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Asymptotic and Robust Mean Square Stability Analysis of Impulsive High-Order BAM Neural Networks with Time-Varying Delays

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Abstract In this paper, the globally asymptotical stability in the mean square for a class of high-order bidirectional associative memory neural networks with time-varying delays and fixed moments of impulsive effect are studied. The proof makes use of Lyapunov–Krasovskii functionals, and the conditions are expressed in terms of linear matrix inequalities. A controller has been derived to robustly stabilize this network. Two illustrative examples are also given at the end of this paper to show the effectiveness of our results.

Keywords Impulsive bidirectional neural networks · Lyapunov functionals · Linear matrix inequality · Globally robust stability

Mathematics Subject Classification 34K20 · 34K13 · 92B20

1 Introduction

The stability of dynamical neural networks with time delays which have been used in many applications such as optimization, control and image processing has received much attention recently (see, such as [2,3,6,17–22,25–27,43,44]). For example, the stability of delay Hopfield neural networks [2,3,5,31,33] and Cohen–

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Grossberg neural networks [20,42,45] has been investigated. In special, BAM neural networks were first proposed by Kosko [29,30], and since then, the BAM neural system has been extensively studied [9,11,13,27,28,34,35,37,39,41,46]. As a class of dynamic systems, BAM neural networks are usually featured by either first-order or high-order forms described in continuous or discrete time. The low-order BAM systems have been widely studied (such as [34,35]) because of its potential for signal and image processing, pattern recognition. Recently, the higher-order BAM systems [12,27,28] display nice properties due to the special structure of connection weights. The existence of a globally asymptotically stable equilibrium state has been studied under a variety of assumptions on the activation functions. Generally, the activation functions have been assumed to be continuously differentiable, monotonic and bounded [2,5,10,13,15,18,20,36,37,39,40]. But in some applications, one is required to use unbounded and non-monotonic activation functions [27,41,46]. It has been shown by Crespi [16], Morita [32] that the capacity of an associative memory network can be significantly improved if the sigmoidal functions are replaced by non-monotonic activation functions. On the other hand, the state of electronic networks is often subject to display impulsive effects [7,22].

To the best of our knowledge, except [27,28] for higher-order BAM neural networks by using the differential inequality with delay and impulse, the stability analysis for impulsive high-order BAM dynamical systems with time-varying delays has seldom been investigated and remains important and challenging. In this paper, we shall study another generation of high-order BAM dynamical neural networks with impulsive by using Lyapunov–Krasovskii functionals, employing linear matrix inequalities (LMIs) and differential inequalities. The organization of this paper is as follows. In Sect. 2, problem formulation and preliminaries are given. In Sect. 3, several sufficient criteria will be established for the equilibrium of the system to be asymptotical stability in the mean square. In Sect. 4, two examples will be given to demonstrate the effectiveness of our results.

2 Preliminaries and Lemmas

In this paper, we will consider the following impulsive neural networks

$$\begin{cases} \dfrac{\mathrm{d}x_i(t)}{\mathrm{d}t} = -d_i x_i(t) + \sum_{j=1}^{m} a_{ij} f_j(y_j(t)) + \sum_{j=1}^{m} b_{ij} f_j(y_j(t-\tau(t))) \\ \qquad + \sum_{j=1}^{m}\sum_{l=1}^{m} c_{ijl} \int_{t-\tau}^{t} f_j(y_j(t-s)) f_l(y_l(s))\mathrm{d}s + \sum_{j=1}^{m_1} r_{ij} u_j(t), \quad t \neq t_k, \\ \dfrac{\mathrm{d}y_j(t)}{\mathrm{d}t} = -\widetilde{d}_j y_j(t) + \sum_{i=1}^{n} \widetilde{a_{ji}} g_i(x_i(t)) + \sum_{i=1}^{n} \widetilde{b_{ji}} g_i(x_i(t-\sigma(t))) \\ \qquad + \sum_{i=1}^{n}\sum_{l=1}^{n} \widetilde{c_{jil}} \int_{t-\sigma}^{t} g_i(x_i(t-s)) g_l(x_l(s))\mathrm{d}s + \sum_{i=1}^{m_2} \widetilde{r}_{ji} \widetilde{u}_i(t), \quad t \neq t_k, \\ \triangle x_i(t) = \sum_{j=1}^{n} (e_{ij}(t) - \delta_{ij}) x_j(t^-) + \sum_{l=1}^{n} m_{il} J_l(t^-), \quad t = t_k, \\ \triangle y_j(t) = \sum_{i=1}^{m} (\widetilde{e_{ji}}(t) - \delta_{ji}) y_i(t^-) + \sum_{l=1}^{m} \widetilde{m_{jl}} \widetilde{J}_l(t^-), \quad t = t_k, \end{cases} \tag{1}$$

or, equivalently,

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = -Dx(t) + Af(y(t)) + Bf(y(t-\tau(t))) + \int_{t-\tau}^{t} \Upsilon_f^T(s)\Gamma_1 f(y(s))\mathrm{d}s + Ru(t), \\ \quad t \neq t_k, \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = -\widetilde{D}y(t) + \widetilde{A}g(x(t)) + \widetilde{B}g(x(t-\sigma(t))) + \int_{t-\sigma}^{t} \Upsilon_g^T(s)\Gamma_2 g(x(s))\mathrm{d}s + \widetilde{R}\widetilde{u}(t), \\ \quad t \neq t_k, \\ \triangle x(t) = (E(t) - I)x(t^-) + MJ(t^-), t = t_k, \\ \triangle y(t) = (\widetilde{E}(t) - \widetilde{I})y(t^-) + \widetilde{M}\widetilde{J}(t^-), t = t_k, \end{cases} \tag{2}$$

