Circuits, Systems, and Signal Processing (2019) 38:3356–3369
https://doi.org/10.1007/s00034-018-1000-8

SHORT PAPER

A Simple Design of Fractional Delay FIR Filter Based on Binomial Series Expansion Theory

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Received: 11 January 2018 / Revised: 23 November 2018 / Accepted: 24 November 2018 /
Published online: 10 December 2018

Abstract

Fractional delay filters modeling non-integer delays are digital filters that ideally have flat group delays. This paper proposes a simple design method of fractional delay FIR filter based on binomial series expansion theory. First, the design technique is based on the binomial series expansion method which is applied to a discrete fractional system to obtain a closed form FIR digital filter which approximates the digital fractional delay operator z^{-m} $(m \in \Re^+)$. Then, the principal differentiation is used to design fractional delay FIR filter with a broader group delay bandwidth. Finally, numerical examples of fractional delay FIR filter design show that the proposed approach yields better performance compared to the existing techniques.

Keywords Fractional delay FIR filters · Binomial series expansion · Discrete fractional system · Broader bandwidth

1 Introduction

In recent years, fractional order signal processing has received considerable interest in many engineering applications including the implementations of the fractional delay/forward filters, the fractional integrators/differentiators, the fractional Fourier transform and the fractional wavelet transform [1, 4, 5, 15, 18].

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In digital signal processing and communications, the main function of some types of digital filters is to delay an input digital signal by a fractional amount of the sampling period. This type of filter is known as fractional delay filters. There are many applications in which the signal delay value is required, and examples of such systems are: speech coding and synthesis [14], beam steering [17], time adjustment in digital receivers [23], time delay estimation [16], modeling of music instruments [21] and analog-to-digital transform [2].

The transfer function of the ideal digital fractional delay is given by:

$$H(z) = z^{-m}, \quad (m \in \Re^{+}) \tag{1}$$

where m is the delay amount that can be split into integer part D and fractional part d.

Despite its apparent simplicity, the above transfer function is not easy to implement. For this reason, many digital finite impulse response (FIR) and infinite impulse response (IIR) filter design techniques have been proposed to approximate the ideal digital fractional delay transfer function of (1). In the literature, some techniques to design fixed digital fractional delay filters such as window method, Lagrange interpolation method, discrete Fourier transform method [2–4, 13, 14, 17, 19, 20, 22–25] and to design digital variable fractional delay filters have been developed. Techniques such weighted least-square, minimax and maximally flat methods have been used for the design of variable fractional delay filters [6–9, 11, 12, 26]. In Ref. [14], we can find a good introductory material of the topic.

In this brief, the binomial series expansion method is proposed to design a fractional delay FIR filter. Recently, the Taylor series method was used to design this type of filter in a maximally flat mode at low frequencies. The maximally flat FIR filter approximation is equivalent to the Lagrange interpolation method [24]. This last method provides its most satisfactory frequency response when it is deployed to implement $m = D + d$ samples of delay, where L is the filter order, $D = L/2$, and d is a fractional amount. The main contribution of this work is that the obtained FIR digital filter approximation of the ideal digital fractional delay operator z^{-m} is a discrete system with a wider group delay bandwidth. In addition, the obtained FIR digital filter coefficients have an explicit formula; so the computation speed to accomplish this design is very fast.

In our previous work [1, 2] and [4, 5], we have considered the design of the digital fractional delay operator z^{-m} $(0 < m < 1)$ and the digital fractional forward operator z^{m} $(0 < m < 1)$ using digital infinite impulse response (IIR) filters based on the approximation, respectively, of the analog fractional power pole $\frac{1}{\left(1+\frac{s}{\omega_c}\right)^m}$ and of the analog fractional power zero $\left(1 + \frac{s}{\omega_c}\right)^m$ and the analog-to-digital transform forward difference generating function ($s = \frac{1-z^{-1}}{Tz^{-1}}$, where T is the sampling period). We have also designed the fractional order integrator s^{-m} and differentiator s^{m} $(0 < m < 1)$ using an adjustable fractional order digital FIR filters and the Tustin generating function for the analog-to-digital transform ($s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$). The design of the fractional delay $z^{-\alpha}$ $(0 < \alpha < 0.5)$ has also been considered using digital infinite impulse response (IIR) filter based on the approximation of analog fractional order systems and the Tustin generating function for analog-to-digital transform ($s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$, where T is the sampling period). The designed fractional delay $z^{-\alpha}$ ($0<\alpha<0.5$) has been used in the implementation of the fractional Euler analog-to-digital transform $s = \frac{(1-z^{-\alpha})}{\alpha T}$ ($0<\alpha<0.5$) which is the regular Euler transform when $\alpha = 1$. The main idea of the current work is the design of the fractional delay operator z^{-m} ($m \in \Re^{+}$) using digital FIR filter by applying the binomial series expansion method to a discrete fractional system.

