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Delay-Dependent Robust Finite-Time H_∞ Control for Uncertain Large Delay Systems Based on a Switching Method

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Abstract The robust finite-time H_∞ control for a kind of uncertain delay systems with large delay period (LDP) is discussed in this paper. First, a switching technique is exploited to transform the original system into a switched delay system. Second, within the limitation of frequency and length rate of LDP, a state feedback controller is designed to guarantee that the closed-loop system is robust finite-time bounded. Third, the finite-time H_∞ performance analysis for the closed-loop system is developed. Finally, two examples are presented to clarify the validity of the proposed approach.

Keywords Finite-time H_∞ control · Large delay period · Switching method · Lyapunov functional

1 Introduction

Switched systems, as a class of hybrid systems, include a family of subsystems and a switching law. Switched systems have received growing attention due to their exten-

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sive applications. Many results related to the stability and stabilization have been derived for the linear or nonlinear switched systems [14–16,19,27,28,35]. For example, adaptive control problem for nonlinear switched systems has been investigated in [14], and different adaptive neural tracking controllers have been designed for uncertain nonlinear switched systems in [15] and [16].

In general, asymptotic stability is enough for practical applications [5,9,36,37]. However, for networked control systems, the bound of the system state trajectories over a fixed finite time interval needs to be considered. To deal with this problem, Dorato introduced the concept of finite-time stability (FTS) in [3] and the definition of finite-time boundedness was proposed in [1] when the exogenous disturbance is involved. From then on, a large number of results on FTS of switched delay systems have been derived, please see the papers [7,10,11] and the references therein. In addition, the finite-time control problem has also obtained a series of results. For instance, the problem of finite-time stabilization has been studied in [12,13,18]. Work [29] has investigated the finite-time H_∞ control of a class of linear switched systems under mode-dependent average dwell time. In [32], the robust finite-time control for switched neutral systems has been dealt with. It should be noted that the FTS can not be got from the Lyapunov asymptotic stability, and vice versa.

On the other hand, a system may be unstable or out of control in the presence of delay [24,25], which brings difficulties to the research of the stability and the stabilization issues of dynamic systems. Thus, the stability and control synthesis problems for delay systems have been highlighted in [7,8,11,18,20,26,31,33,34] by using the traditional Lyapunov functional method, which requires that the time delay is small. That is to say, delay $d(t)$ must satisfy $0 \leq d(t) \leq h_1$, for $\forall\, t \in [t_0, \infty)$, then the stability of the delay systems can be guaranteed. However, in networked control systems, due to the package dropout and the networked induced-delay phenomena, the actual time delay may be greater than the derived bound h_1. That means large delay arises occasionally in some local interval of $[t_0, \infty)$. At this point, it is very important to address the stability of the systems under the influence of large delay period (LDP) and the aforementioned traditional Lyapunov method fails to deal with the problem. Recently, some results on the delay systems with LDP have been reported, such as the stability analysis for a variety of systems with LDP [6,21–23,30], the stabilization for linear delay systems with LDP [2,4], and so on. However, to the best of the authors' knowledge, no attention has been paid to the robust finite-time H_∞ control of uncertain delay systems with LDP, which motivates the present study.

In this paper, the problem of robust finite-time H_∞ control for uncertain delay systems with LDP is investigated. First, when the maximum allowed delay bound increases, the original dynamic system is transformed into a switched delay system with two subsystems. One subsystem is finite-time bounded, while the other may not be finite-time bounded. Then, by restricting the frequency and length rate of LDP, a delay-dependent robust finite-time H_∞ controller is designed to guarantee that the closed-loop system is finite-time bounded with H_∞ performance.

The reminder of this paper is organized as follows. Some definitions and preliminaries are introduced in Sect. 2. In Sect. 3, the main results are presented. Section 4 gives two examples. The conclusions are given in Sect. 5.

Notation We use $P > 0$ to denote positive definite and symmetric matrix P. $\lambda_{\max}(P)$ is used for the maximum eigenvalue of matrix P. Let $\mathbb{N}$ represent the set of all natural numbers. diag$\{\cdots\}$ stands a block-diagonal matrix. The notation $*$ denotes the symmetric term in a matrix. I is an identical matrix with appropriate dimensions.

2 Problem Formulation and Preliminaries

Consider the following uncertain delay system

$$\begin{aligned} \dot{x}(t) &= \hat{A}x(t) + \hat{A}_d x(t - d(t)) + Bu(t) + Dw(t), \\ z(t) &= \hat{F}x(t) + Hu(t) + Gw(t), \\ x(\theta) &= \varphi(\theta), \theta \in [-h_2, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ stand the state vector and the control input, respectively. $w(t) \in \mathbb{R}^p$ is the disturbance input satisfying

$$\int_0^{T_f} w^T(t)w(t)\mathrm{d}t \le d_w^2, \quad d_w > 0, \tag{2}$$

T_f is a time constant. d(t) is the delay and satisfies

$$0 \le d(t) \le h_2, \quad \dot{d}(t) \le d < 1. \tag{3}$$

$z(t) \in \mathbb{R}^q$ is the controlled output. $\varphi(\theta)$ stands a continuously differentiable vector-valued initial function. $\hat{A}$, $\hat{A}_d$, $\hat{F}$ are uncertain real-valued matrices and have the form

$$[\hat{A} \;\; \hat{A}_d \;\; \hat{F}] = [A \;\; A_d \;\; F] + L\Xi(t)[M_{11} \;\; M_{22} \;\; M_{33}], \tag{4}$$

where A, A_d, B, D, F, G, H, L, M_{11}, M_{22}, M_{33} are known real-valued constant matrices with appropriate dimensions, $\Xi(t)$ is unknown and satisfies $\Xi^T(t)\Xi(t) \le I$.

Definition 1 [32] For a given time constant T_f, system (1) with $u(t) \equiv 0$ is said to be finite-time bounded (FTB) with respect to $(e_1, e_2, T_f, d_w^2, R)$ if

$$\begin{aligned} &\sup_{-h_2 \le t_0 \le 0} \{x^T(t_0)Rx(t_0), \dot{x}^T(t_0)R\dot{x}(t_0)\} \le e_1 \\ &\Rightarrow x^T(t)Rx(t) < e_2, \;\; t \in [0, T_f], \end{aligned} \tag{5}$$

where $e_2 > e_1 > 0$, $R > 0$, and $w(t)$ satisfies (2).

The following assumption is adopted:

Assumption 1 System (1) is FTB when delay $d(t)$ satisfies $0 \le d(t) \le h_1$, for $\forall\, t \in [0, T_f]$. But the finite-time boundedness of the system (1) is not assured based on the existing methods or system itself is not FTB if delay $d(t)$ satisfies $h_1 < d(t) \leq h_2$, for $\forall\, t \in [0, T_f]$, where $h_2 > h_1 > 0$ and h_1, h_2 can be obtained based on existing measures.

Definition 2 [22] Time interval $[T_1, T_2)$ is called large delay period (LDP) if for $\forall\, t \in [T_1, T_2)$, it holds that $h_1 < d(t) \leq h_2$. And time interval $[T_3, T_4)$ is known as small delay period (SDP) if for $\forall\, t \in [T_3, T_4)$, it holds that $0 \leq d(t) \leq h_1$.

Assume the LDP appears occasionally, then system (1) can be represented by the following switched delay system

$$\begin{aligned}\dot{x}(t) &= \hat{A}x(t) + \hat{A}_d x(t - d_{\sigma(t)}(t)) + Bu(t) + Dw(t),\\ z(t) &= \hat{F}x(t) + Hu(t) + Gw(t),\\ x(\theta) &= \varphi(\theta), \theta \in [-h_2, 0],\end{aligned} \tag{6}$$

where $\sigma(t) : [0, T_f] \to \{1, 2\}$ is a piecewise constant function and called switching signal, $0 \leq d_1(t) \leq h_1$ and $h_1 < d_2(t) \leq h_2$. When $\sigma(t) = 1$, system (6) is running in SDP, and $\sigma(t) = 2$ illustrates that system (6) is running in LDP.