where $t \geq 0$; $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$; $\triangle x_i(t) = x_i(t) - x_i(t^-)$, $\triangle y_j(t) = y_j(t) - y_j(t^-)$, $\triangle x(t) = x(t) - x(t^-) = (\triangle x_1(t), \ldots, \triangle x_n(t))$, $\triangle y(t) = y(t) - y(t^-) = (\triangle y_1(t), \ldots, \triangle y_m(t))$, $0 \leq t_0 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = \infty$; $x_i(t)$, $y_j(t)$ denote the potential of the cell i and j at time t. $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in R^n$, $y(t) = (y_1(t), y_2(t), \ldots, y_m(t))^T \in R^m$; $f(y(t)) = (f_1(y_1(t)), \ldots, f_m(y_m(t)))^T \in R^m$ and $g(x(t)) = (g_1(x_1(t)), g_2(x_2(t)), \ldots, g_n(x_n(t)))^T \in R^n$ denote the activation functions of the neuron at time t, $u(t) = (u_1(t), \ldots, u_{m_1}(t))^T \in R^{m_1}$, $\widetilde{u}(t) = (\widetilde{u}_1(t), \widetilde{u}_2(t), \ldots, \widetilde{u}_{m_2}(t))^T \in R^{m_2}$ are continuous control input, I and $\widetilde{I}$ denote the identity matrix of size n and m, respectively. $J(t) = (J_1(t), \ldots, J_n(t))^T \in R^n$, $\widetilde{J}(t) = (\widetilde{J}_1(t), \widetilde{J}_2(t), \ldots, \widetilde{J}_m(t))^T \in R^m$ are the impulsive control input at time t, $\Upsilon_f(s) = \mathrm{diag}(f(y(t-s)), f(y(t-s)), \ldots, f(y(t-s)))_{n\times n}$, $\Upsilon_g(s) = \mathrm{diag}(g(x(t-s)), g(x(t-s)), \ldots, g(x(t-s)))_{m\times m}$; $D = \mathrm{diag}(d_1, \ldots, d_n) > 0$, $\widetilde{D} = \mathrm{diag}(\widetilde{d}_1, \ldots, \widetilde{d}_m) > 0$ are positive diagonal matrices, and d_i, $\widetilde{d}_j$ represent the rate of isolation of cells i and j from the other states and inputs, respectively. This means that cells i and j reset their potential to the other state during the isolation. $A = (a_{ij})_{n\times m}$, $B = (b_{ij})_{n\times m}$, $R = (r_{ij})_{n\times m_1}$, $M = (m_{ij})_{n\times n}$, $E = (e_{ij}(t))_{n\times n}$, $\widetilde{A} = (\widetilde{a_{ji}})_{m\times n}$, $\widetilde{B} = (\widetilde{b_{ji}})_{m\times n}$, $\widetilde{R} = (\widetilde{r}_{ji})_{m\times m_2}$, $\widetilde{M} = (\widetilde{m}_{ji})_{m\times m}$, $\widetilde{E} = (\widetilde{e_{ji}}(t))_{m\times m}$ are the feedback matrix and the delayed feedback matrix, respectively. $\Gamma_1 = [C_1^T, C_2^T, \ldots, C_n^T]^T, C_i = (c_{ijl})_{m\times m}; \Gamma_2 = [\widetilde{C}_1^T, \widetilde{C}_2^T, \ldots, \widetilde{C}_m^T]^T, \widetilde{C}_j = (\widetilde{c}_{jil})_{n\times n}$. $E(t)$, $\widetilde{E}(t)$ are matrix functions with time-varying uncertainties, that is, $E(t) = E + \Delta E$, $\widetilde{E}(t) = \widetilde{E} + \Delta\widetilde{E}$, where E, $\widetilde{E}$ are known real constant matrices, ΔE, $\Delta\widetilde{E}$ are unknown matrices representing time-varying parameter uncertainties. We assume that the uncertainties are norm-bounded and can be described as

$$\Delta E = HF(t)D_1, \quad \Delta\widetilde{E} = \widetilde{H}\widetilde{F}(t)\widetilde{D}_1 \tag{3}$$

where H, $\widetilde{H}$, D_1, $\widetilde{D}_1$ are known real constant matrices with appropriate dimensions, and the uncertain matrix $F(t)$, which may be time varying, is unknown and satisfies $F^T(t)F(t) \le I$, $\widetilde{F}^T(t)\widetilde{F}(t) \le \widetilde{I}$ for any given t. It is assumed the elements of $F(t)$ are Lebesgue measurable. When $F(t) = 0$, $\widetilde{F}(t) = 0$, system (2) is referred to as nominal neural impulsive systems. Time delays $\tau(t), \sigma(t)$ are continuous functions, which correspond to the finite speed of axonal signal transmission and $0 \le \tau(t) \le \tau$, $0 \le \sigma(t) \le \sigma$ and $0 < \sigma'(t) \le \sigma_1 < 1, 0 < \tau(t) \le \tau_1 < 1$.

The initial conditions associated with (1) or (2) are of the form

$$\begin{aligned} x_i(t) &= \phi_i(t), \quad t_0 - \sigma \le t \le t_0; \\ y_j(t) &= \varphi_j(t), \quad t_0 - \tau \le t \le t_0, \end{aligned}$$

in which $\phi_i(t), \varphi_j(t) (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m)$ are continuous functions. The notations used in this paper are fairly standard. The matrix $M > \ (\ge, <, \le)\ 0$ denotes a symmetric positive definite (positive semidefinite, negative, negative semidefinite) matrix, respectively. For $x \in R^n$, denote $\|x\| = \sqrt{x^T x}$ and $|x| = \sum_{i=1}^n |x_i|$.