The rest of the work is structured as follows: the design method of the digital fractional delay FIR filter will be explained in Sect. 2. In Sect. 3, this design method is manipulated to obtain a filter with a wider group delay bandwidth. The proposed design has closed form coefficient formulas. Section 4 contains the simulation results, comparisons and discussion of the proposed technique, followed by Sect. 5 where the conclusions are given.

2 Proposed Design Method

The proposed design method of the fractional order delay z^{-m} ($m \in \Re^{+}$) is derived using the binomial series expansion of x^m ($m \in \Re^{+}$), for $-1<x<1$. This binomial series expansion is defined by the following expressions [10]:

$$x^m \cong \sum_{n=0}^{\infty} \binom{m}{n} (x-1)^n \cong \sum_{n=0}^{\infty} \frac{\Gamma(m+1)}{\Gamma(n+1)\cdot\Gamma(m-n+1)} (x-1)^n \tag{2}$$

$$(x-1)^n \cong \sum_{k=0}^{\infty} (-1)^{n-k} \binom{n}{k} x^k \cong \sum_{k=0}^{\infty} (-1)^{n-k} \frac{\Gamma(n+1)}{\Gamma(k+1)\cdot\Gamma(n-k+1)} x^k \tag{3}$$

where $\Gamma(.)$ is the Gamma function.

Then, Eqs. (2) and (3) are linked together by substituting the third equation into the second one and after some manipulations, we can have:

$$\begin{aligned} x^m &\cong \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} (-1)^{n-k} \binom{m}{n}\binom{n}{k} x^k \\ &\cong \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} (-1)^{n-k} \frac{\Gamma(m+1)}{\Gamma(k+1)\cdot\Gamma(n-k+1)\cdot\Gamma(m-n+1)} x^k \end{aligned} \tag{4}$$

Let the function $H(z)$ of (1) be $H(z) = \left(z^{-1}\right)^m$. Hence, by replacing x by z^{-1} in the expression of (4) the ideal digital fractional delay $H(z)$ can be approximated by the following expression:

$$\begin{aligned} H(z) = z^{-m} &\cong \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} (-1)^{n-k} \binom{m}{n}\binom{n}{k} z^{-k} \\ &\cong \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} (-1)^{n-k} \frac{\Gamma(m+1)}{\Gamma(k+1)\cdot\Gamma(n-k+1)\cdot\Gamma(m-n+1)} z^{-k} \end{aligned} \tag{5}$$

The truncated version of expression (5) to L terms of the approximation of $H(z)$ of (5) will lead to its FIR digital filter approximation as follows:

$$H(z) = z^{-m} \cong \sum_{k=0}^{L}\sum_{n=0}^{L}(-1)^{n-k}\binom{m}{n}\binom{n}{k}z^{-k}$$
$$\cong \sum_{k=0}^{L}\sum_{n=0}^{L}(-1)^{n-k}\frac{\Gamma(m+1)}{\Gamma(k+1)\cdot\Gamma(n-k+1)\cdot\Gamma(m-n+1)}z^{-k} \quad (6)$$

Let $i=(n-k)$ then $n=(i+k)$; the expression of (6) becomes:

$$H(z) = z^{-m} \cong \sum_{k=0}^{L}\sum_{i=0}^{L-k}(-1)^{i}\binom{m}{i+k}\binom{i+k}{k}z^{-k}$$
$$\cong \sum_{k=0}^{L}\sum_{i=0}^{L-k}(-1)^{i}\frac{\Gamma(m+1)}{\Gamma(k+1)\cdot\Gamma(i+1)\cdot\Gamma(m-i-k+1)}z^{-k}$$
$$= \sum_{k=0}^{L}h_1(k)z^{-k} \quad (7)$$

where L is the FIR filter order and the terms $h_1(k)$ are its coefficients which are given by the following expressions:

$$h_1(k) = \begin{cases} \sum_{i=0}^{L-k}(-1)^i\binom{m}{i+k}\binom{i+k}{k} = \sum_{i=0}^{L-k}(-1)^i\frac{\Gamma(m+1)}{\Gamma(k+1)\cdot\Gamma(i+1)\cdot\Gamma(m-i-k+1)}, & 0 \le k \le (L) \\ 0, & \text{otherwise} \end{cases}. \quad (8)$$

