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H_∞ Control for Lur'e Singular Systems with Time Delays

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Abstract
This paper considers the H_∞ control problem for Lur'e singular systems with time delays. By using Lyapunov stability theory, sufficient conditions for the system to be exponentially stable and satisfy the performance index of H_∞ are obtained; these conditions are based on the linear matrix inequality method. Then the design of a state feedback controller is given, by applying a more clever approach for a nonlinear matrix with a special format to convert it into the sum of several linear matrices, such that the closed-loop system is also exponentially stable. Finally, numerical examples illustrate the effectiveness of the proposed method and its advantages over existing approaches.

Keywords Time-delay systems · Lur'e systems · Singular systems · Exponentially stable · Lyapunov stability theory · Performance index of H_∞

1 Introduction

A singular system is a kind of dynamic systems with a more general form that has more describe performance characteristics and has a broader form and wider application background than normal systems [10]. The research content regarding singular systems

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is diverse, including stability, dissipation and H_∞ control [17]. However, in many practical control problems, there are multiple time delays in the studied systems, whose existence may induce instability, oscillation and poor performance. System stability is a prerequisite for the normal operation of a control system, and H_∞ control can ensure the stability of the system while suppressing the impact of interference on the system performance to a certain extent. Therefore, it is necessary to study stability and H_∞ control problem of a singular time-delay system [1,10].

A Lur'e system is a type of typical, interval, nonlinear system with widely representative significance. Its nonlinear terms are usually contained in a finite sector interval or an infinite open plane, that is, the linear parts of the system have a fixed matrix and one or more uncertain nonlinear parts are utilized to satisfy sector constraints. Since Lur'e introduced the absolute stability in 1957, many authors have extensively discussed absolute stability of Lur'e control systems based on the Popov frequency domain criteria and the Lyapunov function with a Lur'e form [2,3].

Reference [10] conducted an in-depth study on the delay-dependent H_∞ control problem of Lur'e singular time-delay systems, reference [7] addressed synchronization methods for a specific Lur'e system, and references [6,21] studied the absolute stability problem of Lur'e time-delay control systems, that is, the problem of global asymptotic stability. Reference [18] studied system stability and the compactness of the operators describing the solution trajectories. Based on the linear matrix inequality (LMI) method, Wu proposed delay-range-dependent bounded real lemmas and studied sufficient conditions for a system to be exponentially stable and the existence of the linear H_∞ filter [22]. Then, Park studied H_∞ filtering for a class of Markovian jump systems and successfully proved that there are necessary and sufficient conditions of H_∞ filtering for singular Markovian jump systems (SMJSs) whose transfer rates are partially unknown [15]. Based on LMI theory, Kim used a new design method to study the H_∞ control problem for a singular time-delay system and obtained all solutions including the controller gains [5]. In addition, Long designed a dynamic feedback controller to ensure that the developed closed-loop system was impulse-free and stable under the given performance index of H_∞ [11]. By constructing an enhanced Lyapunov-Krasovskii function with triple integral terms, [4] proposed a bounded real lemma to ensure that a singular state-delay system was stable and designed a static output feedback controller. In addition, based on the bounded real lemma, Yang ensured that the singular time-delay system was regular, impulse-free and stable under the conditions of the performance index of H_∞ [24]. Reference [25] studied the exponential H_∞ control problem for a singular system with time-varying delays, and references [19,23] also studied the control problem for such systems. Furthermore, there are a large number of papers that have studied the stability and H_∞ control problems of discrete-time descriptor systems [9,13], uncertain systems [12,26] and SMJSs [8].

This paper mainly studies the H_∞ control problem of Lur'e singular time-delay systems. By using Lyapunov stability theory, a new Lyapunov function is constructed, which applies not only the upper and lower limits of the time delay, but also the time-delay interval. Based on LMI method, sufficient conditions for systems to be exponentially stable and satisfy the performance index of H_∞ are given, and a state feedback controller is designed to make the closed-loop system exponentially stable.

Notations: R^n denotes the n-dimensional Euclidean space, and $R^{n\times m}$ is the set of all $m \times n$ real matrices. $||\cdot||$ stands for the Euclidean norm of a vector and $||f(t)||_d = \sup_{-d\le t\le 0} ||f(t)||$. $\alpha_1 \bigvee \alpha_2 = \max\{\alpha_1, \alpha_2\}$, and $\alpha_1 \bigwedge \alpha_2 = \min\{\alpha_1, \alpha_2\}$. The superscripts 'T' and '$*$' denote the term that is induced by symmetry, respectively.

2 Problem Formulation

Consider a Lur'e singular time-delay system:

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + A_d x(t-d(t)) - F\varphi(y(t)) + B_w w(t) + B_u u(t) && (1)\\ y(t) &= Cx(t) + C_d x(t-d(t)) + D_w w(t) && (2)\\ z(t) &= Lx(t) + L_d x(t-d(t)) + L_w w(t) + L_u u(t) \\ x(t) &= \phi(t), t \in [-d_2, 0] && (3) \end{aligned}$$

where $x \in R^n$ is the state, $u \in R^m$ is the control input, $w \in R^p$ is the disturbance input that satisfies $\int_0^{+\infty} w^T(t)w(t) < +\infty$, and $z \in R^q$ is the control output of the system. $E, A, A_d, B_w, B_u, C, C_d, D_w, F, L, L_d, L_w, L_u$ are known real constant matrices with appropriate dimensions, and $Rank(E) = r \le n$. $\phi(t) : \mathcal{R} \to \mathcal{R}^n$ is a compatible vector-valued initial function, $y(t) \in R^l$ and the nonlinear function $\varphi(y) \in R^l \to R^l$ is appropriately smooth and satisfy the sector constraint

$$\varphi^T(y)\varphi(y) \le \varphi^T(y)My \tag{4}$$

where $M \in R^{l\times l}$ is a given positive definite matrix. In addition, $d(t)$ is a time-delay continuous function that is time-differentiable at all times, thereby satisfying

$$0 \le d_1 \le d(t) \le d_2, \dot{d}(t) \le \alpha \tag{5}$$

where d_1 and d_2 represent the upper and lower limits of the time delay, respectively, $\dot{d}(t)$ is the corresponding derivative function, and $0 \le \alpha < 1$.

Throughout this paper, the following definitions and lemmas will be used.

Definition 1 [20] A system is exponentially stable, if there exist scalars $\beta_1 > 0$ and $\beta_2 > 0$ such that

$$||x(t)|| \le \beta_1 e^{-\beta_2 t}||x(t)||_{d_2}, t > 0$$

Definition 2 [11] The system (1)–(3) possesses an H_∞ performance of γ, that is, under zero initial conditions, the system satisfies

$$J(t) = \int_0^t [z^T(s)z(s) - \gamma^2 w^T(s)w(s)]\mathrm{d}s < 0$$

for any nonzero $\omega(t)$ that satisfies $\int_0^{+\infty} w^T(t)w(t) < +\infty$, where $\gamma > 0$ is a predefined scalar.

Lemma 1 [20] (Jensen integral inequality) *For any positive definite matrix* $M = M^T > 0$, *scalar quantities* γ_1 *and* γ_2 *and a vector-valued function* $\upsilon : [\gamma_1, \gamma_2] \rightarrow R^n$, *the following inequality holds:*

$$\left(\int_{\gamma_1}^{\gamma_2} \upsilon(s)\mathrm{d}s\right)^T M \left(\int_{\gamma_1}^{\gamma_2} \upsilon(s)\mathrm{d}s\right) \leq (\gamma_2 - \gamma_1)\int_{\gamma_1}^{\gamma_2} \upsilon^T(s)M\upsilon(s)\mathrm{d}s$$

Lemma 2 [11] *Given any real matrices* $Q > 0$, W_1 *and* W_2 *of appropriate dimensions and an umber* $\lambda > 0$, *the following inequality holds:*

$$W_1^T W_2 + W_2^T W_1 \leq \lambda W_1^T Q W_1 + \lambda^{-1} W_2^T Q^{-1} W_2$$

Lemma 3 [22] *Suppose that the positive continuous function* $f(t)$ *satisfies* $f(t) \leq \zeta_1 \sup_{t-d\leq s\leq t} f(s) + \zeta_2 e^{-\varepsilon t}$, *where* $\varepsilon > 0, 0 < \zeta_1 < 1, 0 < \zeta_1 e^{\varepsilon d} < 1, \zeta_2 > 0$. *For* $d > 0$, *the following inequality holds*

$$f(t) \leq e^{-\varepsilon t}||f(s)||_d + \frac{\zeta_2 e^{-\varepsilon t}}{1 - \zeta_1 e^{-\varepsilon d}}$$

3 Main Results

First, we consider the exponential stability of the system (1)–(3) when it satisfies the H_∞ performance requirement. Then, we have the following result.

