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Asynchronous Finite-Time H_∞ Control for Discrete-Time Switched Systems with Admissible Edge-Dependent Average Dwell Time

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Abstract

This paper is concerned with asynchronous finite-time H_∞ control for a class of discrete-time switched linear systems via admissible edge-dependent average dwell time (AED-ADT) approach. Firstly, by considering the switching time delay between the system and the state feedback controller, appropriate Lyapunov functions are constructed for asynchronous and synchronous switching, respectively. Secondly, for the existence of a set of state feedback controllers, a sufficient condition which guarantees the finite-time boundedness of the closed-loop system with AED-ADT is proposed. Thirdly, a sufficient condition for finite-time H_∞ control with a prescribed H_∞ performance is further developed based on the obtained result. Finally, a numerical example is given to verify the validity of the proposed theoretical results.

Keywords Discrete-time switched system · Asynchronous switching · Finite-time boundedness · H_∞ performance · Admissible edge-dependent average dwell time

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1 Introduction

Switched systems are an extremely momentous set of dynamic hybrid systems that can be used to model a large class of controlled objects in the real world, such as physical or man-made systems that show switching characteristics. Generally speaking, a switched system consists of two parts, that is, a family of finite number of continuous or discrete subsystems and a switching signal that determines how to switch between these subsystems [32, 35]. Switched systems have been the subject of intense interest in the past few decades, not only because of their rich and diverse theoretical values [25], but also because of their widespread and profound practical applications [1, 19]. Moreover, there have been fruitful research results on the stability analysis and control synthesis of switched systems under certain constrained switching signals, such as stability and stabilization [31, 34], control and filtering [9, 30].

The so-called asynchronous switching actually refers to the switching behavior in which the switching signal of the controller does not match the one of the subsystems [29]. As we all know, sometimes asynchronous switching will inevitably occur in the process of system operation, which is very likely to reduce the overall performance of the system, even may cause system instability [9]. Moreover, there are many reasons for asynchronous switching, such as time delay, uncertainty and disturbance [21]. In this paper, asynchronous phenomenon due to switching time delay will be mainly considered. If the delay is ignored, the obtained switching control law must be conservative, that is to say, it is crucial to consider the asynchronous phenomenon in the process of studying the stability of the switched system [22]. In addition, in terms of practical value, the application of research results in asynchronous switching theory has a great impact on many practical fields [21].

Finite-time stability is mainly proposed in the research work of Weiss and Infante [20]. A system is called finite-time stable, which means that for a given range of initial conditions, the state will not exceed a certain bound within a specific time interval. This kind of stability and Lyapunov asymptotic stability are two independent concepts, and it has been shown that a system can be finite-time stable but not Lyapunov asymptotically stable, and vice versa [26]. Lyapunov asymptotic stability discusses the system performance in a sufficiently long time interval, while finite-time stability studies it in a finite interval, which is more suitable for practical situations where many state variables do not exceed a given bound in a short interval, and the result is more accurate [2, 8]. In the discrete-time systems, the L_∞ performance refers to the energy-to-peak attenuation of a certain signal, while the H_∞ performance represents the energy-to-energy attenuation of this signal. When the external disturbance $w(k)$ is known, the disturbance suppression performance of H_∞ is better. In the past time, many researchers have developed a strong interest in finite-time stability and H_∞ control of the system, followed by a series of significant research results. For instance, a finite-time H_∞ control design scheme for continuous-time switched systems is proposed in [10], which is extended to discrete-time case subsequently in Ref. [18]. And for discrete-time switched systems, there are finite-time H_∞ control [26], finite-time boundedness and l_2 gain analysis [7], finite-time control [11], etc.

The existence of switching time delay has a crucial impact on the properties of the system. In the switching delay systems, the analysis of stability and the design of the controller have always been of interest to everyone. However, it is very difficult to find a common Lyapunov function for all subsystems in a switching delay system, which motivates the application of the multiple Lyapunov function technique, which is obviously a fairly efficient way to study the stability of switched systems under constrained switching signals [28]. The average dwell time (ADT) [3] indicates that the switching times in a finite interval are limited, and the average time between consecutive switchings is not less than a constant, which generalizes dwell time (DT) [15] to a certain extent. By using the ADT method, sufficient conditions for the globally uniformly exponentially stability of closed-loop systems are deduced in the presence of asynchronous switching controllers [29]. And the design of adaptive output feedback controller [16] and state feedback tracking control [17] for stochastic nonlinear switched systems are studied. However, since the parameters used to calculate the ADT are mode-independent, the results obtained are somewhat conservative. This also prompted the proposal of mode-dependent average dwell time (MDADT) switching, which allows each mode in the underlying system to have its own ADT, which greatly relaxes the constraints of ADT switching [31]. In recent years, the research results obtained by using the MDADT method are also quite rich, such as asynchronous control problem [22], quasi-time-dependent H_∞ controller [12, 33], and finite-time H_∞ control [10].

Subsequently, a new admissible edge-dependent average dwell time (AED-ADT) [6, 23, 24, 27] switching is proposed, which is more flexible and less conservative than MDADT switching, and its switching behavior is based on a directed switching graph, each admissible transition edge represents a directed switch between subsystems. The rational application of this method provides great help to the research of switching, such as time-varying H_∞ control [25], the global uniform exponential stability of discrete-time switched systems [5], input-to-state stability of nonlinear discrete-time switched systems [36], l_2-l_∞ filtering [4], and asynchronous l_2-l_∞ filtering [14]. All in all, the application of the current AED-ADT switching method can effectively reduce the conservativeness of the research results. However, to the best of our knowledge, in the existing research work, there are relatively few results related to analyzing the asynchronous finite-time H_∞ control problem by using the AED-ADT switching, which motivated us to carry out our study.

Inspired by the above literature works, we will mainly study the asynchronous finite-time H_∞ control problem of discrete-time switched linear systems using AED-ADT approach in this paper. There are three main contributions: (1) In the study of discrete-time switched linear systems, the existence of switching time delay is mainly considered. In contrast to those studies where systems and controllers are defaulted to be switched synchronously, this paper is more suitable for practical applications. (2) For the sake of obtaining less conservative research results, the AED-ADT switching signal, which is more flexible and applicable than MDADT switching, is used. (3) An asynchronous finite-time H_∞ control design scheme with AED-ADT switching is proposed for discrete-time switched linear systems.

The remainder of this paper is organized as follows. Section 2 gives the problem statements, along with some necessary definitions and lemmas. In Sect. 3, by using the AED-ADT approach for the resulting closed-loop system, the design of a state feedback controller under asynchronous switching is considered, and a sufficient condition to guarantee its finite-time boundedness is derived. In addition, based on the obtained result, the H_∞ performance analysis of the closed-loop system is also performed. In Sect. 4, the numerical simulation is addressed to demonstrate the reasonability and effectiveness of the proposed method. In the end, some conclusions are given in Sect. 5.

Notations In the paper, the notations used are fairly standard. Let $\mathbb{R}^n$ be the space of n-dimensional real vectors, and $\mathbb{R}^{n\times m}$ be the set of all $(n \times m)$-dimensional real matrices. And $\mathbb{Z}_0^+$ is the nonnegative integer set. For a matrix P, $P > 0$ ($P < 0$) signifies that P is symmetric positive definite (negative definite) matrix, and $P \geq 0$ ($P \leq 0$) signifies that P is symmetric semi-positive definite (semi-negative definite) matrix. P^{-1} and P^{T} denote the inverse and transpose of P, respectively. And the asterisk $(*)$ denotes the symmetrical items in a symmetric matrix. $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ represent the minimum and maximum eigenvalue of matrix P, respectively. I and 0 are identity matrix and zero matrix. $l_2[0, \infty)$ is the space of square-summable infinite sequence over $[0, \infty)$. In addition, if not explicitly stated, it is assumed that the matrices have the compatible dimensions for algebraic operations.

2 Problem Statement and Preliminaries

2.1 Switched System

In this paper, consider the discrete-time switched linear systems as follows:

$$\begin{aligned} x(k+1) &= A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) + D_{\sigma(k)}w(k), x(0) = x_0, \\ y(k) &= C_{\sigma(k)}x(k) + E_{\sigma(k)}u(k) + F_{\sigma(k)}w(k), \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the system state, $x(0) = x_0$ is the initial state, $u(k) \in \mathbb{R}^m$ is the control input, and $w(k) \in \mathbb{R}^p$ is the disturbance input, which belongs to $l_2[0, \infty)$, $y(k) \in \mathbb{R}^q$ is the output. The switching signal of the system (1) is given by $\sigma(k) : \mathbb{Z}_0^+ \to \bar{M} = \{1, 2, \cdots, M\}$, $M > 1$ is the number of switched subsystems. A_i, B_i, D_i, C_i, E_i, F_i, $i \in \bar{M}$, are known constant matrices with appropriate dimensions. For a time sequence $0 = k_0 < k_1 < \cdots < k_i < k_{i+1} < \cdots$, which is the switching instant of $\sigma(k)$, when $k \in [k_i, k_{i+1})$, we say that the $\sigma(k_i)$th subsystem is active, and hence the trajectory $x(k)$ of the system (1) is just the trajectory of this subsystem.

2.2 State Feedback Controller

A state feedback controller can be considered for the discrete-time switched systems (1); the specific form is as follows:

$$u(k) = K_{\sigma(k)}x(k), \tag{2}$$

where $K_{\sigma(k)}$ is the control gain matrix to be determined.

In fact, in the process of system switching, it takes a certain time to identify the activated subsystem, which will cause the system to switch to the next subsystem, but the controller still stays in the current subsystem. Only after the activated subsystem is successfully identified, the controller can switch to the next subsystem, which causes the switching delay of the controller. This situation is common in practical applications, and we will consider it in this paper.

Due to the existence of the controller switching time delay, the state feedback controller in this paper should take the following form:

$$u(k) = K_{\sigma'(k)}x(k), \tag{3}$$

where $K_{\sigma'(k)}$ is the control gain matrix to be determined, $\sigma'(k)$ is the switching signal of the state feedback controller (3), we say that (3) is called the asynchronous switching controller [9, 21].

Due to the existence of the switching time delay of the state feedback controller, the switching of the controller is often later than the switching of the system. Denote $[k_i, k_{i+1}) = [k_i, k_i + \Delta_i) \bigcup [k_i + \Delta_i, k_{i+1})$, where $0 \le \Delta_i \le k_{i+1} - k_i - 1$, $i \in \mathbb{Z}_0^+$ and $\Delta_0 = 0$. And Δ_i is an integer, which denotes the period that the switching instants of the controller have a delay with respect to that of the system. Therefore, when $k \in [k_i, k_i + \Delta_i)$, $\sigma(k) \neq \sigma'(k)$, the switching of the system itself does not match the switching of the controller, and the mismatched period is called the time of asynchronous switching. In addition, $\Delta_i \le k_{i+1} - k_i - 1$ guarantees that there always exists a period which the controller and the system operate synchronously in any switching interval. That is to say, when $k \in [k_i + \Delta_i, k_{i+1})$, $\sigma(k) = \sigma'(k)$, the switching signal of the controller is consistent with that of the system, and this period is called the matched period.

Denote

$$\nu : \{(k_0, \sigma(k_0)), (k_1, \sigma(k_1)), \cdots, (k_{i-1}, \sigma(k_{i-1})), (k_i, \sigma(k_i)), (k_{i+1}, \sigma(k_{i+1})), \cdots\} \tag{4}$$

as the switching sequence of the system, where k_i means the ith switching instant. Then, the switching sequence of the controller can be described as follows:

$$\begin{aligned} \nu' : \{&(k_0, \sigma'(k_0)), (k_1 + \Delta_1, \sigma'(k_1 + \Delta_1)), \cdots, (k_{i-1} + \Delta_{i-1}, \sigma'(k_{i-1} + \Delta_{i-1})), \\ &(k_i + \Delta_i, \sigma'(k_i + \Delta_i)), (k_{i+1} + \Delta_{i+1}, \sigma'(k_{i+1} + \Delta_{i+1})), \cdots\} \end{aligned} \tag{5}$$

where $\sigma(k_0) = \sigma'(k_0)$, which indicates that the system and the controller switch simultaneously at the initial moment. Moreover, denote $\sigma(k_i) = \sigma'(k_i + \Delta_i)$, where $i \in \mathbb{Z}_0^+$.

When the lth subsystem is activated at switching instant k_{i-1}, the ith subsystem is activated at switching instant k_i, and the jth subsystem is activated at switching instant k_{i+1}. Due to the existence of asynchronous switching, the corresponding switches of the controller occur at the instant $k_{i-1} + \Delta_{i-1}$, $k_i + \Delta_i$, and $k_{i+1} + \Delta_{i+1}$, respectively. That is to say, assume $\sigma(k_{i-1}) = l$, $\sigma(k_i) = i$, and $\sigma(k_{i+1}) = j$, then we have $\sigma'(k_{i-1} + \Delta_{i-1}) = l$, $\sigma'(k_i + \Delta_i) = i$, and $\sigma'(k_{i+1} + \Delta_{i+1}) = j$, where $l, i, j \in \bar{M}$, as shown in Fig. 1.

Given any $k > 0$, there are $N_{\sigma(k_0,k)}$ switches for $\sigma(k)$ in the interval $[k_0, k)$, the switching instants of the system are set as $\{k_1, k_2, \cdots k_{N_{\sigma(k_0,k)}}\}$. Denote $0 = k_0 < k_1$,

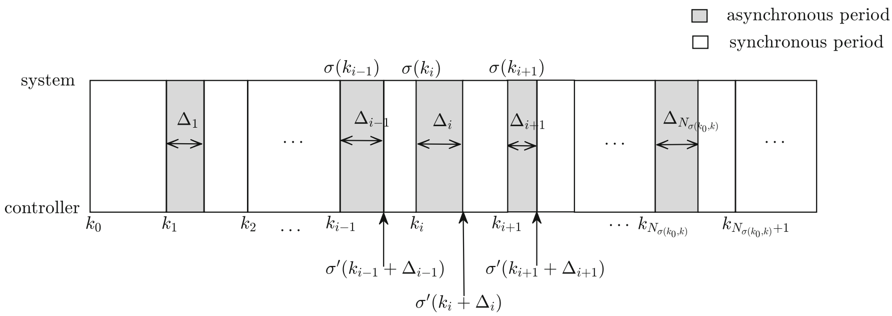

Fig. 1 The illustration of switching sequences for system and controller under asynchronous switching

$k_{N_{\sigma(k_0,k)}} < k < k_{N_{\sigma(k_0,k)}+1}$, and k_0 is not a switching instant. In addition, the switching instants of the controller are set as $\{k_1+\Delta_1, k_2+\Delta_2, \cdots, k_{N_{\sigma(k_0,k)}}+\Delta_{N_{\sigma(k_0,k)}}\}$, where Δ_i, $i = 1, 2, \cdots, N_{\sigma(k_0,k)}$ are the time delays between the system and the controller. We simply denote $N = N_{\sigma(k_0,k)}$.

