10, -20, -3, 5, 2, -24]^T$, the parameters $k_1 = k_2 = 2$.

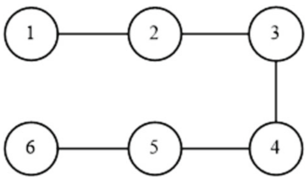
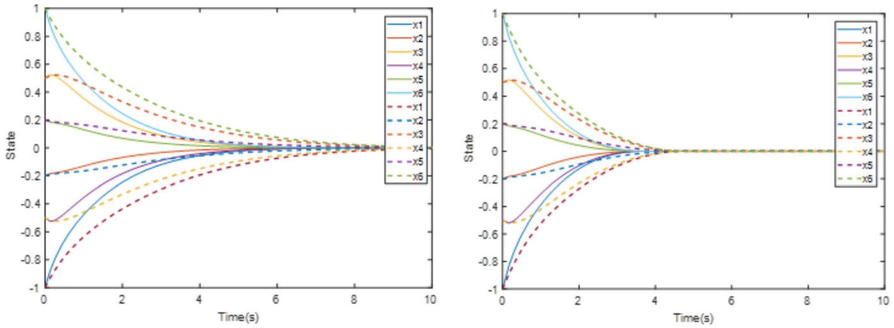


Fig. 1 The undirected communication topology of six agents



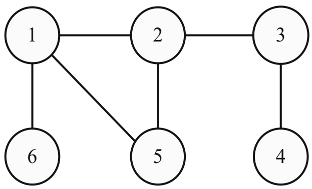
(a) $\alpha = 0.5$ (b) $\alpha = 0.8$

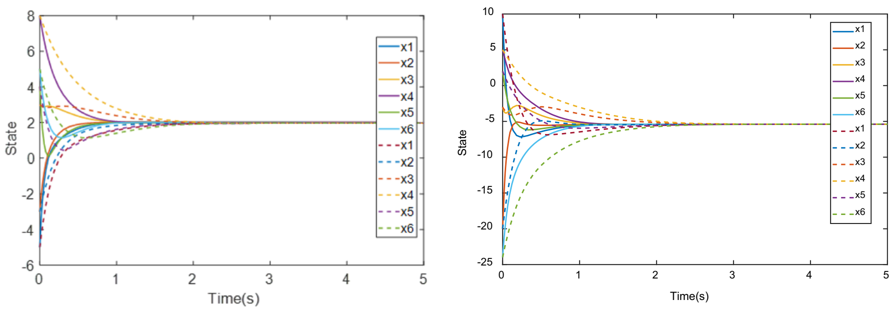
Fig. 2 The state trajectories of agents under protocol (6) with different fractional power (solid line with $k_1 = k_2 = 1$ and dotted line with $k_1 = 0, k_2 = 2$)

The algebraic connectivity of Fig. 3 is 0.83. Figure 4 shows that the settling time of the fast finite-time protocol (7) under different initial conditions are about 1.05 s and 1.22 s, which is less conservative for estimated bounds. The fast finite-time convergence time is shorter than the typical finite-time (dotted line) consensus protocol.

Example 3 [16]: Consider a multi-agent system consists of 5 followers and 1 leader with the controller (15). The communication topology is shown in Fig. 5. The parameters are chosen $k_1 = k_2 = 3$, $\alpha = 0.4$, and a_{ij} is defined in Fig. 5. The initial

Fig. 3 The communication topology of six agents

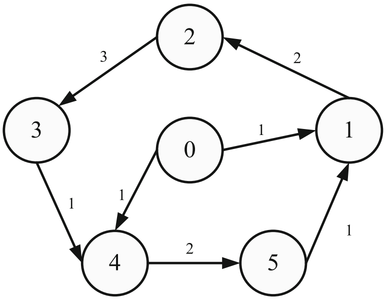




(a) The initial state is $[-5,-3,3,8,4,5]^T$ (b) The initial state is $[10,-20,-3,5,2,-24]^T$

Fig. 4 The state trajectories of agents under protocol (6) with different initial state. (solid line with $k_1 = k_2 = 2$ and dotted line with $k_1 = 0, k_2 = 4$)

Fig. 5 The directed communication topology with 5 followers and 1 leader



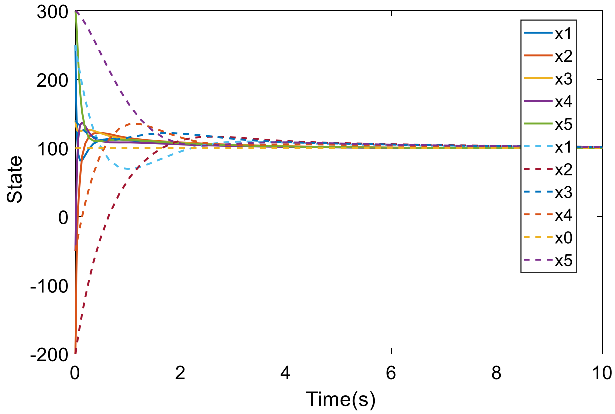


Fig. 6 State trajectories of leader and followers under protocol (15) (solid line with $k_1 = k_2 = 3$ and dotted line with $k_1 = 0, k_2 = 6$)

condition is [250, −200, 140, −50, 300] for the followers, $x_0 = 100$ for the leader. Figures 6 and 7 show that the fast finite-time convergence process of the state and error $e_i = |x_i - x_0|$ trajectories (solid line) is faster than the traditional finite-time (dotted line) respectively.

5 Conclusions

In this paper, the framework of fast finite-time consensus protocols have been investigated under undirected and directed topologies. First, the fast finite-time nonlinear protocol is proposed based on semi-positive definite function to achieve the state agreement under undirected topology. The Lypunov function chooses the states error with neighbors instead of error between states with average. The comparison principle of differential equation is used for finite-time stability. Then it is extended to the leader-following case under the directed networks. The Lypunov function choosing the error between agents and leader, mathematical transformation and inequality is used to prove stability. The effectiveness of the fast finite-time controllers is illustrated in three communication topologies. Future research work will concentrate on

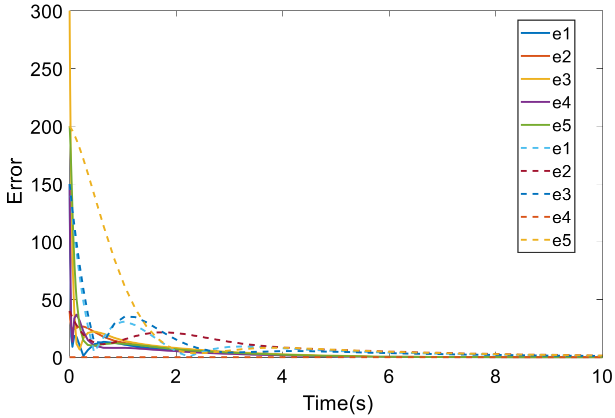


Fig. 7 Error trajectories of followers under protocol (15) (solid line with $k_1 = k_2 = 3$ and dotted line with $k_1 = 0, k_2 = 6$)

the fast finite-time consensus protocols to multi-agent systems with port-Hamiltonian and Euler-Lagrangian dynamics.

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Data Availability The data sets generated during and analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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