Theorem 1 *In the system* (1)–(3), *for given* $0 \leq d_1 \leq d_2$, $0 \leq \alpha < 1$, *if there is a scalar* $\lambda > 0$, *symmetric positive definite matrices* $Q_j, R_j, j = 1, 2, 3$ *and a matrix* P *such that*

$$E^T P = P^T E \geq 0 \tag{6}$$

$$\left[\begin{array}{ccccc}
\Xi_{11} + L^T L & \Xi_{12} + L^T L_d & E^T R_1 E & 0 & -A^T W F + C^T M^T \\
* & \Xi_{22} + L_d^T L_d & E^T R_3 E & \Xi_{24} & -A_d{}^T W F + C_d^T M^T \\
* & * & \Xi_{33} & 0 & 0 \\
* & * & * & -Q_2 - \Xi_{24} & 0 \\
* & * & * & * & -(2-\lambda)I + F^T W F \\
* & * & * & * & * \\
* & * & * & * & *
\end{array}\right.$$

$$\left.\begin{matrix} L^T L_w + P^T B_w + A^T W B_w & P^T F \\ L_d^T L_w + A_d^T W B_w & 0 \\ 0 & 0 \\ 0 & 0 \\ -F^T W B_w + M D_w & 0 \\ L_w^T L_w + B_w W B_w - \gamma^2 I & 0 \\ * & -\lambda \end{matrix}\right] < 0 \tag{7}$$

where $d_{12} = d_2 - d_1, W = d_1^2 R_1 + d_{12} d_2^2 R_2 + d_{12}^2 R_3$, $\Xi_{11} = P^T A + A^T P + \sum_{k=1}^{3} Q_k + A^T W A - E^T R_1 E - d_{12} E^T R_2 E$, $\Xi_{12} = P^T A_d + A^T W A_d + d_{12} E^T R_2 E$, $\Xi_{22} = -(1-\alpha) Q_3 + A_d^T W A_d - E^T((d_{12} + d_2) R_2 + 2R_3)E$, $\Xi_{24} = E^T(d_2 R_2 + R_3)E$, $\Xi_{33} = -Q_1 - E^T R_1 E - E^T R_3 E$, $\Xi_{44} = -Q_2 - E^T(d_2 R_2 + R_3)E$, *then for any time-delay function* $d(t)$ *that satisfies* (5)*, the system is exponentially stable and satisfies the performance index* γ *of* H_∞*.*

Proof First, we prove that the system (1)–(3) is asymptotically stable ($w(t) = u(t) = 0$). We choose the Lyapunov function candidate as

$$\begin{aligned} V(x_t, t) &= x^T(t) E^T P x(t) + \sum_{k=1}^{2} \int_{t-d_k}^{t} x(\alpha)^T Q_k x(\alpha) \mathrm{d}\alpha + \int_{t-d(t)}^{t} x(\alpha)^T Q_3 x(\alpha) \mathrm{d}\alpha \\ &\quad + d_1 \int_{-d_1}^{0} \int_{t+\beta}^{t} \dot{x}^T(\alpha) E^T R_1 E \dot{x}(\alpha) \mathrm{d}\alpha \mathrm{d}\beta \\ &\quad + d_{12} d_2 \int_{-d_2}^{0} \int_{t+\beta}^{t} \dot{x}^T(\alpha) E^T R_2 E \dot{x}(\alpha) \mathrm{d}\alpha \mathrm{d}\beta \\ &\quad + d_{12} \int_{-d_2}^{-d_1} \int_{t+\beta}^{t} \dot{x}^T(\alpha) E^T R_3 E \dot{x}(\alpha) \mathrm{d}\alpha \mathrm{d}\beta \end{aligned}$$

where $x_t = x(t+\theta)$, $-2d_2 \le \theta \le 0$. $\mathcal{A}$ denotes the derivative of a function, so from (5), for the system (1)–(3), we have

$$\begin{aligned} &\mathcal{A} V(x_t, t) \\ &= 2x^T(t) E^T P \dot{x}(t) + \sum_{k=1}^{3} x(t)^T Q_k x(t) - \sum_{k=1}^{2} x(t-d_k)^T Q_k x(t-d_k) \\ &\quad - (1 - \dot{d}(t)) x(t-d(t))^T Q_3 x(t-d(t)) + \dot{x}^T(t) E^T W E \dot{x}(t) \\ &\quad - d_1 \int_{t-d_1}^{t} \dot{x}^T(\alpha) E^T R_1 E \dot{x}(\alpha) \mathrm{d}\alpha - d_{12} \int_{t-d_2}^{t-d_1} \dot{x}^T(\alpha) E^T R_3 E \dot{x}(\alpha) \mathrm{d}\alpha \\ &\quad - d_{12} d_2 \int_{t-d_2}^{t} \dot{x}^T(\alpha) E^T R_2 E \dot{x}(\alpha) \mathrm{d}\alpha \\ &\le 2x^T(t) E^T P \dot{x}(t) + \sum_{k=1}^{3} x(t)^T Q_k x(t) - \sum_{k=1}^{2} x(t-d_k)^T Q_k x(t-d_k) \end{aligned}$$

$$
\begin{aligned}
&-(1-\alpha)x(t-d(t))^T Q_3 x(t-d(t)) + \dot{x}^T(t)E^T W E\dot{x}(t) \\
&-d_1\int_{t-d_1}^{t} \dot{x}^T(\alpha)E^T R_1 E\dot{x}(\alpha)\mathrm{d}\alpha - d_{12}\int_{t-d_2}^{t-d_1} \dot{x}^T(\alpha)E^T R_3 E\dot{x}(\alpha)\mathrm{d}\alpha \\
&-d_{12}d_2\int_{t-d_2}^{t} \dot{x}^T(\alpha)E^T R_2 E\dot{x}(\alpha)\mathrm{d}\alpha
\end{aligned}
$$

Using Lemma 1, the following formula can be obtained:

$$
\begin{aligned}
&-d_1\int_{t-d_1}^{t} \dot{x}^T(\alpha)E^T R_1 E\dot{x}(\alpha)\mathrm{d}\alpha - d_{12}d_2\int_{t-d_2}^{t} \dot{x}^T(\alpha)E^T R_2 E\dot{x}(\alpha)\mathrm{d}\beta \\
&-d_{12}\int_{t-d_2}^{t-d_1} \dot{x}^T(\alpha)E^T R_3 E\dot{x}(\alpha)\mathrm{d}\alpha \\
&\quad = -d_1\int_{t-d_1}^{t} \dot{x}^T(\alpha)E^T R_1 E\dot{x}(\alpha)\mathrm{d}\alpha - d_{12}d_2\int_{t-d(t)}^{t} \dot{x}^T(\alpha)E^T R_2 E\dot{x}(\alpha)\mathrm{d}\beta \\
&-d_{12}\int_{t-d(t)}^{t-d_1} \dot{x}^T(\alpha)E^T R_3 E\dot{x}(\alpha)\mathrm{d}\alpha - d_{12}\int_{t-d_2}^{t-d(t)} \dot{x}^T(\alpha)E^T (d_2R_2 + R_3)E\dot{x}(\alpha)\mathrm{d}\alpha \\
&\quad \le -\left(\int_{t-d_1}^{t} \dot{x}^T(\alpha)E^T\mathrm{d}\alpha\right) R_1 \left(\int_{t-d_1}^{t} E\dot{x}(\alpha)\mathrm{d}\alpha\right) \\
&-d_{12}\left(\int_{t-d(t)}^{t} \dot{x}^T(\alpha)E^T\mathrm{d}\alpha\right) R_2 \left(\int_{t-d(t)}^{t} E\dot{x}(\alpha)\mathrm{d}\alpha\right) \\
&-\left(\int_{t-d(t)}^{t-d_1} \dot{x}^T(\alpha)E^T\mathrm{d}\alpha\right) R_3 \left(\int_{t-d(t)}^{t-d_1} E\dot{x}(\alpha)\mathrm{d}\alpha\right) \\
&-\left(\int_{t-d_2}^{t-d(t)} \dot{x}^T(\alpha)E^T\mathrm{d}\alpha\right) (d_2R_2 + R_3) \left(\int_{t-d_2}^{t-d(t)} E\dot{x}(\alpha)\mathrm{d}\alpha\right)
\end{aligned}
$$