Remark 1 Although system (1) may not be FTB if LDP arises in the total time interval $[0, T_f]$, system (1) may be FTB while LDP only occurs regionally in $[0, T_f]$. The switching signal $\sigma(t)$ relies on the size of the delay.

We use time sequence $0 = t_0 < t_1 < t_2 < \cdots < t_l = T_f$ to denote switching sequence of the switching signal $\sigma(t)$. Suppose for switching signal $\sigma(t)$, there exists time sequence

$$t_0 = p_0 < p_1 < p_2 < \cdots < p_{l'} = T_f, \tag{7}$$

which is one subsequence of $t_0 < t_1 < t_2 < \cdots < t_l$, and satisfies

$$p_{m+1} - p_m \leq \eta_m \leq \eta < T_f, \forall\, m \in \overline{\mathbb{N}} = \{0, 1, 2, \ldots, l' - 1\},$$

for positive constants η_m and η.

Remark 2 Since $p_0 < p_1 < p_2 < \cdots < p_{l'}$ is one subsequence of $t_0 < t_1 < t_2 < \cdots < t_l$, we have $l' \leq l$.

Definition 3 [22] For any $T_2 > T_1 \geq 0$, let $N_l(T_1, T_2)$ denote the number of LDP in time interval $[T_1, T_2)$. $F_l(T_1, T_2) = \dfrac{N_l(T_1, T_2)}{T_2 - T_1}$ is called frequency of LDP in time interval $[T_1, T_2)$.

It is assumed that $[t_{2k}, t_{2k+1})$ and $[t_{2k+1}, t_{2k+2})$ denote SDP and LDP, respectively, where $k \in \mathbb{N}$.

If $N_\sigma(T_1, T_2)$ stands the number of switchings of $\sigma(t)$ in time interval $[T_1, T_2)$, we have

$$N_\sigma(t_0, t) \leq 2N_l(t_0, t). \tag{8}$$

Definition 4 [22] For time interval $[T_1, T_2)$, denote the total time length of LDP during $[T_1, T_2)$ by $T^+(T_1, T_2)$, and denote the total time length of SDP during $[T_1, T_2)$ by $T^-(T_1, T_2)$. We call $\frac{T^+(p_m, p_{m+1})}{T^-(p_m, p_{m+1})}$ the length rate of LDP in time interval $[p_m, p_{m+1})$.

In this paper, the control signal going into the plant is of the form

$$u(t) = K_{\sigma(t)}x(t), \ t \in [0, T_f]. \tag{9}$$

Hence, the corresponding closed-loop system is given by

$$\begin{aligned} \dot{x}(t) &= \hat{A}_{\sigma(t)}x(t) + \hat{A}_d x(t - d_{\sigma(t)}(t)) + Dw(t), \\ z(t) &= \hat{F}_{\sigma(t)}x(t) + Gw(t), \\ x(\theta) &= \varphi(\theta), \theta \in [-h_2, 0], \end{aligned} \tag{10}$$

where $\hat{A}_{\sigma(t)} = \hat{A} + BK_{\sigma(t)}, \ \hat{F}_{\sigma(t)} = \hat{F} + HK_{\sigma(t)}$.

Definition 5 [32] For a given time constant T_f, system (6) with $u(t) \equiv 0$ and $w(t) \equiv 0$ is said to be finite-time stable (FTS) with respect to $(e_1, e_2, T_f, R, \sigma(t))$ if (5) holds, where $e_2 > e_1 > 0$, $R > 0$, $\sigma(t)$ is a switching signal.

Definition 6 [32] For a given time constant T_f, system (6) is said to be robust finite-time stabilizable with H_∞ performance γ, if there exists a controller $u(t) = K_{\sigma(t)}x(t)$, where $t \in [0, T_f]$, such that

(i) the closed-loop system (10) is FTB with respect to $(e_1, e_2, T_f, d_w^2, R, \sigma(t))$;
(ii) under zero initial condition, the following inequality holds

$$\int_0^{T_f} z^T(s)z(s)\mathrm{d}s \le \gamma^2 \int_0^{T_f} w^T(s)w(s)\mathrm{d}s, \tag{11}$$

where $e_2 > e_1 > 0$, $\gamma > 0$, $R > 0$, $\sigma(t)$ is a switching signal and $w(t)$ satisfies (2).

Lemma 1 [17] *Suppose L, M and $\Xi(t)$ are real matrices of appropriate dimensions and $\Xi(t)$ satisfies $\Xi^T(t)\Xi(t) \le I$. Then for any scalar $\varepsilon > 0$,*

$$L\Xi(t)M + M^T\Xi^T(t)L^T \le \varepsilon LL^T + \varepsilon^{-1}M^TM.$$

3 Main Results

The main target of this section is to construct a state feedback controller (9) such that the system (10) is FTB with H_∞ performance.

3.1 Finite-Time Boundedness Analysis

In this subsection, the FTB for the following delay system is considered

$$\begin{aligned}\dot{x}(t) &= \hat{A}_{\sigma(t)}x(t) + \hat{A}_d x(t - d_{\sigma(t)}(t)) + Dw(t),\\ x(\theta) &= \varphi(\theta), \theta \in [-h_2, 0].\end{aligned} \tag{12}$$

Before we refer to the prime development of this paper, two lemmas will be given first.

Consider the following delay system

$$\begin{aligned}\dot{x}(t) &= \hat{A}_1 x(t) + \hat{A}_d x(t - d_1(t)) + Dw(t),\\ x(\theta) &= \varphi(\theta), \theta \in [-h_1, 0].\end{aligned} \tag{13}$$

Choose the Lyapunov functional

$$V_1(t) = V_{11}(t) + V_{12}(t) + V_{13}(t) + V_{14}(t), \tag{14}$$

where

$$\begin{aligned}
V_{11}(t) &= x^T(t)P_1x(t),\\
V_{12}(t) &= \int_{-h_1}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)e^{\alpha(s-t)}Q_1\dot{x}(s)\mathrm{d}s\mathrm{d}\theta,\\
V_{13}(t) &= \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} \dot{x}^T(s)e^{\alpha(s-t)}Q_2\dot{x}(s)\mathrm{d}sd\theta,\\
V_{14}(t) &= \int_{t-d_1(t)}^{t} x^T(s)e^{\alpha(s-t)}Z_1x(s)\mathrm{d}s,\\
h_2 &> h_1 > 0, \quad P_1 > 0,\ Q_1 > 0,\ Q_2 > 0,\ \text{and } Z_1 > 0.
\end{aligned}$$

Lemma 2 *Consider the system* (13)*, for given constants* $\varepsilon > 0$, $\alpha > 0$, $h_2 > h_1 > 0$, *if there exist positive definite symmetric matrices* X_1, $\overline{Q}_i$ $(i = 1, 2)$, $\overline{Z}_1$, T_1, *and any matrices* M_{11}, M_{22}, Y_1, $\overline{M}_j$ $(j = 1, 2, 3)$ *with appropriate dimensions such that*

$$\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 & \Delta_5\\ * & -T_1 & c_0\overline{M}_3 & \Delta_6 & 0\\ * & * & c_0(\overline{Q}_1 - 2X_1) & 0 & 0\\ * & * & * & \Delta_7 & 0\\ * & * & * & * & \Delta_8 \end{bmatrix} \leq 0, \tag{15}$$

then under the state feedback controller $u(t) = K_1x(t)$ *with* $K_1 = Y_1X_1^{-1}$, *we have*

$$V_1(t) \leq e^{-\alpha(t-t_0)}V_1(t_0) + \int_{t_0}^{t} e^{-\alpha(t-s)}w^T(s)T_1w(s)\mathrm{d}s, \tag{16}$$

where

$$\Delta_1 = \begin{bmatrix} \Delta_{111} & \Delta_{112} \\ * & \Delta_{122} \end{bmatrix},$$