In this paper, let $\mathfrak{M}_s$ and $\mathfrak{M}_u$ be the set of all time intervals in which the system and the state feedback controller switch synchronously and asynchronously, respectively. Then, we have $\mathfrak{M}_s = (k_{N_{\sigma(k_0,k)}} + \Delta_{N_{\sigma(k_0,k)}}, k) \bigcup (\bigcup_{i=0}^{N_{\sigma(k_0,k)}-1} [k_i + \Delta_i, k_{i+1}))$ and $\mathfrak{M}_u = \bigcup_{i=1}^{N_{\sigma(k_0,k)}} [k_i, k_i + \Delta_i)$. Let $T_\downarrow(k_0, k)$ and $T_\uparrow(k_0, k)$ denote unions of the time intervals during which Lyapunov functional candidate $V(x(k))$ is decreasing and increasing within the time interval $[k_0, k)$, respectively. In other words, $T_\downarrow(k_0, k)$ and $T_\uparrow(k_0, k)$ denote unions of the time intervals in which the system itself and the state feedback controller switch synchronously and asynchronously, respectively. $T^s(k_0, k)$ and $T^u(k_0, k)$ represent the length of $T_\downarrow(k_0, k)$ and $T_\uparrow(k_0, k)$, respectively. Then, in the interval $[k_0, k)$, the total time for synchronous switching of the system and the controller is $T^s(k_0, k) = \sum_{i=0}^{N_{\sigma(k_0,k)}} [k_i + \Delta_i, k_{i+1}) + (k - (k_{N_{\sigma(k_0,k)}} + \Delta_{N_{\sigma(k_0,k)}}))$, where $\Delta_0 = 0$, and the total time for their asynchronous switching is $T^u(k_0, k) = \sum_{i=1}^{N_{\sigma(k_0,k)}} [k_i, k_i + \Delta_i) = \sum_{i=1}^{N_{\sigma(k_0,k)}} \Delta_i$, and $T^s(k_0, k) + T^u(k_0, k) = k - k_0$.

2.3 Switching Signal

Definition 2.1 [10] For a switching signal $\sigma(k)$, let $N_i^\sigma(k_0, k)$ and $T_i(k_0, k)$ denote the switching numbers that the ith subsystem is active and the total running time of the ith subsystem over the interval $[k_0, k)$, respectively. If there exist positive numbers τ_i and N_i^0, such that

$$N_i^\sigma(k_0, k) \leq N_i^0 + \frac{T_i(k_0, k)}{\tau_i^a}, \forall k \geq k_0 \geq 0, i \in \bar{M}, \tag{6}$$

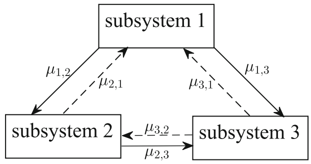

Fig. 2 A directed switching graph $\mathfrak{G}$ with $\bar{M} = 3$

we say that the switching signal $\sigma(k)$ has a mode-dependent average dwell time (MDADT) τ_i^a and a corresponding mode-dependent chatter bound N_i^0.

Definition 2.2 [6] Given a directed graph $\mathfrak{G}$ and $\forall i, j \in \bar{M}$ $(i \neq j)$, we define $\mathcal{I}(i, j)$ as an admissible transition edge (ATE) of $\mathfrak{G}$ if the directed edge from i to j is admissible. The set of all admissible transition edges (ATEs) is represented by Ω. An ATE $\mathcal{I}(i, j)$ has an admissible transition edge-dependent weight (ATEDW) μ_{ij}, which describes the switching property from the subsystem i to subsystem j. And the set of all admissible transition edge-dependent weights (ATEDWs) is signified by Γ.

Remark 2.1 The subsystem will switch from i to j when switching occurs, where i, $j \in \bar{M}$ and $i \neq j$. Assuming that there are three subsystems in a discrete-time switched system, then in this directed switching graph $\mathfrak{G}$, the sets of ATEs and ATEDWs between these three subsystems can be represented as $\Omega = \{\mathcal{I}(1, 2), \mathcal{I}(1, 3), \mathcal{I}(2, 1), \mathcal{I}(2, 3), \mathcal{I}(3, 1), \mathcal{I}(3, 2)\}$ and $\Gamma = \{\mu_{1,2}, \mu_{1,3}, \mu_{2,1}, \mu_{2,3}, \mu_{3,1}, \mu_{3,2}\}$, respectively, as shown in Fig. 2[6, 26].

Definition 2.3 [26]For a switching signal $\sigma(k) : \mathbb{Z}_0^+ \to \bar{M}$, and $\forall (i, j) \in \bar{M} \times \bar{M}$, $i \neq j$, let $N_{ij}^\sigma(k_0, k)$ and $T_{ij}(k_0, k)$ denote the switching numbers from subsystem i to j and the total running time of subsystem j over the interval $[k_0, k)$, respectively. If there exist positive numbers τ_{ij}^a and N_{ij}^0 such that

$$N_{ij}^\sigma(k_0, k) \leq N_{ij}^0 + \frac{T_{ij}(k_0, k)}{\tau_{ij}^a}, \forall k \geq k_0 \geq 0, \tag{7}$$

we say that the switching signal $\sigma(k)$ has an admissible edge-dependent average dwell time (AED-ADT) τ_{ij}^a and a corresponding admissible edge-dependent chatter bound N_{ij}^0.

Remark 2.2 Comparing (6) of Definition 2.1 and (7) of Definition 2.3, it is obvious that the above concept of AED-ADT is more general than that of the mode-dependent case. Let $\varepsilon(\bar{M})$ be the set of all admissible ordered pairs (i, j), $\bar{J}(i) = \{j \in \bar{M} \mid (i, j) \in \varepsilon(\bar{M})\}$. Note that $N_i^\sigma(k_0, k) = \sum_{j\in\bar{J}(i)} N_{ij}^\sigma(k_0, k)$ and $T_i(k_0, k) = \sum_{j\in\bar{J}(i)} T_{ij}^\sigma(k_0, k)$. Obviously, we have $T_{ij}(k_0, k) = T_{ij}^s(k_0, k) + T_{ij}^u(k_0, k)$ and $T^s(k_0, k) = \sum_{i\in\mathfrak{M}_s} \sum_{j\in\bar{J}(i)} T_{ij}(k_0 , k)$, $T^u(k_0, k) = \sum_{i\in\mathfrak{M}_u} \sum_{j\in\bar{J}(i)} T_{ij}(k_0, k)$.

As considered in AED-ADT, it is assumed that the activation time of each subsystem is also edge-dependent.

Assumption 2.1 [36] Suppose there exist scalars $\nu_{ij}^s \in (0, 1]$, $\nu_{ij}^u \in [0, 1)$ and constants $\bar{T}_{ij}^s \geq 0$, $\bar{T}_{ij}^u \geq 0$ such that $\forall k > k_0 \geq 0$,

$$T_{ij}(k_0, k) \geq -\bar{T}_{ij}^s + \nu_{ij}^s(k - k_0), i \in \mathfrak{M}_s, \tag{8a}$$

$$T_{ij}(k_0, k) \leq \bar{T}_{ij}^u + \nu_{ij}^u(k - k_0), i \in \mathfrak{M}_u, \tag{8b}$$

$$\sum_{i\in\mathfrak{M}_s} \sum_{j\in\bar{J}(i)} \bar{T}_{ij}^s = \sum_{i\in\mathfrak{M}_u} \sum_{j\in\bar{J}(i)} \bar{T}_{ij}^u, \tag{8c}$$

$$\sum_{i\in\mathfrak{M}_s} \sum_{j\in\bar{J}(i)} \nu_{ij}^s + \sum_{i\in\mathfrak{M}_u} \sum_{j\in\bar{J}(i)} \nu_{ij}^u = 1. \tag{8d}$$

Assumption 2.2 [36] Whenever $(i, j) \in \varepsilon(\bar{M})$, there exist constants $\mu_{ij} \geq 1$, such that Lyapunov functions V_i satisfy inequalities as follows:

$$V_j(x(k)) \leq \mu_{ij} V_i(x(k)), (i, j) \in \varepsilon(\bar{M}). \tag{9}$$

2.4 System Stability

Assumption 2.3 [26] It is assumed that the external disturbance $w(k)$ is time-varying and satisfies the following constraint:

$$\sum_{k=0}^{N} w^{\mathrm{T}}(k)w(k) \leq d, d > 0, \tag{10}$$

where N is a positive integer and d is a positive constant.

Definition 2.4 [26] Given a matrix $R > 0$, two scalars $c_2 > c_1 > 0$, an integer $N > 0$, and a switching signal σ, the discrete-time switched linear system (1) with $u(k) \equiv 0$ and $w(k) \equiv 0$ is said to be finite-time stable with respect to (c_1, c_2, R, N, σ), if $x^{\mathrm{T}}(0)Rx(0) \leq c_1 \Rightarrow x^{\mathrm{T}}(k)Rx(k) < c_2, \forall k \in \{1, 2, \cdots, N\}$.

Definition 2.5 [26] Given a matrix $R > 0$, two scalars $c_2 > c_1 > 0$, an integer $N > 0$, and a switching signal σ, the discrete-time switched linear system (1) subject to an exogenous disturbance $w(k)$ satisfying (10) is said to be finite-time bounded w.r.t. $(c_1, c_2, R, d, N, \sigma)$, if $x^{\mathrm{T}}(0)Rx(0) \leq c_1 \Rightarrow x^{\mathrm{T}}(k)Rx(k) < c_2, \forall k \in \{1, 2, \cdots, N\}$.

Definition 2.6 [18] Given a matrix $R > 0$, two scalars $c_2 > c_1 > 0$, an integer $N > 0$, and a switching signal σ, the discrete-time switched linear system (1) is said to be finite-time bounded with a H_∞ performance index γ w.r.t. $(c_1, c_2, R, d, \gamma, N, \sigma)$, if the system (1) is finite-time bounded w.r.t. $(c_1, c_2, R, d, N, \sigma)$ and under initial condition $x(0) = 0$, it holds that

$$\sum_{k=0}^{N} y^{\mathrm{T}}(k)y(k) < \gamma^2 \sum_{k=0}^{N} w^{\mathrm{T}}(k)w(k). \tag{11}$$

Lemma 2.1 *[13] For the system (1) and Lyapunov function of the ith subsystem of the form* $V_i(x(k)) = x^{\mathrm{T}}(k)P_i x(k)$, $i \in \bar{M}$, *let* $\lambda_{\max}(P_i)$ *and* $\lambda_{\min}(P_i)$ *be the maximal and minimal eigenvalue of the positive definite matrix* P_i*, respectively. Then, for any* $i \in \bar{M}$, $\lambda_{\max}(P_i) > 0$, $\lambda_{\min}(P_i) > 0$ *and* $\lambda_{\min}(P_i)x^{\mathrm{T}}(k)x(k) \leq V_i(x(k)) \leq \lambda_{\max}(P_i)x^{\mathrm{T}}(k)x(k)$.

Under the asynchronous switching controller (3), the corresponding closed-loop system is given by

$$\begin{aligned} x(k+1) &= (A_{\sigma(k)} + B_{\sigma(k)}K_{\sigma'(k)})x(k) + D_{\sigma(k)}w(k), x(0) = x_0, \\ y(k) &= (C_{\sigma(k)} + E_{\sigma(k)}K_{\sigma'(k)})x(k) + F_{\sigma(k)}w(k). \end{aligned} \tag{12}$$

In the resulting closed-loop system (12), the asynchronous switching between the system and the controller may damage the performance of the system to a certain extent. Therefore, in this paper, we aim to design a set of state feedback controllers formed in (3) and a switching signal with AED-ADT such that (12) is finite-time bounded.

3 Main Results

In this section, we will design a state feedback controller of the form (3) for the system (1), such that the resulting closed-loop system (12) is finite-time bounded, and in the presence of a controller, the corresponding state gain matrix is derived. Then, based on the obtained result, the H_∞ performance will be analyzed. In addition, to reduce the conservatism of the results, we only require the subsystems to be stable during the matched period, and allow the subsystems to be unstable within a bounded mismatched time interval, while allowing the Lyapunov function to increase during the mismatched period.

3.1 Finite-Time Boundedness

At the beginning, the problem of finite-time boundedness for the closed-loop system (12) will be considered under AED-ADT switching.

Theorem 3.1 *Given a matrix* $R > 0$*, an integer* $N > 0$*, for specified constants* $c_2 > c_1 > 0$, $d > 0$, $\gamma > 0$, $0 < \alpha_j < 1$, $\beta_{ij} > 0$, $\mu_{ij} \geq 1$*, and suppose that there exist matrices* $X_i > 0$, $X_j > 0$, $X_{ij} > 0$ *and* $Q_i > 0$, $Q_j > 0$, $Q_{ij} > 0$, Y_i *such that* $\forall i, j \in \bar{M}, i \neq j$,

$$X_{ij} \leq \mu_{ij}X_j, Q_j \leq \mu_{ij}Q_{ij}, i \neq j \in \bar{M}, \tag{13a}$$

$$X_i \leq \mu_{ij}X_{ij}, Q_{ij} \leq \mu_{ij}Q_i, i \neq j \in \bar{M}, \tag{13b}$$

$$\begin{bmatrix} -(1-\alpha_j)X_j & 0 & X_jA_j^{\mathrm{T}} + Y_j^{\mathrm{T}}B_j^{\mathrm{T}} \\ * & -\gamma Q_j & D_j^{\mathrm{T}} \\ * & * & -X_j \end{bmatrix} \leq 0, \tag{13c}$$

$$\begin{bmatrix} -(1+\beta_{ij})X_{ij} & 0 & X_{ij}A_j^{\mathrm{T}} + Y_i^{\mathrm{T}} B_j^{\mathrm{T}} \\ * & -\gamma Q_{ij} & D_j^{\mathrm{T}} \\ * & * & -X_{ij} \end{bmatrix} \leq 0, \tag{13d}$$

$$\frac{c_1(\lambda_1 - \lambda_2\phi_0)}{\lambda_1\lambda_2} < \frac{\phi_0 \eta \lambda_3 \gamma d}{1-\lambda}, \tag{13e}$$

where

$$\eta_1 = (1-\alpha_{\sigma(k_n)})^{(k_n+\Delta_n)-k_{n+1}} \geq 1, n = 0, 1, 2, \cdots, N,$$
$$\eta_2 = (1+\beta_{\sigma(k_{n-1})\sigma(k_n)})^{\Delta_n - 1} \geq 1, n = 1, 2, \cdots, N,$$
$$\eta = \max\{\eta_1, \eta_2\} \geq 1, \quad \varphi_0 = \exp(\sum_{(i,j)\in\varepsilon(\mathcal{L})} 2N_{ij}^0 \ln \mu_{ij}),$$

$$\phi_0 = \varphi_0 \prod_{i\in \mathrm{M}_s} \prod_{j\in \mathrm{J}(i)} ((1-\alpha_j)\mu_{ij}^{\frac{2}{\tau_{ij}^a}})^{-\bar{T}_{ij}^s} \times \prod_{i\in \mathrm{M}_u} \prod_{j\in \mathrm{J}(i)} ((1+\beta_{ij})\mu_{ij}^{\frac{2}{\tau_{ij}^a}})^{\bar{T}_{ij}^u},$$
$$\lambda_1 = \min_{j\in\mathcal{L}}(\lambda_{\min}(\bar{X}_j)), \quad \lambda_2 = \max_{j\in\mathcal{L}}(\lambda_{\max}(\bar{X}_j)),$$
$$\lambda_3 = \max_{j\in\mathcal{L}}(\lambda_{\max}(Q_j)), \quad \bar{X}_{\sigma(k)} = R^{1/2} X_{\sigma(k)} R^{1/2}.$$

