Closed-Form Estimator for Frequency Estimation of Complex Sinusoidal Signals in Multiplicative and Additive Noise

Reza Ashkiani[1] · Alimorad Mahmoudi[1] · Mahmood Karimi[2] · Karim Ansari-Asl[1]

Abstract

In this paper, a new method is proposed for estimating the frequency of a complex sinusoidal signal in multiplicative and additive noise. The proposed method is based on the linear prediction property of the signals' autocorrelation function. This frequency estimation method is a closed-form algorithm which can directly estimate the frequency without peak searching. The performance of the proposed frequency estimator is investigated by computer simulations. Simulation results demonstrate that the performance of the proposed technique is reasonable and is close to Cramer–Rao bound.

Keywords Multiplicative noise · Linear prediction · Generalized eigenvalue problem · Frequency estimation · Yule–Walker equations · Autoregressive processes

1 Introduction

Frequency estimation in multiplicative and additive noise is an important step in many applications in signal processing and communications. For example, frequency estimation is vital in many systems such as sonars and radars. In radars, for

✉ Alimorad Mahmoudi
a.mahmoudi@scu.ac.ir

Reza Ashkiani
r-ashkiani@stu.scu.ac.ir

Mahmood Karimi
karimi@shirazu.ac.ir

Karim Ansari-Asl
karim.ansari@scu.ac.ir

1 Electrical Engineering Department, Faculty of Engineering, Shahid Chamran University of Ahvaz, Ahvaz, Iran

2 Department of Communications and Electronics, School of Electrical and Computer Engineering, Shiraz University, Shiraz, Iran

instance, the Doppler frequency should be estimated which depends on the velocity of the target. Similarly, in array processing, the spatial frequency should be estimated that depends on the direction of arrival (DOA) of the source [11, 14, 15]. In addition, in the field of speech processing, frequency estimation can be used to estimate maximum voiced frequency (MVF) which can separate the periodic and aperiodic components of sounds [8]. To observe time variations in the vocal tract system characteristics, frequency estimation can be applied to the tracking of speech formant frequencies [21]. Modal analysis is discussed in [19] as another application of frequency estimation.

On the other hand, multiplicative noise is used for modeling some important processes in signal processing applications. In wireless communications, the fading process is modeled as multiplicative noise [1, 2, 12, 10, 3, 27]. This fading is due to the local scatterers around the mobile unit [22]. In radars, the constant amplitude model is insufficient for pulse-to-pulse fluctuating targets; therefore, the random amplitude is typically modeled by multiplicative noise. In addition, the multiplicative noise is suitable for modeling the jammers in radar systems [25, 18]. Furthermore, investigation of the multiplicative noise is necessary in backscattered acoustic signals [1] in array processing [26], synthetic aperture radar (SAR) imagery systems [28], and some recent applications such as vehicle speed estimation and pseudo-maximum likelihood estimation of ballistic missiles [5, 16].

The problem of frequency estimation in multiplicative noise has been investigated in [20, 23, 29, 9, 4]. The estimation algorithm used in [23] is based on maximizing the likelihood function and estimates the frequency by an iterative method. This algorithm does not give a closed-form estimate and cannot estimate the frequency directly. The algorithm in [23] estimates the frequency by the estimation of signal parameters via rotation invariance technique (ESPRIT) method. Another method is presented in [24], which is based on a subspace algorithm and finds the frequency by multidimensional search. The nonlinear least squares (NLS) method [4, 20] is based on peak searching and cannot estimate the frequency directly. Another method used for frequency estimation is the multiple signal classification (MUSIC) method [9].

In this paper, we propose a closed-form algorithm for frequency estimation of complex sinusoidal signals in multiplicative and additive noise. The proposed method is based on the linear prediction property of the autocorrelation function (ACF) of the signal. The predictor is order one, and its coefficient can be obtained by over determined linear equations. These equations can be considered as a generalized eigenvalue problem. By solving the generalized problem using the method in [7], the predictor coefficient is estimated. Finally, we can find the frequency using the angle of predictor coefficient. Five cases are considered for the multiplicative noise. In the first case, the multiplicative noise is assumed to be white having a nonzero mean. In the second case, the multiplicative noise is assumed to be white zero mean. In the third case, the noise is considered as a colored noise with exponential ACF. In the fourth case, we extend the proposed method to multi-component signal frequency estimation. In the last case, we apply the proposed method to a real application of radial velocity estimation for fluctuating targets.

The main contribution of this paper is frequency estimation using linear prediction instead of autoregressive (AR) modeling. It is well known that the estimates obtained by AR modeling may be biased. The estimates obtained by linear prediction approach can be unbiased, as the quasi-Yule–Walker equations proposed by the authors start from zero lag instead of one lag in the Yule–Walker equations.

The rest of the paper is organized as follows: The signal model is given in Sect. 2. In Sect. 3, the methodology is presented. Simulation results and comparisons are provided in Sect. 4. Conclusions are presented in Sect. 5.