Hence, the proposed approximation of the ideal fractional order delay z^{-m} ($m \in \Re^{+}$) is a closed form digital FIR filter.

3 Differentiation of the Proposed Fractional Delay FIR Filter

Our second contribution in this paper is the application of the principle of differentiation to design a fractional delay FIR filter with wider group delay bandwidth. Applying the derivative of the frequency response of Eq. (7) leads to the following expression:

$$\frac{\mathrm{d}H(e^{-j\omega})}{\mathrm{d}\omega} = \frac{\mathrm{d}}{\mathrm{d}\omega}\left[e^{-j\omega m}\right] = \frac{\mathrm{d}}{\mathrm{d}\omega}\left[\sum_{k=0}^{L}h_1(k)\,\mathrm{e}^{-j\omega k}\right]. \quad (9)$$

By taking the derivative of both sides of Eq. (9), we will get:

$$-jm\left[e^{-j\omega m}\right]=\sum_{k=0}^{L}-jk\left[h_1(k)\,e^{-j\omega k}\right] \tag{10}$$

We can write then:

$$e^{-j\omega m}=\sum_{k=0}^{L}\frac{k}{m}\left[h_1(k)\,e^{-j\omega k}\right] \tag{11}$$

Therefore, the transfer function of Eq. (7) of this proposed design is given as:

$$z^{-m}=\sum_{k=0}^{L}\frac{k}{m}h_1(k)z^{-k}=\sum_{k=0}^{L}h(k)z^{-k} \tag{12}$$

where the coefficients $h(k)$ of the proposed digital FIR filter design of the ideal fractional order delay z^{-m} ($m\in\Re^{+}$) are given in closed form in terms of the coefficients $h_1(k)$ of Eq. (8) and the fractional delay m as follows:

$$h(k)=\begin{cases}\frac{k}{m}h_1(k), & 0\le k\le L\\ 0, & \text{otherwise}\end{cases} \tag{13}$$

The implementation in direct form structure of the digital fractional delay operator z^{-m} in terms of its approximation by the digital FIR filter of Eq. (13) is shown in Fig. 1.

4 Simulation Results and Comparison

In this section, the design is implemented in MATLAB. Numerical examples are presented to demonstrate the effectiveness of the proposed design method. First, we

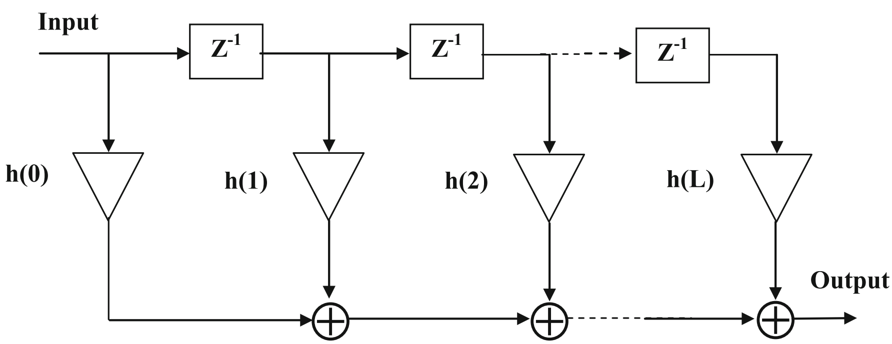

Fig. 1 Direct form implementation of the proposed $(L+1)$th order fractional delay FIR filter