Then, there exists a set of controllers (3) such that the closed-loop system (12) is finite-time bounded for any AED-ADT switching signal $\sigma(k)$ and coefficients $\nu_{ij}^s \in (0, 1]$, $\nu_{ij}^u \in [0, 1)$ in Assumption 2.1 satisfying

$$\tau_{ij}^a > \tau_{ij}^{a*} = -\frac{2\ln\mu_{ij}}{\ln(1-\alpha_j)}, \forall i, j \in \bar{M}, \tag{14a}$$

$$\lambda = \prod_{i\in\mathfrak{M}_s, i\neq j} \prod_{j\in \bar{J(i)}} ((1-\alpha_j)\mu_{ij}^{\frac{2}{\tau_{ij}^a}})^{\nu_{ij}^s} \times \prod_{i\in\mathfrak{M}_u, i\neq j} \prod_{j\in \bar{J(i)}} (1+\beta_{ij})\mu_{ij}^{\frac{2}{\tau_{ij}^a}})^{\nu_{ij}^u} < 1. \tag{14b}$$

Moreover, if the controllers exist, the controller gains are given by

$$K_i = \begin{cases} Y_i X_i^{-1}, & i \in \mathfrak{M}_s, \\ Y_i X_{ij}^{-1}, & i \in \mathfrak{M}_u. \end{cases} \tag{15}$$

Proof The whole proof process is divided into four steps, which is shown as follows.

Step 1 First, for any $k \in [k_{i+1}, k_{i+2}) = [k_{i+1}, k_{i+1} + \Delta_{i+1}) \bigcup [k_{i+1} + \Delta_{i+1}, k_{i+2})$, the switching situation of a subsystem is analyzed.

(i) When $k \in [k_{i+1} + \Delta_{i+1}, k_{i+2})$, one has that $\sigma(k) = \sigma'(k) = j$, and the closed-loop system (12) can be written as

$$x(k+1) = (A_j + B_j K_j)x(k) + D_j w(k), x(0) = x_0. \tag{16}$$

Consider the following Lyapunov functional candidate:

$$V_{\sigma(k)}(x(k)) = x^{\mathrm{T}}(k)P_{\sigma(k)}x(k), \tag{17}$$

where $P_{\sigma(k)} = (X_{\sigma(k)})^{-1}$ is a symmetric positive definite matrix, and $X_{\sigma(k)}$ satisfies conditions (13a)–(13d). So the difference of the $V_j(x(k))$ along the trajectory of the switched system (16) is

$$\begin{aligned}
&\Delta V_j(x(k))\\
&\quad = V_j(x(k+1)) - V_j(x(k))\\
&\quad = x^{\mathrm{T}}(k+1)P_jx(k+1) - x^{\mathrm{T}}(k)P_jx(k)\\
&\quad = x^{\mathrm{T}}(k)[(A_j + B_jK_j)^{\mathrm{T}}P_j(A_j + B_jK_j) - P_j]x(k) + x^{\mathrm{T}}(k)(A_j\\
&\qquad + B_jK_j)^{\mathrm{T}}P_jD_jw(k) + w^{\mathrm{T}}(k)D_j^{\mathrm{T}}P_j(A_j + B_jK_j)x(k) + w^{\mathrm{T}}(k)D_j^{\mathrm{T}}P_jD_jw(k)\\
&\quad = \xi^{\mathrm{T}}(k)\begin{bmatrix} (A_j + B_jK_j)^{\mathrm{T}}P_j(A_j + B_jK_j) - P_j & (A_j + B_jK_j)^{\mathrm{T}}P_jD_j \\ D_j^{\mathrm{T}}P_j(A_j + B_jK_j) & D_j^{\mathrm{T}}P_jD_j \end{bmatrix}\xi(k),
\end{aligned} \tag{18}$$

where $\xi^{\mathrm{T}}(k) = [x^{\mathrm{T}}(k), w^{\mathrm{T}}(k)]$.

Meanwhile, by multiplying (13c) from both sides by $\mathrm{diag}\{P_j, I, P_j\}$, and using (15), we have

$$\begin{bmatrix} -(1-\alpha_j)P_j & 0 & (A_j + B_jK_j)^{\mathrm{T}}P_j \\ * & -\gamma Q_j & D_j^{\mathrm{T}}P_j \\ * & * & -P_j \end{bmatrix} \leq 0, \tag{19}$$

by using Schur complement, inequality (19) is equivalent to

$$\begin{aligned}
&\begin{bmatrix} (A_j + B_jK_j)^{\mathrm{T}}P_j(A_j + B_jK_j) - (1-\alpha_j)P_j & (A_j + B_jK_j)^{\mathrm{T}}P_jD_j \\ D_j^{\mathrm{T}}P_j(A_j + B_jK_j) & D_j^{\mathrm{T}}P_jD_j - \gamma Q_i \end{bmatrix}\\
&= \begin{bmatrix} (A_j + B_jK_j)^{\mathrm{T}}P_j(A_j + B_jK_j) - P_j & (A_j + B_jK_j)^{\mathrm{T}}P_jD_j \\ D_j^{\mathrm{T}}P_j(A_j + B_jK_j) & D_j^{\mathrm{T}}P_jD_j \end{bmatrix}\\
&\quad + \begin{bmatrix} \alpha_jP_j & 0 \\ 0 & -\gamma Q_j \end{bmatrix} \leq 0.
\end{aligned} \tag{20}$$

Thus, by combining (18) and (20), we obtain

$$\Delta V_j(x(k)) = V_j(x(k+1)) - V_j(x(k)) \leq -\alpha_j x^{\mathrm{T}}(k)P_jx(k) + \gamma w^{\mathrm{T}}(k)Q_jw(k), \tag{21}$$

obviously we get

$$V_j(x(k+1)) \leq (1-\alpha_j)V_j(x(k)) + \gamma w^{\mathrm{T}}(k)Q_jw(k). \tag{22}$$

Then for any $k \in [k_{i+1}+\Delta_{i+1}, k_{i+2}), \sigma(k) = \sigma(k-1) = \cdots = \sigma(k_{i+1}+\Delta_{i+1}) = j$, and $\sigma'(k) = \sigma(k) = j$, by iterating (22), it gives

$$V_j(x(k)) \leq (1-\alpha_j)^{k-(k_{i+1}+\Delta_{i+1})} V_j(x(k_{i+1}+\Delta_{i+1})) + \sum_{s=k_{i+1}+\Delta_{i+1}}^{k-1} (1-\alpha_j)^{k-1-s} \gamma w^{\mathrm{T}}(s) Q_j w(s). \tag{23}$$

(ii) When $k \in [k_{i+1}, k_{i+1}+\Delta_{i+1})$, we have $\sigma(k) = j$, and $\sigma'(k) = i$, so the closed-loop system (12) can be written as

$$x(k+1) = (A_j + B_j K_i)x(k) + D_j w(k), x(0) = x_0. \tag{24}$$

Consider the following Lyapunov functional candidate

$$V_{\sigma'(k)\sigma(k)}(x(k)) = x^{\mathrm{T}}(k) P_{\sigma'(k)\sigma(k)} x(k), \tag{25}$$

where $P_{\sigma'(k)\sigma(k)} = (X_{\sigma'(k)\sigma(k)})^{-1}$ is a symmetric positive definite matrix, and it satisfies conditions (13a)–(13d). So the difference of the $V_{ij}(x(k))$ along the trajectory of the switched system (24) is

$$\begin{aligned}
&\Delta V_{ij}(x(k)) \\
&= V_{ij}(x(k+1)) - V_{ij}(x(k)) \\
&= x^{\mathrm{T}}(k+1) P_{ij} x(k+1) - x^{\mathrm{T}}(k) P_{ij} x(k) \\
&= x^{\mathrm{T}}(k)[(A_j+B_jK_i)^{\mathrm{T}} P_{ij} (A_j+B_jK_i) - P_{ij}]x(k) + x^{\mathrm{T}}(k)(A_j+B_jK_i)^{\mathrm{T}} P_{ij} D_j w(k) \\
&\quad + B_jK_i)^{\mathrm{T}} P_{ij} D_j w(k) + w^{\mathrm{T}}(k) D_j^{\mathrm{T}} P_{ij}(A_j + B_jK_i)x(k) + w^{\mathrm{T}}(k) D_j^{\mathrm{T}} P_{ij} D_j w(k) \\
&= \xi^{\mathrm{T}}(k) \begin{bmatrix} (A_j+B_jK_i)^{\mathrm{T}} P_{ij}(A_j+B_jK_i) - P_{ij} & (A_j+B_jK_i)^{\mathrm{T}} P_{ij} D_j \\ D_j^{\mathrm{T}} P_{ij}(A_j+B_jK_i) & D_j^{\mathrm{T}} P_{ij} D_j \end{bmatrix} \xi(k),
\end{aligned} \tag{26}$$

where $\xi^{\mathrm{T}}(k) = [x^{\mathrm{T}}(k), w^{\mathrm{T}}(k)]$.

Meanwhile, by multiplying (13d) from both sides by $\mathrm{diag}\{P_{ij}, I, P_{ij}\}$, and using (15), we have

$$\begin{bmatrix} -(1+\beta_{ij})P_{ij} & 0 & (A_j+B_jK_i)^{\mathrm{T}} P_{ij} \\ * & -\gamma Q_{ij} & D_j^{\mathrm{T}} P_{ij} \\ * & * & -P_{ij} \end{bmatrix} \leq 0, \tag{27}$$

by using Schur complement, inequality (27) is equivalent to

$$\begin{aligned}
&\begin{bmatrix} (A_j+B_jK_i)^{\mathrm{T}} P_{ij}(A_j+B_jK_i) - (1+\beta_{ij})P_{ij} & (A_j+B_jK_i)^{\mathrm{T}} P_{ij} D_j \\ D_j^{\mathrm{T}} P_{ij}(A_j+B_jK_i) & D_j^{\mathrm{T}} P_{ij} D_j - \gamma Q_{ij} \end{bmatrix} \\
&= \begin{bmatrix} (A_j+B_jK_i)^{\mathrm{T}} P_{ij}(A_j+B_jK_i) - P_{ij} & (A_j+B_jK_i)^{\mathrm{T}} P_{ij} D_j \\ D_j^{\mathrm{T}} P_{ij}(A_j+B_jK_i) & D_j^{\mathrm{T}} P_{ij} D_j \end{bmatrix} \\
&\quad + \begin{bmatrix} -\beta ij P_{ij} & 0 \\ 0 & -\gamma Q_{ij} \end{bmatrix} \leq 0.
\end{aligned} \tag{28}$$

Thus, by combining (26) and (28), we obtain

$$\Delta V_{ij}(x(k)) = V_{ij}(x(k+1)) - V_{ij}(x(k)) \leq \beta_{ij} x^{\mathrm{T}}(k) P_{ij} x(k) + \gamma w^{\mathrm{T}}(k) Q_{ij} w(k), \tag{29}$$

obviously we get

$$V_{ij}(x(k+1)) \leq (1+\beta_{ij}) V_{ij}(x(k)) + \gamma w^{\mathrm{T}}(k) Q_{ij} w(k). \tag{30}$$

Then for any $k \in [k_{i+1}, k_{i+1} + \Delta_{i+1})$, $\sigma(k) = \sigma(k-1) = \cdots = \sigma(k_{i+1}) = j$, and $\sigma'(k) = i$, by iterating (30), it gives

$$V_{ij}(x(k_{i+1} + \Delta_{i+1})) \leq (1+\beta_{ij})^{\Delta_{i+1}} V_{ij}(x(k_{i+1})) + \sum_{s=k_{i+1}}^{k_{i+1}+\Delta_{i+1}-1} (1+\beta_{ij})^{k_{i+1}+\Delta_{i+1}-1-s} \gamma w^{\mathrm{T}}(s) Q_{ij} w(s). \tag{31}$$

(iii) Now, by considering the iterative case of the Lyapunov functional candidate on the interval $k \in [k_{i+1}, k_{i+2}) = [k_{i+1}, k_{i+1} + \Delta_{i+1}) \bigcup [k_{i+1} + \Delta_{i+1}, k)$, using (13a) and (13b), and combining (23) and (31), we have

$$\begin{aligned}
V_j(x(k)) \leq{}& (1-\alpha_j)^{k-(k_{i+1}+\Delta_{i+1})} V_j(x(k_{i+1} + \Delta_{i+1})) \\
&+ \sum_{s=k_{i+1}+\Delta_{i+1}}^{k-1} (1-\alpha_j)^{k-1-s} \gamma w^{\mathrm{T}}(s) Q_j w(s) \\
\leq{}& \mu_{ij} (1-\alpha_j)^{k-(k_{i+1}+\Delta_{i+1})} [(1+\beta_{ij})^{\Delta_{i+1}} V_{ij}(x(k_{i+1})) \\
&+ \sum_{s=k_{i+1}}^{k_{i+1}+\Delta_{i+1}-1} (1+\beta_{ij})^{k_{i+1}+\Delta_{i+1}-1-s} \gamma w^{\mathrm{T}}(s) Q_{ij} w(s)] \\
&+ \mu_{ij} \sum_{s=k_{i+1}+\Delta_{i+1}}^{k-1} (1-\alpha_j)^{k-1-s} \gamma w^{\mathrm{T}}(s) Q_{ij} w(s) \\
\leq{}& \mu_{ij} (1-\alpha_j)^{k-(k_{i+1}+\Delta_{i+1})} (1+\beta_{ij})^{\Delta_{i+1}} (\mu_{ij} V_i(x(k_{i+1}))) + \mu_{ij} \\
&(1-\alpha_j)^{k-(k_{i+1}+\Delta_{i+1})} \sum_{s=k_{i+1}}^{k_{i+1}+\Delta_{i+1}-1} (1+\beta_{ij})^{k_{i+1}+\Delta_{i+1}-1-s} \gamma w^{\mathrm{T}}(s) \\
&(\mu_{ij} Q_i) w(s) + \mu_{ij} \sum_{s=k_{i+1}+\Delta_{i+1}}^{k-1} (1-\alpha_j)^{k-1-s} \gamma w^{\mathrm{T}}(s) (\mu_{ij} Q_i) w(s) \\
={}& \mu_{ij}^2 (1-\alpha_j)^{k-(k_{i+1}+\Delta_{i+1})} (1+\beta_{ij})^{\Delta_{i+1}} V_i(x(k_{i+1})) \\
&+ \mu_{ij}^2 (1-\alpha_j)^{k-(k_{i+1}+\Delta_{i+1})} \sum_{s=k_{i+1}}^{k_{i+1}+\Delta_{i+1}-1} (1+\beta_{ij})^{k_{i+1}+\Delta_{i+1}-1-s} \\
&\gamma w^{\mathrm{T}}(s) Q_i w(s) + \mu_{ij}^2 \sum_{s=k_{i+1}+\Delta_{i+1}}^{k-1} (1-\alpha_j)^{k-1-s} \gamma w^{\mathrm{T}}(s) Q_i w(s).
\end{aligned} \tag{32}$$