$$\Delta_{111} = X_1A^T + AX_1 + BY_1 + Y_1^TB^T + \alpha X_1 + \overline{Z}_1 + \overline{M}_1 + \overline{M}_1^T + 2\varepsilon LL^T,$$

$$\Delta_{112} = A_dX_1 + \overline{M}_2^T - \overline{M}_1,$$

$$\Delta_{122} = -(1-d)e^{-\alpha h_1}\overline{Z}_1 - \overline{M}_2 - \overline{M}_2^T,$$

$$\Delta_2 = \begin{bmatrix} D^T + \overline{M}_3 & -\overline{M}_3 \end{bmatrix}^T,$$

$$\Delta_3 = \begin{bmatrix} c_0\overline{M}_1^T & c_0\overline{M}_2^T \end{bmatrix}^T,$$

$$\Delta_4 = \begin{bmatrix} h_1\left(X_1A^T + Y_1^TB^T\right) & h_{21}\left(X_1A^T + Y_1^TB^T\right) \\ h_1X_1A_d^T & h_{21}X_1A_d^T \end{bmatrix},$$

$$\Delta_5 = \mathrm{diag}\{2X_1M_{11}^T, 2X_1M_{22}^T\},$$

$$\Delta_6 = \begin{bmatrix} h_1D^T & h_{21}D^T \end{bmatrix},$$

$$\Delta_7 = \begin{bmatrix} 2\varepsilon h_1^2LL^T - h_1\overline{Q}_1 & 2\varepsilon h_1h_{21}LL^T \\ * & 2\varepsilon h_{21}^2LL^T - h_{21}\overline{Q}_2 \end{bmatrix},$$

$$\Delta_8 = \mathrm{diag}\{-2\varepsilon I, -2\varepsilon I\},\ c_0 = \frac{e^{\alpha h_1} - 1}{\alpha},\ h_{21} = h_2 - h_1.$$

Proof Under the conditions of Lemma 2, we set

$$P_1 = X_1^{-1},\quad Q_1 = \overline{Q}_1^{-1},\quad Q_2 = \overline{Q}_2^{-1},\quad Z_1 = P_1\overline{Z}_1P_1,$$
$$M_1 = P_1\overline{M}_1P_1,\quad M_2 = P_1\overline{M}_2P_1,\quad M_3 = \overline{M}_3P_1.$$

From $\overline{Q}_1 > 0$ and $X_1 > 0$, we can get

$$(\overline{Q}_1 - X_1)^T\overline{Q}_1^{-1}(\overline{Q}_1 - X_1) \geq 0.$$

Then by simplifying, we have

$$\overline{Q}_1 - 2X_1 \geq -X_1\overline{Q}_1^{-1}X_1. \tag{17}$$

Substituting (17) into (15), then implementing a congruent transformation by $\mathrm{diag}\{P_1, P_1, I, P_1, I, I, I, I\}$, the following inequality is got

$$\begin{bmatrix} \overline{\Delta}_1 & \overline{\Delta}_2 & \overline{\Delta}_3 & \overline{\Delta}_4 & \overline{\Delta}_5 \\ * & -T_1 & c_0M_3 & \Delta_6 & 0 \\ * & * & -c_0Q_1 & 0 & 0 \\ * & * & * & \Delta_7 & 0 \\ * & * & * & * & \Delta_8 \end{bmatrix} \leq 0, \tag{18}$$

where

$$
\begin{aligned}
\overline{\Delta}_1 &= \begin{bmatrix} \overline{\Delta}_{111} & \overline{\Delta}_{112} \\ * & \overline{\Delta}_{122} \end{bmatrix}, \\
\overline{\Delta}_{111} &= A^T P_1 + P_1 A + P_1 B K_1 + K_1^T B^T P_1 + \alpha P_1 + Z_1 + M_1 + M_1^T + 2\varepsilon P_1 L L^T P_1, \\
\overline{\Delta}_{112} &= P_1 A_d + M_2^T - M_1, \\
\overline{\Delta}_{122} &= -(1-d)e^{-\alpha h_1} Z_1 - M_2 - M_2^T, \\
\overline{\Delta}_2 &= \begin{bmatrix} D^T P_1 + M_3 & -M_3 \end{bmatrix}^T, \\
\overline{\Delta}_3 &= \begin{bmatrix} c_0 M_1^T & c_0 M_2^T \end{bmatrix}^T, \\
\overline{\Delta}_4 &= \begin{bmatrix} h_1 \left(A^T + K_1^T B^T\right) & h_{21} \left(A^T + K_1^T B^T\right) \\ h_1 A_d^T & h_{21} A_d^T \end{bmatrix}, \\
\overline{\Delta}_5 &= \mathrm{diag}\{2M_{11}^T, 2M_{22}^T\}.
\end{aligned}
$$

Using Schur complement lemma, it can be concluded

$$
\begin{bmatrix} \widetilde{\Delta}_1 & \overline{\Delta}_2 & \overline{\Delta}_3 & \overline{\Delta}_4 \\ * & -T_1 & c_0 M_3 & \Delta_6 \\ * & * & -c_0 Q_1 & 0 \\ * & * & * & \widetilde{\Delta}_7 \end{bmatrix} + \varepsilon^{-1} \Pi_1 \Pi_1^T + \varepsilon^{-1} \Pi_2 \Pi_2^T + 2\varepsilon \Pi_3 \Pi_3^T \le 0, \quad (19)
$$

where

$$
\begin{aligned}
\widetilde{\Delta}_1 &= \begin{bmatrix} \widetilde{\Delta}_{111} & \overline{\Delta}_{112} \\ * & \widetilde{\Delta}_{122} \end{bmatrix}, \\
\widetilde{\Delta}_{111} &= (A + BK_1)^T P_1 + P_1(A + BK_1) + \alpha P_1 \\
&\quad + \ Z_1 + M_1 + M_1^T + 2\varepsilon P_1 L L^T P_1 + \varepsilon^{-1} M_{11}^T M_{11}, \\
\widetilde{\Delta}_{122} &= -(1-d)e^{-\alpha h_1} Z_1 - M_2 - M_2^T + \varepsilon^{-1} M_{22}^T M_{22}, \\
\widetilde{\Delta}_7 &= \mathrm{diag}\left\{-h_1 Q_1^{-1}, -h_{21} Q_2^{-1}\right\}, \\
\Pi_1 &= \begin{bmatrix} M_{11} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
\Pi_2 &= \begin{bmatrix} 0 & M_{22} & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
\Pi_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 & h_1 L^T & h_{21} L^T \end{bmatrix}^T.
\end{aligned}
$$

According to Lemma 1, the following inequalities hold

$$
\begin{aligned}
\Pi_1 \Xi(t)^T \Pi_3^T + \Pi_3 \Xi(t) \Pi_1^T &\le \varepsilon^{-1} \Pi_1 \Pi_1^T + \varepsilon \Pi_3 \Pi_3^T, \\
\Pi_2 \Xi(t)^T \Pi_3^T + \Pi_3 \Xi(t) \Pi_2^T &\le \varepsilon^{-1} \Pi_2 \Pi_2^T + \varepsilon \Pi_3 \Pi_3^T, \\
\Pi_4 \Xi(t) \Pi_2^T + \Pi_2 \Xi(t)^T \Pi_4^T &\le \varepsilon^{-1} \Pi_2 \Pi_2^T + \varepsilon \Pi_4 \Pi_4^T, \\
P_1 L \Xi(t) M_{11} + M_{11}^T \Xi^T(t) L^T P_1 &\le \varepsilon^{-1} M_{11}^T M_{11} + \varepsilon P_1 L L^T P_1,
\end{aligned}
$$

where

$$\Pi_4 = \begin{bmatrix} L^T P_1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T.$$