use an example to compare the behavior of the proposed method with the conventional Lagrange maximally flat method in [22] and the Fourier transform interpolation method in [19]. In this comparison, we have adopted the same design parameters for the proposed method and the two other methods to design the digital FIR filter equivalent to the ideal digital fractional delay $H(z) = z^{-15.5}$. The chosen parameters are: $L=30$, $m=15.5$ and a design frequency band of $[0, 0.8\pi]$. In addition, to evaluate the performance and the effectiveness of the proposed method, the magnitude and the group delay (GD) absolute error functions defined, respectively, by the following equations were used:

$$\mathrm{Ea} = \left|\left|H(\mathrm{e}^{j\omega})\right| - \left|\mathrm{e}^{-j\omega m}\right|\right|, \quad \mathrm{Eg} = \left|GD\left[H(\mathrm{e}^{j\omega})\right] - GD\left[\mathrm{e}^{-j\omega m}\right]\right| \tag{14}$$

Before comparing the behavior of the proposed method with the methods in [22] and in [19], we will first compare the magnitude and the group delay responses of the designed digital FIR fractional delay filters defined by Eqs. (8) and (13) equivalent to the ideal digital fractional delay $H(z) = z^{-15.5}$ to highlight the benefits of the application of the principle of the differentiation of the frequency response to design a fractional delay FIR filter with wider group delay bandwidth. Figure 2 shows the magnitude and the group delay responses of the designed digital FIR fractional delay filters equivalent to the ideal digital fractional delay $H(z) = z^{-15.5}$ before and after applying the differentiation of the frequency response defined, respectively, by Eqs. (8) and (13).

From Fig. 2, we note the improvement in the group delay response realized after applying the differentiation of the frequency response to design the digital FIR frac-

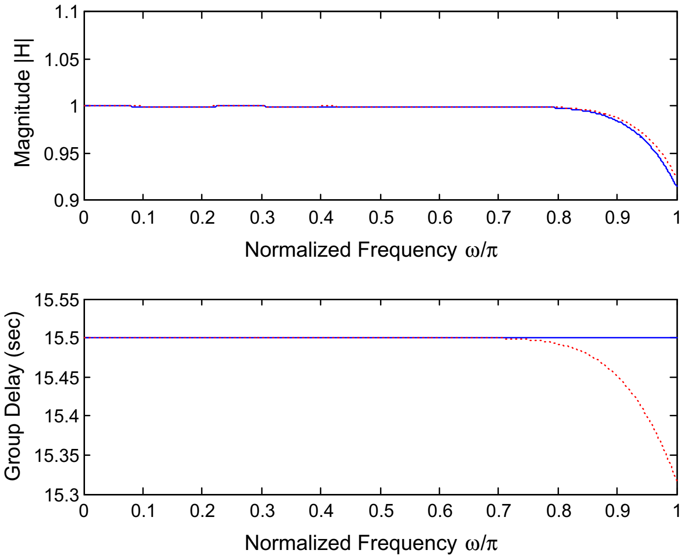

Fig. 2 Magnitude and group delay responses of the proposed designed digital FIR fractional delay filters before (dotted line) and after (solid line) applying the differentiation of the frequency response

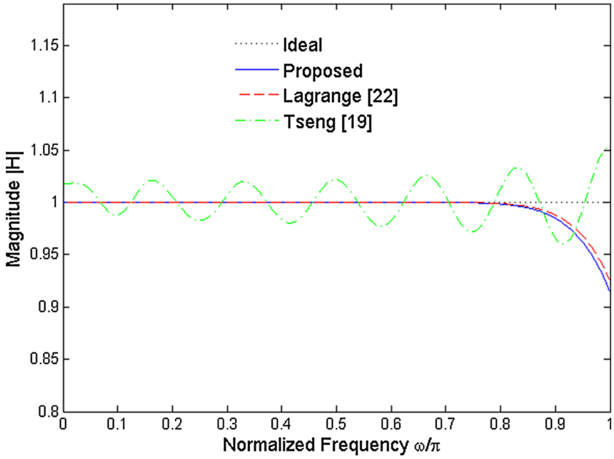

Fig. 3 Magnitude responses of the ideal fractional delay $z^{-15.5}$ (dotted line), the proposed digital FIR filter (solid line), the Lagrange maximally flat method of [22] (dashed line) and the Fourier transform interpolation method of [19] (dash-dot line)

tional delay filter defined by Eq. (13) equivalent to the ideal digital fractional delay $H(z) = z^{-m}(m > 0)$.