Remark 1 BAM is a type of recurrent neural network. Giving a pattern, it can return another pattern which is potentially of a different size. It is similar to the Hopfield network [2,3,5,31,33] in that they are both forms of associative memory. However, Hopfield networks return patterns of the same size. As for competitive neural networks [1,4], they can model the dynamics of cortical cognitive maps with unsupervised synaptic modifications.

Throughout this paper, the activation functions $f(\cdot), g(\cdot), J(\cdot), \widetilde{J}(\cdot)$ are assumed to possess the following properties:

(H1) There exist matrices $K \in R^{m\times m}$, $U \in R^{n\times n}$ such that, for all $y, z \in R^m$; $x, s \in R^n$,

$$|f(y) - f(z)| \le |K(y - z)|; \quad |g(x) - g(s)| \le |U(x - s)|.$$

(H2) $f(0) = g(0) = J(0) = \widetilde{J}(0) = 0$.

(H3) There exist positive numbers O_j, $\widetilde{O}_j$ such that

$$|f_j(x)| \le O_j, \ |g_i(x)| \le \widetilde{O}_i;$$

for all $x \in R (i = 1, 2, \ldots, n;\ j = 1, 2, ..., m)$.

Remark 2 Under assumption (H2), we have that the equilibrium point of system (2) is the trivial solution of system (2). In fact, when the equilibrium point (x^*, y^*) of system (2) in the engineering background isn't the trivial solution of system (2), one can transfer the equilibrium point (x^*, y^*) to (0,0) by the transformation $u = x - x^*$, $v = y - y^*$. Then, by the transformation $\widetilde{f}(v) = f(v + y^*) - f(y^*)$, $\widetilde{g}(u) = g(u + x^*) - g(x^*)$, assumption (H2) is always satisfied(see [19,32]), where $J(0) = 0,\ \widetilde{J}(0) = 0$ mean that the impulsive control inputs at time 0 do not work in the engineering background.

Let $x(t; \phi),\ y(t; \varphi)$ denote the state trajectory of neural network (1) or (2) from the initial data $x(s) = \phi(s) \in PC([t_0 - \sigma, t_0]; R^n),\ y(s) = \varphi(s) \in PC([t_0 - \tau, t_0]; R^m)$, respectively, where $PC([t_0 - r, t_0]; R^n)$ denote the set of piecewise right continuous function $\phi : [-r, 0] \to R^p$ with the norm defined by $\|\phi\|_r = \sup_{-r \le \theta \le 0} \|\phi(\theta)\|$. It can be easily seen that system (2) admits a trivial solution $x(t; 0) = 0,\ y(t; 0) = 0$ corresponding to the initial data $\phi = 0,\ \varphi = 0$.

Definition 1 ([46]) For system (1) or (2) and every $\xi_1 \in PC([t_0 - \sigma, t_0]; R^n)$ and $\xi_2 \in PC([t_0 - \tau, t_0]; R^m)$, the trivial solution (equilibrium point) is robustly, globally, asymptotically stable in the mean square if the following holds:

$$\lim_{t \to \infty} \left(|x(t; \xi_1)|^2 + |y(t; \xi_2)|^2 \right) = 0.$$

Lemma 1 *For any vectors $a, b \in R^n$, the inequality*

$$2a^T b \le \rho a^T a + \frac{1}{\rho} b^T b$$

holds for $\forall\ \rho > 0$.

Lemma 2 *For any vectors $a, b \in R^n$, the inequality*

$$2a^T b \le a^T X^{-1} a + b^T X b$$

holds for any matrices $X > 0$.

Lemma 3 ([9]) *Given constant matrices $\Sigma_1, \Sigma_2, \Sigma_3$ where $\Sigma_1 = \Sigma_1^T$ and $0 < \Sigma_2 = \Sigma_2^T$, then*

$$\Sigma_1 + \Sigma_3^T \Sigma_2^{-1} \Sigma_3 < 0$$

if and only if

$$\begin{pmatrix} \Sigma_1 & \Sigma_3^T \\ \Sigma_3 & -\Sigma_2 \end{pmatrix} < 0 \ \text{ or } \ \begin{pmatrix} -\Sigma_2 & \Sigma_3 \\ \Sigma_3^T & \Sigma_1 \end{pmatrix} < 0.$$

Lemma 4 ([38]) *Let A, D, E, F and P be real matrices of appropriate dimensions with $P > 0$ and F satisfying $F^T F \leq I$. Then, for any scalar $\varepsilon > 0$ satisfying $P^{-1} - \varepsilon^{-1} DD^T > 0$, we have*

$$(A + DFE)^T P(A + DFE) \leq A^T (P^{-1} - \varepsilon^{-1} DD^T)^{-1} A + \varepsilon E^T E.$$

3 Impulsively Exponential Stability

Now, we shall present and prove our main results. Our results complement and improve some of the known results found in the literature.

$$\begin{aligned}
&\widetilde{\Lambda}_1 = -Q\widetilde{D} - \widetilde{D}^T Q + Q_1 + W_1 + W_1^T,\\
&\widetilde{\Lambda}_2 = -(1 - \tau_1)Q_1 - W_2 - W_2^T + \rho_{Z_2} K^T K,\ \ \widetilde{\Lambda}_3 = -\sigma P_2 + \epsilon_1 \sigma \lambda_2 I,\\
&\Lambda_1 = -PD - D^T P + N_1 + N_1^T + P_1,\\
&\Lambda_2 = -(1 - \sigma_1)P_1 - N_2 - N_2^T + \rho_{Z_1} U^T U,\ \ \Lambda_3 = -\tau Q_2 + \epsilon_2 \tau \lambda_1 I.
\end{aligned}$$

where $\lambda_1 = \rho_{\Gamma_1^T \Gamma_1}$, $\lambda_2 = \rho_{\Gamma_2^T \Gamma_2}$.