Step 2 Next, the iterative condition of the Lyapunov functional candidate over the interval $k \in [k_i, k_{i+1}) = [k_i, k_i + \Delta_i) \bigcup [k_i + \Delta_i, k_{i+1})$ is considered, according to (32), we can get

$$\begin{aligned} V_i(x(k)) \le\ & \mu_{li}^2 (1-\alpha_i)^{k-(k_i+\Delta_i)} (1+\beta_{li})^{\Delta_i} V_l(x(k_i)) \\ & + \mu_{li}^2 (1-\alpha_i)^{k-(k_i+\Delta_i)} \sum_{s=k_i}^{k_i+\Delta_i-1} (1+\beta_{li})^{k_i+\Delta_i-1-s} \gamma w^{\mathrm{T}}(s) Q_l w(s) \\ & + \mu_{li}^2 \sum_{s=k_i+\Delta_i}^{k-1} (1-\alpha_i)^{k-1-s} \gamma w^{\mathrm{T}}(s) Q_l w(s). \end{aligned} \tag{33}$$

Step 3 Then, according to (22), (23), (30), and (31), the iteration of the Lyapunov functional candidate in a small switching interval $[k_i, k_{i+1}) = [k_i, k_i + \Delta_i) \bigcup [k_i + \Delta_i, k_{i+1})$ is summarized and organized. Furthermore, using (32) and (33), we can derive the iterative result on the entire time interval $[k_0, k)$.

(i) For any $k \in [k_i + \Delta_i, k_{i+1})$ (Synchronous switching period), we derive

$$V_{\sigma(k)}(x(k+1)) \le (1-\alpha_{\sigma(k)}) V_{\sigma(k)}(x(k)) + \gamma w^{\mathrm{T}}(k) Q_{\sigma(k)} w(k), \tag{34a}$$

$$\begin{aligned} V_{\sigma(k)}(x(k)) \le\ & (1-\alpha_{\sigma(k_i)})^{k-(k_i+\Delta_i)} V_{\sigma(k)}(x(k_i+\Delta_i)) \\ & + \sum_{s=k_i+\Delta_i}^{k-1} (1-\alpha_{\sigma(k_i)})^{k-1-s} \gamma w^{\mathrm{T}}(s) Q_{\sigma(k)} w(s). \end{aligned} \tag{34b}$$

(ii) For any $k \in [k_i, k_i + \Delta_i)$ (Asynchronous switching period), we receive

$$\begin{aligned} V_{\sigma'(k)\sigma(k)}(x(k+1)) \le\ & (1+\beta_{\sigma(k_{i-1})\sigma(k_i)}) V_{\sigma'(k)\sigma(k)}(x(k)) \\ & + \gamma w^{\mathrm{T}}(k) Q_{\sigma'(k)\sigma(k)} w(k), \end{aligned} \tag{35a}$$

$$\begin{aligned} V_{\sigma'(k)\sigma(k)}(x(k)) \le\ & (1+\beta_{\sigma(k_{i-1})\sigma(k_i)})^{k-k_i} V_{\sigma'(k)\sigma(k)}(x(k_i)) \\ & + \sum_{s=k_i}^{k-1} (1+\beta_{\sigma(k_{i-1})\sigma(k_i)})^{k-1-s} \gamma w^{\mathrm{T}}(s) Q_{\sigma'(k)\sigma(k)} w(s). \end{aligned} \tag{35b}$$

(iii) Since the switch between the system and the controller is synchronous at first, for any $k \in [k_0, k_1)$, on the basis of (34b), we obtain

$$\begin{aligned} V_{\sigma(k_0)}(x(k_1)) \le\ & (1-\alpha_{\sigma(k_0)})^{k_1-k_0} V_{\sigma(k_0)}(x(k_0)) \\ & + \sum_{s=k_0}^{k_1-1} (1-\alpha_{\sigma(k_0)})^{k_1-1-s} \gamma w^{\mathrm{T}}(s) Q_{\sigma(k_0)} w(s). \end{aligned} \tag{36}$$

(iv) For any $k \in [k_0, k) = [k_0, k_1) \bigcup [k_1, k)$, where $k \ge k_i + \Delta_i$, $\sigma(k) = \sigma(k_i) = i$, by using (33), (34b), (35b), and (36), we can get

$$
\begin{aligned}
V_{\sigma(k)}(x(k)) \leq{}& \mu^2_{\sigma(k_{i-1})\sigma(k_i)}(1-\alpha_{\sigma(k_i)})^{k-(k_i+\Delta_i)}(1+\beta_{\sigma(k_{i-1})\sigma(k_i)})^{\Delta_i} \\
& V_{\sigma(k_{i-1})}(x(k_i)) + \mu^2_{\sigma(k_{i-1})\sigma(k_i)}(1-\alpha_{\sigma(k_i)})^{k-(k_i+\Delta_i)} \sum_{s=k_i}^{k_i+\Delta_i-1} \\
& (1+\beta_{\sigma(k_{i-1})\sigma(k_i)})^{k_i+\Delta_i-1-s}\gamma w^{\mathrm{T}}(s) Q_{\sigma(k_{i-1})} w(s) + \mu^2_{\sigma(k_{i-1})\sigma(k_i)} \\
& \sum_{s=k_i+\Delta_i}^{k-1} (1-\alpha_{\sigma(k_i)})^{k-1-s}\gamma w^{\mathrm{T}}(s) Q_{\sigma(k_{i-1})} w(s) \\
\leq{}& \mu^2_{\sigma(k_{i-1})\sigma(k_i)}\mu^2_{\sigma(k_{i-2})\sigma(k_{i-1})}(1-\alpha_{\sigma(k_i)})^{k-(k_i+\Delta_i)} \\
& (1-\alpha_{\sigma(k_{i-1})})^{k_i-(k_{i-1}+\Delta_{i-1})}(1+\beta_{\sigma(k_{i-1})\sigma(k_i)})^{\Delta_i} \\
& (1+\beta_{\sigma(k_{i-2})\sigma(k_{i-1})})^{\Delta_{i-1}} V_{\sigma(k_{i-2})}(x(k_{i-1})) + \Big[\mu^2_{\sigma(k_{i-1})\sigma(k_i)} \\
& \mu^2_{\sigma(k_{i-2})\sigma(k_{i-1})}(1-\alpha_{\sigma(k_i)})^{k-(k_i+\Delta_i)}(1-\alpha_{\sigma(k_{i-1})})^{k_i-(k_{i-1}+\Delta_{i-1})} \\
& (1+\beta_{\sigma(k_{i-1})\sigma(k_i)})^{\Delta_i} \sum_{s=k_{i-1}}^{k_{i-1}+\Delta_{i-1}-1} (1+\beta_{\sigma(k_{i-2})\sigma(k_{i-1})})^{k_{i-1}+\Delta_{i-1}-1-s} \\
& \gamma w^{\mathrm{T}}(s) Q_{\sigma(k_{i-2})} w(s) + \mu^2_{\sigma(k_{i-1})\sigma(k_i)}(1-\alpha_{\sigma(k_i)})^{k-(k_i+\Delta_i)} \sum_{s=k_i}^{k_i+\Delta_i-1} \\
& (1+\beta_{\sigma(k_{i-1})\sigma(k_i)})^{k_i+\Delta_i-1-s}\gamma w^{\mathrm{T}}(s) Q_{\sigma(k_{i-1})} w(s)\Big] + \Big[\mu^2_{\sigma(k_{i-1})\sigma(k_i)} \\
& \mu^2_{\sigma(k_{i-2})\sigma(k_{i-1})}(1-\alpha_{\sigma(k_i)})^{k-(k_i+\Delta_i)}(1+\beta_{\sigma(k_{i-1})\sigma(k_i)})^{\Delta_i} \sum_{s=k_{i-1}+\Delta_{i-1}}^{k_i-1} \\
& (1-\alpha_{\sigma(k_{i-1})})^{k_i-1-s}\gamma w^{\mathrm{T}}(s) Q_{\sigma(k_{i-2})} w(s) + \mu^2_{\sigma(k_{i-1})\sigma(k_i)} \sum_{s=k_i+\Delta_i}^{k-1} \\
& (1-\alpha_{\sigma(k_i)})^{k-1-s}\gamma w^{\mathrm{T}}(s) Q_{\sigma(k_{i-1})} w(s)\Big] \\
\leq{}& \cdots \\
\leq{}& \mu^2_{\sigma(k_{i-1})\sigma(k_i)}\mu^2_{\sigma(k_{i-2})\sigma(k_{i-1})}\cdots\mu^2_{\sigma(k_0)\sigma(k_1)} \times (1-\alpha_{\sigma(k_i)})^{k-(k_i+\Delta_i)} \\
& (1-\alpha_{\sigma(k_{i-1})})^{k_i-(k_{i-1}+\Delta_{i-1})}\ldots(1-\alpha_{\sigma(k_1)})^{k_2-(k_1+\Delta_1)} \\
& \times(1+\beta_{\sigma(k_{i-1})\sigma(k_i)})^{\Delta_i}(1+\beta_{\sigma(k_{i-2})\sigma(k_{i-1})})^{\Delta_{i-1}}\cdots(1+\beta_{\sigma(k_0)\sigma(k_1)})^{\Delta_1} \\
& V_{\sigma(k_0)}(x(k_1)) + \Big[\mu^2_{\sigma(k_{i-1})\sigma(k_i)}\mu^2_{\sigma(k_{i-2})\sigma(k_{i-1})}\cdots\mu^2_{\sigma(k_0)\sigma(k_1)} \\
& \times(1-\alpha_{\sigma(k_i)})^{k-(k_i+\Delta_i)}(1-\alpha_{\sigma(k_{i-1})})^{k_i-(k_{i-1}+\Delta_{i-1})}\ldots \\
& (1-\alpha_{\sigma(k_1)})^{k_2-(k_1+\Delta_1)}\times(1+\beta_{\sigma(k_{i-1})\sigma(k_i)})^{\Delta_i}(1+\beta_{\sigma(k_{i-2})\sigma(k_{i-1})})^{\Delta_{i-1}} \\
& \cdots(1+\beta_{\sigma(k_1)\sigma(k_2)})^{\Delta_2} \sum_{s=k_1}^{k_1+\Delta_1-1}(1+\beta_{\sigma(k_0)\sigma(k_1)})^{k_1+\Delta_1-1-s}\gamma w^{\mathrm{T}}(s)
\end{aligned}
$$

$$
Q_{\sigma(k_0)}w(s) + \cdots + \mu^2_{\sigma(k_{i-1})\sigma(k_i)}\mu^2_{\sigma(k_{i-2})\sigma(k_{i-1})}(1-\alpha_{\sigma(k_i)})^{k-(k_i+\Delta_i)}
(1-\alpha_{\sigma(k_{i-1})})^{k_i-(k_{i-1}+\Delta_{i-1})}(1+\beta_{\sigma(k_{i-1})\sigma(k_i)})^{\Delta_i}\sum_{s=k_{i-1}}^{k_{i-1}+\Delta_{i-1}-1}
(1+\beta_{\sigma(k_{i-2})\sigma(k_{i-1})})^{k_{i-1}+\Delta_{i-1}-1-s}\gamma w^{\mathrm{T}}(s)Q_{\sigma(k_{i-2})}w(s) + \mu^2_{\sigma(k_{i-1})\sigma(k_i)}
(1-\alpha_{\sigma(k_i)})^{k-(k_i+\Delta_i)}\sum_{s=k_i}^{k_i+\Delta_i-1}(1+\beta_{\sigma(k_{i-1})\sigma(k_i)})^{k_i+\Delta_i-1-s}\gamma w^{\mathrm{T}}(s)
Q_{\sigma(k_{i-1})}w(s)\Big] + \Big[\mu^2_{\sigma(k_{i-1})\sigma(k_i)}\mu^2_{\sigma(k_{i-2})\sigma(k_{i-1})}\cdots\mu^2_{\sigma(k_0)\sigma(k_1)}
\times(1-\alpha_{\sigma(k_i)})^{k-(k_i+\Delta_i)}(1-\alpha_{\sigma(k_{i-1})})^{k_i-(k_{i-1}+\Delta_{i-1})}\ldots
(1-\alpha_{\sigma(k_2)})^{k_3-(k_2+\Delta_2)}\times(1+\beta_{\sigma(k_{i-1})\sigma(k_i)})^{\Delta_i}(1+\beta_{\sigma(k_{i-2})\sigma(k_{i-1})})^{\Delta_{i-1}}
\cdots(1+\beta_{\sigma(k_1)\sigma(k_2)})^{\Delta_2}\sum_{s=k_1+\Delta_1}^{k_2-1}(1-\alpha_{\sigma(k_1)})^{k_2-1-s}\gamma w^{\mathrm{T}}(s)Q_{\sigma(k_0)}w(s)
+\cdots+\mu^2_{\sigma(k_{i-1})\sigma(k_i)}\mu^2_{\sigma(k_{i-2})\sigma(k_{i-1})}(1-\alpha_{\sigma(k_i)})^{k-(k_i+\Delta_i)}
(1+\beta_{\sigma(k_{i-1})\sigma(k_i)})^{\Delta_i}\sum_{s=k_{i-1}+\Delta_{i-1}}^{k_i-1}(1-\alpha_{\sigma(k_{i-1})})^{k_i-1-s}\gamma w^{\mathrm{T}}(s)
Q_{\sigma(k_{i-2})}w(s) + \mu^2_{\sigma(k_{i-1})\sigma(k_i)}\sum_{s=k_i+\Delta_i}^{k-1}(1-\alpha_{\sigma(k_i)})^{k-1-s}
\gamma w^{\mathrm{T}}(s)Q_{\sigma(k_{i-1})}w(s)\Big]. \tag{37}
$$