2 Signal Model

The complex sinusoidal signal in multiplicative and additive noise is modeled as:

$$x(n) = z(n)e^{j(\omega_0 n + \theta)} + v(n) \tag{1}$$

where ω_0 is the normalized angular frequency to be estimated ($\omega_0 \in [-\pi, \pi)$) and θ is the phase of the signal. The observation noise $v(n)$ is additive complex white Gaussian noise with zero mean. The multiplicative noise $z(n)$ is a random process with mean μ:

$$z(n) = \mu + w(n) \tag{2}$$

where $w(n)$ is white noise with zero mean. It is also assumed that $w(n)$ and $v(n)$ are mutually independent with variances σ_v^2 and σ_w^2, respectively. It is worth noting that parameters $\omega_0, \mu, \theta, \sigma_v^2$ and σ_w^2 are unknown. The main goal of this paper is to estimate ω_0 from the observations $x(n); n = 0, 1, \ldots, N - 1$. All of the random processes are under Wide-Sense Stationary (WSS) hypothesis.

3 Proposed Method

The proposed method is based on the linear prediction property of the autocorrelation sequence of observations. Five cases for multiplicative noise are considered. In each case, the autocorrelation sequence of observations is computed by an unbiased relation and is then used to derive the estimator.

If the signal is defined as:

$$s(n) = z(n)e^{j(\omega_0 n + \theta)} \tag{3}$$

Then the ACF of the signal is:

$$\begin{aligned}
r_s(k) &= E\{s(n)s^*(n-k)\} \\
&= E\left\{ z(n)e^{j(\omega_0 n + \theta)} z^*(n-k)\left(e^{-j(\omega_0(n-k)+\theta)} \right) \right\} \\
&= e^{j\omega_0 k} E\{z(n)z^*(n-k)\} \\
&= e^{j\omega_0 k}\left(\mu^2 + \sigma_w^2 \delta(k) \right)
\end{aligned} \tag{4}$$

$$\rightarrow r_s(k) = \mu^2 e^{j\omega_0 k} + \sigma_w^2 \delta(k) \tag{5}$$

The signal in the presence of both multiplicative and additive noises is as follows:

$$x(n) = s(n) + v(n) \tag{6}$$

Finally, the ACF of the observed signal is:

$$\begin{aligned}
r_x(k) &= E\{x(n)x^*(n-k)\} \\
&= E\{(s(n)+v(n))(s^*(n-k)+v^*(n-k))\} \\
&= E\{s(n)s^*(n-k)\} + E\{s(n)v^*(n-k)\} + E\{v(n)s^*(n-k)\} + E\{v(n)v^*(n-k)\} \\
r_x(k) &= r_s(k) + r_v(k)
\end{aligned} \tag{7}$$

where

$$r_v(k) = \sigma_v^2 \delta(k) \tag{8}$$

It follows that:

$$r_x(k) = \mu^2 e^{j\omega_0 k} + \left(\sigma_w^2 + \sigma_v^2\right)\delta(k) \tag{9}$$

Using the prediction property of $r_{ps}(k) = \mu^2 e^{j\omega_0 k}$, we have:

$$r_{ps}(k) - a r_{ps}(k-1) = 0 \tag{10}$$

where

$$a = e^{j\omega_0} \tag{11}$$

Therefore, we can see that the frequency can be estimated by calculating the phase of the perdition coefficient a.

Case 1 The signal with additive and multiplicative white noise and $\mu \neq 0$

It follows from (9) and (10) that:

$$r_x(k) - a r_x(k-1) = \left(\sigma_w^2 + \sigma_v^2\right)\delta(k) - a\left(\sigma_w^2 + \sigma_v^2\right)\delta(k-1) \tag{12}$$

Evaluating (12) for $k=0, \dots q$ and arranging equations in matrix form give

$$(\boldsymbol{R} - \lambda \boldsymbol{G})\mathbf{v} = 0 \tag{13}$$

where

$$\boldsymbol{R} = \left(\begin{bmatrix} r_x(0) & r_x(-1) \\ r_x(1) & r_x(0) \\ r_x(2) & r_x(1) \\ \vdots & \vdots \\ r_x(q) & r_x(q-1) \end{bmatrix}\right), \quad \boldsymbol{G} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}\right), \quad \mathbf{v} = \begin{bmatrix} 1 \\ -a \end{bmatrix} \text{ and } \lambda = \left(\sigma_w^2 + \sigma_v^2\right) \tag{14}$$

The $\mathbf{R}$ and $\mathbf{G}$ are $(q + 1) \times 2$ matrices, and $\mathbf{v}$ is $a(2 \times 1)$ vector.