The nonlinear function $\varphi(y)$ satisfies inequality (4), and based on Lemma 2, from (1) and (4), we have

$$
\begin{aligned}
&\mathcal{A}V(x_t, t) \\
&\quad \le 2x^T(t)P^T(Ax(t) + A_d x(t-d(t)) - F\varphi(y(t))) + \sum_{k=1}^{3} x(t)^T Q_k x(t) \\
&-\sum_{k=1}^{2} x(t-d_k)^T Q_k x(t-d_k) - (1-\alpha)x(t-d(t))^T Q_3 x(t-d(t)) \\
&+\dot{x}^T(t)E^T W E\dot{x}(t) - \left(\int_{t-d_1}^{t} \dot{x}^T(\alpha)E^T\mathrm{d}\alpha\right) R_1 \left(\int_{t-d_1}^{t} E\dot{x}(\alpha)\mathrm{d}\alpha\right) \\
&-d_{12}\left(\int_{t-d(t)}^{t} \dot{x}^T(\alpha)E^T\mathrm{d}\alpha\right) R_2 \left(\int_{t-d(t)}^{t} E\dot{x}(\alpha)\mathrm{d}\alpha\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\int_{t-d(t)}^{t-d_1} \dot{x}^T(\alpha)E^T \mathrm{d}\alpha\right) R_3 \left(\int_{t-d(t)}^{t-d_1} E\dot{x}(\alpha)\mathrm{d}\alpha\right) \\
&-\left(\int_{t-d_2}^{t-d(t)} \dot{x}^T(\alpha)E^T \mathrm{d}\alpha\right) (d_2R_2+R_3) \left(\int_{t-d_2}^{t-d(t)} E\dot{x}(\alpha)\mathrm{d}\alpha\right) \\
&+2(\varphi(y(t))^T My(t) - \varphi(y(t))^T\varphi(y(t)))
\end{aligned} \tag{8}
$$

$$
\leq \xi^T \begin{bmatrix} \Xi_{11}+\frac{1}{\lambda}P^TFF^TP & \Xi_{12} & E^TR_1E & 0 & -A^TWF+\mu C^TM^T \\ * & \Xi_{22} & E^TR_3E & \Xi_{24} & -A_d^TWF+\mu C_d^TM^T \\ * & * & \Xi_{33} & 0 & 0 \\ * & * & * & \Xi_{44} & 0 \\ * & * & * & * & -(2-\lambda)I+F^TWF \end{bmatrix} \xi \tag{9}
$$

where $\xi(t) = \begin{bmatrix} x^T(t) & x^T(t-d(t)) & x^T(t-d_1) & x^T(t-d_2) & \varphi^T(t) \end{bmatrix}^T$.

Next, the following matrix decomposition is executed. Since $Rank(E) = r \leq n$, there are nonsingular matrices G, H such that

$$
GEH = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}
$$

Denoting

$$
GAH = \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix}, G^{-T}PH = \begin{bmatrix} \hat{P}_1 & \hat{P}_2 \\ \hat{P}_3 & \hat{P}_4 \end{bmatrix}
$$

we can obtain $\hat{P}_2 = 0$ from (6). We can obtain $\hat{A}_4^T\hat{P}_4 + \hat{P}_4^T\hat{A}_4 < 0$ by pre-multiplying and post-multiplying $\Xi_{11} < 0$ by H^T and H, respectively, so $\hat{A}_4$ is nonsingular. Setting

$$
\hat{G} = \begin{bmatrix} I_r & -\hat{A}_2\hat{A}_4^{-1} \\ 0 & \hat{A}_4^{-1} \end{bmatrix} G
$$

we have

$$
\hat{G}EH = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \hat{G}AH = \begin{bmatrix} A_1 & 0 \\ A_3 & I \end{bmatrix}, \hat{G}^{-T}PH = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}
$$

where $A_1 = \hat{A}_1 - \hat{A}_2\hat{A}_4^{-1}\hat{A}_3$, $A_3 = \hat{A}_4^{-1}\hat{A}_3$.

Denoting

$$
\hat{G}A_dH = \begin{bmatrix} A_{d1} & A_{d2} \\ A_{d3} & A_{d4} \end{bmatrix}, H^TQ_3H = \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix}, \hat{G}FH = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}
$$

Inequality (6) implies that $P_2 = 0$ and $P_1 \geq 0$. We define $\eta(t) = \begin{bmatrix} \eta_1(t) & \eta_2(t) \end{bmatrix}^T = H^{-1}x(t)$, $\psi(y(t)) = \begin{bmatrix} \psi_1(y(t)) & \psi_2(y(t)) \end{bmatrix}^T = H^{-1}\varphi(y(t))$. We can obtain

$$\begin{aligned} \hat{G}EH\dot{\eta}(t) &= \hat{G}E\dot{x}(t) = \hat{G}[Ax(t) + A_d x(t-d(t)) - F\varphi(y(t))] \\ &= \hat{G}AH\eta(t) + \hat{G}A_d H\eta(t-d(t)) - \hat{G}FH\psi(y(t)) \end{aligned}$$

Then, the system (1)–(3) with $\omega(t) \equiv 0$ is a restricted system that is equivalent to

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1(t) \\ \dot{\eta}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ A_3 & I \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} + \begin{bmatrix} A_{d1} & A_{d2} \\ A_{d3} & A_{d4} \end{bmatrix} \begin{bmatrix} \eta_1(t-d(t)) \\ \eta_2(t-d(t)) \end{bmatrix} - \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix} \begin{bmatrix} \psi_1(y(t)) \\ \psi_2(y(t)) \end{bmatrix}$$

In other words, we have that

$$\begin{aligned} \dot{\eta}_1(t) &= A_1\eta_1(t) + A_{d1}\eta_1(t-d(t)) \\ &\quad + A_{d2}\eta_2(t-d(t)) - F_1\psi_1(y(t)) - F_2\psi_2(y(t)) \qquad (10) \\ 0 &= A_3\eta_1(t) + \eta_2(t) + A_{d3}\eta_1(t-d(t)) + A_{d4}\eta_2(t-d(t)) \\ &\quad - F_3\psi_1(y(t)) - F_4\psi_2(y(t)) \\ \eta(t) &= H^{-1}\phi(t), t \in [-d_2, 0] \qquad (11) \end{aligned}$$

Defining a new function $W(x_t, t) = e^{\varepsilon t}V(x_t, t)$, we can obtain a scalar $\sigma > 0$ such that $\mathcal{A}V(x_t, t) \leq -\sigma||x(t)||^2$, so $W(x_t, t) \leq W(x_0, 0) + \int_0^t e^{\varepsilon s}(\varepsilon V(x_s, s) - \sigma||x(s)||^2)\mathrm{d}s$. Due to

$$\begin{aligned} x^T(t)E^T Px(t) &= \eta(t)^T H^T E^T \hat{G}^T \hat{G}^{-T} PH\eta(t) \\ &= \begin{bmatrix} \eta_1(t)^T & \eta_2(t)^T \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix} \\ &= \eta_1(t)^T P_1 \eta_1(t) \end{aligned}$$

we can obtain

$$\begin{aligned} &\lambda_{min}(P_1)||\eta_1(t)||^2 \leq x^T(t)E^T Px(t) \leq V(x_t, t) = e^{-\varepsilon t}W(x_t, t) \\ &\leq e^{-\varepsilon t}\{W(x_0, 0) + \int_0^t e^{\varepsilon s}[\varepsilon V(x_t, t) - \sigma||x(s)||^2]\mathrm{d}s\} \end{aligned}$$