Theorem 1 *Consider system (2) with the impulsive control inputs $J(\cdot) = 0$, $\widetilde{J}(\cdot) = 0$. Under assumptions (H1)–(H3), the equilibrium point of system (2) is robustly, globally, asymptotically stable in the mean square if there exist some scalars $\epsilon_i > 0 (i = 1, 2, 3, 4)$, $\rho_X > 0 (X = Q_2, P_2, X_1, X_2, Z_1, Z_2)$ and matrices $N_i (i = 1, 2, 3)$, $W_i (i = 1, 2, 3)$, $X_1 > 0, X_2 > 0, Z_1 > 0, Z_2 > 0, P > 0, Q > 0, P_1 > 0, Q_1 > 0, P_2 > 0, Q_2 > 0$ such that*

$$X < \rho_X I, \tag{4}$$

$$\Psi_1 = \begin{bmatrix}
\widetilde{\Lambda}_1 & -W_1^T + W_2 & -W_1^T + W_3 & 0 & Q\widetilde{A} & (\rho_{X_2} + \rho_{Q_2})K^T & Q\widetilde{B} & \sqrt{\sigma\alpha}Q\\
* & \widetilde{\Lambda}_2 & -W_2^T - W_3 & 0 & 0 & 0 & 0 & 0\\
* & * & -W_3^T - W_3 & 0 & 0 & 0 & 0 & 0\\
* & * & * & \widetilde{\Lambda}_3 & 0 & 0 & 0 & 0\\
* & * & * & * & -X_1 & 0 & 0 & 0\\
* & * & * & * & * & -(\rho_{X_2} + \rho_{Q_2})I & 0 & 0\\
* & * & * & * & * & * & -Z_1 & 0\\
* & * & * & * & * & * & * & -\epsilon_1 I
\end{bmatrix} < 0, \tag{5}$$

$$\Psi_2 = \begin{bmatrix}
\Lambda_1 & -N_1^T + N_2 & -N_1^T + N_3 & 0 & PA & (\rho_{X_1} + \rho_{P_2})U^T & PB & \sqrt{\tau\widetilde{\alpha}}P\\
* & \Lambda_2 & -N_2^T - N_3 & 0 & 0 & 0 & 0 & 0\\
* & * & -N_3^T - N_3 & 0 & 0 & 0 & 0 & 0\\
* & * & * & \Lambda_3 & 0 & 0 & 0 & 0\\
* & * & * & * & -X_2 & 0 & 0 & 0\\
* & * & * & * & * & -(\rho_{X_1} + \rho_{P_2})I & 0 & 0\\
* & * & * & * & * & * & -Z_2 I & 0\\
* & * & * & * & * & * & * & -\epsilon_2 I
\end{bmatrix} < 0, \tag{6}$$

$$\begin{bmatrix} -P & E^T P & \epsilon_3 D_1^T & 0 \\ * & -P & 0 & PH \\ * & * & -\epsilon_3 I & 0 \\ * & * & * & -\epsilon_3 I \end{bmatrix} \le 0, \tag{7}$$

$$\begin{bmatrix} -Q & \widetilde{E}^T Q & \epsilon_4 \widetilde{D}_1^T & \\ * & -Q & 0 & Q\widetilde{H} \\ * & * & -\epsilon_4 I & 0 \\ * & * & * & -\epsilon_4 I \end{bmatrix} \le 0, \tag{8}$$

$$\begin{bmatrix} -P & PH \\ * & -\epsilon_3 I \end{bmatrix} < 0, \tag{9}$$

$$\begin{bmatrix} -Q & Q\widetilde{H} \\ * & -\epsilon_4 I \end{bmatrix} < 0, \tag{10}$$

hold.

Proof Let

$$\begin{aligned} V(t) = {} & x^T(t)Px(t) + y^T(t)Qy(t) + \int_{t-\sigma(t)}^{t} x^T(s)P_1x(s)\mathrm{d}s \\ & + \int_{t-\tau(t)}^{t} y^T(s)Q_1y(s)\mathrm{d}s + \int_{-\tau}^{0}\int_{t+s}^{t} f^T(y(\eta))Q_2f(y(\eta))\mathrm{d}\eta\mathrm{d}s \\ & + \int_{-\sigma}^{0}\int_{t+s}^{t} g^T(x(\eta))P_2g(x(\eta))\mathrm{d}\eta\mathrm{d}s. \end{aligned}$$

(I) We consider the case of $t \neq t_k$. Calculate the derivative of $V(t)$ along the solutions of (2), and we obtain

$$\begin{aligned} \dot{V}(t) = {} & 2x^T(t)P\left(-Dx(t) + Af(y(t)) + Bf(y(t-\tau(t)))\right. \\ & \left. + C\int_{t-\tau}^{t} \Upsilon_f^T\Gamma_1 f(y(s))\mathrm{d}s\right) \\ & + 2y^T(t)Q\Big(-\widetilde{D}y(t) + \widetilde{A}g(x(t)) + \widetilde{B}g(x(t-\sigma(t))) \\ & + \widetilde{C}\int_{t-\sigma}^{t} \Upsilon_g^T\Gamma_2 g(x(s))\mathrm{d}s\Big) \\ & + x^T(t)P_1x(t) - (1-\sigma'(t))x^T(t-\sigma(t))P_1x(t-\sigma(t)) \\ & + y^T(t)Q_1y(t) - (1-\tau'(t))y^T(t-\tau(t))Q_1y(t-\tau(t)) \\ & + f^T(y(t))Q_2f(y(t)) - \int_{t-\tau}^{t} f^T(y(s))Q_2f(y(s))\mathrm{d}s \\ & + g^T(x(t))P_2g(x(t)) - \int_{t-\sigma}^{t} g^T(x(s))P_2g(x(s))\mathrm{d}s. \end{aligned}$$