Using generalized eigenvalue problem in [6], we can find $\hat{a}$, and then, $\hat{\omega}_0 = $ phase $(\hat{a})$. The steps for solving the generalized eigenvalue problem are as follows [7]:

1. Construct the following matrices

$$\mathbf{A}_o = \mathbf{R}^H . \mathbf{R}, \mathbf{A}_1 = \mathbf{R}^H \mathbf{G} + \mathbf{G}^H \mathbf{R}, \mathbf{A}_2 = \mathbf{G}^H \mathbf{G} \tag{15}$$

2. Using the matrices in step 1 form:

$$\mathbf{P} = \begin{bmatrix} \mathbf{A}_o & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{bmatrix}, \mathbf{Q} = \begin{bmatrix} \mathbf{A}_1 & -\mathbf{A}_2 \\ \mathbf{I} & \mathbf{O} \end{bmatrix} \tag{16}$$

where O and I are the zeros matrix and the identity matrix, respectively.
3. By solving the generalized eigenvalue problem as $\mathbf{Px} = \lambda\mathbf{Qx}$, the eigenvalues and eigenvectors can be obtained.
4. λ and $\mathbf{x}$ choose to be the minimum eigenvalue and its associated eigenvector, respectively, of this generalized eigenvalue problem. This choice is supported by the simulation results given in Sect. 4.
5. We select two first elements of $\mathbf{x}$ to obtain $\mathbf{v}$. The second element of $\mathbf{v}$ gives us an estimate of $-a$.

Case 2 The signal with additive and multiplicative white noise and $\mu = 0$

In this case, the term related to the signal in autocorrelation is removed. Therefore, we cannot directly use linear prediction.

In the case where $\mu = 0$, the observed signal is:

$$\mu = 0 \rightarrow x(n) = w(n)e^{j(\omega_0 n + \theta)} + v(n) \tag{17}$$

By squaring the observed signal, we have:

$$y(n) = x^2(n) = w^2(n)e^{j2(\omega_0 n + \theta)} + v^2(n) + 2v(n)w(n)e^{j(\omega_0 n + \theta)} \tag{18}$$

The expectation of $y(n)$ is equal to:

$$E\{y(n)\} = \sigma_w^2 e^{j2(\omega_0 n + \theta)} + \sigma_v^2 \tag{19}$$

An approximation of (19) is given by:

$$y(n) \approx \sigma_w^2 e^{j2(\omega_0 n + \theta)} + e(n) \tag{20}$$

The randomness $e(n)$ is due to omitting of the expectation in (20). We can roughly approximate $e(n)$ as a white noise with variance σ_e^2. It follows that:

$$y(n) - ay(n - 1) = e(n) - ae(n - 1) \tag{21}$$

$$r_y(k) - ar_y(k - 1) = \sigma_e^2 \delta(k) - a\sigma_e^2 \delta(k - 1) \tag{22}$$

where the a parameter is $e^{j2\omega_o}$.

Similar to Case 1, the equation can be considered as a quadratic eigenvalue problem. By solving the quadratic eigenvalue problem [7], we can find $\hat{a}$ so $\hat{\omega}_0 = (0.5) * \text{phase}(\hat{a})$. In this case, the estimated frequency must be multiplied by 0.5, so the range of frequencies must be between $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Case 3.1 The signal with additive white and multiplicative colored AR(1) noise

At first, we assume that the multiplicative colored noise is generated by [17]:

$$z(n) = bz(n-1) + e(n) \tag{23}$$

where b is supposed to be a positive real value and $e(n)$ is a white Gaussian noise with zero mean and variance σ_e^2. It is worth noting that in the case of $b=0$, $z(n)=e(n)$, so the colored noise is converted to zero mean white noise (see Case 2).

The ACF of the colored multiplicative noise is:

$$r_z(k) = \frac{\sigma_e^2}{1-b^2} b^{|k|} \tag{24}$$

Here, the ACF of the observed data is:

$$r_x(k) = r_z(k)e^{j\omega_0 k} + \sigma_v^2 \delta(k) \tag{25}$$

For $k \geq 0$, we can find out:

$$r_x(k) - ar_x(k-1) = \sigma_v^2 \delta(k) - a\sigma_v^2 \delta(k-1) \tag{26}$$

where

$$a = be^{j\omega_0} \tag{27}$$

Assuming b is a positive number, we can find $\hat{a}$ and $\hat{\omega}_0 = \text{phase}(\hat{a})$.

Case 4 Multi-component signal spectral estimation

If we suppose that two fluctuating targets exist, the observation is modeled as:

$$x(n) = z(n)e^{j\omega_1 n} + \beta(n)e^{j\omega_2 n} + v(n) \tag{28}$$

where $z(n)$ is a multiplicative white Gaussian noise with mean μ_1 and variance σ_z^2, $\beta(n)$ is a multiplicative white Gaussian noise with mean μ_2 and variance σ_β^2 and $v(n)$ is an additive white Gaussian noise with zero mean and variance σ_v^2. It is assumed that $z(n)$ and $\beta(n)$ are mutually uncorrelated.