Because $w(t) = u(t) = 0$, Lemma 2 implies that we have

$$\begin{aligned} &d_1 \int_{-d_1}^{0} \int_{t+\beta}^{t} \dot{x}^T(\alpha)E^T R_1 E\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\beta \\ &= d_1 \int_{-d_1}^{0} \int_{t+\beta}^{t} [Ax(\alpha) + A_d x(\alpha - d(\alpha)) - F\varphi(y(\alpha))]^T R_1[Ax(\alpha) \end{aligned}$$

$$
\begin{aligned}
&+A_d x(\alpha - d(\alpha)) - F\varphi(y(\alpha))]\mathrm{d}\alpha\mathrm{d}\beta\\
&= d_1 \int_{-d_1}^{0}\int_{t+\beta}^{t} [x^T(\alpha)A^T R_1 A x(\alpha) + x^T(\alpha - d(\alpha))A_d^T R_1 A_d x(\alpha - d(\alpha))\\
&+\varphi(y(\alpha))^T F^T R_1 F\varphi(y(\alpha))\\
&+x^T(\alpha)A^T R_1 A_d x(\alpha - d(\alpha)) + x^T(\alpha - d(\alpha))A_d^T R_1 A x(\alpha)\\
&-x^T(\alpha - d(\alpha))A_d^T R_1 F\varphi(y(\alpha))\\
&-\varphi(y(\alpha))^T F^T R_1 A_d x(\alpha - d(\alpha)) - x^T(\alpha)A^T R_1 F\varphi(y(\alpha))\\
&-\varphi(y(\alpha))^T F^T R_1 A x(\alpha)]\mathrm{d}\alpha\mathrm{d}\beta\\
&\le d_1 \int_{-d_1}^{0}\int_{t+\beta}^{t} [x^T(\alpha)A^T R_1 A x(\alpha) + x^T(\alpha - d(\alpha))A_d^T R_1 A_d x(\alpha - d(\alpha))\\
&+\varphi(y(\alpha))^T F^T R_1 F\varphi(y(\alpha))\\
&+x^T(\alpha)A^T R_1 A_d x(\alpha - d(\alpha)) + x^T(\alpha - d(\alpha))A_d^T R_1 A x(\alpha)]\mathrm{d}\alpha\mathrm{d}\beta\\
&\le d_1 \int_{-d_1}^{0}\int_{t+\beta}^{t} [x^T(\alpha)A^T R_1 A x(\alpha) + x^T(\alpha - d(\alpha))A_d^T R_1 A_d x(\alpha - d(\alpha))\\
&+\lambda_{max}(F^T R_1 F)\varphi(y(\alpha))^T\varphi(y(\alpha))\\
&+x^T(\alpha)A^T R_1^2 A x(\alpha) + x^T(\alpha - d(\alpha))A_d^T A_d x(\alpha - d(\alpha))]\mathrm{d}\alpha\mathrm{d}\beta\\
&\le d_1 \int_{-d_1}^{0}\int_{t+\beta}^{t} \{[\lambda_{max}(A^T R_1 A) + \lambda_{max}(A^T R_1^2 A)]||x(\alpha)||^2\\
&+[\lambda_{max}(A_d^T R_1 A_d) + \lambda_{max}(A_d^T A_d)]||x(\alpha - d(\alpha))||^2\\
&+\lambda_{max}(F^T R_1 F)y(\alpha)^T M^2 y(\alpha)\}\mathrm{d}\alpha\mathrm{d}\beta\\
&\le d_1 \int_{-d_1}^{0}\int_{t+\beta}^{t} \{[\lambda_{max}(A^T R_1 A) + \lambda_{max}(A^T R_1^2 A)]||x(\alpha)||^2\\
&+[\lambda_{max}(A_d^T R_1 A_d) + \lambda_{max}(A_d^T A_d)]||x(\alpha - d(\alpha))||^2\\
&+\lambda_{max}(F^T R_1 F)\lambda_{max}(M^2)[Cx(t)\\
&+C_d x(t - d(t))]^T[Cx(t) + C_d x(t - d(t))]\}\mathrm{d}\alpha\mathrm{d}\beta\\
&\le d_1 \int_{-d_1}^{0}\int_{t+\beta}^{t} \{[\lambda_{max}(A^T R_1 A) + \lambda_{max}(A^T R_1^2 A)\\
&+2\lambda_{max}(F^T R_1 F)\lambda_{max}(M^2)\lambda_{max}(C^T C)]||x(\alpha)||_{d_2}^2\\
&+[\lambda_{max}(A_d^T R_1 A_d) + \lambda_{max}(A_d^T A_d)\\
&+2\lambda_{max}(F^T R_1 F)\lambda_{max}(M^2)\lambda_{max}(C_d^T C_d)]||x(\alpha - d(\alpha))||_{d_2}^2\}\mathrm{d}\alpha\mathrm{d}\beta\\
&\le -\frac{{d_1}^3}{2}K_1||x(\alpha)||_{d_2}^2
\end{aligned}
$$

where $K_i = \lambda_{max}(A^T R_i A)+\lambda_{max}(A^T R_i^2 A)+2\lambda_{max}(F^T R_i F)\lambda_{max}(M^2)\lambda_{max}(C^T C) + \lambda_{max}(A_d^T R_i A_d) + \lambda_{max}(A_d^T A_d) + 2\lambda_{max}(F^T R_i F)\lambda_{max}(M^2)\lambda_{max}(C_d^T C_d)$, $i = 1, 2, 3$. By the same way,
$d_{12}d_2 \int_{-d_2}^{0}\int_{t+\beta}^{t} \dot{x}^T(\alpha)E^T R_2 E\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\beta \le -\frac{d_{12}{d_2}^2}{2}K_2||x(\alpha)||_{d_2}^2$,

$d_{12}\int_{-d_2}^{-d_1}\int_{t+\beta}^{t}\dot{x}^T(\alpha)E^T R_3 E\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\beta \leq -\frac{d_{12}({d_1}^2-{d_2}^2)}{2}K_3||x(\alpha)||_{d_2}^2$. Therefore,

$$
\begin{aligned}
&\lambda_{min}(P_1)||\eta_1(t)||^2\\
&\leq e^{-\varepsilon t}\{W(x_0,0)+\int_0^t[\varepsilon e^{\varepsilon s}K_0||x(s)||_{d_2}^2\\
&+e^{\varepsilon s}\varepsilon K\int_{-d_2}^{0}||x(s+\theta)||^2 d\theta-\sigma||x(s)||^2]\mathrm{d}s\}\\
&\leq e^{-\varepsilon t}[W(x_0,0)+\int_0^t\varepsilon e^{\varepsilon s}K_0||x(s)||_{d_2}^2\mathrm{d}s+\varepsilon K\int_0^t e^{\varepsilon s}\int_{-d_2}^{0}||x(s+\theta)||^2 d\theta\mathrm{d}s]
\end{aligned}
\tag{12}
$$

where $K_0=\lambda_{max}(E^T P)-\frac{{d_1}^3}{2}K_1-\frac{d_{12}{d_2}^2}{2}K_2-\frac{d_{12}({d_1}^2-{d_2}^2)}{2}K_3$, $K=\sum_{k=1}^{3}\lambda_{max}(Q_k)$, and we have

$$
\begin{aligned}
&\int_0^t e^{\varepsilon s}\int_{-d_2}^{0}||x(s+\theta)||^2 d\theta\mathrm{d}s=\int_0^t e^{\varepsilon s}\int_{t-d_2}^{t}||x(\theta)||^2 d\theta\mathrm{d}s\\
&\leq\int_{-d_2}^{t}\left(\int_{\theta\bigvee 0}^{(\theta+d_2)\bigwedge t}e^{\varepsilon s}\mathrm{d}s\right)||x(\theta)||^2 d\theta\leq\int_{-d_2}^{t}d_2 e^{\varepsilon(s+d_2)}||x(s)||^2\mathrm{d}s\\
&\leq d_2 e^{\varepsilon d_2}\int_0^t e^{\varepsilon s}||x(s)||^2\mathrm{d}s+d_2 e^{\varepsilon d_2}\int_{-d_2}^{0}||\phi(s)||^2\mathrm{d}s
\end{aligned}
\tag{13}
$$

Therefore, from inequalities (12) and (13), if the scalar ε is sufficiently small, there exists a scalar $k=W(x_0,0)(||\phi(t)||^2{}_{d_2})^{-1}+K_0 e^{\varepsilon d_2}+\varepsilon K{d_2}^2 e^{\varepsilon d_2}>0$, such that

$$
\lambda_{min}(P_1)||\eta_1(t)||^2\leq ke^{-\varepsilon t}||\phi(t)||^2{}_{d_2}
\tag{14}
$$

Therefore, $\eta_1(t)$ is exponentially stable.