Let

$$
\begin{aligned}
\Xi_1 &= \mu^2_{\sigma(k_{i-1})\sigma(k_i)}\mu^2_{\sigma(k_{i-2})\sigma(k_{i-1})}\cdots\mu^2_{\sigma(k_0)\sigma(k_1)}\times(1-\alpha_{\sigma(k_i)})^{k-(k_i+\Delta_i)} \\
&\quad (1-\alpha_{\sigma(k_{i-1})})^{k_i-(k_{i-1}+\Delta_{i-1})}\ldots(1-\alpha_{\sigma(k_1)})^{k_2-(k_1+\Delta_1)}\times(1+\beta_{\sigma(k_{i-1})\sigma(k_i)})^{\Delta_i} \\
&\quad (1+\beta_{\sigma(k_{i-2})\sigma(k_{i-1})})^{\Delta_{i-1}}\cdots(1+\beta_{\sigma(k_0)\sigma(k_1)})^{\Delta_1}V_{\sigma(k_0)}(x(k_1)) \\
&= (1-\alpha_{\sigma(k_N)})^{k-(k_N+\Delta_N)}\prod_{s=1}^{N}\mu^2_{\sigma(k_{s-1})\sigma(k_s)}\times\Big[\prod_{s=1}^{N-1}(1-\alpha_{\sigma(k_s)})^{k_{s+1}-(k_s+\Delta_s)} \\
&\quad \prod_{s=0}^{N-1}(1+\beta_{\sigma(k_s)\sigma(k_{s+1})})^{\Delta_{s+1}}\Big]V_{\sigma(k_0)}(x(k_1)) \\
&\le (1-\alpha_{\sigma(k_N)})^{k-(k_N+\Delta_N)}\prod_{s=1}^{N}\mu^2_{\sigma(k_{s-1})\sigma(k_s)}\times\Big[\prod_{s=1}^{N-1}(1-\alpha_{\sigma(k_s)})^{k_{s+1}-(k_s+\Delta_s)} \\
&\quad \prod_{s=0}^{N-1}(1+\beta_{\sigma(k_s)\sigma(k_{s+1})})^{\Delta_{s+1}}\Big]\times\Big[(1-\alpha_{\sigma(k_0)})^{k_1-k_0}V_{\sigma(k_0)}(x(k_0))
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{s=k_0}^{k_1-1}(1-\alpha_{\sigma(k_0)})^{k_1-1-s}\gamma w^{\mathrm{T}}(s)Q_{\sigma(k_0)}w(s)\Big]\\
&=(1-\alpha_{\sigma(k_N)})^{k-(k_N+\Delta_N)}\prod_{s=1}^{N}\mu^2_{\sigma(k_{s-1})\sigma(k_s)}\times\Big[\prod_{s=0}^{N-1}(1-\alpha_{\sigma(k_s)})^{k_{s+1}-(k_s+\Delta_s)}\\
&(1+\beta_{\sigma(k_s)\sigma(k_{s+1})})^{\Delta_{s+1}}\Big]V_{\sigma(k_0)}(x(k_0))+(1-\alpha_{\sigma(k_N)})^{k-(k_N+\Delta_N)}\\
&\prod_{s=1}^{N}\mu^2_{\sigma(k_{s-1})\sigma(k_s)}\times\Big[\prod_{s=0}^{N-1}(1-\alpha_{\sigma(k_s)})^{k_{s+1}-(k_s+\Delta_s)}(1+\beta_{\sigma(k_s)\sigma(k_{s+1})})^{\Delta_{s+1}}\Big]\\
&\times(1-\alpha_{\sigma(k_0)})^{k_0-k_1}\sum_{s=k_0}^{k_1-1}(1-\alpha_{\sigma(k_0)})^{k_1-1-s}\gamma w^{\mathrm{T}}(s)Q_{\sigma(k_0)}w(s),
\end{aligned}
$$

and by using $0<\alpha_{\sigma(k_N)}<1$, $\beta_{\sigma(k_N)\sigma(k_{N+1})}>0$, we derive

$$
\begin{aligned}
V_{\sigma(k)}(x(k))\le\Big\{&(1-\alpha_{\sigma(k_N)})^{k-(k_N+\Delta_N)}\prod_{s=1}^{N}\mu^2_{\sigma(k_{s-1})\sigma(k_s)}\\
&\times\Big[\prod_{s=0}^{N-1}(1-\alpha_{\sigma(k_s)})^{k_{s+1}-(k_s+\Delta_s)}(1+\beta_{\sigma(k_s)\sigma(k_{s+1})})^{\Delta_{s+1}}\Big]\\
&V_{\sigma(k_0)}(x(k_0))+(1-\alpha_{\sigma(k_N)})^{k-(k_N+\Delta_N)}\prod_{s=1}^{N}\mu^2_{\sigma(k_{s-1})\sigma(k_s)}\\
&\times\Big[\prod_{s=0}^{N-1}(1-\alpha_{\sigma(k_s)})^{k_{s+1}-(k_s+\Delta_s)}(1+\beta_{\sigma(k_s)\sigma(k_{s+1})})^{\Delta_{s+1}}\Big]\\
&\times(1-\alpha_{\sigma(k_0)})^{k_0-k_1}\sum_{s=k_0}^{k_1-1}(1-\alpha_{\sigma(k_0)})^{k_1-1-s}\gamma w^{\mathrm{T}}(s)Q_{\sigma(k_0)}w(s)\Big\}\\
&+\Big\{\mu^2_{\sigma(k_{N-1})\sigma(k_N)}(1-\alpha_{\sigma(k_N)})^{k-(k_N+\Delta_N)}\times\sum_{n=1}^{N-1}\Big[\prod_{p=n}^{N-1}\mu^2_{\sigma(k_{p-1})\sigma(k_p)}\\
&\times(\prod_{p=n}^{N-1}(1-\alpha_{\sigma(k_p)})^{k_{p+1}-(k_p+\Delta_p)}(1+\beta_{\sigma(k_p)\sigma(k_{p+1})})^{\Delta_{p+1}})\\
&\times\sum_{s=k_n}^{k_n+\Delta_n-1}(1+\beta_{\sigma(k_{n-1})\sigma(k_n)})^{k_n+\Delta_n-1-s}\gamma w^{\mathrm{T}}(s)Q_{\sigma(k_{n-1})}w(s)\Big]\\
&+\mu^2_{\sigma(k_{N-1})\sigma(k_N)}(1-\alpha_{\sigma(k_N)})^{k-(k_N+\Delta_N)}\sum_{s=k_N}^{k_N+\Delta_N-1}
\end{aligned}
$$

$$
(1+\beta_{\sigma(k_{N-1})\sigma(k_N)})^{k_N+\Delta_N-1-s}\gamma w^{\mathrm{T}}(s)Q_{\sigma(k_{N-1})}w(s)\Big\}
$$

$$
+\Big\{\mu^2_{\sigma(k_{N-1})\sigma(k_N)}(1-\alpha_{\sigma(k_N)})^{k-(k_N+\Delta_N)}\times\sum_{n=1}^{N-1}\Big[\prod_{p=n}^{N-1}\mu^2_{\sigma(k_{p-1})\sigma(k_p)}
$$

$$
\times(\prod_{p=n}^{N-1}(1-\alpha_{\sigma(k_p)})^{k_{p+1}-(k_p+\Delta_p)}(1+\beta_{\sigma(k_p)\sigma(k_{p+1})})^{\Delta_{p+1}})
$$

$$
\times(1-\alpha_{\sigma(k_n)})^{(k_n+\Delta_n)-k_{n+1}}\times\sum_{s=k_n+\Delta_n}^{k_{n+1}-1}(1-\alpha_{\sigma(k_n)})^{k_{n+1}-1-s}\gamma w^{\mathrm{T}}(s)
$$

$$
Q_{\sigma(k_{n-1})}w(s)\Big]+\mu^2_{\sigma(k_{N-1})\sigma(k_N)}\sum_{s=k_N+\Delta_N}^{k-1}(1-\alpha_{\sigma(k_N)})^{k-1-s}\gamma w^{\mathrm{T}}(s)
$$

$$
Q_{\sigma(k_{N-1})}w(s)\Big\}
$$

$$
\leq\prod_{s=1}^{N}\mu^2_{\sigma(k_{s-1})\sigma(k_s)}\times\Big[\prod_{s=0}^{N-1}(1-\alpha_{\sigma(k_s)})^{k_{s+1}-(k_s+\Delta_s)}
$$

$$
(1+\beta_{\sigma(k_s)\sigma(k_{s+1})})^{\Delta_{s+1}}\Big]V_{\sigma(k_0)}(x(k_0))+\prod_{s=1}^{N}\mu^2_{\sigma(k_{s-1})\sigma(k_s)}
$$

$$
\times\Big[\prod_{s=0}^{N-1}(1-\alpha_{\sigma(k_s)})^{k_{s+1}-(k_s+\Delta_s)}(1+\beta_{\sigma(k_s)\sigma(k_{s+1})})^{\Delta_{s+1}}\Big]
$$

$$
\times(1-\alpha_{\sigma(k_0)})^{k_0-k_1}\sum_{s=k_0}^{k_1-1}(1-\alpha_{\sigma(k_0)})^{k_1-1-s}\gamma w^{\mathrm{T}}(s)Q_{\sigma(k_0)}w(s)
$$

$$
+\sum_{n=1}^{N}\Big\{\prod_{p=n}^{N}\mu^2_{\sigma(k_{p-1})\sigma(k_p)}\times\Big[\prod_{p=n}^{N}(1-\alpha_{\sigma(k_p)})^{k_{p+1}-(k_p+\Delta_p)}
$$

$$
(1+\beta_{\sigma(k_p)\sigma(k_{p+1})})^{\Delta_{p+1}}\Big]\times\Big[\sum_{s=k_n}^{k_n+\Delta_n-1}(1+\beta_{\sigma(k_{n-1})\sigma(k_n)})^{k_n+\Delta_n-1-s}
$$

$$
\gamma w^{\mathrm{T}}(s)Q_{\sigma(k_{n-1})}w(s)+(1-\alpha_{\sigma(k_n)})^{(k_n+\Delta_n)-k_{n+1}}
$$

$$
\times\sum_{s=k_n+\Delta_n}^{k_{n+1}-1}(1-\alpha_{\sigma(k_n)})^{k_{n+1}-1-s}\gamma w^{\mathrm{T}}(s)Q_{\sigma(k_{n-1})}w(s)\Big]\Big\}. \tag{38}
$$

Step 4 Finally, according to iteration result (38), we analyze the finite-time boundedness of system (12). Denote $\Phi(s,k)=\prod_{(i,j)\in\varepsilon(\bar{M})}\mu_{ij}^{2N_{ij}^{\sigma}(s,k)}$, $\Psi(s,k)=\prod_{i\in\mathfrak{M}_s,i\neq j}\prod_{j\in\bar{J}(i)}(1-\alpha_j)^{T_{ij}(s,k)}\times\prod_{i\in\mathfrak{M}_u,i\neq j}\prod_{j\in\bar{J}(i)}(1+\beta_{ij})^{T_{ij}(s,k)}$, and let $\eta_1=(1-$

$\alpha_{\sigma(k_n)})^{(k_n+\Delta_n)-k_{n+1}}, n = 0, 1, 2, \cdots, N, \ \eta_2 = (1 + \beta_{\sigma(k_{n-1})\sigma(k_n)})^{\Delta_{n-1}}, n = 1, 2, \cdots, N$, and $\eta = \max\{\eta_1, \eta_2\} \geq 1$, where $\eta_1 \geq 1, \eta_2 \geq 1$, and $\lambda_3 = \max\limits_{j\in\bar{M}}(\lambda_{\max}(Q_j))$. Then, by using (10), we obtain

$$\begin{aligned}
V_{\sigma(k)}(x(k)) \leq{}& \Phi(0,k)\Psi(0,k-1)V_{\sigma(k_0)}(x(k_0)) + \eta_1\Phi(0,k)\Psi(0,k-1)\sum_{s=k_0}^{k_1-1}\gamma w^{\mathrm{T}}(s) \\
& Q_{\sigma(k_0)}w(s) + \sum_{n=1}^{N}\Big\{\Phi(n,k)\Psi(n,k)\Big[\sum_{s=k_n}^{k_n+\Delta_n-1}\eta_2\gamma w^{\mathrm{T}}(s)Q_{\sigma(k_{n-1})}w(s) \\
& + \eta_1\sum_{s=k_n+\Delta_n}^{k_{n+1}-1}\gamma w^{\mathrm{T}}(s)Q_{\sigma(k_{n-1})}w(s)\Big]\Big\} \\
\leq{}& \Phi(0,k)\Psi(0,k-1)V_{\sigma(k_0)}(x(k_0)) + \eta_1\lambda_3\gamma\Phi(0,k)\Psi(0,k-1)\sum_{s=k_0}^{k_1-1} \\
& w^{\mathrm{T}}(s)w(s) + \sum_{n=1}^{N}\Big\{\Phi(n,k)\Psi(n,k)\Big[\eta_2\lambda_3\gamma\sum_{s=k_n}^{k_n+\Delta_n-1}w^{\mathrm{T}}(s)w(s) \\
& + \eta_1\lambda_3\gamma\sum_{s=k_n+\Delta_n}^{k_{n+1}-1}w^{\mathrm{T}}(s)w(s)\Big]\Big\} \\
={}& \Phi(0,k)\Psi(0,k-1)V_{\sigma(k_0)}(x(k_0)) + \eta_1\lambda_3\gamma\sum_{n=0}^{N}\Big[\Phi(n,k)\Psi(n,k)\sum_{s=k_n+\Delta_n}^{k_{n+1}-1} \\
& w^{\mathrm{T}}(s)w(s)\Big] + \eta_2\lambda_3\gamma\sum_{n=1}^{N}\Big[\Phi(n,k)\Psi(n,k)\sum_{s=k_n}^{k_n+\Delta_n-1}w^{\mathrm{T}}(s)w(s)\Big] \\
\leq{}& \Phi(0,k)\Psi(0,k-1)V_{\sigma(k_0)}(x(k_0)) + \eta\lambda_3\gamma\sum_{n=0}^{N}\Big\{\Phi(n,k)\Psi(n,k) \\
& \Big[\sum_{s=k_n+\Delta_n}^{k_{n+1}-1}w^{\mathrm{T}}(s)w(s) + \sum_{s=k_n}^{k_n+\Delta_n-1}w^{\mathrm{T}}(s)w(s)\Big]\Big\} \\
\leq{}& \Phi(0,k)\Psi(0,k)V_{\sigma(k_0)}(x(k_0)) + \eta\lambda_3\gamma d\sum_{n=0}^{N}\Big[\Phi(n,k)\Psi(n,k)\Big] \\
\leq{}& \Upsilon(0,k)V_{\sigma(0)}(x(0)) + \eta\lambda_3\gamma d\sum_{n=0}^{k}\Upsilon(n,k), \qquad (39)
\end{aligned}$$

where $\Upsilon(s,k) = \Phi(s,k)\Psi(s,k)$, $k_0 = 0$.