Figures 3 and 4 show, respectively, the magnitude and the group delay responses of the ideal fractional delay and its digital FIR filter version that was obtained by the proposed method, the Lagrange maximally flat method of [22] and the Fourier transform interpolation method of [19]. Figure 5 shows the magnitude and the group delay absolute error functions Ea and Eg of Eq. (14) of the ideal fractional delay, the proposed design method, the Lagrange maximally flat method [22] and the Fourier transform interpolation method [19].

From Fig. 3, we remark that, in the frequency band [0, 0.8π], the magnitude responses of the proposed method and Lagrange method of [22] are almost equal and smoother than the oscillatory magnitude response in the interpolation method of [19]. From Fig. 4, we can also see that the group delay response of the proposed method is almost the ideal one and it is much better than the group delay responses of the methods of [22] and [19]. These remarks are quantified in Fig. 5 where, in the frequency band [0, 0.8π], the magnitude absolute error function Ea of the proposed method and Lagrange maximally flat method of [22] are smaller than the magnitude absolute error function Ea of the transform interpolation method of [19]. But, in the frequency band [0.8π, π], both magnitude absolute error functions Ea of the proposed method and Lagrange maximally method of [22] are higher than the magnitude absolute error function Ea of the transform interpolation method of [19]. However, in the full frequency band [0, π], the group delay absolute error function Eg of the proposed method is almost zero and it is much smaller than the group delay absolute error function of both methods of [22] and [19]. Hence, we conclude that the proposed design outperforms the other two designs of [22] and [19] and realizes a wider group delay bandwidth system.

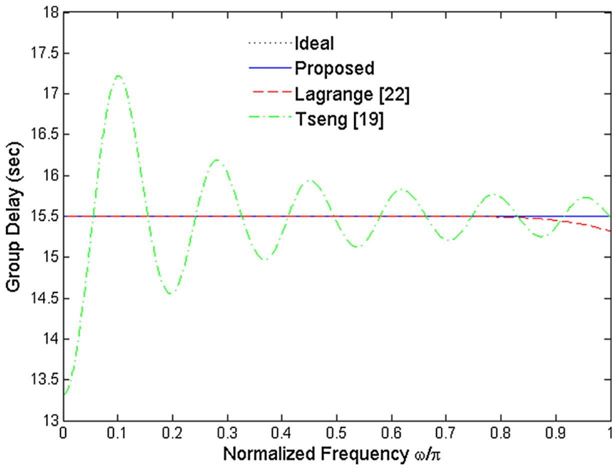

Fig. 4 Group delay responses of the ideal fractional delay $z^{-15.5}$(dotted line), the proposed digital FIR filter (solid line), Lagrange maximally flat method of [22] (dashed line) and the Fourier transform interpolation method of [19] (dash-dot line)