We consider the signal in the absence of the observation noise as the following:

$$s(n) = z(n)e^{j\omega_1 n} + \beta(n)e^{j\omega_2 n} \tag{29}$$

The ACF of the signal can be computed as:

$$r_s(k) = \mu_1^2 e^{j\omega_1 k} + \mu_2^2 e^{j\omega_2 k} + \left(\sigma_z^2 + \sigma_\beta^2\right)\delta(k) \tag{30}$$

It can be considered as follows:

$$r_x(k) = r_{ps1}(k) + r_{ps2}(k) + \left(\sigma_w^2 + \sigma_v^2\right)\delta(k) \tag{31}$$

where $r_{ps1}(k) = \mu_1^2 e^{j\omega_1 k}$ and $r_{ps2}(k) = \mu_2^2 e^{j\omega_2 k}$.

Using the prediction property of $r_{ps1}(k) = \mu_1^2 e^{j\omega_1 k}$, we have:

$$r_{ps1}(k) - a r_{ps1}(k - 1) = 0 \tag{32}$$

where

$$a = e^{j\omega_1} \tag{33}$$

Similar to (32), we use linear prediction for $r_{ps2}(k) = \mu_2^2 e^{j\omega_2 k}$:

$$r_{ps2}(k) - b r_{ps2}(k - 1) = 0 \tag{34}$$

where

$$b = e^{j\omega_2} \tag{35}$$

Taking Z transform on Eqs. (32) and (34), we obtain:

$$\left(1 - az^{-1}\right) r_{ps1}(z) = 0 \tag{36}$$

$$\left(1 - bz^{-1}\right) r_{ps2}(z) = 0 \tag{37}$$

Multiplying (36) by $\left(1 - bz^{-1}\right)$ and (37) by $\left(\left(1 - az^{-1}\right)\right)$, we get:

$$\left(1 - bz^{-1}\right)\left(1 - az^{-1}\right) r_{ps1}(z) = 0 \tag{38}$$

$$\left(1 - az^{-1}\right)\left(1 - bz^{-1}\right) r_{ps2}(z) = 0 \tag{39}$$

Adding Eqs. (38) and (39) yields:

$$\left(1 - az^{-1}\right)\left(1 - bz^{-1}\right)(r_{ps1}(z) + r_{ps2}(z)) = 0 \tag{40}$$

It results

$$\left(1 - az^{-1}\right)\left(1 - bz^{-1}\right)(r_{ps}(z)) = 0 \tag{41}$$

By taking inverse Z transform on the above equation, we have:

$$r_{ps}(k) - a r_{ps}(k - 1) - b r_{ps}(k - 1) + ab r_{ps}(k - 2) = 0 \tag{42}$$

Substituting Eq. (31) into (42), we can obtain:

$$r_x(k) - a r_x(k - 1) - b r_x(k - 1) + ab r_x(k - 2) = h(\delta(k) - a\delta(k - 1) - b\delta(k - 1) + ab\delta(k - 2)) \tag{43}$$

where $h = \sigma_z^2 + \sigma_\beta^2 + \sigma_v^2$

Similar to previous cases, Eq. (43) can be arranged in matrix form $((R - \lambda G)\mathbf{v} = 0)$ and solved for a, b and h by constructing a generalized eigenvalue problem as follows:

where

$$R = \left(\begin{bmatrix} r_x(0) & r_x(-1) & r_x(-2) \\ r_x(1) & r_x(0) & r_x(-1) \\ \vdots & \vdots & \vdots \\ r_x(q) & r_x(q-1) & r_x(q-2) \end{bmatrix} \right), \quad G = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \right), \quad \mathbf{v} = \begin{bmatrix} 1 \\ -a-b \\ ab \end{bmatrix} \tag{44}$$

and $\lambda = h = \sigma_z^2 + \sigma_\beta^2 + \sigma_v^2$

The R and G are $(q+1) \times 3$ matrices, and $\mathbf{v}$ is (3×1) vector. In comparison with the previous cases, in this case the number of columns of matrices is increased by one.

Using the estimates, the spectrum of observations can be obtained as follows:

$$\hat{\varphi}(\omega) = \frac{1}{\left| \left(1 - ae^{-j\omega}\right)\left(1 - be^{-j\omega}\right) \right|^2} \tag{45}$$

Case 5 Real application example: Doppler frequency estimation for fluctuating moving targets in radar systems

The linear frequency modulated (LFM) signals are common in radar systems. One of their important applications is to estimate the range and radial velocity of the targets. This goal is achieved by transmitting the known LFM signal as follows [13]:

$$x_T(n) = e^{j2\pi f_o n} e^{j\pi \alpha n^2} \tag{46}$$

where f_o is the carrier frequency and α denotes the frequency rate.

In the case of a fluctuating target, the received signal in the absence of the additive noise is given by:

$$x_R(n) = u(n)e^{j2\pi f_o(n-n_d)}e^{j\pi \alpha(n-n_d)^2} \tag{47}$$

where $u(n)$ is considered as a complex white Gaussian noise due to target fluctuation. Time delay n_d for the target is obtained by:

$$n_d = \frac{2R}{c} = \frac{2}{c}(R_o - v_o n) \tag{48}$$

where R, R_o and v_o are the range, the initial range and the velocity of the target, respectively.