Based on (7) in Definition 2.3, we can obtain that

$$\prod_{s=1}^{N}\mu^2_{\sigma(k_{s-1})\sigma(k_s)} = \prod_{(i,j)\in\varepsilon(\bar{M})}\mu_{ij}^{2N_{ij}^{\sigma}(k_0,k)} = \exp\Big(\sum_{(i,j)\in\varepsilon(\bar{M})}2N_{ij}^{\sigma}(k_0,k)\ln\mu_{ij}\Big)$$

$$
\begin{aligned}
&\le \exp\Big(\sum_{(i,j)\in\varepsilon(\bar{M})} 2(N_{ij}^0 + \frac{T_{ij}(k_0,k)}{\tau_{ij}^a}) \ln \mu_{ij}\Big)\\
&= \varphi_0 \exp\Big(\sum_{i\in\mathfrak{M}_s, i\neq j} \sum_{j\in \bar{J(i)}} \frac{2\ln \mu_{ij}}{\tau_{ij}^a} T_{ij}(k_0,k)\Big)\\
&\times \exp\Big(\sum_{i\in\mathfrak{M}_u, i\neq j} \sum_{j\in \bar{J(i)}} \frac{2\ln \mu_{ij}}{\tau_{ij}^a} T_{ij}(k_0,k)\Big),
\end{aligned}
\tag{40}
$$

where $\varphi_0 = \exp(\sum_{(i,j)\in\varepsilon(\bar{M})} 2N_{ij}^0 \ln \mu_{ij})$.

Note that

$$
\prod_{s=0}^{N-1} (1-\alpha_{\sigma(k_s)})^{k_{s+1}-(k_s+\Delta_s)} = \prod_{i\in\mathfrak{M}_s, i\neq j} \prod_{j\in \bar{J(i)}} (1-\alpha_j)^{T_{ij}(k_0,k)},
\tag{41}
$$

$$
\prod_{s=0}^{N-1} (1+\beta_{\sigma(k_s)\sigma(k_{s+1})})^{\Delta_{s+1}} = \prod_{i\in\mathfrak{M}_u, i\neq j} \prod_{j\in \bar{J(i)}} (1+\beta_{ij})^{T_{ij}(k_0,k)},
\tag{42}
$$

then, according to (8a), (8b), (40), (41), and (42), and (14a) that implies $(1-\alpha_j)\mu_{ij}^{\frac{2}{\tau_{ij}^a}} < 1$, we can get

$$
\begin{aligned}
\Upsilon(s,k) &= \Phi(s,k)\Psi(s,k)\\
&\le \varphi_0 \prod_{i\in\mathfrak{M}_s, i\neq j} \prod_{j\in \bar{J(i)}} ((1-\alpha_j)\mu_{ij}^{\frac{2}{\tau_{ij}^a}})^{T_{ij}(s,k)}\\
&\times \prod_{i\in\mathfrak{M}_u, i\neq j} \prod_{j\in \bar{J(i)}} ((1+\beta_{ij})\mu_{ij}^{\frac{2}{\tau_{ij}^a}})^{T_{ij}(s,k)}\\
&\le \varphi_0 \prod_{i\in\mathfrak{M}_s, i\neq j} \prod_{j\in \bar{J(i)}} ((1-\alpha_j)\mu_{ij}^{\frac{2}{\tau_{ij}^a}})^{-\bar{T}_{ij}^s+\nu_{ij}^s(k-s)}\\
&\times \prod_{i\in\mathfrak{M}_u, i\neq j} \prod_{j\in \bar{J(i)}} ((1+\beta_{ij})\mu_{ij}^{\frac{2}{\tau_{ij}^a}})^{\bar{T}_{ij}^u+\nu_{ij}^u(k-s)}\\
&= \Big[\varphi_0 \prod_{i\in\mathfrak{M}_s, i\neq j} \prod_{j\in \bar{J(i)}} ((1-\alpha_j)\mu_{ij}^{\frac{2}{\tau_{ij}^a}})^{-\bar{T}_{ij}^s} \prod_{i\in\mathfrak{M}_u, i\neq j} \prod_{j\in \bar{J(i)}} ((1+\beta_{ij})\mu_{ij}^{\frac{2}{\tau_{ij}^a}})^{\bar{T}_{ij}^u}\Big]\\
&\times \prod_{i\in\mathfrak{M}_s, i\neq j} \prod_{j\in \bar{J(i)}} ((1-\alpha_j)\mu_{ij}^{\frac{2}{\tau_{ij}^a}})^{\nu_{ij}^s(k-s)}
\end{aligned}
$$

$$\times \prod_{i\in\mathfrak{M}_u, i\neq j} \prod_{j\in \bar{J(i)}} ((1+\beta_{ij})\mu_{ij}^{\frac{2}{\tau_{ij}^a}})^{\nu_{ij}^u(k-s)}$$
$$= \phi_0 \lambda^{k-s}, \tag{43}$$

where

$$\phi_0 = \varphi_0 \prod_{i\in\mathfrak{M}_s, i\neq j} \prod_{j\in \bar{J(i)}} ((1-\alpha_j)\mu_{ij}^{\frac{2}{\tau_{ij}^a}})^{-\bar{T_{ij}^s}} \times \prod_{i\in\mathfrak{M}_u, i\neq j} \prod_{j\in \bar{J(i)}} ((1+\beta_{ij})\mu_{ij}^{\frac{2}{\tau_{ij}^a}})^{\bar{T_{ij}^u}},$$
$$\lambda = \prod_{i\in\mathfrak{M}_s, i\neq j} \prod_{j\in \bar{J(i)}} ((1-\alpha_j)\mu_{ij}^{\frac{2}{\tau_{ij}^a}})^{\nu_{ij}^s} \times \prod_{i\in\mathfrak{M}_u, i\neq j} \prod_{j\in \bar{J(i)}} ((1+\beta_{ij})\mu_{ij}^{\frac{2}{\tau_{ij}^a}})^{\nu_{ij}^u}.$$

Therefore, (39) can be rewritten as

$$\begin{aligned} V_{\sigma(k)}(x(k)) &\leq \Upsilon(0,k)V_{\sigma(0)}(x(0)) + \eta\lambda_3\gamma d \sum_{n=0}^{k} \Upsilon(n,k) \\ &\leq \phi_0\lambda^k V_{\sigma(0)}(x(0)) + \eta\lambda_3\gamma d \sum_{n=0}^{k} \phi_0\lambda^{k-n} \\ &= \phi_0\lambda^k V_{\sigma(0)}(x(0)) + \phi_0\eta\lambda_3\gamma d \times \frac{1-\lambda^{k+1}}{1-\lambda} \\ &< \phi_0 V_{\sigma(0)}(x(0)) + \phi_0\eta\lambda_3\gamma d \times \frac{1}{1-\lambda}. \end{aligned} \tag{44}$$

On the other hand, considering $\bar{X}_{\sigma(k)} = R^{1/2} X_{\sigma(k)} R^{1/2}$, from Lemma 2.1, we can follow from (17) that

$$\begin{aligned} V_{\sigma(0)}(x(0)) &= x^{\mathrm{T}}(0)P_{\sigma(0)}x(0) = x^{\mathrm{T}}(0)R^{1/2}(\bar{X}_{\sigma(0)})^{-1}R^{1/2}x(0) \\ &\leq \max(\lambda_{\max}((\bar{X}_{\sigma(0)})^{-1}))x^{\mathrm{T}}(0)Rx(0) \leq \frac{c_1}{\lambda_1}, \\ V_{\sigma(k)}(x(k)) &= x^{\mathrm{T}}(k)P_{\sigma(k)}x(k) = x^{\mathrm{T}}(k)R^{1/2}(\bar{X}_{\sigma(k)})^{-1}R^{1/2}x(k) \\ &\geq \min(\lambda_{\min}((\bar{X}_{\sigma(k)})^{-1}))x^{\mathrm{T}}(k)Rx(k) = \frac{1}{\lambda_2}x^{\mathrm{T}}(k)Rx(k). \end{aligned} \tag{45}$$

Hence, combining inequalities (44) and (45) yields

$$\begin{aligned} x^{\mathrm{T}}(k)Rx(k) &\leq \lambda_2 V_{\sigma(k)}(x(k)) < \lambda_2(\phi_0 V_{\sigma(0)}(x(0)) + \phi_0\eta\lambda_3\gamma d \times \frac{1}{1-\lambda}) \\ &< \lambda_2(\phi_0\frac{c_1}{\lambda_1} + \phi_0\eta\lambda_3\gamma d \times \frac{1}{1-\lambda}) \triangleq c_2. \end{aligned} \tag{46}$$

From Definition 2.5, the closed-loop system (12) is finite-time bounded w.r.t. $(c_1, c_2, R, d, N, \sigma)$. Hence, the theorem is proved. □

3.2 H_∞ Performance Analysis

In this section, we will further give a sufficient condition which guarantees that the closed-loop system (12) is finite-time bounded with a specified H_∞ performance index via AED-ADT switching based on Theorem 3.1.

Theorem 3.2 *Given a matrix $R > 0$, an integer $N > 0$, for specified constants $c_2 > c_1 > 0$, $d > 0$, $\gamma > 0$, $0 < \alpha_j < 1$, $\beta_{ij} > 0$, $\mu_{ij} \geq 1$, and suppose that there exist matrices $X_i > 0$, $X_j > 0$, $X_{ij} > 0$ and Y_i such that $\forall i, j \in \bar{M}, i \neq j$,*

$$X_{ij} \leq \mu_{ij} X_j, i \neq j \in \bar{M}, \tag{47a}$$

$$X_i \leq \mu_{ij} X_{ij}, i \neq j \in \bar{M}, \tag{47b}$$

$$\begin{bmatrix} -(1-\alpha_j)X_j & 0 & X_j A_j^{\mathrm{T}} + Y_j^{\mathrm{T}} B_j^{\mathrm{T}} & X_j C_j^{\mathrm{T}} + Y_j^{\mathrm{T}} E_j^{\mathrm{T}} \\ * & -\gamma^2 I & D_j^{\mathrm{T}} & F_j^{\mathrm{T}} \\ * & * & -X_j & 0 \\ * & * & * & -I \end{bmatrix} \leq 0, \tag{47c}$$

$$\begin{bmatrix} -(1+\beta_{ij})X_{ij} & 0 & X_{ij} A_j^{\mathrm{T}} + Y_i^{\mathrm{T}} B_j^{\mathrm{T}} & X_{ij} C_j^{\mathrm{T}} + Y_i^{\mathrm{T}} E_j^{\mathrm{T}} \\ * & -\gamma^2 I & D_j^{\mathrm{T}} & F_j^{\mathrm{T}} \\ * & * & -X_{ij} & 0 \\ * & * & * & -I \end{bmatrix} \leq 0, \tag{47d}$$

$$\frac{c_1(\lambda_1 - \lambda_2\phi_0)}{\lambda_1\lambda_2} < \frac{\phi_0 \eta \gamma^2 d}{1-\lambda}, \tag{47e}$$

where

$$\eta_1 = (1-\alpha_{\sigma(k_n)})^{(k_n+\Delta_n)-k_{n+1}} \geq 1, n = 0, 1, 2, \cdots, N,$$
$$\eta_2 = (1+\beta_{\sigma(k_{n-1})\sigma(k_n)})^{\Delta_n - 1} \geq 1, n = 1, 2, \cdots, N,$$
$$\eta = \max\{\eta_1, \eta_2\} \geq 1, \quad \varphi_0 = \exp\Big(\sum_{(i,j)\in\varepsilon(\mathcal{L})} 2N_{ij}^0 \ln \mu_{ij}\Big),$$

$$\phi_0 = \varphi_0 \prod_{i\in \mathrm{M}_s} \prod_{j\in \mathrm{J}(i)} \big((1-\alpha_j)\mu_{ij}^{\frac{2}{\tau_{ij}^a}}\big)^{-\bar{T}_{ij}^s} \times \prod_{i\in \mathrm{M}_u} \prod_{j\in \mathrm{J}(i)} \big((1+\beta_{ij})\mu_{ij}^{\frac{2}{\tau_{ij}^a}}\big)^{\bar{T}_{ij}^u},$$
$$\lambda_1 = \min_{j\in\mathcal{L}}(\lambda_{\min}(\bar{X}_j)), \quad \lambda_2 = \max_{j\in\mathcal{L}}(\lambda_{\max}(\bar{X}_j)),$$
$$\lambda_3 = \max_{j\in\mathcal{L}}(\lambda_{\max}(Q_j)), \quad \bar{X}_{\sigma(k)} = R^{1/2} X_{\sigma(k)} R^{1/2}.$$

Then, there exists a set of controllers (3) such that the closed-loop system (12) is finite-time bounded with a H_∞ performance index $\tilde{\gamma}$ w.r.t. $(c_1, c_2, R, d, \tilde{\gamma}, N, \sigma)$ for any AED-ADT switching signal $\sigma(k)$ and coefficients $\nu_{ij}^s \in (0, 1]$, $\nu_{ij}^u \in [0, 1)$ in

Assumption 2.1 *satisfying*

$$\tau_{ij}^{a} > \tau_{ij}^{a*} = -\frac{2\ln\mu_{ij}}{\ln(1-\alpha_j)}, \forall i, j \in \bar{M}, \tag{48a}$$

$$1 - \eta\phi_0 < \lambda < 1, \tag{48b}$$

where $\lambda = \prod_{i\in\mathfrak{M}_s, i\neq j}\prod_{j\in\bar{J(i)}}((1-\alpha_j)\mu_{ij}^{\frac{2}{\tau_{ij}^a}})^{\nu_{ij}^s} \times \prod_{i\in\mathfrak{M}_u, i\neq j}\prod_{j\in\bar{J(i)}}(1+\beta_{ij})\mu_{ij}^{\frac{2}{\tau_{ij}^a}})^{\nu_{ij}^u}$ *and* $\tilde{\gamma} = \sqrt{\frac{\gamma^2\eta\phi_0}{1-\lambda}} > 0$. *Moreover, if the controllers exist, the controller gains are given by (15).*

Proof First, according to Theorem 3.1, we analyze the finite-time boundedness of the closed-loop system (12) combined with the conditions (47a)–(47e). Obviously, by using Schur complement, we can get from (47c) and (47d) that

$$\begin{bmatrix} -(1-\alpha_j)X_j & 0 & X_jA_j^{\mathrm{T}} + Y_j^{\mathrm{T}}B_j^{\mathrm{T}} \\ * & -\gamma^2 I & D_j^{\mathrm{T}} \\ * & * & -X_j \end{bmatrix} \leq 0, \tag{49a}$$

$$\begin{bmatrix} -(1+\beta_{ij})X_{ij} & 0 & X_{ij}A_j^{\mathrm{T}} + Y_i^{\mathrm{T}}B_j^{\mathrm{T}} \\ * & -\gamma^2 I & D_j^{\mathrm{T}} \\ * & * & -X_{ij} \end{bmatrix} \leq 0, \tag{49b}$$

which imply (13c) and (13d) by setting $Q_i = Q_{ij} = \gamma I$. So, by analogy with Theorem 3.1, it can be concluded that the closed-loop system (12) is finite-time bounded w.r.t. $(c_1, c_2, R, d, N, \sigma)$ for any AED-ADT switching signal $\sigma(k)$ and coefficients $\nu_{ij}^s \in (0, 1]$, $\nu_{ij}^u \in [0, 1)$ in Assumption 2.1 satisfying (48a) and (48b).