These together with (19) give

$$\begin{bmatrix} \overrightarrow{\Delta}_1 & \overline{\Delta}_2 & \overline{\Delta}_3 & \overrightarrow{\Delta}_4 \\ * & -T_1 & c_0 M_3 & \Delta_6 \\ * & * & -c_0 Q_1 & 0 \\ * & * & * & \widetilde{\Delta}_7 \end{bmatrix} \leq 0, \tag{20}$$

where

$$\begin{aligned}
\overrightarrow{\Delta}_1 &= \begin{bmatrix} \overrightarrow{\Delta}_{111} & \overrightarrow{\Delta}_{112} \\ * & \overrightarrow{\Delta}_{122} \end{bmatrix}, \\
\overrightarrow{\Delta}_{111} &= \hat{A}_1^T P_1 + P_1 \hat{A}_1 + \alpha P_1 + Z_1 + M_1 + M_1^T, \\
\overrightarrow{\Delta}_{112} &= P_1 \hat{A}_d - M_1 + M_2^T, \\
\overrightarrow{\Delta}_{122} &= -(1-d)e^{-\alpha h_1} Z_1 - M_2 - M_2^T, \\
\overrightarrow{\Delta}_4 &= \begin{bmatrix} h_1 \hat{A}_1^T & h_{21} \hat{A}_1^T \\ h_1 \hat{A}_d^T & h_{21} \hat{A}_d^T \end{bmatrix}, \\
\hat{A}_1 &= \hat{A} + BK_1.
\end{aligned}$$

Using Schur complement lemma to (20), it follows

$$\Delta' + c_0 M Q_1^{-1} M^T \leq 0, \tag{21}$$

where

$$\begin{aligned}
\Delta' &= \begin{bmatrix} \Delta'_{11} & \Delta'_{12} & \Delta'_{13} \\ * & \Delta'_{22} & \Delta'_{23} \\ * & * & \Delta'_{33} \end{bmatrix}, \\
\Delta'_{11} &= \hat{A}_1^T P_1 + P_1 \hat{A}_1 + \alpha P_1 + Z_1 + M_1 + M_1^T + h_1 \hat{A}_1^T Q_1 \hat{A}_1 + h_{21} \hat{A}_1^T Q_2 \hat{A}_1, \\
\Delta'_{12} &= P_1 \hat{A}_d - M_1 + M_2^T + h_1 \hat{A}_1^T Q_1 \hat{A}_d + h_{21} \hat{A}_1^T Q_2 \hat{A}_d, \\
\Delta'_{13} &= P_1 D + M_3^T + h_1 \hat{A}_1^T Q_1 D + h_{21} \hat{A}_1^T Q_2 D, \\
\Delta'_{22} &= -(1-d)e^{-\alpha h_1} Z_1 - M_2 - M_2^T + h_1 \hat{A}_d^T Q_1 \hat{A}_d + h_{21} \hat{A}_d^T Q_2 \hat{A}_d, \\
\Delta'_{23} &= -M_3^T + h_1 \hat{A}_d^T Q_1 D + h_{21} \hat{A}_d^T Q_2 D, \\
\Delta'_{33} &= h_1 D^T Q_1 D + h_{21} D^T Q_2 D - T_1, \\
M &= \begin{bmatrix} M_1^T & M_2^T & M_3^T \end{bmatrix}^T,
\end{aligned}$$

and c_0 is defined in Lemma 2. □

On the other hand, for matrix M, we get

$$2\xi_1^T(t)M \times [x(t) - x(t-d_1(t)) - \int_{t-d_1(t)}^{t} \dot{x}(s)\mathrm{d}s] = 0,$$

where $\xi_1(t) = \left[x^T(t)\ x^T(t-d_1(t))\ w^T(t) \right]^T$.
Calculating the derivative of Lyapunov functional (14) along the trajectory of the system (13), we have

$$\begin{aligned}
&\dot{V}_1(t) + \alpha V_1(t) - w^T(t)T_1 w(t) \\
&\quad \le 2\dot{x}^T(t)P_1 x(t) + h_1\dot{x}^T(t)Q_1\dot{x}(t) + \alpha x^T(t)P_1 x(t) \\
&\quad - \int_{t-h_1}^{t} \dot{x}^T(s)e^{\alpha(s-t)}Q_1\dot{x}(s)\mathrm{d}s + h_{21}\dot{x}^T(t)Q_2\dot{x}(t) \\
&\quad - \int_{t-h_2}^{t-h_1} \dot{x}^T(s)e^{\alpha(s-t)}Q_2\dot{x}(s)\mathrm{d}s + x^T(t)Z_1 x(t) \\
&\quad - (1-d)x^T(t-d_1(t))e^{-\alpha h_1}Z_1 x(t-d_1(t)) - w^T(t)T_1 w(t) \\
&\quad \le \xi_1^T(t)[\Delta' + c_0 M Q_1^{-1} M^T]\xi_1(t).
\end{aligned} \tag{22}$$

Thus, it follows from (21) and (22) that

$$\dot{V}_1(t) + \alpha V_1(t) - w^T(t)T_1 w(t) \le 0. \tag{23}$$

Using differential inequality theory, we easily obtain (16).
Now, consider the following delay system

$$\begin{aligned}
&\dot{x}(t) = \hat{A}_2 x(t) + \hat{A}_d x(t-d_2(t)) + Dw(t), \\
&x(\theta) = \varphi(\theta), \theta \in [-h_2, 0].
\end{aligned} \tag{24}$$

Choose the Lyapunov functional

$$V_2(t) = V_{21}(t) + V_{22}(t) + V_{23}(t) + V_{24}(t), \tag{25}$$

where

$$\begin{aligned}
V_{21}(t) &= x^T(t)P_2 x(t), \\
V_{22}(t) &= \int_{-h_1}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)e^{\beta(t-s)}Q_3\dot{x}(s)\mathrm{d}s\mathrm{d}\theta, \\
V_{23}(t) &= \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} \dot{x}^T(s)e^{\beta(t-s)}Q_4\dot{x}(s)\mathrm{d}s\mathrm{d}\theta, \\
V_{24}(t) &= \int_{t-d_2(t)}^{t} x^T(s)e^{\beta(t-s)}Z_2 x(s)\mathrm{d}s, \\
&\quad h_2 > h_1 > 0, \quad P_2 > 0,\ Q_3 > 0,\ Q_4 > 0, \text{ and } Z_2 > 0.
\end{aligned}$$

Lemma 3 *Consider the system (24), for given constants $\varepsilon > 0$, $\beta > 0$, $h_2 > h_1 > 0$, if there exist positive definite symmetric matrices X_2, $\overline{Q}_i$ $(i = 3, 4)$, $\overline{Z}_2$, T_2, and any matrices M_{11}, M_{22}, Y_2, $\overline{N}_j$, $\overline{R}_j$, $\overline{S}_j$ $(j = 1, 2, 3, 4, 5)$ with appropriate dimensions such that*

$$\begin{bmatrix} \Omega_1 & \Omega_2 & \Omega_3 & \Omega_4 & \Omega_5 & \Omega_6 \\ * & \Omega_7 & \Omega_8 & \Omega_9 & 0 & 0 \\ * & * & -T_2 & \Omega_{10} & \Omega_{11} & 0 \\ * & * & * & \Omega_{12} & 0 & 0 \\ * & * & * & * & \Omega_{13} & 0 \\ * & * & * & * & * & \Omega_{14} \end{bmatrix} \leq 0, \tag{26}$$

then under the state feedback controller $u(t) = K_2 x(t)$ with $K_2 = Y_2 X_2^{-1}$, we have

$$V_2(t) \leq e^{\beta(t-t_0)} V_2(t_0) + \int_{t_0}^{t} e^{\beta(t-s)} w^T(s) T_2 w(s) \mathrm{d}s, \tag{27}$$