The goal is to estimate the velocity of the moving target in the presence of target fluctuation. Multiplying the conjugate of the received signal by the transmitted signal, we will have:

$$x_b(n) = x_T(n)x_R^*(n) = u^*(n)e^{j2\pi f_o n_d}e^{j2\pi n\alpha n_d}e^{-j\pi\alpha n_d^2} \tag{49}$$

where $(.)^*$ stands for complex conjugate.

With a little approximation, we can neglect the term $e^{-j\pi\alpha n_d^2}$, and by substituting (48) into (49), we will have:

$$x_b(n) = u^*(n)e^{\frac{j4\pi f_o R_o}{c}}e^{\frac{j4\pi}{c}(\alpha R_o - f_o v_o)n}e^{\frac{-j4\pi\alpha v_o n^2}{c}} \tag{50}$$

Now, we have:

$$c(n) = x_b(n-1)x_b^*(n) = z(n)e^{\frac{j8\pi\alpha v_o n}{c}} \tag{51}$$

where

$$z(n) = u^*(n)u(n-1)e^{\frac{-j4\pi\alpha R_o}{c}}e^{\frac{-j4\pi\alpha v_o}{c}}e^{\frac{j4\pi f_o v_o}{c}} \tag{52}$$

We can suppose that $z(n)$ is a complex white noise. Moreover, in (51), the term $\frac{8\pi\alpha v_o}{c}$ is equal to ω_o which can be estimated by the proposed method.

4 Simulations and Comparisons

Several computer simulations have been conducted to illustrate the performance of the proposed method and compare it with the method in [20], the ESPRIT method, the MUSIC method and the CRB.

Under the assumptions that the multiplicative noise $z(n)$ in (3) is a Gaussian stationary random process and $v(n)$ is an additive Gaussian white noise, for Cases 1–2, the Cramer-Rao bound (CRB) for the parameter ω_0 can be approximated by [23]:

$$\text{CRB}\,(\omega_0) = \frac{6}{N(N^2 - 1)\,\text{SNR}} \tag{53}$$

$$\text{SNR} = \frac{\mu^2}{\sigma_v^2}(1 + m_A^2) = \frac{\mu^2 + \sigma_w^2}{\sigma_v^2} \tag{54}$$

$$m_A = \frac{\sigma_w}{\mu} \tag{55}$$

For Case 3, the SNR is defined as:

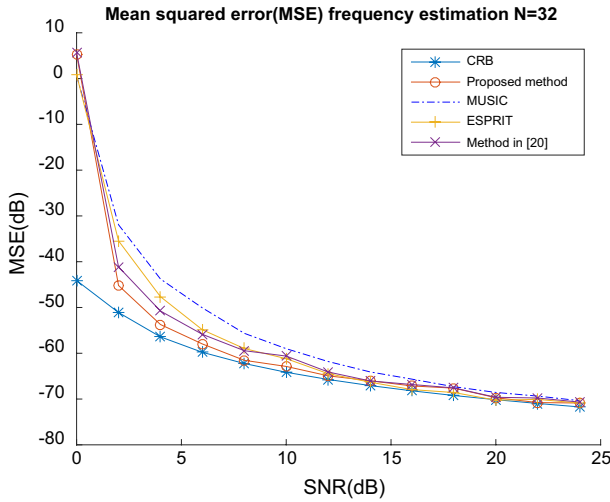$$\text{SNR} = \frac{\sigma_e^2}{(1 - b^2)\sigma_v^2} \tag{56}$$

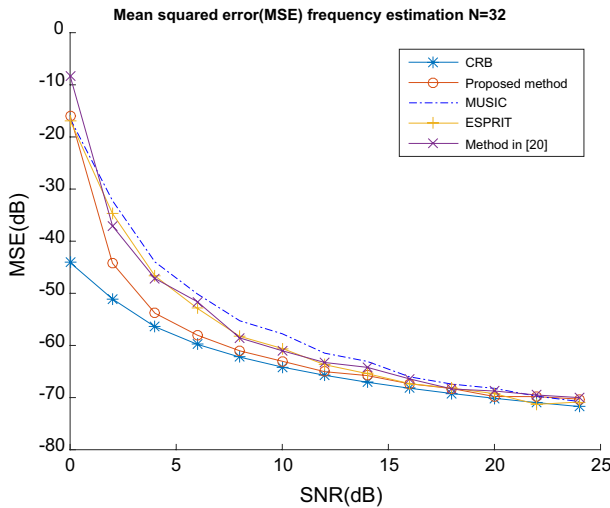**Fig. 1** MSE versus SNR. $N=32$, $\mu=2$, $\omega_0 = 0.3$ radian/sample