Next, letting $\Gamma(k) = \gamma^2 w^{\mathrm{T}}(k)w(k) - y^{\mathrm{T}}(k)y(k)$, to analyze the finite-time H_∞ performance of the closed-loop systems (12).

Step 1 For any $k \in [k_{i+1}, k_{i+2}) = [k_{i+1}, k_{i+1} + \Delta_{i+1}) \bigcup [k_{i+1} + \Delta_{i+1}, k_{i+2})$, we expand the analysis. For $i \in \mathfrak{M}_s$, synchronous switching is performed between the system and the state feedback controller. At this time, we choose the Lyapunov functional as (17). Based on the proof process of Theorem 3.1, when $k \in [k_{i+1} + \Delta_{i+1}, k_{i+2})$, $\sigma(k) = \sigma'(k) = j$, we can obtain the following closed-loop system:

$$\begin{aligned} x(k+1) &= (A_j + B_jK_j)x(k) + D_jw(k), x(0) = x_0, \\ y(k) &= (C_j + E_jK_j)x(k) + F_jw(k). \end{aligned} \tag{50}$$

Since $\Gamma(k) = \gamma^2 w^{\mathrm{T}}(k)w(k) - y^{\mathrm{T}}(k)y(k)$, substituting (50) into it gives

$$\begin{aligned} \Gamma(k) &= \gamma^2 w^{\mathrm{T}}(k)w(k) - y^{\mathrm{T}}(k)y(k) \\ &= \gamma^2 w^{\mathrm{T}}(k)w(k) - ((C_j + E_jK_j)x(k) + F_jw(k))^{\mathrm{T}}((C_j + E_jK_j)x(k) + F_jw(k)) \\ &= -x^{\mathrm{T}}(k)(C_j + E_jK_j)^{\mathrm{T}}(C_j + E_jK_j)x(k) - x^{\mathrm{T}}(k)(C_j + E_jK_j)^{\mathrm{T}}F_jw(k) \end{aligned}$$

$$- w^{\mathrm{T}}(k)F_j^{\mathrm{T}}(C_j + E_jK_j)x(k) + w^{\mathrm{T}}(k)(\gamma^2 I - F_j^{\mathrm{T}}F_j)w(k)$$
$$= \xi^{\mathrm{T}}(k)\begin{bmatrix} -(C_j+E_jK_j)^{\mathrm{T}}(C_j+E_jK_j) & -(C_j+E_jK_j)^{\mathrm{T}}F_j \\ -F_j^{\mathrm{T}}(C_j+E_jK_j) & \gamma^2 I - F_j^{\mathrm{T}}F_j \end{bmatrix}\xi(k), \tag{51}$$

where $\xi^{\mathrm{T}}(k) = [x^{\mathrm{T}}(k), w^{\mathrm{T}}(k)]$.

Meanwhile, by multiplying (47c) from both sides by diag$\{P_j, I, P_j, I\}$, and using (15), we have

$$\begin{bmatrix} -(1-\alpha_j)P_j & 0 & (A_j+B_jK_j)^{\mathrm{T}}P_j & (C_j+E_jK_j)^{\mathrm{T}} \\ * & -\gamma^2 I & D_j^{\mathrm{T}}P_j & F_j^{\mathrm{T}} \\ * & * & -P_j & 0 \\ * & * & * & -I \end{bmatrix} \le 0, \tag{52}$$

by using Schur complement, inequality (52) can be rewritten as

$$\begin{bmatrix} -(1-\alpha_j)P_j + (A_j+B_jK_j)^{\mathrm{T}}P_j(A_j+B_jK_j) & (A_j+B_jK_j)^{\mathrm{T}}P_jD_j \\ D_j^{\mathrm{T}}P_j(A_j+B_jK_j) & D_j^{\mathrm{T}}P_jD_j \end{bmatrix}$$
$$+ \begin{bmatrix} (C_j+E_jK_j)^{\mathrm{T}}(C_j+E_jK_j) & (C_j+E_jK_j)^{\mathrm{T}}F_j \\ F_j^{\mathrm{T}}(C_j+E_jK_j) & -\gamma^2 I + F_j^{\mathrm{T}}F_j \end{bmatrix} \le 0. \tag{53}$$

Thus, by combining (51) and (53), we obtain

$$\Delta V_j(x(k)) = V_j(x(k+1)) - V_j(x(k)) \le -\alpha_j x^{\mathrm{T}}(k)P_jx(k) + \Gamma(k), \tag{54}$$

so we get

$$V_j(x(k+1)) \le (1-\alpha_j)V_j(x(k)) + \Gamma(k). \tag{55}$$

Hence, by iterating inequality (55) for any $k \in [k_{i+1} + \Delta_{i+1}, k_{i+2})$, it holds that

$$V_j(x(k)) \le (1-\alpha_j)^{k-(k_{i+1}+\Delta_{i+1})}V_j(x(k_{i+1}+\Delta_{i+1})) + \sum_{s=k_{i+1}+\Delta_{i+1}}^{k-1}(1-\alpha_j)^{k-1-s}\Gamma(s). \tag{56}$$

Similarly, for any $k \in [k_{i+1}, k_{i+1} + \Delta_{i+1})$, it holds that

$$V_{ij}(x(k_{i+1}+\Delta_{i+1})) \le (1+\beta_{ij})^{\Delta_{i+1}}V_{ij}(x(k_{i+1}))$$
$$+ \sum_{s=k_{i+1}}^{k_{i+1}+\Delta_{i+1}-1}(1+\beta_{ij})^{k_{i+1}+\Delta_{i+1}-1-s}\Gamma(s). \tag{57}$$

Step 2 Because of the great similarity with Theorem 3.1, the iterative result of $V_{\sigma(k)}(x(k))$ over the entire interval $[k_0, k)$ can be obtained corresponding to its proof process, and according to (56) and (57), we have the following results.

(i) For any $k \in [k_i+\Delta_i, k_{i+1}), \sigma(k) = \sigma'(k) = \sigma(k_i) = i$, (Synchronous switching period), we have

$$V_{\sigma(k)}(x(k)) \leq (1-\alpha_{\sigma(k_i)})^{k-(k_i+\Delta_i)} V_{\sigma(k)}(x(k_i+\Delta_i)) + \sum_{s=k_i+\Delta_i}^{k-1} (1-\alpha_{\sigma(k_i)})^{k-1-s} \Gamma(s). \tag{58}$$

(ii) For any $k \in [k_i, k_i + \Delta_i)$, $\sigma(k) = \sigma(k_i), \sigma'(k) = \sigma(k_{i-1})$, (Asynchronous switching period), we have

$$\begin{aligned} V_{\sigma'(k)\sigma(k)}(x(k)) \leq\ & (1 + \beta_{\sigma(k_{i-1})\sigma(k_i)})^{k-k_i} V_{\sigma'(k)\sigma(k)}(x(k_i)) \\ & + \sum_{s=k_i}^{k-1} (1 + \beta_{\sigma(k_{i-1})\sigma(k_i)})^{k-1-s} \Gamma(s). \end{aligned} \tag{59}$$

(iii) Since the switch between the system and the controller is synchronous at first, for any $k \in [k_0, k_1)$, we obtain

$$V_{\sigma(k_0)}(x(k_1)) \leq (1 - \alpha_{\sigma(k_0)})^{k_1-k_0} V_{\sigma(k_0)}(x(k_0)) + \sum_{s=k_0}^{k_1-1} (1 - \alpha_{\sigma(k_0)})^{k_1-1-s} \Gamma(s). \tag{60}$$

(iv) For any $k \in [k_0, k) = [k_0, k_1) \bigcup [k_1, k)$, where $k \geq k_i + \Delta_i$, by using(58), (59), and (60) and analogizing (37), (38), and (39), we can get

$$\begin{aligned} V_{\sigma(k)}(x(k)) \leq & \prod_{s=1}^{N} \mu^2_{\sigma(k_{s-1})\sigma(k_s)} \times \Big[\prod_{s=0}^{N-1} (1 - \alpha_{\sigma(k_s)})^{k_{s+1}-(k_s+\Delta_s)} \\ & (1 + \beta_{\sigma(k_s)\sigma(k_{s+1})})^{\Delta_{s+1}} \Big] V_{\sigma(k_0)}(x(k_0)) + \prod_{s=1}^{N} \mu^2_{\sigma(k_{s-1})\sigma(k_s)} \\ & \times \Big[\prod_{s=0}^{N-1} (1 - \alpha_{\sigma(k_s)})^{k_{s+1}-(k_s+\Delta_s)} (1 + \beta_{\sigma(k_s)\sigma(k_{s+1})})^{\Delta_{s+1}} \Big] \\ & \times (1 - \alpha_{\sigma(k_0)})^{k_0-k_1} \sum_{s=k_0}^{k_1-1} (1 - \alpha_{\sigma(k_0)})^{k_1-1-s} \Gamma(s) + \sum_{n=1}^{N} \Big\{ \prod_{p=n}^{N} \\ & \mu^2_{\sigma(k_{p-1})\sigma(k_p)} \times \Big[\prod_{p=n}^{N} (1 - \alpha_{\sigma(k_p)})^{k_{p+1}-(k_p+\Delta_p)} (1 + \beta_{\sigma(k_p)\sigma(k_{p+1})})^{\Delta_{p+1}} \Big] \\ & \times \Big[\sum_{s=k_n}^{k_n+\Delta_n-1} (1 + \beta_{\sigma(k_{n-1})\sigma(k_n)})^{k_n+\Delta_n-1-s} \Gamma(s) \\ & + (1 - \alpha_{\sigma(k_n)})^{(k_n+\Delta_n)-k_{n+1}} \times \sum_{s=k_n+\Delta_n}^{k_{n+1}-1} (1 - \alpha_{\sigma(k_n)})^{k_{n+1}-1-s} \Gamma(s) \Big] \Big\} \end{aligned}$$

$$
\begin{aligned}
&\le \Upsilon(0,k)V_{\sigma(0)}(x(0)) + \eta\sum_{n=0}^{k}\Upsilon(n,k)\times\sum_{s=0}^{N}\Gamma(s)\\
&\le \phi_0\lambda^k V_{\sigma(0)}(x(0)) + \eta\sum_{n=0}^{k}\phi_0\lambda^{k-n}\times\sum_{s=0}^{N}\Gamma(s)\\
&= \phi_0\lambda^k V_{\sigma(0)}(x(0)) + \eta\phi_0\times\frac{1-\lambda^{k+1}}{1-\lambda}\times\sum_{s=0}^{N}\Gamma(s)\\
&< \phi_0 V_{\sigma(0)}(x(0)) + \eta\phi_0\times\frac{1}{1-\lambda}\times\sum_{s=0}^{N}\Gamma(s).
\end{aligned}\tag{61}
$$

Step 3 Finally, according to iteration result (61), the finite-time H_∞ performance of system (12) can be analyzed. When $w(k)\neq 0$, under zero initial conditions $x(0)=0$, we can get $V_{\sigma(0)}(x(0)) = x^{\mathrm{T}}(0)P_{\sigma(k)}x(0) = 0$, $V_{\sigma(k)}(x(k)) \ge 0$. From $\Gamma(k) = \gamma^2 w^{\mathrm{T}}(k)\,w(k) - y^{\mathrm{T}}(k)y(k)$, (61) can be expressed as

$$
V_{\sigma(k)}(x(k)) < \frac{\eta\phi_0}{1-\lambda}\sum_{s=0}^{N}\Gamma(s) = \frac{\eta\phi_0}{1-\lambda}\sum_{s=0}^{N}\left(\gamma^2 w^{\mathrm{T}}(s)w(s) - y^{\mathrm{T}}(s)y(s)\right), \tag{62}
$$

then (62) yields that

$$
\frac{\eta\phi_0}{1-\lambda}\sum_{s=0}^{N}y^{\mathrm{T}}(s)y(s) \le \gamma^2\frac{\eta\phi_0}{1-\lambda}\sum_{s=0}^{N}w^{\mathrm{T}}(s)w(s). \tag{63}
$$

Let $\lambda > 1-\eta\phi_0$, then there has

$$
\sum_{s=0}^{N}y^{\mathrm{T}}(s)y(s) \le \frac{\gamma^2\eta\phi_0}{1-\lambda}\sum_{s=0}^{N}w^{\mathrm{T}}(s)w(s) \le \tilde{\gamma}^2\sum_{s=0}^{N}w^{\mathrm{T}}(s)w(s), \tag{64}
$$

where $\tilde{\gamma} = \sqrt{\frac{\gamma^2\eta\phi_0}{1-\lambda}} > 0$.

According to Definition 2.5 and Definition 2.6, the system (12) is finite-time bounded with a H_∞ performance index $\tilde{\gamma}$ for any AED-ADT switching signal satisfying (48a). Hence, the proof is completed. □

When $\Delta_i \equiv 0$, $i = 1, 2, \cdots, N$, that is, the switching of the system and the controller is completely synchronous, then we can get the following corollary from Theorem 3.2.