From Lemmas 1, 2 and 4, we have

$$\begin{aligned}
&2y^T(t)Q\widetilde{A}g(x(t))\\
&\quad\le y^T(t)Q\widetilde{A}X_1^{-1}\widetilde{A}^TQy(t)+g^T(x(t))X_1g(x(t))\\
&\quad\le y^T(t)Q\widetilde{A}X_1^{-1}\widetilde{A}^TQy(t)+\rho_{X_1}x^T(t)U^TUx(t);\\
&2x^T(t)PAf(y(t))\\
&\quad\le x^T(t)PAX_2^{-1}A^TPx(t)+f^T(y(t))X_2f(y(t))\\
&\quad\le x^T(t)PAX_2^{-1}A^TPx(t)+\rho_{X_2}y^T(t)K^TKy(t);\\
&2y^T(t)Q\widetilde{B}g(x(t-\sigma(t)))\\
&\quad\le y^T(t)Q\widetilde{B}Z_1^{-1}\widetilde{B}^TQy(t)+g^T(x(t-\sigma(t)))Z_1g(x(t-\sigma(t)))\\
&\quad\le y^T(t)Q\widetilde{B}Z_1^{-1}\widetilde{B}^TQy(t)+\rho_{Z_1}x^T(t-\sigma(t))U^TUx(t-\sigma(t));\\
&2x^T(t)PBf(y(t-\tau(t)))\\
&\quad\le x^T(t)PBZ_2^{-1}B^TPx(t)+f^T(y(t-\tau(t)))Z_2f(y(t-\tau(t)))\\
&\quad\le x^T(t)PBZ_2^{-1}B^TPx(t)+\rho_{Z_2}y^T(t-\tau(t))K^TKy(t-\tau(t));\\
&2y^T(t)Q\int_{t-\sigma}^{t}\Upsilon_g^T\Gamma_2g(x(s))\mathrm{d}s\\
&\quad\le\int_{t-\sigma}^{t}\left[\epsilon_1^{-1}y^T(t)Q\Upsilon_g^T\Upsilon_gQy(t)+\epsilon_1g^T(x(s))\Gamma_2^T\Gamma_2g(x(s))\right]\mathrm{d}s.\\
&2x^T(t)P\int_{t-\tau}^{t}\Upsilon_f^T\Gamma_1f(y(s))\mathrm{d}s\\
&\quad\le\int_{t-\tau}^{t}\left[\epsilon_2^{-1}x^T(t)P\Upsilon_f^T\Upsilon_fPx(t)+\epsilon_2f^T(y(s))\Gamma_1^T\Gamma_1f(y(s))\right]\mathrm{d}s.
\end{aligned}$$

Since $\Upsilon_g^T\Upsilon_g=\|g(x(t-s)\|^2I$ and $\|g(x(t-s)\|^2\le\sum_{j=1}^m O_j=\alpha$, it follows that

$$y^T(t)Q\Upsilon_g^T\Upsilon_gQy(t)\le\alpha y^T(t)QQy(t).$$

Since $\Upsilon_f^T\Upsilon_f=\|f(y(t-s)\|^2I$ and $\|f(y(t-s)\|^2\le\sum_{i=1}^n\widetilde{O}_i=\widetilde{\alpha}$, it follows that

$$x^T(t)P\Upsilon_f^T\Upsilon_fPx(t)\le\widetilde{\alpha}x^T(t)PPx(t).$$

Noting that $x(t)-x(t-\sigma(t))-\int_{t-\sigma(t)}^{t}\dot{x}(s)\mathrm{d}s=0$, $y(t)-y(t-\tau(t))-\int_{t-\tau(t)}^{t}\dot{y}(s)\mathrm{d}s=0$, then, there exist matrices $N_1, N_2, N_3, W_1, W_2, W_3$ such that

$$\begin{aligned}
&\left[2x^T(t)N_1^T+2x^T(t-\sigma(t))N_2^T+2\left(\int_{t-\sigma(t)}^{t}\dot{x}(s)\mathrm{d}s\right)^TN_3^T\right]\\
&\quad\times\left(x(t)-x(t-\sigma(t))-\int_{t-\sigma(t)}^{t}\dot{x}(s)\mathrm{d}s\right)=0,
\end{aligned}$$

$$\left[2y^T(t)W_1^T + 2y^T(t-\tau(t))W_2^T + 2\left(\int_{t-\tau(t)}^{t}\dot{y}(s)\mathrm{d}s\right)^T W_3^T\right]$$
$$\times\left(y(t) - y(t-\tau(t)) - \int_{t-\tau(t)}^{t}\dot{y}(s)\mathrm{d}s\right) = 0.$$

Moreover,

$$f^T(y(t))Q_2 f(y(t)) \le \rho_{Q_2} y^T(t)K^T K y(t),\ g^T(x(t))P_2 g(x(t))$$
$$\le \rho_{P_2} x^T(t)U^T U x(t).$$