Next, defining $e(t)=A_3\eta_1(t)+A_{d3}\eta_1(t-d(t))$, it can be seen from (14) that, if there exists a scalar $m>0$,

$$
||e(t)||^2\leq me^{-\varepsilon t}||\phi(t)||^2{}_{d_2}
\tag{15}
$$

We construct the function $L(t)=\eta_2(t)^T Q_{22}\eta_2(t)-\eta_2(t-d(t))^T Q_{22}\eta_2(t-d(t))$. We can obtain following formula by pre-multiplying (11) by $\eta_2(t)^T {P_4}^T$:

$$
\begin{aligned}
0=&\ \eta_2(t)^T {P_4}^T\eta_2(t)+\eta_2(t)^T {P_4}^T A_{d4}\eta_2(t-d(t))+\eta_2(t)^T {P_4}^T e(t)\\
&+\eta_2(t)^T {P_4}^T[-F_3\psi_1(y(t))-F_4\psi_2(y(t))]
\end{aligned}
\tag{16}
$$

Then, we can obtain

$$
\begin{aligned}
&L(t)=\eta_2(t)^T({P_4}^T+P_4+Q_{22})\eta_2(t)\\
&+2\eta_2(t)^T {P_4}^T A_{d4}\eta_2(t-d(t))+2\eta_2(t)^T {P_4}^T e(t)
\end{aligned}
$$

$$
\begin{aligned}
&+2\eta_2(t)^T P_4{}^T[-F_3\psi_1(y(t)) - F_4\psi_2(y(t))] - \eta_2(t-d(t))^T Q_{22}\eta_2(t-d(t)) \\
\le &\begin{bmatrix} \eta_2(t) \\ \eta_2(t-d(t)) \\ \psi_1(y(t)) \\ \psi_2(y(t)) \end{bmatrix}^T \begin{bmatrix} P_4{}^T + P_4 + Q_{22} & P_4{}^T A_{d4} & -P_4{}^T F_3 & -P_4{}^T F_4 \\ * & -Q_{22} & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix} \\
&\begin{bmatrix} \eta_2(t) \\ \eta_2(t-d(t)) \\ \psi_1(y(t)) \\ \psi_2(y(t)) \end{bmatrix} \\
&+\varepsilon_1\eta_2(t)^T\eta_2(t) + \varepsilon_1{}^{-1}e(t)^T P_4 P_4{}^T e(t) \qquad (17)
\end{aligned}
$$

where ε_1 is an any positive real number.

From (7), we can obtain

$$
\begin{bmatrix} \Xi_{11} & \Xi_{12} \\ * & \Xi_{22} \end{bmatrix} < 0
$$

Pre-multiplying and post-multiplying the above formula, respectively, by

$$
\begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}^T \textit{and} \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}
$$

we can derive

$$
\begin{bmatrix} P_4{}^T + P_4 + Q_{22} & P_4{}^T A_{d4} \\ * & -Q_{22} \end{bmatrix} < 0
$$

Then,

$$
\Omega_1 = \begin{bmatrix} P_4{}^T + P_4 + Q_{22} & P_4{}^T A_{d4} & -P_4{}^T F_3 & -P_4{}^T F_4 \\ * & -Q_{22} & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix} < 0
$$

so there exists a scalar $\varepsilon_2 > 0$ such that

$$
\Omega_1 \le -\begin{bmatrix} \varepsilon_2 I & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix} \qquad (18)
$$

Choosing a sufficiently small ε_1, such that $\varepsilon_2 - \varepsilon_1 > 0$. Then we can obtain an $\varepsilon_3 > 0$ such that

$$
\varepsilon_3 Q_{22} \le Q_{22} - (\varepsilon_1 - \varepsilon_2)I \qquad (19)
$$

It follows from (17), (18) and (19) that

$$L(t) \leq -(\varepsilon_2 - \varepsilon_1)\eta_2(t)^T \eta_2(t) + \varepsilon_1{}^{-1} e(t)^T P_4 P_4{}^T e(t) \tag{20}$$

From (19) and (20), we have that

$$\begin{aligned}
\eta_2(t)^T Q_{22}\eta_2(t) \leq\ & \varepsilon_3{}^{-1}\eta_2(t)^T[Q_{22} - (\varepsilon_1 - \varepsilon_2)I]\eta_2(t) \\
=\ & \varepsilon_3{}^{-1}[\eta_2(t)^T Q_{22}\eta_2(t) - (\varepsilon_1 - \varepsilon_2)\eta_2(t)^T \eta_2(t)] \\
=\ & \varepsilon_3{}^{-1}[\eta_2(t - d(t))^T Q_{22}\eta_2(t - d(t)) \\
& \quad + L(t) - (\varepsilon_1 - \varepsilon_2)\eta_2(t)^T \eta_2(t)] \\
\leq\ & \varepsilon_3{}^{-1}[\eta_2(t - d(t))^T Q_{22}\eta_2(t - d(t)) - (\varepsilon_2 - \varepsilon_1)\eta_2(t)^T \eta_2(t) \\
& + \varepsilon_1{}^{-1} e(t)^T P_4 P_4{}^T e(t) - (\varepsilon_1 - \varepsilon_2)\eta_2(t)^T \eta_2(t)] \\
=\ & \varepsilon_3{}^{-1}\eta_2(t - d(t))^T Q_{22}\eta_2(t - d(t)) + (\varepsilon_1\varepsilon_3)^{-1} e(t)^T P_4 P_4{}^T e(t) \\
\leq\ & \varepsilon_3{}^{-1}\eta_2(t - d(t))^T Q_{22}\eta_2(t - d(t)) \\
& \quad + (\varepsilon_1\varepsilon_3)^{-1} m e^{-\delta t} ||P_4||^2 ||\phi(t)||^2{}_{d_2}
\end{aligned} \tag{21}$$

Because $0 < \delta < \min(\varepsilon, d_2{}^{-1} \ln \varepsilon_3)$ and $0 < \varepsilon_3{}^{-1} < 1$, we can obtain that for $0 < \varepsilon_3{}^{-1} e^{-\delta d_2} < 1, \zeta = (\varepsilon_1\varepsilon_3)^{-1} m ||P_4||^2 ||\phi(t)||^2{}_{d_2} > 0$, where $d_2 > 0$. In addition, Lemma 3 implies that we have

$$\eta_2(t)^T Q_{22}\eta_2(t) \leq e^{-\delta t} \lambda_{\max}(Q_{22}) ||\eta_2(t)||^2{}_{d_2} + \frac{\zeta e^{-\delta t}}{1 - \varepsilon_3{}^{-1} e^{-\delta d_2}}$$

$\lambda_{\min}(Q_{22})||\eta_2(t)||^2 \leq \eta_2(t)^T Q_{22}\eta_2(t)$, so we can obtain

$$||\eta_2(t)||^2 \leq \lambda_{\min}(Q_{22})^{-1} \lambda_{\max}(Q_{22}) e^{-\delta t} ||\eta_2(t)||^2{}_{d_2} + \frac{\lambda_{\min}(Q_{22})^{-1} \zeta e^{-\delta t}}{1 - \varepsilon_3{}^{-1} e^{-\delta d_2}} \tag{22}$$

so $\eta_2(t)$ is exponentially stable.

Because of $||\eta(t)||^2 = ||\eta_1(t)||^2 + ||\eta_2(t)||^2$ and $x(t) = H\eta(t)$, we can obtain from (14), (22) and Definition 1 that the system is exponentially stable for any time-delay function $d(t)$ that satisfies (5).

Next, we study the H_∞ performance of the system, from (9) we can obtain

$$\begin{aligned}
J(t) &= \int_0^t [z^T(s)z(s) - \gamma^2 w^T(s)w(s)]\mathrm{d}s \\
&\leq \int_0^t [z^T(s)z(s) - \gamma^2 w^T(s)w(s) + \mathcal{A}V(x_s, s)]\mathrm{d}s \\
&\leq \int_0^t \begin{bmatrix} \xi(t) \\ w(t) \end{bmatrix}^T \Omega \begin{bmatrix} \xi(t) \\ w(t) \end{bmatrix}
\end{aligned} \tag{23}$$

where

$$\Omega = \begin{bmatrix} \Xi_{11} + \frac{1}{\lambda} P^T F F^T P + L^T L & \Xi_{12} + L^T L_d & E^T R_1 E & 0 & -A^T W F + C^T M^T & L^T L_w + P^T B_w + A^T W B_w \\ * & \Xi_{22} + L_d^T L_d & E^T R_3 E & \Xi_{24} & -A_d W F + C_d^T M^T & L_d^T L_w + A_d^T W B_w \\ * & * & \Xi_{33} & 0 & 0 & 0 \\ * & * & * & \Xi_{44} & 0 & 0 \\ * & * & * & * & -(2-\lambda) I + F^T W F & -F^T W B_w + M D_w \\ * & * & * & * & * & L_w^T L_w + B_w^T W B_w - \gamma^2 I \end{bmatrix}$$

By the Schur complements, Formula (7) implies that $\Omega < 0$, so $J(t) < 0$ for $\forall t > 0$; therefore, $||z(t)||_2 \leq \gamma ||w(t)||_2, \forall w(t) \in \mathcal{L}_2[0, +\infty]$. □

Remark 1 In Theorem 1, a new Lyapunov–Krasovskii functional is constructed, which not only uses the information of the upper limit d_2 of the time-delay; but also uses the information of the lower limit d_1 of the time-delay and the time-delay interval d_{12}. By using the Jensen Integral inequality and the Schur complements, the system is proven to be exponentially stable and satisfies the performance index γ of H_∞.