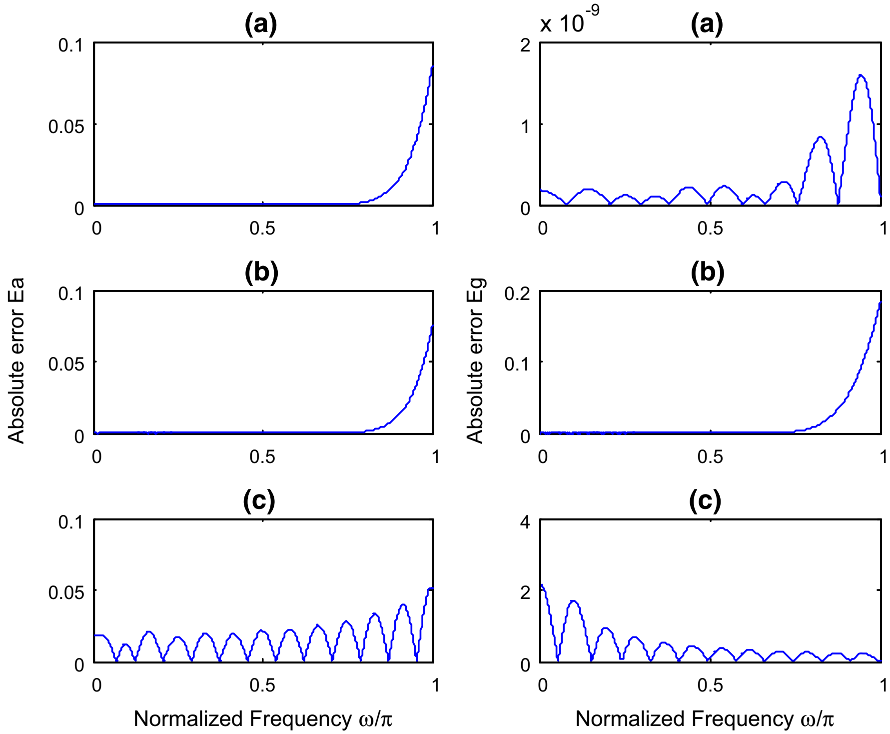

Fig. 5 Magnitude and group delay absolute error functions Ea and Eg of the proposed method (**a**), Lagrange maximally flat method of [22] (**b**) and the Fourier transform interpolation method of [19] (**c**)

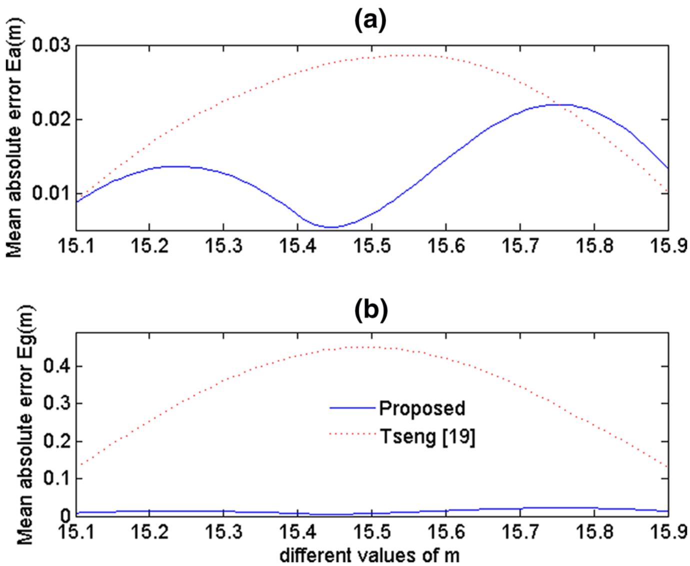

Fig. 6 Mean absolute errors of the magnitude Ea(m) (**a**) and the group delay Eg(m) (**b**) of the digital FIR filter design of the ideal fractional delay of the proposed method (solid line) and the method of [19] (dotted line), for different values *m*

To compare the accuracy of the proposed method and the method of [19], for different values of the fractional delay m, the mean absolute errors Ea(m) of the magnitude and Eg(m) of the group delay are used. These mean absolute errors are given by the following expressions:

$$\mathrm{Ea}(m) = \frac{1}{N_\mathrm{p}} \sum_{i=1}^{N_\mathrm{p}} \left| \left| H(\mathrm{e}^{j\omega_i}) \right| - \left| \mathrm{e}^{-j\omega_i m} \right| \right| \tag{15}$$

$$\mathrm{Eg}(m) = \frac{1}{N_\mathrm{p}} \sum_{i=1}^{N_\mathrm{p}} \left| \text{Group Delay}\left[H(\mathrm{e}^{j\omega_i}) \right] - \text{Group Delay}\left[\mathrm{e}^{-j\omega_i m} \right] \right| \tag{16}$$

where N_p is the number of points ω_i in the frequency band [0, 0.9π].

Figure 6 shows the mean absolute errors Ea(m) of the magnitude and Eg(m) of the group delay of the proposed method and the method of [19] the digital FIR filter design of the ideal fractional delay for different values of the fractional delay m in the range of [15.1, 15.9].

From Fig. 6, we can easily see that the mean absolute errors Ea(m) of the magnitude and Eg(m) of the group delay of the digital FIR filter design of the ideal fractional delay of the proposed method are smaller than the ones of the method of [19]. This result shows that the effectiveness of the proposed digital FIR filter design of the ideal fractional delay is not restricted to only one fractional delay m.