Thus,

$$\dot{V}(t) \le \frac{1}{\sigma}\int_{t-\sigma}^{t}\xi_1^T \Pi_1 \xi_1 \mathrm{d}s + \frac{1}{\tau}\int_{t-\tau}^{t}\xi_2^T \Pi_2 \xi_2 \mathrm{d}s,$$

where $\xi_1 = [y^T(t)\ y^T(t-\tau(t))\ (\int_{t-\tau(t)}^{t}\dot{y}(s)\mathrm{d}s)^T\ \ g^T(x(s))]^T$, $\xi_2 = [x^T(t)\ x^T(t-\sigma(t))\ (\int_{t-\sigma(t)}^{t}\dot{x}(s)\mathrm{d}s)^T\ \ f^T(y(s))]^T$, and

$$\Pi_1 = \begin{bmatrix} \widetilde{\Omega}_1 & -W_1^T + W_2 & -W_1^T + W_3 & 0 \\ * & \widetilde{\Omega}_2 & -W_2^T - W_3 & 0 \\ & * & -W_3^T - W_3 & 0 \\ * & * & * & \widetilde{\Omega}_3 \end{bmatrix},\ \Pi_2 = \begin{bmatrix} \Omega_1 & -N_1^T + N_2 & -N_1^T + N_3 & 0 \\ * & \Omega_2 & -N_2^T - N_3 & 0 \\ & * & -N_3^T - N_3 & 0 \\ * & * & * & \Omega_3 \end{bmatrix}.$$

$$\begin{aligned}
\widetilde{\Omega}_1 &= -Q\widetilde{D} - \widetilde{D}^T Q + Q_1 + W_1 + W_1^T + Q\widetilde{A}X_1^{-1}\widetilde{A}^T Q + (\rho_{X_2} + \rho_{Q_2})K^T K \\
&\quad + Q\widetilde{B}Z_1^{-1}\widetilde{B}^T Q + \sigma\epsilon_1^{-1}\alpha QQ, \\
\widetilde{\Omega}_2 &= -(1-\tau_1)Q_1 - W_2 - W_2^T + \rho_{Z_2}K^T K,\ \widetilde{\Omega}_3 = -\sigma P_2 + \epsilon_1\sigma\lambda_2 I, \\
\Omega_1 &= -PD - D^T P + N_1 + N_1^T + P_1 + PAX_2^{-1}A^T P + (\rho_{X_1} + \rho_{P_2})U^T U \\
&\quad + PBZ_2^{-1}B^T P + \tau\epsilon_2^{-1}\widetilde{\alpha}PP, \\
\Omega_2 &= -(1-\sigma_1)P_1 - N_2 - N_2^T + \rho_{Z_1}U^T U,\ \Omega_3 = -\tau Q_2 + \epsilon_2\tau\lambda_1 I.
\end{aligned}$$

By Lemma 3, it is obvious from (5) and (6) that $\Pi_1 < 0$, $\Pi_2 < 0$. There must exist scalars $\eta_1 > 0$, $\eta_2 > 0$ such that

$$\Pi_1 + \begin{bmatrix} \eta_1 I & 0\,0\,0 \\ 0 & 0\,0\,0 \\ 0 & 0\,0\,0 \\ 0 & 0\,0\,0 \end{bmatrix} < 0;\ \Pi_2 + \begin{bmatrix} \eta_2 I & 0\,0\,0 \\ 0 & 0\,0\,0 \\ 0 & 0\,0\,0 \\ 0 & 0\,0\,0 \end{bmatrix} < 0.$$

So

$$\dot{V}(t) \le -\eta_1\|x(t)\|^2 - \eta_2\|y(t)\|^2. \tag{11}$$

(II) We consider the case of $t = t_k$. By (4), we have

$$
\begin{aligned}
V(t_k) - V(t_k^-) &= x^T(t_k)Px(t_k) - x^T(t_k^-)Px(t_k^-) + y^T(t_k)Qy(t_k) - y^T(t_k^-)Qy(t_k^-) \\
&\quad + \int_{t_k-\sigma(t_k)}^{t_k} x^T(s)P_1x(s)\mathrm{d}s - \int_{t_k^- -\sigma(t_k^-)}^{t_k^-} x^T(s)P_1x(s)\mathrm{d}s \\
&\quad + \int_{t_k-\tau(t_k)}^{t_k} y^T(s)Q_1y(s)\mathrm{d}s - \int_{t_k^- -\tau(t_k^-)}^{t_k^-} y^T(s)Q_1y(s)\mathrm{d}s \\
&\quad + \tau\int_{-\tau}^{0}\int_{t_k+s}^{t_k} f^T(y(t_k-\eta))\Gamma_1^T\Upsilon_f(\eta)Q_2\Upsilon_f^T(\eta)\Gamma_1 f(y(t_k \\
&\qquad - \eta))\mathrm{d}\eta\mathrm{d}s \\
&\quad -\tau\int_{-\tau}^{0}\int_{t_k^-+s}^{t_k^-} f^T(y(t_k^- - \eta))\Gamma_1^T\Upsilon_f(\eta)Q_2\Upsilon_f^T(\eta)\Gamma_1 f(y(t_k^- \\
&\qquad -\eta))\mathrm{d}\eta\mathrm{d}s \\
&\quad +\sigma\int_{-\sigma}^{0}\int_{t_k+s}^{t_k} g^T(x(t_k-\eta))\Gamma_2^T\Upsilon_g(\eta)P_2\Upsilon_g^T(\eta)\Gamma_2 g(x(t_k \\
&\qquad - \eta))\mathrm{d}\eta\mathrm{d}s \\
&\quad -\sigma\int_{-\sigma}^{0}\int_{t_k^-+s}^{t_k^-} g^T(x(t_k^- - \eta))\Gamma_2^T\Upsilon_g(\eta)P_2\Upsilon_g^T(\eta)\Gamma_2 g(x(t_k^- \\
&\qquad - \eta))\mathrm{d}\eta \\
&\le x^T(t_k^-)[(E+HF(t_k)D_1)^TP(E+HF(t_k)D_1) - P]x(t_k^-) \\
&\qquad + y^T(t_k^-)[(\widetilde{E}+\widetilde{H}\widetilde{F}(t_k)\widetilde{D}_1)^TQ(\widetilde{E}+\widetilde{H}\widetilde{F}(t_k)\widetilde{D}_1) - Q]y(t_k^-) \\
&\le x^T(t_k^-)[E^T(P^{-1} - \epsilon_3^{-1}HH^T)^{-1}E + \epsilon_3 D_1^TD_1 - P]x(t_k^-) \\
&\qquad + y^T(t_k^-)[\widetilde{E}^T(Q^{-1} - \epsilon_4^{-1}\widetilde{H}\widetilde{H}^T)^{-1}\widetilde{E} + \epsilon_4\widetilde{D}_1^T\widetilde{D}_1 - Q]y(t_k^-)
\end{aligned}
$$