We design a controller for the Lur'e singular system. By considering the state feedback controller $u(t) = Kx(t)$, we obtain the following closed-loop system:

$$E\dot{x}(t) = (A + B_u K)x(t) + A_d x(t - d(t)) - F\varphi(y(t)) + B_w w(t) \tag{24}$$
$$y(t) = Cx(t) + C_d x(t - d(t)) + D_w w(t) \tag{25}$$
$$z(t) = (L + L_u K)x(t) + L_d x(t - d(t)) + L_w w(t) \tag{26}$$
$$x(t) = \phi(t), t \in [-d_2, 0]$$

Corollary 1 *In the system (24)–(26), for given* $0 \leq d_1 \leq d_2, 0 \leq \alpha < 1$*, if there is a positive number* $\lambda > 0$*, symmetric positive definite matrices* $N_j, \bar{R}_j, j = 1, 2, 3, V_1, V_2$ *and matrices* X, Y *such that*

$$XE^T = EX^T \geq 0 \tag{27}$$

$$\left[\begin{array}{cccccccccccc} \Xi_1 & A_d X^T & EX^T & 0 & XC^T M^T & B_\omega & FX^T & XL^T + YL_u{}^T & d_1XA^T + d_1YB_u{}^T & d_2XA^T + d_2YB_u{}^T & d_{12}XA^T + d_{12}YB_u{}^T \\ * & \Xi_2 & 0 & EX^T & XC_d^T M^T & 0 & 0 & XL_d{}^T & d_1XA_d{}^T & d_2XA_d{}^T & d_{12}XA_d{}^T \\ * & * & V_1 + V_2 - N_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & V_2 - N_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -(2-\lambda)I & MD_\omega & 0 & 0 & -d_1F^T & -d_2F^T & -d_{12}F^T \\ * & * & * & * & * & -\gamma^2 I & 0 & L_\omega{}^T & d_1B_\omega{}^T & d_2B_\omega{}^T & d_{12}B_\omega{}^T \\ * & * & * & * & * & * & -\lambda I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -\bar{R}_1 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -\bar{R}_2 & 0 \\ * & * & * & * & * & * & * & * & * & * & -\bar{R}_3 \end{array}\right] < 0 \tag{28}$$

where $\Xi_1 = XA^T + AX^T + \sum_{k=1}^{3} N_k + YB_u{}^T + B_uY^T - EX^T - XE^T + \bar{R}_1$, $\Xi_2 = -(1-\alpha)N_3 - EX^T - XE^T + V_1 + \bar{R}_2$, *then for any time-delay function* $d(t)$ *that satisfies* (5), *the closed-loop system is exponentially stable and satisfies the performance index* γ *of* H_∞.

Proof Setting $\bar{A} = A + B_uK$, $\bar{L} = L + L_uK$, it can be seen from Theorem 1 that (6) is established and

$$S = \left[\begin{array}{ccccccc} \bar{\Xi}_{11} + \bar{L}^T\bar{L} & \bar{\Xi}_{12} + \bar{L}^T L_d & E^T R_1 E & 0 & -\bar{A}^T WF + C^T M^T & \bar{L}^T L_w + P^T B_w + \bar{A}^T W B_w & P^T F \\ * & \bar{\Xi}_{22} + L_d^T L_d & E^T R_3 E & \Xi_{24} & -A_d{}^T WF + C_d^T M^T & L_d^T L_w + A_d^T W B_w & 0 \\ * & * & \Xi_{33} & 0 & 0 & 0 & 0 \\ * & * & * & -Q_2 - \Xi_{24} & 0 & 0 & 0 \\ * & * & * & * & -(2-\lambda)I + F^T WF & -F^T W B_w + MD_w & 0 \\ * & * & * & * & * & L_w^T L_w + B_w W B_w - \gamma^2 I & 0 \\ * & * & * & * & * & * & -\lambda \end{array}\right] < 0$$

where $\bar{\Xi}_{11} = P^T\bar{A} + \bar{A}^T P + \sum_{k=1}^{3} Q_k + \bar{A}^T W\bar{A} - E^T R_1 E - d_{12}E^T R_2 E$, $\bar{\Xi}_{12} = P^T A_d + \bar{A}^T W A_d + d_{12}E^T R_2 E$. At this time, the closed-loop system (24)–(26) is stable and satisfies the performance index γ of H_∞.

Setting

$$S_1 = \begin{bmatrix} S_{11} & S_{12} & E^T R_1 E & 0 & -\bar{A}^T W F + C^T M^T & \bar{L}^T L_w + P^T B_w + \bar{A}^T W B_w & P^T F \\ * & S_{22} & 0 & d_2 E^T R_2 E & -A_d{}^T W F + C_d^T M^T & L_d^T L_w + A_d^T W B_w & 0 \\ * & * & -Q_1 - E^T R_1 E & 0 & 0 & 0 & 0 \\ * & * & * & -Q_2 - d_2 E^T R_2 E & 0 & 0 & 0 \\ * & * & * & * & -(2-\lambda)I + F^T W F & -F^T W B_w + M D_w & 0 \\ * & * & * & * & * & L_w^T L_w + B_w W B_w - \gamma^2 I & 0 \\ * & * & * & * & * & * & -\lambda \end{bmatrix}$$

$$S_2 = \begin{bmatrix} -d_{12}E^T R_2 E & d_{12}E^T R_2 E & 0 & 0 & 0 & 0 & 0 \\ * & -d_{12}E^T R_2 E - 2E^T R_3 E & E^T R_3 E & E^T R_3 E & 0 & 0 & 0 \\ * & * & -E^T R_3 E & 0 & 0 & 0 & 0 \\ * & * & * & -E^T R_3 E & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & 0 \end{bmatrix}$$

where $S_{11} = P^T\bar{A} + \bar{A}^T P + \sum_{k=1}^{3} Q_k + \bar{A}^T W\bar{A} - E^T R_1 E + \bar{L}^T\bar{L}$, $S_{12} = P^T A_d + \bar{A}^T W A_d + \bar{L}^T L_d$, $S_{22} = -(1-\alpha)Q_3 + A_d^T W A_d - d_2 E^T R_2 E + L_d^T L_d$, it is easy to see that $S = S_1 + S_2$ and $S_2 \leq 0$ are naturally established, so $S_1 < 0$ is the sufficient condition of $S < 0$.

Pre-multiplying S_1 by $H_1 = diag[P^{-T}, P^{-T}, P^{-T}, P^{-T}, I, I, P^{-T}]$ and post-multiplying it by H_1^T, we can obtain $S_3 = H_1 S_1 H_1^T$:

$$S_3 = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & P^{-T}E^T R_1 E P^{-1} & 0 & -P^{-T}\bar{A}^T W F + P^{-T}C^T M^T \\ * & \bar{S}_{22} & 0 & d_2 P^{-T}E^T R_2 E P^{-1} & -P^{-T}A_d{}^T W F + P^{-T}C_d^T M^T \\ * & * & S_{33} & 0 & 0 \\ * & * & * & S_{44} & 0 \\ * & * & * & * & -(2-\lambda)I + F^T W F \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$

$$\begin{matrix} P^{-T}\bar{L}^TL_w + B_w + P^{-T}\bar{A}^TWB_w & FP^{-1} \\ P^{-T}L_d^TL_w + P^{-T}A_d^TWB_w & 0 \\ 0 & 0 \\ 0 & 0 \\ -F^TWB_w + MD_w & 0 \\ L_w^TL_w + B_wWB_w - \gamma^2 I & 0 \\ * & -\lambda \end{matrix}\right]$$

where $\bar{S}_{11} = \bar{A}P^{-1} + P^{-T}\bar{A}^T + \sum_{k=1}^{3} P^{-T}Q_kP^{-1} + P^{-T}\bar{A}^TW\bar{A}P^{-1} - P^{-T}E^TR_1EP^{-1} + P^{-T}\bar{L}^T\bar{L}P^{-1}$, $\bar{S}_{12} = A_dP^{-1} + P^{-T}\bar{A}^TWA_dP^{-1} + P^{-T}\bar{L}^TL_dP^{-1}$, $\bar{S}_{22} = -(1-\alpha)P^{-T}Q_3P^{-1} + P^{-T}A_d^TWA_dP^{-1} - d_2P^{-T}E^TR_2EP^{-1} + P^{-T}L_d^TL_dP^{-1}$, $S_{33} = -P^{-T}Q_1P^{-1} - P^{-T}E^TR_1EP^{-1}$, $S_{44} = -P^{-T}Q_2P^{-1} - d_2P^{-T}E^TR_2EP^{-1}$.