where

$$\begin{aligned}
\Omega_1 &= \begin{bmatrix} \Omega_{111} & \Omega_{112} \\ * & \Omega_{122} \end{bmatrix}, \\
\Omega_{111} &= AX_2 + X_2 A^T + BY_2 + Y_2^T B^T - \beta X_2 + \overline{Z}_2 + \overline{N}_1 + \overline{N}_1^T + 2\varepsilon LL^T, \\
\Omega_{112} &= A_d X_2 + \overline{N}_2^T - \overline{R}_1 + \overline{S}_1, \\
\Omega_{122} &= -(1-d)e^{\beta h_1} \overline{Z}_2 - \overline{R}_2 - \overline{R}_2^T + \overline{S}_2 + \overline{S}_2^T, \\
\Omega_2 &= \begin{bmatrix} \overline{N}_3^T - \overline{N}_1 + \overline{R}_1 & \overline{N}_4^T - \overline{S}_1 \\ \overline{S}_3^T - \overline{R}_3^T - \overline{N}_2 + \overline{R}_2 & \overline{S}_4^T - \overline{R}_4^T - \overline{S}_2 \end{bmatrix}, \\
\Omega_3 &= \begin{bmatrix} D^T + \overline{N}_5 & \overline{S}_5 - \overline{R}_5 \end{bmatrix}^T, \\
\Omega_4 &= \begin{bmatrix} c_1 \overline{N}_1 & c_2 \overline{R}_1 & c_2 \overline{S}_1 \\ c_1 \overline{N}_2 & c_2 \overline{R}_2 & c_2 \overline{S}_2 \end{bmatrix}, \\
\Omega_5 &= \begin{bmatrix} h_1(X_2 A^T + Y_2^T B^T) & h_{21}(X_2 A^T + Y_2^T B^T) \\ h_1 X_2 A_d^T & h_{21} X_2 A_d^T \end{bmatrix}, \\
\Omega_6 &= \mathrm{diag}\{2X_2 M_{11}^T, 2X_2 M_{22}^T\}, \\
\Omega_7 &= \begin{bmatrix} \overline{R}_3^T + \overline{R}_3 - \overline{N}_3^T - \overline{N}_3 & -\overline{N}_4^T - \overline{S}_3 + \overline{R}_4^T \\ * & -\overline{S}_4^T - \overline{S}_4 \end{bmatrix}, \\
\Omega_8 &= \begin{bmatrix} \overline{R}_5 - \overline{N}_5 & -\overline{S}_5 \end{bmatrix}^T, \\
\Omega_9 &= \begin{bmatrix} c_1 \overline{N}_3 & c_2 \overline{R}_3 & c_2 \overline{S}_3 \\ c_1 \overline{N}_4 & c_2 \overline{R}_4 & c_2 \overline{S}_4 \end{bmatrix}, \\
\Omega_{10} &= \begin{bmatrix} c_1 \overline{N}_5 & c_2 \overline{R}_5 & c_2 \overline{S}_5 \end{bmatrix}, \\
\Omega_{11} &= \begin{bmatrix} h_1 D^T & h_{21} D^T \end{bmatrix}, \\
\Omega_{12} &= \mathrm{diag}\{c_1(\overline{Q}_3 - 2X_2), c_2(\overline{Q}_4 - 2X_2), c_2(\overline{Q}_4 - 2X_2)\},
\end{aligned}$$

$$\Omega_{13} = \begin{bmatrix} 2\varepsilon h_1^2 LL^T - h_1\overline{Q}_3 & 2\varepsilon h_1 h_{21} LL^T \\ * & 2\varepsilon h_{21}^2 LL^T - h_{21}\overline{Q}_4 \end{bmatrix},$$
$$\Omega_{14} = \text{diag}\{-2\varepsilon I, -2\varepsilon I\},$$
$$c_1 = \frac{e^{\beta h_1} - 1}{\beta},\ c_2 = \frac{e^{\beta h_2} - e^{\beta h_1}}{\beta}.$$

Proof Following the similar proof of Lemmas 2, 3 can be derived, it is omitted. □

Theorem 1 *Consider the system (12), for given constants* $\varepsilon > 0$, $\alpha > 0$, $\beta > 0$, $h_2 > h_1 > 0$, *if there exist positive definite symmetric matrices* X_i, $\overline{Z}_i$, T_i $(i = 1, 2)$, $\overline{Q}_j$ $(j = 1, 2, 3, 4)$, *and any matrices* M_{11}, M_{22}, Y_i, $\overline{M}_i$ $(i = 1, 2, 3)$, $\overline{N}_j$, $\overline{R}_j$, $\overline{S}_j$ $(j = 1, 2, 3, 4, 5)$ *with appropriate dimensions such that matrix inequalities (15) and (26) hold, then under the state feedback controller* $u(t) = K_i x(t)$ *with* $K_i = Y_i X_i^{-1}$ $(i = 1, 2)$, *the system (12) is FTB with respect to* $(e_1, e_2, T_f, d_w^2, R, \sigma(t))$, *where*

$$e_2 = \frac{\lambda e_1 + \lambda_5 d_w^2}{\lambda_6} e^{\alpha_1 T_f + 2c + \alpha_1 \eta},$$
$$c = \frac{(\beta + \alpha^*)(\alpha - \alpha^*)}{\alpha + \beta}\eta,$$
$$\lambda_1 = \lambda_{\max}\left(R^{-\frac{1}{2}} X_1^{-1} R^{-\frac{1}{2}}\right),$$
$$\lambda_2 = \lambda_{\max}\left(R^{-\frac{1}{2}} \overline{Q}_1^{-1} R^{-\frac{1}{2}}\right),$$
$$\lambda_3 = \lambda_{\max}\left(R^{-\frac{1}{2}} \overline{Q}_2^{-1} R^{-\frac{1}{2}}\right),$$
$$\lambda_4 = \lambda_{\max}\left(R^{-\frac{1}{2}} X_1^{-1} \overline{Z}_1 X_1^{-1} R^{-\frac{1}{2}}\right),$$
$$\lambda_5 = max_{i=1,2}\{\lambda_{\max}(T_i)\},$$
$$\lambda_6 = min_{i=1,2}\left\{\lambda_{\min}\left(R^{-\frac{1}{2}} X_i^{-1} R^{-\frac{1}{2}}\right)\right\},$$
$$\lambda = \lambda_1 + \frac{h_1^2}{2}\lambda_2 + \frac{h_2^2 - h_1^2}{2}\lambda_3 + h_1\lambda_4.$$

And switching signal $\sigma(t)$ *satisfies the following two conditions:*
(C1) the length rate of LDP satisfies $\frac{T^+(p_m, p_{m+1})}{T^-(p_m, p_{m+1})} \leq \frac{\alpha - \alpha^*}{\beta + \alpha^*}$, $\alpha^* \in (0, \alpha)$,
(C2) the frequency of LDP satisfies $F_l(p_m, p_{m+1}) \leq \frac{\alpha_1}{\ln(\mu^2 \mu_1)}$, $\alpha_1 \in (0, \alpha^*)$, $\forall$ $m \in \overline{\mathbb{N}}$, *where* $\mu \geq 1$ *satisfies*

$$\begin{aligned} & X_i \leq \mu X_j, \overline{Z}_i \leq \mu \overline{Z}_j, \quad \forall i, j \in \{1, 2\}, \\ & \overline{Q}_m \leq \mu \overline{Q}_n, \quad \forall \{m, n\} \in \{\{1, 3\}, \{2, 4\}\}, \end{aligned} \tag{28}$$

and

$$\mu_1 = e^{(\alpha + \beta) h_2}. \tag{29}$$

Proof Construct the following piecewise Lyapunov functional

$$V(t) = V_{\sigma(t)}(t) = \begin{cases} V_1(t), & t \in [t_{2k}, t_{2k+1}), \\ V_2(t), & t \in [t_{2k+1}, t_{2k+2}), k \in \mathbb{N}, \end{cases} \tag{30}$$

where $V_1(t)$ and $V_2(t)$ are defined in (14) and (25), respectively.
From (28) and (29), we have

$$V_1(t) \leq \mu V_2(t), V_2(t) \leq \mu\mu_1 V_1(t). \tag{31}$$

For the Lyapunov functional (30), based on Lemma 2 and Lemma 3, it is easy to see that

$$V(t) \leq \begin{cases} e^{-\alpha(t-t_{2k})} V_1(t_{2k}) \\ \quad + \int_{t_{2k}}^{t} e^{-\alpha(t-s)} w^T(s) T_1 w(s) \mathrm{d}s, & t \in [t_{2k}, t_{2k+1}), \\ e^{\beta(t-t_{2k+1})} V_2(t_{2k+1}) \\ \quad + \int_{t_{2k+1}}^{t} e^{\beta(t-s)} w^T(s) T_2 w(s) \mathrm{d}s, & t \in [t_{2k+1}, t_{2k+2}), k \in \mathbb{N}. \end{cases} \tag{32}$$