By (7)–(10) and combining with Schur complements (Lemma 3) yield

$$
\begin{aligned}
&E^T(P^{-1} - \epsilon_3^{-1}HH^T)^{-1}E + \epsilon_3 D_1^TD_1 - P \le 0, \\
&\widetilde{E}^T(Q^{-1} - \epsilon_4^{-1}\widetilde{H}\widetilde{H}^T)^{-1}\widetilde{E} + \epsilon_4\widetilde{D}_1^T\widetilde{D}_1 - Q \le 0;
\end{aligned}
$$

then, $V(t_k) - V(t_k^-) \le 0$, that is,

$$V(t_k) \le V(t_k^-). \tag{12}$$

This and (I) implie that the equilibrium point of system (2) is robustly, globally, asymptotically stable in the mean square. The proof is complete.

Let us now consider to design a state feedback memory control law of the form

$$
\begin{aligned}
&u(t) = K_c x(t) + K_{c1} x(t-\sigma(t)), \widetilde{u}(t) = \widetilde{K}_c y(t) + \widetilde{K}_{c1} y(t-\tau(t)),\\
&J(t_k^-) = K_d x(t_k^-),\\
&\widetilde{J}(t_k^-) = \widetilde{K}_d y(t_k^-)
\end{aligned}
\tag{13}
$$

to stabilize system (2), where $K_c, K_{c1} \in R^{m_1 \times n}$, $\widetilde{K}_c, \widetilde{K}_{c1} \in R^{m_2 \times m}$, $K_d \in R^{n \times n}$, $\widetilde{K}_d \in R^{m \times m}$ are constant gains to be designed.

Substituting (13) into (2) and applying Theorem 1, it is easy to obtain the next theorem.

Theorem 2 *Consider system* (2). *Under assumptions (H1)–(H3), if there exist scalars* $\epsilon_i > 0 (i = 1, 2, 3, 4)$, $\rho_X > 0 (X = Q_2, P_2, X_1, X_2, Z_1, Z_2)$ *and matrices* $N_i (i = 1, 2, 3)$, $W_i (i = 1, 2, 3)$, $Y_c, Y_{c1}, Y_d, \widetilde{Y}_c, \widetilde{Y}_{c1}, \widetilde{Y}_d, X_1 > 0, X_2 > 0, Z_1 > 0, Z_2 > 0, P > 0, Q > 0, P_1 > 0, Q_1 > 0, P_2 > 0, Q_2 > 0$ *such that* (4), (9), (10) *and*

$$
\begin{bmatrix}
\widetilde{\Phi}_1 & -W_1^T + W_2 + \widetilde{R}\widetilde{Y}_{c1} & -W_1^T + W_3 & 0 & Q\widetilde{A} & (\rho_{X_2} + \rho_{Q_2})K^T & Q\widetilde{B} & \sqrt{\sigma\alpha}Q \\
* & \widetilde{\Lambda}_2 & -W_2^T - W_3 & 0 & 0 & 0 & 0 & 0 \\
* & * & -W_3^T - W_3 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \widetilde{\Lambda}_3 & 0 & 0 & 0 & 0 \\
* & * & * & * & -X_1 & 0 & 0 & 0 \\
* & * & * & * & * & -(\rho_{X_2} + \rho_{Q_2})I & 0 & 0 \\
* & * & * & * & * & * & -Z_1 & 0 \\
* & * & * & * & * & * & * & -\epsilon_1 I
\end{bmatrix} < 0, \tag{14}
$$

$$
\begin{bmatrix}
\Phi_1 & -N_1^T + N_2 + RY_{c1} & -N_1^T + N_3 & 0 & PA & (\rho_{X_1} + \rho_{P_2})U^T & PB & \sqrt{\tau\widetilde{\alpha}}P \\
* & \Lambda_2 & -N_2^T - N_3 & 0 & 0 & 0 & 0 & 0 \\
* & * & -N_3^T - N_3 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \Lambda_3 & 0 & 0 & 0 & 0 \\
* & * & * & * & -X_2 & 0 & 0 & 0 \\
* & * & * & * & * & -(\rho_{X_1} + \rho_{P_2})I & 0 & 0 \\
* & * & * & * & * & * & -Z_2 I & 0 \\
* & * & * & * & * & * & * & -\epsilon_2 I
\end{bmatrix} < 0, \tag{15}
$$

$$
\begin{bmatrix}
-P & E^T P + M^T Y_d & \epsilon_3 D_1^T & 0 \\
* & -P & 0 & PH \\
* & * & -\epsilon_3 I & 0 \\
* & * & * & -\epsilon_3 I
\end{bmatrix} \le 0, \tag{16}
$$