It can be seen that for a matrix with the following format, there are positive definite matrices V_1 and V_2 such that

$$\begin{bmatrix} -P^{-T}E^TR_1EP^{-1} & 0 & P^{-T}E^TR_1EP^{-1} \\ * & 0 & 0 \\ * & * & -P^{-T}E^TR_1EP^{-1} \end{bmatrix} \leq -\begin{bmatrix} P^{-T}E^T & 0 & 0 \\ 0 & 0 & 0 \\ -P^{-T}E^T & 0 & 0 \end{bmatrix} - \begin{bmatrix} EP^{-1} & 0 & -EP^{-1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} {R_1}^{-1} & 0 & 0 \\ 0 & V_1 & 0 \\ 0 & 0 & V_2 \end{bmatrix} \tag{29}$$

$$\begin{bmatrix} -d_2P^{-T}E^TR_2EP^{-1} & 0 & d_2P^{-T}E^TR_2EP^{-1} \\ * & 0 & 0 \\ * & * & -d_2P^{-T}E^TR_2EP^{-1} \end{bmatrix} \leq -\begin{bmatrix} P^{-T}E^T & 0 & 0 \\ 0 & 0 & 0 \\ -P^{-T}E^T & 0 & 0 \end{bmatrix} - \begin{bmatrix} EP^{-1} & 0 & -EP^{-1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} (d_2R_2)^{-1} & 0 & 0 \\ 0 & V_1 & 0 \\ 0 & 0 & V_2 \end{bmatrix} \tag{30}$$

Substituting $W = d_1^2R_1 + d_{12}d_2^2R_2 + d_{12}^2R_3$, Formulas (29) and (30) into matrix S_3, by the Schur complements, we can obtain matrix S_4:

$$S_4 = \left[\begin{matrix} \Theta_1 & A_dP^{-1} & EP^{-1} & 0 & P^{-T}C^TM^T & B_\omega \\ * & \Theta_2 & 0 & EP^{-1} & P^{-T}C_d^TM^T & 0 \\ * & * & V_1 + V_2 - P^{-T}Q_1P^{-1} & 0 & 0 & 0 \\ * & * & * & V_2 - P^{-T}Q_2P^{-1} & 0 & 0 \\ * & * & * & * & -(2-\lambda)I & MD_\omega \\ * & * & * & * & * & -\gamma^2 I \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{matrix}\right.$$

$$\begin{bmatrix} FP^{-1} & P^{-T}\bar{L}^T & d_1P^{-T}\bar{A}^T & d_2P^{-T}\bar{A}^T & d_{12}P^{-T}\bar{A}^T \\ 0 & P^{-T}{L_d}^T & d_1P^{-T}{A_d}^T & d_2P^{-T}{A_d}^T & d_{12}P^{-T}{A_d}^T \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -d_1F^T & -d_2F^T & -d_{12}F^T \\ 0 & {L_\omega}^T & d_1{B_\omega}^T & d_2{B_\omega}^T & d_{12}{B_\omega}^T \\ -\lambda I & 0 & 0 & 0 & 0 \\ * & -I & 0 & 0 & 0 \\ * & * & -{R_1}^{-1} & 0 & 0 \\ * & * & * & -(d_{12}R_2)^{-1} & 0 \\ * & * & * & * & -{R_3}^{-1} \end{bmatrix} < 0 \tag{31}$$

where $\Theta_1 = \bar{A}P^{-1} + P^{-T}\bar{A}^T + \sum_{k=1}^{3} P^{-T}Q_kP^{-1} - P^{-T}E^T - EP^{-1} + {R_1}^{-1}$, $\Theta_2 = -(1-\alpha)P^{-T}Q_3P^{-1} - P^{-T}E^T - EP^{-1} + (d_2R_2)^{-1} + V_1$.

We substitute $\bar{A} = A + B_uK$ and $\bar{L} = L + L_uK$ into matrix S_4, and set $P^{-T} = X$, $P^{-T}K^T = Y$, $P^{-T}Q_kP^{-1} = N_k$, ${R_1}^{-1} = \bar{R}_1$, $(d_{12}R_2)^{-1} = \bar{R}_2$, ${R_3}^{-1} = \bar{R}_3$, so that $(d_2R_2)^{-1} = \frac{d_{12}}{d_2}\bar{R}_2 = \frac{d_2-d_1}{d_2}\bar{R}_2 \le \bar{R}_2$. From Formula (28), we can obtain $S_4 < 0$, so $S_3 < 0$, which implies that $S_1 < 0$. $S_2 \le 0$, and $S = S_1 + S_2 < 0$ is established; therefore, the closed-loop system is exponentially stable. □

Remark 2 In Corollary 1, we propose a more clever approach to deal with matrices with a special format, where a nonlinear matrix is converted into the sum of several linear matrices. For example, in Formula (29), the nonlinear matrix $\begin{bmatrix} -P^{-T}E^TR_1EP^{-1} & 0 & P^{-T}E^TR_1EP^{-1} \\ * & 0 & 0 \\ * & * & -P^{-T}E^TR_1EP^{-1} \end{bmatrix}$ is converted into the sum of three linear matrices through scaling: $-\begin{bmatrix} P^{-T}E^T & 0 & 0 \\ 0 & 0 & 0 \\ -P^{-T}E^T & 0 & 0 \end{bmatrix}$, $-\begin{bmatrix} EP^{-1} & 0 & -EP^{-1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} {R_1}^{-1} & 0 & 0 \\ 0 & V_1 & 0 \\ 0 & 0 & V_2 \end{bmatrix}$. By applying the result of Theorem 1 and repeatedly applying the Schur complements, the closed-loop system is proven to be exponentially stable and to satisfy the performance index γ of H_∞ .

4 Numerical Examples

In this section, several numerical examples are presented to illustrate the effectiveness of the proposed method, especially regarding the responses of $x(t)$,$y(t)$ and $z(t)$. Through a comparison with existing results,the advantages of new method are demonstrated.

Example 1 Consider the linear singular time-delay system of Example 1 (shown in reference [14]), where the coefficient matrices and parameters are

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} -0.9 & 0.2 \\ -0.1 & -0.9 \end{bmatrix}, A_d = \begin{bmatrix} -1.1 & 0.2 \\ -0.1 & -1.1 \end{bmatrix}$$

Table 1 The maximum allowed time delay d_2 for the different values of α

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
[16]	1.14	0.89	0.81	0.79	0.78	0.76	0.75	0.75	0.74
[14]	1.71	1.60	1.49	1.39	1.29	1.20	1.10	1.00	0.90
Theorem 1	2.61	2.32	2.08	1.88	1.70	1.55	1.42	1.30	1.22

Table 2 The maximum allowed time delay d_2 for the different values of d_1

d_1	2.5	3	3.5
[22]	3.0621	3.2234	3.6393
Theorem 1	3.0627	3.2260	3.6407

Table 3 The maximum allowed time delay d_2 for the different values of γ

γ	1	2	3	4	5
[22]	1.8899	2.3430	2.5403	2.6506	2.7209
Theorem 1	1.9697	2.3971	2.5806	2.6826	2.7475

$$F = \begin{bmatrix} -0.2 & 0.1 \\ -0.45 & -0.3 \end{bmatrix}, d_1 = 0$$

By choosing different values for the derivative upper limits α of the delay function, the maximum allowed time delay d_2 of the delay function, which ensures the stability of the system, is obtained. Then, by comparing the obtained results with those of references [16] and [14], we obtain Table 1.

Remark 3 It is clear that the structure of the Lyapunov function in our paper is simpler than that in reference [14]. However, as shown in Table 1, when the derivative α of the delay function takes different values, we derive larger maximum allowed time delays than those in references [16] and [14], which ensures that the system is stable.

Example 2 Consider the linear singular time-delay system of Example 1 (shown in reference [22]), where the coefficient matrices and parameters are

$$E = \begin{bmatrix} 9 & 3 \\ 6 & 2 \end{bmatrix}, A = \begin{bmatrix} -13.1 & -13.7 \\ -15.4 & -23.8 \end{bmatrix}, A_d = \begin{bmatrix} -18.6 & -10.4 \\ -25.2 & -16.8 \end{bmatrix}$$
$$B_w = \begin{bmatrix} 1.9 \\ 1.8 \end{bmatrix}, L = \begin{bmatrix} 0.4 \\ -0.8 \end{bmatrix}^T, \alpha = 0.2$$

(1) Suppose that $w(t) = 0$; by comparing the values of the maximum allowed time delay d_2 for the different values of d_1, we can obtain Table 2.