To show the merits and the efficiency of the proposed method, the obtained results are also compared to those of the method proposed by Tseng in [20] where the opti-

Table 1 Different errors of the magnitude and group delay of the four designs

Method	$\mathbf{e_{m,mag}}$	$\mathbf{e_{m,gd}}$	Ea(m)	Eg(m)
Proposed method	0.08	1.6×10^{-9}	0.02	4.55×10^{-10}
Least squares method with fractional derivative constraints [20]	0.01	0.04	0.002	0.01
Lagrange method [22]	0.08	0.18	0.02	0.055
Fourier transform interpolation method [19]	0.05	2	0.019	0.387

mal least squares method with fractional derivative constraints has been used. The comparison with the work of Tseng of [20] is made because the author has done a lot of work in the last decade and his work in [20] is his most recent work on digital FIR filter design of the fixed ideal fractional delay. In this comparison, the design parameters $L = 30$, $m = 15.5$ and the frequency band $[0, 0.9\pi]$ of example (1) of [20] are adopted. The magnitude mean absolute error Ea(m) of (15), the group delay mean absolute error Eg(m) of (16), the maximum magnitude response error $\mathbf{e_{m,mag}}$ and the maximum group delay error $\mathbf{e_{m,gd}}$ are used as comparison performances of the digital FIR filter design of the fixed ideal fractional delay using the proposed method, the least squares method with fractional derivative constraints of [20], Lagrange maximally flat method of [22] and Fourier transform interpolation method of [19]. The obtained comparison results are reported in Table 1.

As shown in Table 1, the magnitude response errors for all methods are approximately the same, whereas the group delay error for the proposed method is significantly lower than the other designs used in the comparison. Hence, we conclude that the proposed method has achieved a design with an acceptable higher accuracy compared to the methods of Table 1.

To show the relationship between the length L of the fractional delay FIR filter design of the fixed ideal fractional delay and the approximation accuracy of the proposed method, the mean relative errors Ea(L) and Eg(L) versus L, for different values of the order m for $T = 1$ s in the frequency band $[0, 0.9\pi]$ are, respectively, plotted in Fig. 7a–d.

From Fig. 7, we note that the errors Ea(L) and Eg(L) have the smallest values for the values of the length L around 2 m. These observations confirm the compatibility of the proposed method with the Lagrange interpolation method [22] where the authors have stated that their proposed design provides the best approximation when the fractional delay m is close to $L/2$.

Figure 8 depicts the magnitude and the group delay absolute errors functions Ea and Eg of the proposed designed digital FIR fractional delay filters before and after applying the differentiation of the frequency response for $L = 30$ and $m = 15.5$.

From Fig. 8, we can easily see the amelioration in the group delay absolute error function Eg and the deterioration of the magnitude absolute error function Ea after the application of the differentiation of the frequency response of the proposed digital FIR fractional delay filter. To highlight the trade-off between the amelioration of the group