Fig. 2 MSE versus SNR. $N = 32$, $\mu = 0$, $\omega_0 = 0.3$ radian/sample

All processes are generated with Gaussian distribution. The number of simulation runs is 500. Five cases are considered, and the results are shown in Figs. 1, 2, 3, 4, 5 and 6, respectively. In Figs. 1, 2, 3 and 4, the number of data samples is 32. In Fig. 5, the number of data samples is 100, in Fig. 6, this number is 32, and in the last figure this number is different. The mean squared errors (MSEs) of the frequency estimation for different SNRs are presented in Figs. 1, 2 and 3. In Fig. 1, the multiplicative noise is a white Gaussian noise with $\mu=2$. In Fig. 2,
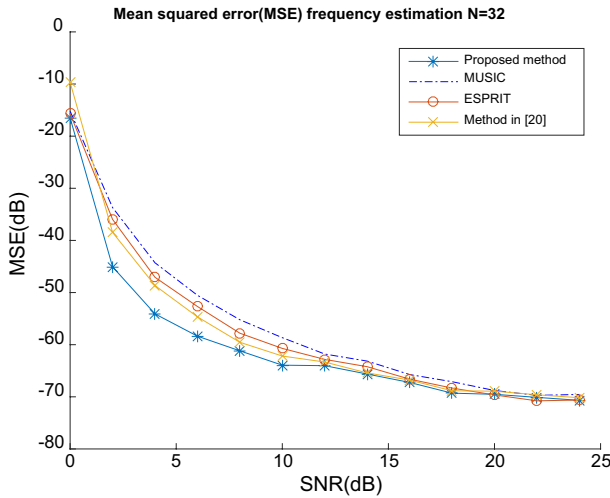
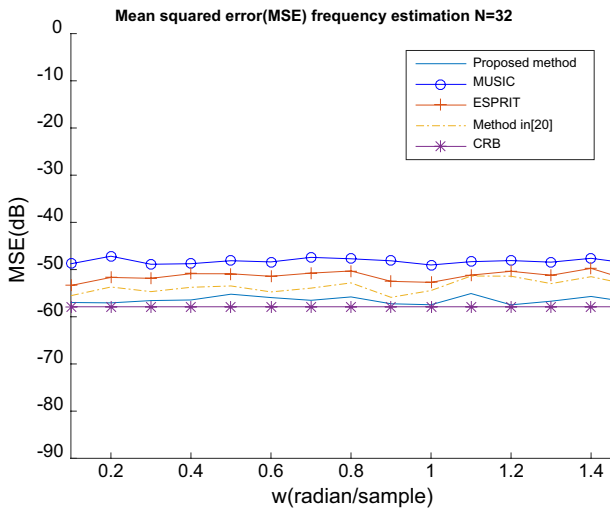**Fig. 3** MSE versus SNR. $N=32$, $\mu=0$, $b=0.8$, $\omega_0 = 0.3$ radian/sample

Fig. 4 MSE versus frequency. $N=32$, $\mu=2$, $b=0.8$

the multiplicative noise is a white Gaussian noise with $\mu=0$. In Fig. 3, the multiplicative noise is a colored noise generated by (23), in which the parameters are set to $\sigma_e^2 = 1$ and b$=0.8$. In Figs. 1, 2 and 3, the SNR varies from 0 to 25 dB with step size 2 dB. In the proposed method, the parameter q is set to 8.

It can be seen from Figs. 1 and 2 that the performance of the proposed method is close to CRB (except for low SNRs) and better than the algorithm presented in [20], the ESPRIT method and the MUSIC method.
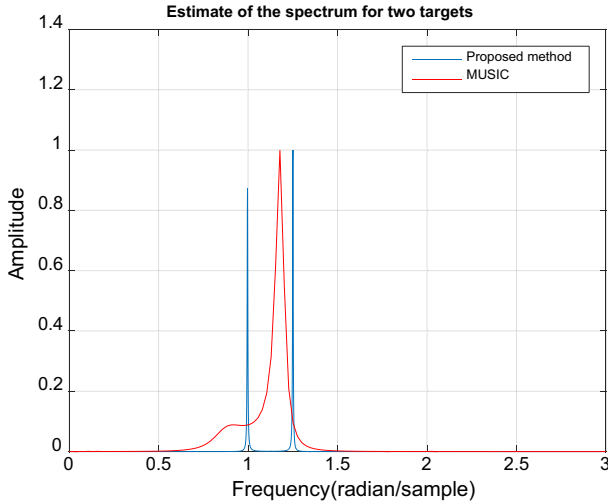
Fig. 5 Spectrum versus frequency. $N = 100$, $\omega_1 = 1$ (radian/sample) and $\omega_2 = 1.25$ (radian/sample)
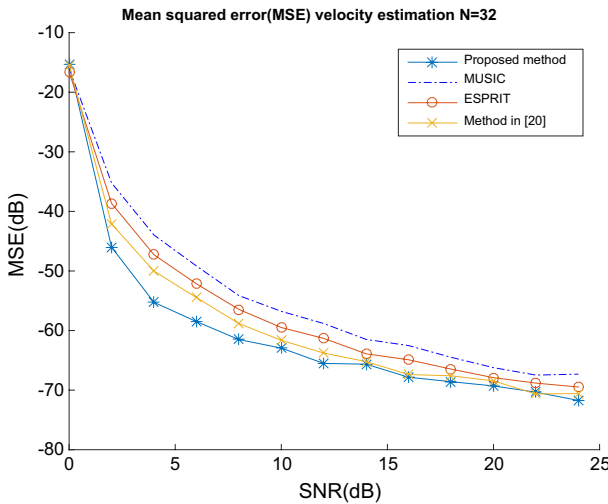

Fig. 6 MSE versus velocity. $N = 32$, $R_o = 3$ km, $v_o = 500$ m/s, $\alpha = 5$

According to Fig. 3, in the case of colored multiplicative noise, the proposed method can still estimate the frequency better than the other algorithms.