Corollary 1 *Given a matrix $R > 0$, an integer $N > 0$, for specified constants $c_2 > c_1 > 0$, $d > 0$, $\gamma > 0$, $0 < \alpha_j < 1$, $\mu_{ij} \ge 1$, and suppose that there exist matrices $X_i > 0$, $X_j > 0$ and Y_i such that $\forall i, j \in \bar{M}, i \neq j$,*

$$
X_i \le \mu_{ij}X_j, i \neq j \in \bar{M}, \tag{65a}
$$

$$\begin{bmatrix} -(1-\alpha_j)X_j & 0 & X_jA_j^{\mathrm{T}}+Y_j^{\mathrm{T}}B_j^{\mathrm{T}} & X_jC_j^{\mathrm{T}}+Y_j^{\mathrm{T}}E_j^{\mathrm{T}} \\ * & -\gamma^2 I & D_j^{\mathrm{T}} & F_j^{\mathrm{T}} \\ * & * & -X_j & 0 \\ * & * & * & -I \end{bmatrix} \leq 0, \tag{65b}$$

$$\left(\frac{c_1}{\lambda_1}+\eta\gamma^2 d\right)\exp\left[N_{ij}^0\ln(\mu_{ij})+N\ln(1-\alpha_{\min})\right] < \frac{c_2}{\lambda_2}. \tag{65c}$$

Then, there exists a set of controllers (3) such that the closed-loop system (12) is finite-time bounded with a H_∞ performance index $\tilde{\gamma}$ w.r.t. $(c_1, c_2, R, d, \tilde{\gamma}, N, \sigma)$ for any AED-ADT switching signal $\sigma(k)$ satisfying

$$\tau_{ij}^a > \tau_{ij}^{a*} = \max\left\{\frac{N\ln\mu_{ij}}{\ln(\frac{c_2}{\lambda_2})-\ln(\frac{c_1}{\lambda_1}+\eta\gamma^2 d)-\left[N_{ij}^0\ln(\mu_{ij})+N\ln(1-\alpha_{\min})\right]}, -\frac{\ln\mu_{ij}}{\ln(1-\alpha_j)}\right\}, \tag{66}$$

where

$$\eta_n = (1-\alpha_{\sigma(k_{n-1})})^{k_{n-1}-k_n} > 1, n = 1, 2, \cdots, N, \quad \eta = \max_{n\geq 1}\{\eta_n\} > 1,$$

$$\lambda_1 = \min_{j\in\mathcal{L}}(\lambda_{\min}(\bar{X}_j)), \quad \lambda_2 = \max_{j\in\mathcal{L}}(\lambda_{\max}(\bar{X}_j)), \quad \bar{X}_{\sigma(k)} = R^{1/2}X_{\sigma(k)}R^{1/2},$$

$$\tilde{\gamma} = \gamma\sqrt{\eta\exp\left\{\sum_{i\in\mathcal{L}}\sum_{j\in \mathrm{J}(i)}\left[N_{ij}^0\ln(\mu_{ij})\right]\right\}} > 0.$$

Moreover, if the controllers exist, the controller gains are given by $K_j = Y_j(X_j)^{-1}$.

4 Numerical Simulation

The numerical example is given to show the validity of the proposed asynchronous finite-time H_∞ control approach under AED-ADT switching.

Example Consider the given parameters of discrete-time switched linear system (1) including two subsystems as follows:

$$A_1 = \begin{bmatrix} 0.35 & 0 \\ -0.2 & -0.32 \end{bmatrix}, \ A_2 = \begin{bmatrix} -0.42 & 0.1 \\ 0.3 & 0.7 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.25 \\ 0.4 \end{bmatrix}, \ B_2 = \begin{bmatrix} 0.45 \\ 0.15 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} -0.2 \\ 0.4 \end{bmatrix}^{\mathrm{T}}, \ C_2 = \begin{bmatrix} 0 \\ 0.25 \end{bmatrix}^{\mathrm{T}}, \ D_1 = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}, \ D_2 = \begin{bmatrix} 0.15 \\ -0.2 \end{bmatrix},$$

$$E_1 = 0.2, \ E_2 = -0.32, \ F_1 = 0.1, \ F_2 = 0.25.$$

Suppose that $N = 20$, $c_1 = 0$, $c_2 = 10$, $d = 1$, $R = I$. Consider the initial condition $x(0) = [0\ 0]^{\mathrm{T}}$ and the disturbance input $w(k) = \cos(k)\exp(-0.8k)$. Let $\alpha_1 = 0.62$, $\alpha_2 = 0.4$, $\beta_{12} = 1.01$, $\beta_{21} = 1.2$, $\mu_{12} = 1$, $\mu_{21} = 1.3$. Then by utilizing MATLAB LMI Toolbox and applying Theorem 3.2, we can get

$$X_1 = \begin{bmatrix} 2.0446 & -0.2724 \\ -0.2724 & 0.9115 \end{bmatrix}, X_2 = \begin{bmatrix} 2.7523 & -0.5863 \\ -0.5863 & 1.2483 \end{bmatrix},$$
$$X_{12} = \begin{bmatrix} 2.4287 & -0.4512 \\ -0.4512 & 1.0992 \end{bmatrix}, X_{21} = \begin{bmatrix} 2.4094 & -0.3846 \\ -0.3846 & 1.0843 \end{bmatrix},$$
$$Y_1 = \begin{bmatrix} 0.3910\ 0.2177 \end{bmatrix}, Y_2 = \begin{bmatrix} 0.1081\ -0.5065 \end{bmatrix}.$$

Hence, according to (15), we can obtain the controller gains

$$K_1 = \begin{cases} Y_1 X_1^{-1} = \begin{bmatrix} 0.2323\ 0.3082 \end{bmatrix}, & i \in [k_1 + \Delta_1, k_2), \\ Y_1 X_{12}^{-1} = \begin{bmatrix} 0.2141\ 0.2859 \end{bmatrix}, & i \in [k_2, k_2 + \Delta_2), \end{cases}$$
$$K_2 = \begin{cases} Y_2 X_2^{-1} = \begin{bmatrix} -0.0524\ -0.4304 \end{bmatrix}, & i \in [k_2 + \Delta_2, k_3), \\ Y_2 X_{21}^{-1} = \begin{bmatrix} -0.0315\ -0.4783 \end{bmatrix}, & i \in [k_1, k_1 + \Delta_1). \end{cases}$$

In terms of (48a), the corresponding AED-ADT are $\tau_{12}^a > \tau_{12}^{a*} = 0$, $\tau_{21}^a > \tau_{21}^{a*} = 0.5423$. On account of $\bar{X}_{\sigma(k)} = R^{1/2} X_{\sigma(k)} R^{1/2}$, we can get $\lambda_1 = 0.8494$, $\lambda_2 = 2.9539$. Let $\eta = 1.1$, $\gamma = 0.5$, $\nu_{12}^s = 0.997$, $\nu_{12}^u = 0.001$, $\nu_{21}^s = 0.001$, $\nu_{21}^u = 0.001$, $N_{ij}^0 = 0$, $\bar{T}_{ij}^s = 0$, $\bar{T}_{ij}^u = 0$, $\forall (i, j) \in \bar{M}$ $(i \neq j)$, we have $-0.1 < \lambda = 0.6024 < 1$ from (48b). From this, we can verify that (47e) holds, and get $\tilde{\gamma} = 0.8317 > 0$.

On the other hand, assume that $\alpha_1 = 0.62$, $\alpha_2 = 0.4$, $\mu_{12} = 2.5$, $\mu_{21} = 2.8$, $\eta = 1.5$, $N_{ij}^0 = 1$. Then, by utilizing MATLAB LMI Toolbox and applying Corollary 1, we can get

$$X_1 = \begin{bmatrix} 4.0019 & 0.2633 \\ 0.2633 & 1.0909 \end{bmatrix}, X_2 = \begin{bmatrix} 4.3918 & -1.0590 \\ -1.0590 & 1.2564 \end{bmatrix},$$
$$Y_1 = \begin{bmatrix} 2.1879\ 0.4307 \end{bmatrix}, Y_2 = \begin{bmatrix} -0.1144\ -0.5256 \end{bmatrix}.$$

Hence, we can obtain the controller gains

$$K_1 = Y_1 X_1^{-1} = \begin{bmatrix} 0.5291\ 0.2672 \end{bmatrix}, K_2 = Y_2 X_2^{-1} = \begin{bmatrix} -0.1593\ -0.5526 \end{bmatrix},$$

and the corresponding AED-ADT are $\tau_{12}^a > \tau_{12}^{a*} = 1.7937$, $\tau_{21}^a > \tau_{21}^{a*} = 1.8859$, the H_∞ performance index $\tilde{\gamma} = 1.6202 > 0$.

From Table 1, we can clearly see that the AED-ADT τ_{ij}^{a*} in Theorem 3.2 is smaller than that in Corollary 1, and the H_∞ performance index $\tilde{\gamma}$ in Theorem 3.2 is also smaller than that in Corollary 1, which means that the finite-time H_∞ controller in Theorem 3.2 can make a better performance than the one in Corollary 1.

Table 1 Comparison results between Corollary 1 ($\Delta_i \equiv 0$) and Theorem 3.2

Criteria	Corollary 1 ($\Delta_i \equiv 0$)	Theorem 3.2
Parameters	$\alpha_1 = 0.62, \alpha_2 = 0.4,$ $\mu_{12} = 2.5$, $\mu_{21} = 2.8$	$\alpha_1 = 0.62, \alpha_2 = 0.4, \beta_{12} = 1.01,$ $\beta_{21} = 1.2$, $\mu_{12} = 1$, $\mu_{21} = 1.3$
AED-ADT bounds	$\tau_{12}^{a*} = 1.7937$, $\tau_{21}^{a*} = 1.8859$	$\tau_{12}^{a*} = 0$, $\tau_{21}^{a*} = 0.5423$
H_∞ Performance index	$\tilde{\gamma} = 1.6202$	$\tilde{\gamma} = 0.8317$

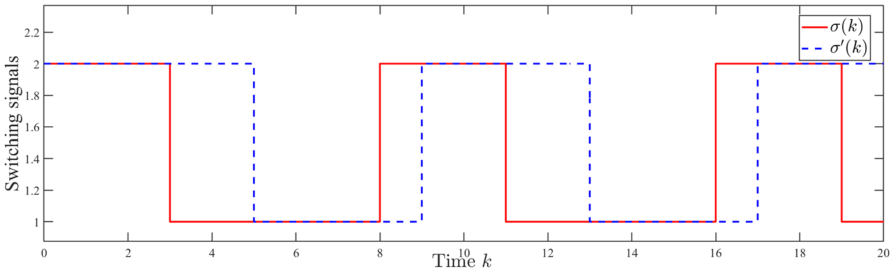

Fig. 3 Switching signals $\sigma(k)$ and $\sigma'(k)$

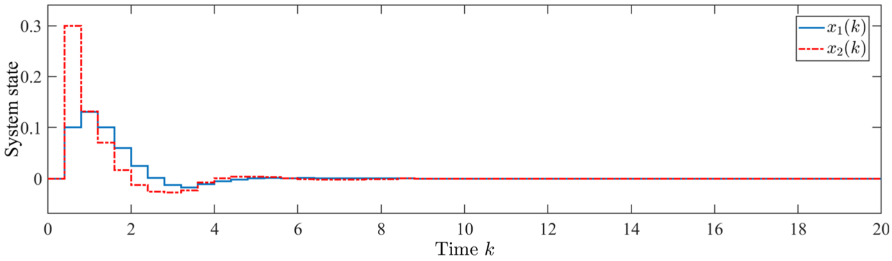

Fig. 4 State trajectory $x(k)$ under AED-ADT switching

Additionally, according to Theorem 3.2, it can be obtained the AED-ADT conditions such that the closed-loop system (12) is finite-time bounded as $\tau_{12}^a = 3 \geq \tau_{12}^{a*} = 0$, $\tau_{21}^a = 5 \geq \tau_{21}^{a*} = 0.5423$. Asynchronous switching of $\sigma(k)$ and $\sigma'(k)$ occurs when $\Delta_1 = 2$, $\Delta_2 = 1$ in this example. Figure 3 is drawn to show the evolutions of $\sigma(k)$ and $\sigma'(k)$. And Fig. 4 is used to show the trajectories of state responses $x_1(k)$ and $x_2(k)$ of system. The output responses and $w(k)$ are displayed in Fig. 5. Then, it is obvious that $x^{\mathrm{T}}(k)Rx(k) \ll c_2$, $\forall k \in [1, 20]$, as shown in Fig. 6. That signifies the closed-loop system (12) is finite-time bounded. Consequently, the proposed method of this work is valid.

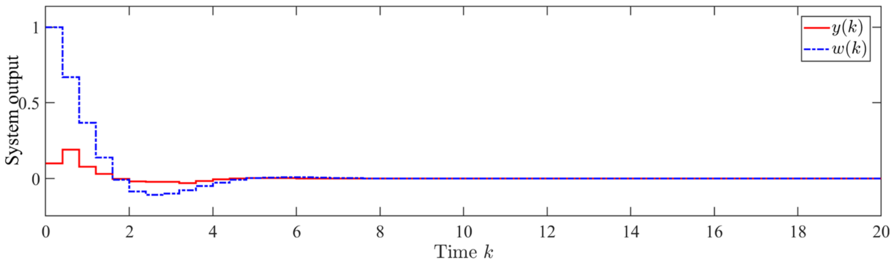

Fig. 5 System output $y(k)$ and $w(k)$ under AED-ADT switching

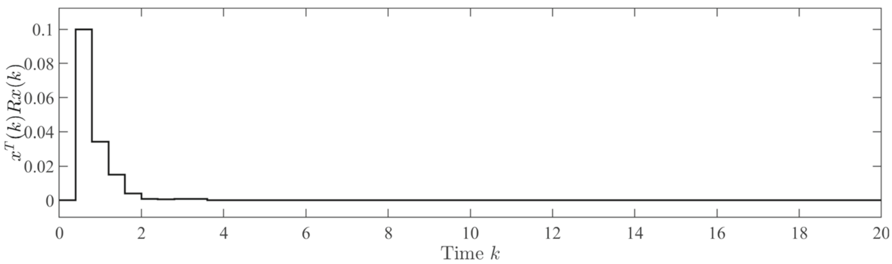

Fig. 6 History of $x^{\mathrm{T}}(k)Rx(k)$ of the closed-loop system (12)

5 Conclusions

This paper has dealt with asynchronous finite-time H_∞ control problem for a class of discrete-time switched linear systems with switching time delay. By using the AED-ADT method, multiple Lyapunov functions, and linear matrix inequalities, a asynchronous state feedback controller is designed, and a sufficient condition to guarantee that the closed-loop system is finite-time bounded is derived. Then, in view of the obtained result, a sufficient condition for finite-time H_∞ control is deduced, which can ensure not only the finite-time boundedness of the closed-loop system, but also the H_∞ performance. Finally, the rationality of the proposed method is verified by a numerical example.

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Data Availability The authors declare that the data supporting the findings of this study are available within the article.

Declarations

Conflict of interest No potential conflict of interest was reported by the authors.

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