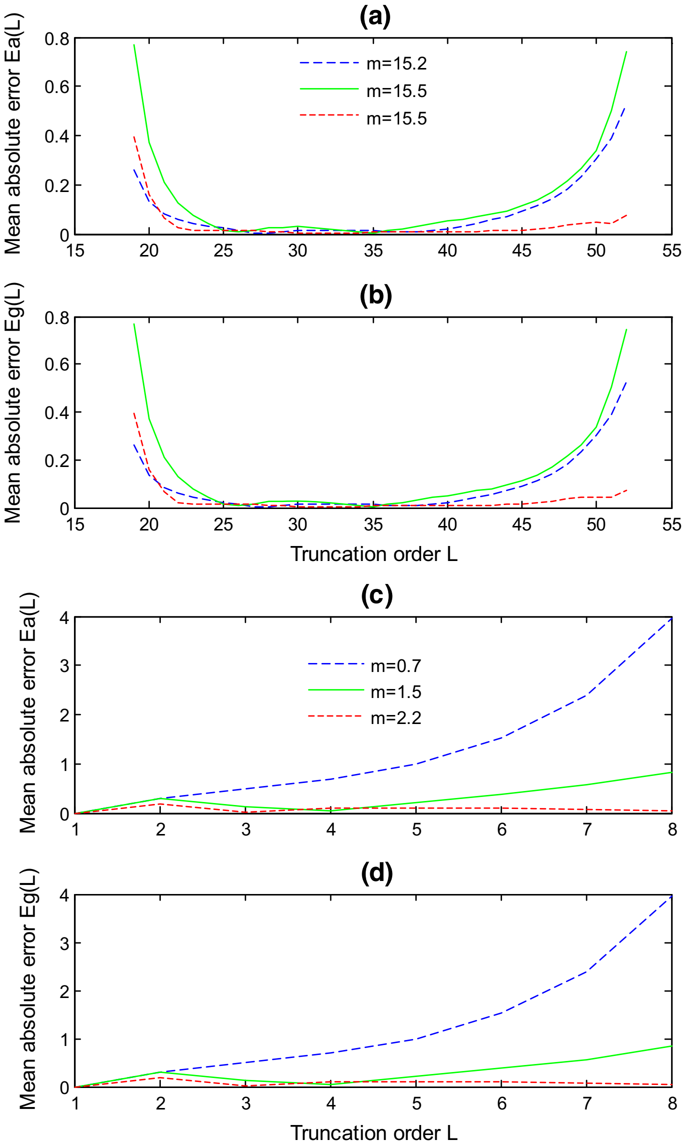

Fig. 7 Plots of the mean relative errors Ea(L) and Eg(L) versus the FIR filter length L for different values of the fractional order m

delay absolute error function Eg and the deterioration of the magnitude absolute error function Ea after the application of the differentiation of the frequency response of the proposed digital FIR fractional delay filter, the maximum magnitude error $\mathbf{e_{m,mag}}$ and the maximum group delay error $\mathbf{e_{m,gd}}$ of the proposed digital FIR fractional delay filter before and after applying the differentiation of the frequency response for $L = 30$ and $m = 15.5$ are reported in Table 2.

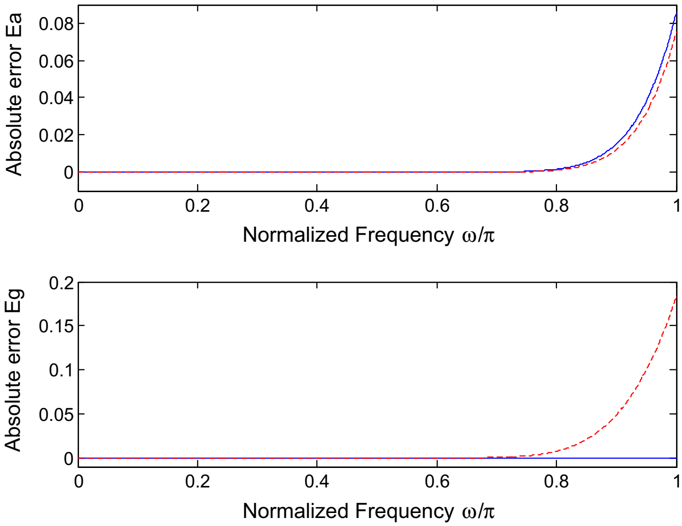

Fig. 8 Magnitude and group delay absolute error functions Ea and Eg before (dotted line) and after (solid line) applying the differentiation of the frequency response of the digital FIR fractional delay filters

Table 2 Maximum magnitude and group delay errors $\mathbf{e_{m,mag}}$ and $\mathbf{e_{m,gd}}$ before and after differentiation of the frequency response of the digital FIR fractional delay filters

Proposed FIR fractional delay filter	$\mathbf{e_{m,mag}}$	$\mathbf{e_{m,gd}}$
Before differentiation	0.075	0.183
After differentiation	0.08	1.6×10^{-9}

From Table 2, we note that a very small maximum group delay error is achieved after applying the differentiation of the frequency response of the proposed digital FIR fractional delay filter. However, the very good improvement realized in the group delay is obtained at the cost of the increase in the proposed digital FIR fractional delay filter maximum magnitude error. Thus, it is interesting to extend the proposed technique to design a digital FIR filter fractional delay with a broader magnitude bandwidth.

5 Conclusion

In this paper, the binomial series expansion formula has been used to design a new fractional delay FIR filter to approximate the ideal fractional delay operator. Further, the differentiation principle is applied to obtain a fractional delay FIR filter with a wider group delay bandwidth. The filter coefficients are in closed form formulas leading to an easy computation. Numerical design example has been presented to demonstrate the performance and the effectiveness of the proposed digital FIR fractional delay filter design method. Errors functions have also been used to compare the obtained results with the ideal digital fractional delay operator. The obtained results have confirmed the superior performance of the proposed method compared to the some recent and efficient digital FIR fractional delay filter design methods.

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