To show the uniformity of performance in the proposed method, the frequency is changed from 0 to 1.4 with step size 0.1. Other simulation parameters are the same as Fig. 1. The results are shown in Fig. 4. It can be seen from this figure that the performance of the proposed method is uniform in this frequency range.

To show the advantage of the proposed method regarding resolution performance in the multi-target case, we consider two signals at frequencies
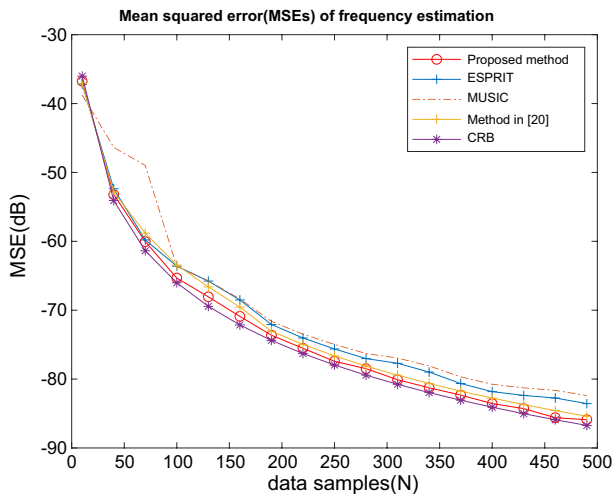
Fig. 7 MSE versus data samples (N). SNR $= 7$ dB, $\mu = 2$, $\omega_0 = 0.3$ radian/sample

$\omega_1 = 1$(radian/sample) and $\omega_2 = 1.25$(radian/sample). The number of data samples is 100, SNR $=5$ dB and $q=80$. The multiplicative noise is generated by a complex white Gaussian process with $\mu=2$. In this case, we compare the performance of the proposed method with the MUSIC method. The result is shown in Fig. 5. From Fig. 5, it is understood that the resolution of the proposed method is better than that of the MUSIC method. In other words, the proposed method can precisely separate the two targets while the MUSIC method fails to separate the targets.

Figure 6 shows the performance of the proposed method and the other methods in velocity estimation in Case 5. It can be seen from Fig. 6 that the performance of the proposed method in velocity estimation is better than the other methods in terms of the mean squared error. Moreover, it is worth emphasizing that the proposed method can estimate the velocity in this case without knowing parameters R_o, f_o and α.

Figure 7 shows the performance of the proposed method and the other methods in frequency estimation in Case 1 versus data samples. It can be seen from Fig. 7 that the performance of the proposed method in frequency estimation is better than the other methods in terms of the mean squared error in several data samples.

5 Conclusion

In this paper, the frequency estimation of complex sinusoidal signals in multiplicative and additive noise is addressed. The proposed method is based on the linear prediction property of autocorrelation function. The problem of frequency estimation is converted into a noisy autoregressive parameter estimation one. Five cases of signal and noise are considered in autocorrelation domain. For each case, the resulting

equations are solved by generalized eigenvalue problem. This frequency estimator can directly estimate the frequency without peak searching.

Simulation results have confirmed that the performance of the proposed method is better than the other existing methods. Monte Carlo simulation results confirm the effective performance of the proposed method. Furthermore, it has been shown that the performance of the proposed method is close to CRB at high SNRs. It is worth noting that simulation results match with the mathematical model.