Without loss of generality, we assume that $t \in [t_{2k+1}, t_{2k+2}) \subseteq [p_m, p_{m+1}) \subseteq [t_0, T_f]$, where $k \in \mathbb{N}, m \in \overline{\mathbb{N}}$. From (31) and (32), along the trajectory of system (12), the Lyapunov functional (30) satisfies

$$\begin{aligned} V(t) &\leq e^{\beta(t-t_{2k+1})} V_2(t_{2k+1}) + \int_{t_{2k+1}}^{t} e^{\beta(t-s)} w^T(s) T_2 w(s) \mathrm{d}s \\ &\leq \mu\mu_1 e^{\beta(t-t_{2k+1})} V_1(t_{2k+1}^-) + \int_{t_{2k+1}}^{t} e^{\beta(t-s)} w^T(s) T_2 w(s) \mathrm{d}s \\ &\leq \mu\mu_1 e^{\beta T^+(t_{2k},t)-\alpha T^-(t_{2k},t)} V_1(t_{2k}) \\ &\quad + \mu\mu_1 \int_{t_{2k}}^{t_{2k+1}} e^{\beta T^+(s,t)-\alpha T^-(s,t)} w^T(s) T_1 w(s) \mathrm{d}s \\ &\quad + \int_{t_{2k+1}}^{t} e^{\beta T^+(s,t)} w^T(s) T_2 w(s) \mathrm{d}s \\ &\leq \cdots \leq \mu^{N_\sigma(t_0,t)} \mu_1^{N_l(t_0,t)} e^{\beta T^+(t_0,t)-\alpha T^-(t_0,t)} V_1(t_0) \\ &\quad + \sum_{j=0}^{k} \int_{t_{2j}}^{t_{2j+1}} \mu^{N_\sigma(s,t)} \mu_1^{N_l(s,t)} e^{\beta T^+(s,t)-\alpha T^-(s,t)} w^T(s) T_1 w(s) \mathrm{d}s \\ &\quad + \sum_{j=0}^{k-1} \int_{t_{2j+1}}^{t_{2j+2}} \mu^{N_\sigma(s,t)} \mu_1^{N_l(s,t)} e^{\beta T^+(s,t)-\alpha T^-(s,t)} w^T(s) T_2 w(s) \mathrm{d}s \\ &\quad + \int_{t_{2k+1}}^{t} \mu^{N_\sigma(s,t)} \mu_1^{N_l(s,t)} e^{\beta T^+(s,t)-\alpha T^-(s,t)} w^T(s) T_2 w(s) \mathrm{d}s \end{aligned}$$

$$\leq \mu^{N_\sigma(t_0,t)}\mu_1^{N_l(t_0,t)}e^{\beta T^+(t_0,t)-\alpha T^-(t_0,t)}V_1(t_0)+max_{i=1,2}\{\lambda_{\max}(T_i)\}\cdot \int_{t_0}^{t}\mu^{N_\sigma(t_0,t)}\mu_1^{N_l(t_0,t)}e^{\beta T^+(s,t)-\alpha T^-(s,t)}w^T(s)w(s)\mathrm{d}s. \tag{33}$$

We get from (C1) and (C2) that

$$\beta T^+(t_0,t)-\alpha T^-(t_0,t)\leq -\alpha^*(t-t_0)+c, \tag{34}$$

$$\mu^{N_\sigma(t_0,t)}\mu_1^{N_l(t_0,t)}\leq e^{\alpha_1(t-t_0)+\alpha_1\eta}, \tag{35}$$

and

$$\beta T^+(s,t)-\alpha T^-(s,t)\leq -\alpha^*(t-s)+2c\leq 2c. \tag{36}$$

Suppose $t_0=0$. Substituting (34), (35) and (36) to (33) leads to

$$V(t)\leq e^{(\alpha_1-\alpha^*)t}e^{c}e^{\alpha_1\eta}V_1(0)+max_{i=1,2}\{\lambda_{\max}(T_i)\}e^{\alpha_1 t}e^{2c}e^{\alpha_1\eta}d_w^2, \tag{37}$$

and

$$\begin{aligned} V_1(0)&\leq \lambda_1 x^T(0)Rx(0)+\lambda_2\int_{-h_1}^{0}\int_{\theta}^{0}\dot{x}^T(s)R\dot{x}(s)\mathrm{d}s\mathrm{d}\theta \\ &\quad+\lambda_3\int_{-h_2}^{-h_1}\int_{\theta}^{0}\dot{x}^T(s)R\dot{x}(s)\mathrm{d}s\mathrm{d}\theta+\lambda_4\int_{-d_1(0)}^{0}x^T(s)Rx(s)\mathrm{d}s \\ &\leq [\lambda_1+\frac{h_1^2}{2}\lambda_2+\frac{h_2^2-h_1^2}{2}\lambda_3+h_1\lambda_4]\sup_{-h_2\leq t\leq 0}\left\{x^T(t)Rx(t),\dot{x}^T(t)R\dot{x}(t)\right\} \\ &\leq \lambda e_1. \end{aligned} \tag{38}$$

It implies that

$$V(t)\leq e^{(\alpha_1-\alpha^*)t}e^{c}e^{\alpha_1\eta}\lambda e_1+\lambda_5 e^{\alpha_1 t}e^{2c}e^{\alpha_1\eta}d_w^2. \tag{39}$$

Hence, $\forall\, t\in[0,T_f]$,

$$\begin{aligned} x^T(t)Rx(t)&\leq \frac{V(t)}{min_{i=1,2}\left\{\lambda_{\min}\left(R^{-\frac{1}{2}}X_i^{-1}R^{-\frac{1}{2}}\right)\right\}} \\ &\leq \frac{e^{(\alpha_1-\alpha^*)t}e^{c}e^{\alpha_1\eta}\lambda e_1+\lambda_5 e^{\alpha_1 t}e^{2c}e^{\alpha_1\eta}d_w^2}{min_{i=1,2}\left\{\lambda_{\min}\left(R^{-\frac{1}{2}}X_i^{-1}R^{-\frac{1}{2}}\right)\right\}} \\ &< \frac{\left(\lambda e_1+\lambda_5 d_w^2\right)}{\lambda_6}e^{\alpha_1 T_f+2c+\alpha_1\eta} \\ &= e_2. \end{aligned} \tag{40}$$

Thus, the system (12) is FTB. □

Corollary 1 *Consider the system (12) with* $w(t)=0$*, for given constants* $\varepsilon>0$*,* $\alpha>0$*,* $\beta>0$*,* $h_2>h_1>0$*, if there exist positive definite symmetric matrices* X_i*,*

$\overline{Z}_i$ $(i = 1, 2)$, $\overline{Q}_j$ $(j = 1, 2, 3, 4)$, *and any matrices* M_{11}, M_{22}, Y_i, $\overline{M}_i$ $(i = 1, 2)$, $\overline{N}_j$, $\overline{R}_j$, $\overline{S}_j$ $(j = 1, 2, 3, 4)$ *with appropriate dimensions such that*

$$\begin{bmatrix} \varDelta_1 & \varDelta_3 & \varDelta_4 & \varDelta_5 \\ * & c_0(\overline{Q}_1 - 2X_1) & 0 & 0 \\ * & * & \varDelta_7 & 0 \\ * & * & * & \varDelta_8 \end{bmatrix} \leq 0,$$

$$\begin{bmatrix} \varOmega_1 & \varOmega_2 & \varOmega_4 & \varOmega_5 & \varOmega_6 \\ * & \varOmega_7 & \varOmega_9 & 0 & 0 \\ * & * & \varOmega_{12} & 0 & 0 \\ * & * & * & \varOmega_{13} & 0 \\ * & * & * & * & \varOmega_{14} \end{bmatrix} \leq 0,$$

then under the state feedback controller $u(t) = K_i x(t)$ *with* $K_i = Y_i X_i^{-1}$ $(i = 1, 2)$, *the system* (12) *is FTS with respect to* $(e_1, e_2, T_f, R, \sigma(t))$, *where switching signal* $\sigma(t)$ *satisfies (C1) and (C2),* $\mu \geq 1$ *satisfies* (28), μ_1 *satisfies* (29), *and* $\varDelta_i$, $\varOmega_j$ *are defined in Theorem* 1, $i = 1, 3, 4, 5, 7, 8,$ $j = 1, 2, 4, 5, 6, 7, 9, 12, 13, 14$.