(2) Suppose that $d_1(t) = 0$; by comparing the values of the maximum allowed time delay d_2 for the different values of γ, we can obtain Table 3.

It can be seen from Tables 2 and 3 that compared with those in reference [22], the values of the maximum allowed time delay d_2 that we obtain are larger, which means that the system is more stable.

Remark 4 In Example 2, we illustrate how we choose the design parameters to affect the control performance. First, in Table 2, through the LMI method, we obtain different values of the maximum allowed time delay d_2 by choosing different values of d_1 while $w(t) = 0$. The values of d_2 are larger when that of d_1 is chosen to be larger. Then, in Table 3, we choose different values of γ when $d_1(t) = 0$ and obtain the corresponding values of the maximum allowed time delay d_2. It is clear that when we choose larger values of γ, we can obtain larger values of d_2.

Example 3 We design the state feedback controller for the system (1)–(3), and choose scalars as follows:

$$
\begin{aligned}
&E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \Phi = \begin{bmatrix} -5 \\ 20 \end{bmatrix}, A = \begin{bmatrix} -32.1 & 23.6 \\ -45 & 100 \end{bmatrix}, A_d = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}, M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&B_u = \begin{bmatrix} -25 & 10 \\ -0.3 & 0 \end{bmatrix}, B_w = \begin{bmatrix} -1 & 2 \\ -0.2 & 0 \end{bmatrix}, F = \begin{bmatrix} -1.2 & -10 \\ 1 & -10 \end{bmatrix}, C = \begin{bmatrix} -24 & 0.1 \\ 0 & -2 \end{bmatrix} \\
&C_d = \begin{bmatrix} -0.5 & -16 \\ 3 & 0 \end{bmatrix}, D_w = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, L = \begin{bmatrix} -2 & -2.5 \\ 0 & -1.4 \end{bmatrix}, L_u = \begin{bmatrix} -0.1 & 0 \\ 0 & -3 \end{bmatrix} \\
&L_d = \begin{bmatrix} -1 & -2 \\ 1 & -3 \end{bmatrix}, L_w = \begin{bmatrix} 0 & 0 \\ 0 & -0.3 \end{bmatrix}, \varphi(y(t)) = \omega(t) = \begin{bmatrix} e^{-t} sin(t) \\ e^{-t} cos(t) \end{bmatrix} \\
&d(t) = 0.1 sin(t) + 0.1, d_1 = 0, d_2 = 0.2, d_{12} = d_2 - d_1, \alpha = 0.2, \gamma = 1, \lambda = 0.1
\end{aligned}
$$

By using Corollary 1 in this paper and the linear matrix inequality (LMI) method, we can obtain the following results

$$
X = \begin{bmatrix} 0.1433 & 0.0154 \\ 0.0143 & -0.0574 \end{bmatrix}, K = \begin{bmatrix} 20.3834 & 5.0959 \\ 5.4887 & -50.9204 \end{bmatrix}
$$

which means the state feedback controller we design is

$$
u(t) = \begin{bmatrix} 20.3834 & 5.0959 \\ 5.4887 & -50.9204 \end{bmatrix} x(t)
$$

so that the closed-loop system (24)–(26) is exponentially stable and satisfies the performance index γ of H_∞.

By applying simulink software, we can obtain the response figures of the state $x(t)$, control input $y(t)$ and control output $z(t)$, as shown in Figs. 1, 2 and 3.

In the figures, we can see that, at first, the values of the state $x(t)$, control input $y(t)$ and control output $z(t)$ in the chosen numerical system are not equal to zero. However they rapidly approach zero with the state feedback controller, and after two seconds, they steadily approach zero, which implies that the numerical system is exponentially stable and satisfies the performance index γ of H_∞. Therefore, this numerical example illustrates that our method is effective.

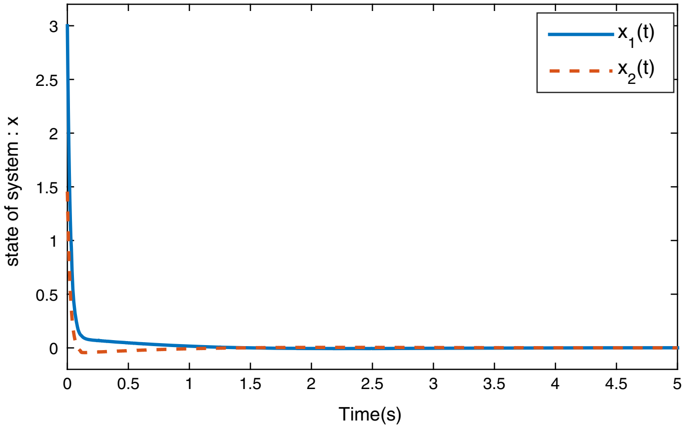

Fig. 1 State responses of $x(t)$ in Example 3

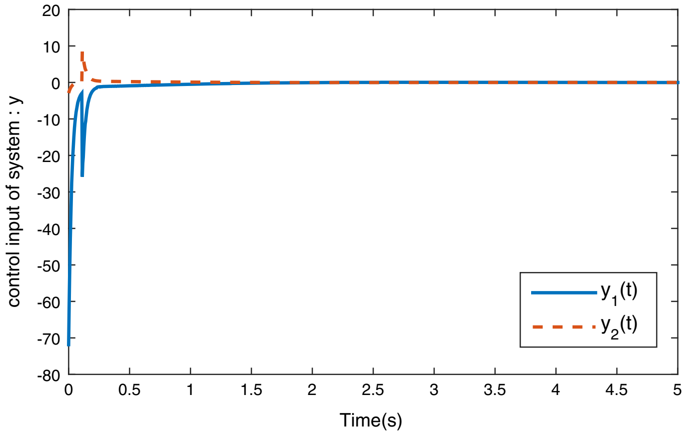

Fig. 2 Input responses of $y(t)$ in Example 3

5 Conclusion

This paper mainly studies the H_∞ control problem for Lur'e singular time-delay systems. By using Lyapunov stability theory, a new Lyapunov function is constructed. Compared with that developed in a previous paper, our function has one more double integral term $d_1 \int_{-d_1}^{0} \int_{t+\beta}^{t} \dot{x}^T(\alpha) E^T R_1 E \dot{x}(\alpha) \mathrm{d}\alpha \mathrm{d}\beta$, which makes our obtained result more conservative. Based on the linear matrix inequality (LMI) method, sufficient conditions for the designed system to be exponentially stable and to satisfy the performance index of H_∞ are obtained. During this process, the main difficulty is confirming the exponential stability of the system. Subsequently, a design method for the state feedback controller of the system is given, and by applying a more clever approach for a nonlinear matrix with a special format to convert it into the sum of several linear matrices, the closed-loop system is also exponentially stable and satisfies the

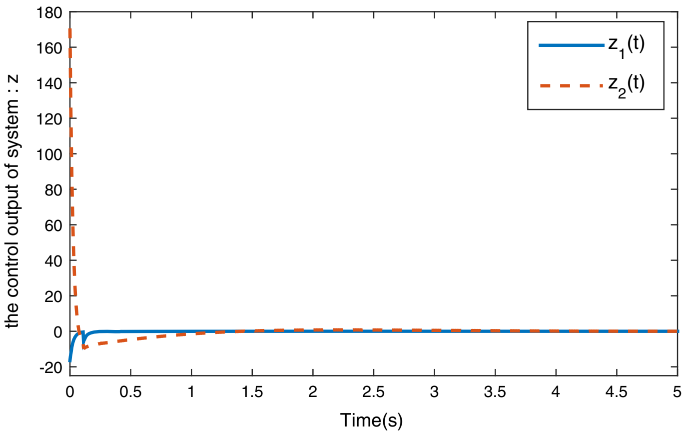

Fig. 3 Output responses of $z(t)$ in Example 3

H_∞ performance index. Finally, numerical examples illustrate the effectiveness of the proposed method and its advantages over the existing results and the response figures clearly reflect the stability of the system.

In the future, we will consider more normal systems, such as Lur'e singular systems with uncertainties or Markov process, and we will study the problems of stability, H_∞ control and finite-time H_∞ control for these systems.

Data Availability The data supporting the conclusions of this article is included within the article.

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