References

1. S.S. Abeysekera, Performance of pulse-pair method of Doppler estimation. IEEE Trans. Aerosp. Electron. Syst. **34**(2), 520–531 (1998)
2. J.A. Alzubi, O.A. Alzubi, T.M. Chen, *Forward Error Correction Based On Algebraic-Geometric Theory* (Springer, Berlin, 2014). **ISBN-978-3319082929**
3. O.A. Alzubi, T.M. Chen, J.A. Alzubi, H. Rashaideh, N. Al-Najdawi, Secure channel coding schemes based on algebraic-geometric codes over hermitian curves. J. Univer. Comput. Sci. **22**(4), 552–566 (2016)
4. O. Besson, P. Stoica, Nonlinear least-squares approach to frequency estimation and detection for sinusoidal signals with arbitrary envelope. Digit. Signal Proc. **9**(1), 45–56 (1999)
5. V. Cevher, R. Chellappa, J.H. McClellan, Vehicle speed estimation using acoustic wave patterns. IEEE Trans. Signal Process. **57**(1), 30–47 (2009)
6. F. Chatelin, *Eigenvalues of Matrices* (Wiley, New York, 1993)
7. C.E. Davila, A subspace approach to estimation of autoregressive parameters from noisy measurements. IEEE Trans. Signal Process. **46**(2), 531–534 (1998)
8. T. Drugman, Y. Stylianou, Maximum voiced frequency estimation: exploiting amplitude and phase spectra. IEEE Signal Process. Lett. **21**(10), 1230–1234 (2014)
9. F.-X. Ge, Q. Wan, L. Guo, B. Sun, Frequency estimation of sinusoidal signals in multiplicative and additive noise. IEEE J. Oceanic Eng. **41**(4), 810–819 (2016)
10. M. Ghogho, A.K. Nandi, A. Swami, Cramer-Rao bounds and maximum likelihood estimation for random amplitude phase-modulated signals. IEEE Trans. Signal Process. **47**(11), 2905–2916 (1999)
11. R. Gui, W.-Q. Wang, Y. Pan, J. Xu, Cognitive target tracking via angle-range-Doppler estimation with transmit subaperturing FDA radar. IEEE J. Sel. Top. Signal Process. **12**(1), 76–89 (2018)
12. M. Ghogho, A. Swami. Cramer-Rao bounds and estimators for the parameters of time selective fading channels. in *Signal Processing Advances in Wireless Communications, 1999. SPAWC'99. 1999 2nd IEEE Workshop on* (pp. 407–410, 1999). IEEE
13. S. Gholami, A. Mahmoudi, E. Farshidi, Two-stage estimator for frequency rate and initial frequency in LFM signal using linear prediction approach. Circuits Syst. Signal Process. **38**(1), 105–117 (2019)
14. S. Liu, W. Wang, L. Fu, Q. Lu, Linear prediction-based DOA estimation for directional borehole radar 3-D imaging. IEEE Trans. Geosci. Remote Sens. (2019)
15. A. Lenz, M.S. Stein, A.L. Swindlehurst, Joint transmit and receive filter optimization for sub-Nyquist delay-Doppler estimation. IEEE Trans. Signal Process. **66**(10), 2542–2556 (2018)
16. L. Liu, M. Ghogho, D. McLernon, W. Hu, Pseudo-maximum likelihood estimation of ballistic missile precession frequency. Signal Process. **92**(9), 2018–2028 (2012)
17. A. Mahmoudi, M. Karimi, H. Amindavar, Parameter estimation of autoregressive signals in presence of colored AR (1) noise as a quadratic eigenvalue problem. Sig. Process. **92**(4), 1151–1156 (2012)
18. S. Minhong, T. Bin, Noise amplitude modulation jamming signal suppression based on weighted-matching pursuit. J. Syst. Eng. Electron. **20**(5), 962–967 (2009)
19. J.L. Ntakpe, G.-R. Gillich, I.C. Mituletu, Z.-I. Praisach, N. Gillich, An accurate frequency estimation algorithm with application in modal analysis. Rom. J. Acoust. Vib. **13**(2), 98 (2016)

20. H. Peng, S. Yu, J. Bian, Y. Zhang, H. Li, Statistical analysis of non linear least squares estimation for harmonic signals in multiplicative and additive noise. Commun. Stat.-Theory Methods **44**(2), 217–240 (2015)
21. P. Rao, A.Das Barman, Speech formant frequency estimation: evaluating a nonstationary analysis method. Sig. Process. **80**(8), 1655–1667 (2000)
22. T.S. Rappaport, *Wireless Communications: Principles and Practice*, vol. 2 (Prentice Hall PTR, New Jersey, 1996)
23. J. Ren, X. Dai, N. Wang, H. Li, Parameters estimation of noise amplitude modulation signal. IET Radar Sonar Navig. **11**(1), 161–170 (2016)
24. R. Schmidt, Multiple emitter location and signal parameter estimation. IEEE Trans. Antennas Propag. **34**(3), 276–280 (1986)
25. C. Shi, F. Wang, M. Sellathurai, J. Zhou, Low probability of intercept-based distributed MIMO radar waveform design against barrage jamming in signal-dependent clutter and coloured noise. IET Signal Proc. **13**(4), 415–423 (2019)
26. T. Trump, B. Ottersten, Estimation of nominal direction of arrival and angular spread using an array of sensors. Signal Process. **50**(1–2), 57–69 (1996)
27. I.C. Wong, B.L. Evans, Sinusoidal modeling and adaptive channel prediction in mobile OFDM systems. IEEE Trans. Signal Process. **56**(4), 1601–1615 (2008)
28. J. Xu, G. Li, Y.-N. Peng, X.-G. Xia, Y.-L. Wang, Parametric velocity synthetic aperture radar: Signal modeling and optimal methods. IEEE Trans. Geosci. Remote Sens. **46**(9), 2463–2480 (2008)
29. S. Yang, Frequency estimation of harmonics with multiplicative noise using ESPRIT, in *Image and Signal Processing (CISP), 2015 8th International Congress on* (pp. 1251–1255). IEEE