Remark 3 If there is no a LDP, the system (1) is identical with a general delay system. Thus, the obtained results in Corollary 1 are more universal than the existing ones in [18].

Remark 4 If the LDPs appear, we know that $h_1 < d(t) \leq h_2$. In other words, we have $d(t) = h_1 + d_2(t)$ in LDPs, where $0 < d_2(t) \leq h_2 - h_1$. Thus, Corollary 1 has wider applicability than the conclusions in [12]. In addition, the switching method is used in this paper, which is different from the studies in [12]. Furthermore, the time length of the LDPs can be calculated so that more accurate results can be obtained in some degree.

3.2 Robust Finite-Time H_∞ Control

Theorem 2 *Consider the system* (10), *for given constants* $\varepsilon > 0$, $\alpha > 0$, $\beta > 0$, $h_2 > h_1 > 0$, *if there exist positive definite symmetric matrices* X_i, $\overline{Z}_i$ $(i = 1, 2)$, $\overline{Q}_j$ $(j = 1, 2, 3, 4)$, *and any matrices* M_{11}, M_{22}, M_{33}, Y_i, $\overline{M}_i$ $(i = 1, 2, 3)$, $\overline{N}_j$, $\overline{R}_j$, $\overline{S}_j$ $(j = 1, 2, 3, 4, 5)$ *with appropriate dimensions such that*

$$\begin{bmatrix} \varDelta_1 & \varDelta_2 & \varDelta_3 & \check{\varDelta}_1 & \varDelta_4 & \check{\varDelta}_2 \\ * & -\gamma^2 I & c_0\overline{M}_3 & G^T & \varDelta_6 & 0 \\ * & * & c_0(\overline{Q}_1 - 2X_1) & 0 & 0 & 0 \\ * & * & * & \check{\varDelta}_3 & 0 & 0 \\ * & * & * & * & \varDelta_7 & 0 \\ * & * & * & * & * & \check{\varDelta}_4 \end{bmatrix} \leq 0, \tag{41}$$

$$\begin{bmatrix} \Omega_1 & \Omega_2 & \Omega_3 & \Omega_4 & \check{\Omega}_1 & \Omega_5 & \check{\Omega}_2 \\ * & \Omega_7 & \Omega_8 & \Omega_9 & 0 & 0 & 0 \\ * & * & -\gamma^2 I & \Omega_{10} & G^T & \Omega_{11} & 0 \\ * & * & * & \Omega_{12} & 0 & 0 & 0 \\ * & * & * & * & \check{\Delta}_3 & 0 & 0 \\ * & * & * & * & * & \Omega_{13} & 0 \\ * & * & * & * & * & * & \check{\Omega}_3 \end{bmatrix} \le 0, \tag{42}$$

then under the state feedback controller $u(t) = K_i x(t)$ *with* $K_i = Y_i X_i^{-1}$ $(i = 1, 2)$*, the system* (10) *is FTB with* H_∞ *performance with respect to* $(0, e_2, T_f, d_w^2, \gamma, R, \sigma(t))$*,* H_∞ *performance index is* $\bar{\gamma} = \gamma e^{\frac{\alpha T_f + \alpha_1 \eta}{2} + c}$*, where switching signal* $\sigma(t)$ *satisfies (C1) and (C2),* $\mu \ge 1$ *satisfies* (28)*,* μ_1 *satisfies* (29)*, and*

$$\begin{aligned}
\check{\Delta}_1 &= \begin{bmatrix} FX_1 + HY_1 & 0 \end{bmatrix}^T, \\
\check{\Delta}_2 &= \begin{bmatrix} 2X_1 M_{11}^T & X_1 M_{33}^T & 0 \\ 0 & 0 & 2X_1 M_{22}^T \end{bmatrix}, \\
\check{\Delta}_3 &= \varepsilon L L^T - I, \\
\check{\Delta}_4 &= \mathrm{diag}\{-2\varepsilon I, -\varepsilon I, -2\varepsilon I\}, \\
\check{\Omega}_1 &= \begin{bmatrix} FX_2 + HY_2 & 0 \end{bmatrix}^T, \\
\check{\Omega}_2 &= \begin{bmatrix} 2X_2 M_{11}^T & X_2 M_{33}^T & 0 \\ 0 & 0 & 2X_2 M_{22}^T \end{bmatrix}, \\
\check{\Omega}_3 &= \mathrm{diag}\{-2\varepsilon I, -\varepsilon I, -2\varepsilon I\},
\end{aligned}$$

$c_0, c_1, c_2, h_{21}, c, \Delta_i, \Omega_j$ *are defined in Theorem* 1*,* $i = 1, 2, \ldots, 7$*,* $j = 1, 2, \ldots, 13$.

Proof Choose the same Lyapunov functional as in Theorem 1, after some mathematical manipulation, we can get

$$\begin{cases} \dot{V}_1(t) + \alpha V_1(t) \le -(z^T(t)z(t) - \gamma^2 w^T(t)w(t)), \ \sigma(t) = 1, \\ \dot{V}_2(t) - \beta V_2(t) \le -(z^T(t)z(t) - \gamma^2 w^T(t)w(t)), \ \sigma(t) = 2. \end{cases} \tag{43}$$

Denoting $\Gamma(t) = z^T(t)z(t) - \gamma^2 w^T(t)w(t)$, and integrating both sides of the two inequalities in (43) from t_0 to t, it follows that

$$\begin{cases} V_1(t) \le e^{-\alpha(t-t_0)} V_1(t_0) - \displaystyle\int_{t_0}^{t} e^{-\alpha(t-s)} \Gamma(s) \mathrm{d}s, \\ V_2(t) \le e^{\beta(t-t_0)} V_2(t_0) - \displaystyle\int_{t_0}^{t} e^{\beta(t-s)} \Gamma(s) \mathrm{d}s. \end{cases} \tag{44}$$

Without loss of generality, we suppose that $t \in [t_{2k+1}, t_{2k+2}) \subseteq [p_m, p_{m+1}) \subseteq [t_0, T_f]$, where $k \in \mathbb{N}$, $m \in \overline{\mathbb{N}}$. Combining (28), (29), (43) and (44), we obtain

$$\begin{aligned}
V(t) &\leq e^{\beta(t-t_{2k+1})} V_2(t_{2k+1}) - \int_{t_{2k+1}}^{t} e^{\beta(t-s)} \Gamma(s)\mathrm{d}s \\
&\leq \mu\mu_1 e^{\beta(t-t_{2k+1})} V_1(t_{2k+1}^{-}) - \int_{t_{2k+1}}^{t} e^{\beta(t-s)} \Gamma(s)\mathrm{d}s \\
&\leq \mu\mu_1 e^{\beta(t-t_{2k+1})} e^{-\alpha(t_{2k+1}-t_{2k})} V_1(t_{2k}) \\
&\quad - \mu\mu_1 e^{\beta(t-t_{2k+1})} \int_{t_{2k}}^{t_{2k+1}} e^{-\alpha(t_{2k+1}-s)} \Gamma(s)\mathrm{d}s - \int_{t_{2k+1}}^{t} e^{\beta(t-s)} \Gamma(s)\mathrm{d}s \\
&= \mu\mu_1 e^{\beta T^+(t_{2k},t) - \alpha T^-(t_{2k},t)} V_1(t_{2k}) \\
&\quad - \mu\mu_1 \int_{t_{2k}}^{t_{2k+1}} e^{\beta T^+(s,t) - \alpha T^-(s,t)} \Gamma(s)\mathrm{d}s - \int_{t_{2k+1}}^{t} e^{\beta T^+(s,t)} \Gamma(s)\mathrm{d}s \\
&\leq \cdots \leq \mu^{N_\sigma(t_0,t)} \mu_1^{N_l(t_0,t)} e^{\beta T^+(t_0,t) - \alpha T^-(t_0,t)} V(t_0) \\
&\quad - \int_{t_0}^{t} \mu^{N_\sigma(s,t)} \mu_1^{N_l(s,t)} e^{\beta T^+(s,t) - \alpha T^-(s,t)} \Gamma(s)\mathrm{d}s.
\end{aligned} \tag{45}$$

Furthermore, under the zero initial condition, (45) becomes

$$\int_{t_0}^{t} \mu^{N_\sigma(s,t)} \mu_1^{N_l(s,t)} e^{\beta T^+(s,t) - \alpha T^-(s,t)} \Gamma(s)\mathrm{d}s \leq 0. \tag{46}$$

Assume $t_0 = 0$, then multiplying both sides of (46) by $e^{-N_l(0,t)\ln(\mu^2\mu_1)}$ yields

$$\begin{aligned}
&\int_0^t e^{-N_l(0,s)\ln(\mu^2\mu_1) + \beta T^+(s,t) - \alpha T^-(s,t)} z^T(s)z(s)\mathrm{d}s \\
&\quad \leq \int_0^t e^{-N_l(0,s)\ln(\mu^2\mu_1) + \beta T^+(s,t) - \alpha T^-(s,t)} \gamma^2 w^T(s)w(s)\mathrm{d}s.
\end{aligned}$$

It follows from (C1) and (C2) that $N_l(0,s)\ln(\mu^2\mu_1) \leq \alpha_1 s + \alpha_1 \eta$ and $\beta T^+(s,t) - \alpha T^-(s,t) \leq -\alpha^*(t-s) + 2c$ hold. Thus,

$$\int_0^t e^{-\alpha s - \alpha_1\eta - \alpha(t-s)} z^T(s)z(s)\mathrm{d}s \leq \int_0^t e^{-\alpha^*(t-s)+2c} \gamma^2 w^T(s)w(s)\mathrm{d}s.$$

Let $t = T_f$, it can be obtained

$$\int_0^{T_f} z^T(s)z(s)\mathrm{d}s \leq \bar{\gamma}^2 \int_0^{T_f} w^T(s)w(s)\mathrm{d}s,$$

where $\bar{\gamma}$ is defined in Theorem 2.

Combining Theorem 1, we know that the system (6) is finite-time stabilizable with H_∞ performance. □

4 Numerical Examples

Example 1 Consider the system (12) with the parameters as follows:

$$A=\begin{bmatrix}1.4 & -1\\ -0.2 & 1\end{bmatrix},\ A_d=\begin{bmatrix}1.1 & 0.1\\ 0.1 & 0.1\end{bmatrix},\ D=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\ B=\begin{bmatrix}0.3 & 0.8\\ -0.1 & 1.9\end{bmatrix},$$
$$L=\begin{bmatrix}1 & 0.1\\ 0.1 & 0.1\end{bmatrix},\ R=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix},\ M_{11}=\begin{bmatrix}1 & 0\\ 0 & 0.1\end{bmatrix},\ M_{22}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

Choosing $T_f=16$, $h_1=0.01$, $h_2=0.12$, $\alpha=0.34$, $\beta=0.9$, $\mu=2.6$, $d=0.58$, $\varepsilon=0.19$, $e_1=0.06$, $d_w^2=0.01$, $\Xi(t)=I$. By solving linear matrix inequalities (15), (26) and (28), the state feedback gain matrices are given

$$K_1=\begin{bmatrix}-87.9478 & 10.1476\\ -5.9780 & -6.4146\end{bmatrix},\ K_2=\begin{bmatrix}-50.3907 & 6.7613\\ -8.2238 & -1.1791\end{bmatrix}.$$

Let $\alpha^*=0.31$, $\alpha_1=0.3$, it holds that $\dfrac{T^+(p_m,p_{m+1})}{T^-(p_m,p_{m+1})}\le 0.0248$ according to (C1), and it holds that $F_l(p_m,p_{m+1})\le 0.1456$ according to (C2). Thus, if $\dfrac{T^+(p_m,p_{m+1})}{T^-(p_m,p_{m+1})}\le 0.0248$ and $F_l(p_m,p_{m+1})\le 0.1456$ hold for switching signal $\sigma(t)$, the system (12) is FTB. Suppose that $p_{m+1}-p_m=\eta$, $\forall m\in\overline{\mathbb{N}}$, and let $\eta=8s$. It can be seen that $T^+(p_m,p_{m+1})\le 0.1935$, and $N_l(p_m,p_{m+1})=F_l(p_m,p_{m+1})\times\eta\le 1.1648$. It means that the LDP can arise once in each $8s$, and the permitted length of LDP can reach to 0.1935.

Example 2 Consider the system (6) with the parameters as follows:

$$A=\begin{bmatrix}1.4 & -1\\ -0.2 & 1\end{bmatrix},\ A_d=\begin{bmatrix}1.1 & 0.1\\ 0.1 & 0.1\end{bmatrix},\ B=\begin{bmatrix}0.3 & 0.8\\ -0.1 & 1.9\end{bmatrix},\ D=F=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},$$
$$M_{11}=\begin{bmatrix}1 & 0\\ 0 & 0.1\end{bmatrix},\ M_{22}=M_{33}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ L=\begin{bmatrix}1 & 0.1\\ 0.1 & 0.1\end{bmatrix},$$
$$G=H=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},\ R=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}.$$

Choosing $T_f=16$, $h_1=0.01$, $h_2=0.12$, $\alpha=0.34$, $\beta=1.9$, $\mu=3.5$, $d=0.58$, $\varepsilon=0.19$, $e_1=0.06$, $d_w^2=0.01$, $\Xi(t)=I$. By solving linear matrix inequalities (41), (42) and (28), the controller gain is

$$K_1=\begin{bmatrix}-111.4323 & 16.3923\\ -10.0020 & -9.3632\end{bmatrix},\ K_2=\begin{bmatrix}-62.5018 & 11.3196\\ -17.8095 & -0.6246\end{bmatrix}.$$

Let $\alpha^* = 0.31$, $\alpha_1 = 0.3$, it holds that $\frac{T^+(p_m, p_{m+1})}{T^-(p_m, p_{m+1})} \leq 0.0200$ according to (C1), and it holds that $F_l(p_m, p_{m+1}) \leq 0.1116$ according to (C2). Thus, if $\frac{T^+(p_m, p_{m+1})}{T^-(p_m, p_{m+1})} \leq 0.0200$ and $F_l(p_m, p_{m+1}) \leq 0.1116$ hold for switching signal $\sigma(t)$, the system (10) is FTB. Moreover, we choose $\gamma = 0.1$, then it can be obtained that H_∞ performance index is $\bar{\gamma} = 9.5978$.

5 Conclusions

The robust finite-time H_∞ control for the uncertain delay systems with LDP has been investigated by using a switching method. Under the limitation of frequency and length rate of LDP, a controller has been designed to ensure the FTB with H_∞ performance. We have also illustrated the effectiveness of the proposed results by two numerical examples. Our future work will focus on extending the results in this paper to the uncertain neutral systems with LDP.

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