$$
\begin{bmatrix}
-Q & \widetilde{E}^T Q + \widetilde{M}^T \widetilde{Y}_d & \epsilon_4 \widetilde{D}_1^T \\
* & -Q & 0 & Q\widetilde{H} \\
* & * & -\epsilon_4 I & 0 \\
* & * & * & -\epsilon_4 I
\end{bmatrix} \le 0, \tag{17}
$$

hold. Then, for any bounded time delay $\tau(t), \sigma(t)$, *the control law* (13) *with* $K_c = Y_c P^{-1}$, $\widetilde{K}_c = \widetilde{Y}_c Q^{-1}$, $K_{c1} = Y_{c1} P^{-1}$, $\widetilde{K}_{c1} = \widetilde{Y}_{c1} Q^{-1}$, $K_d = Y_d P^{-1}$ *and* $\widetilde{K}_d = \widetilde{Y}_d Q^{-1}$ *robustly stabilizes system* (1) *or* (2) *in the mean square for any impulsive time sequence* $\{t_k\}$ *where*

$$\widetilde{\Phi}_1 = -Q\widetilde{D} - \widetilde{D}Q + \widetilde{R}\widetilde{Y}_c + \widetilde{Y}_c^T\widetilde{R}^T + Q_1 + W_1 + W_1^T,$$
$$\widetilde{\Lambda}_2 = -(1-\tau_1)Q_1 - W_2 - W_2^T + \mu_2 K^T K,\ \widetilde{\Lambda}_3 = -\tau^2\lambda_1^2(Q_2 - Z_2),$$
$$\Phi_1 = -PD - DP + RY_c + Y_c^T R^T + N_1 + N_1^T + P_1,$$
$$\Lambda_2 = -(1-\sigma_1)P_1 - N_2 - N_2^T + \mu_1 U^T U,\ \Lambda_3 = -\sigma^2\lambda_2^2(P_2 - Z_1),$$
$$\lambda_1 = \max_{1\le i\le n}\{\lambda_{\max}(C_i)\},\ \lambda_2 = \max_{1\le i\le n}\{\lambda_{\max}(\widetilde{C}_i)\}.$$

Remark 3 Some stability analysis of higher-order BAM neural networks with delays is presented [12,27,28,37]. However, simple and efficient conditions for the design and robustness analysis of such controllers were missing. The present paper fills this gap through introducing simple LMIs for robust stability analysis of the BAM systems with multiple delays and justifying that these LMIs are always feasible for small enough delays. Moreover, all of these criteria in this paper are easy to verify.

Remark 4 In this paper, the methods are by using Lyapunov–Krasovskii functionals, employing linear matrix inequalities and differential inequalities. In [31,33], the authors mainly used the differential inequality with delay and impulse, coincidence degree theory as well as a priori estimates and Lyapunov functional, respectively. Moreover, the methods derived in this paper can be used to analyze, design and control some other high-order artificial neural networks.

Remark 5 In this paper, the essential difficulties encountered are how to choose suitable Lyapunov–Krasovskii functionals, and how to design a state feedback memory control law. The first is because that the time-varying delay functions $\tau(t), \sigma(t)$ exist; the second is because designing the delay feedback impulsive control is a new theory in the impulsive stabilization. And, they were solved by adding the four terms

$$\int_{t-\tau(t)}^{t} y^T(s)Q_1y(s)\mathrm{d}s,\ \int_{-\sigma}^{0}\int_{t+s}^{t} g^T(x(\eta))P_2g(x(\eta))\mathrm{d}\eta\mathrm{d}s,$$
$$\int_{-\tau}^{0}\int_{t+s}^{t} f^T(y(\eta))Q_2f(y(\eta))\mathrm{d}\eta\mathrm{d}s,\ \int_{t-\sigma(t)}^{t} x^T(s)P_1x(s)\mathrm{d}s$$

to Lyapunov–Krasovskii functionals and designing a linear delay feedback impulsive control law (13) (since f, g is global Lipschitz continuous), respectively.

Remark 6 When the system is running with impulses, our results show that impulses make contribution to the stability of differential systems with time delay even if they are unstable, which is shown in Figs. 2 and 3.

4 Numerical Examples

Example 1 Consider system (2) with:

$$D=\begin{bmatrix}2.9&0&0\\0&3.2&0\\0&0&3\end{bmatrix};\quad A=\begin{bmatrix}-1&2\\2&1\\-2&1\end{bmatrix};\quad E=\begin{bmatrix}0.3&0&0\\0&0.3&0\\0&0&0.3\end{bmatrix};$$

$$C_1=\begin{bmatrix}0.1210&0.3159\\0.3508&0.2028\end{bmatrix};\quad C_2=\begin{bmatrix}0.2731&0.3656\\0.2548&0.2324\end{bmatrix};\quad C_3=\begin{bmatrix}0.1049&0.2319\\0.2084&0.2393\end{bmatrix};$$

$$B=\begin{bmatrix}-1&1\\1&1\\1&-1\end{bmatrix};\quad K=\begin{bmatrix}0.09&0\\0&0.01\end{bmatrix};$$

$$U=\begin{bmatrix}0.3&0&0\\0&0.1&0\\0&0&0.5\end{bmatrix};\quad D1=\begin{bmatrix}1&-1&0\\-1&-1&0\\0&-2&1\end{bmatrix};\quad H=\begin{bmatrix}0.02&0&0\\0&0.03&0\\0&0&0.03\end{bmatrix};$$

$$\widetilde{D}=\begin{bmatrix}2.6&0\\0&2.9\end{bmatrix};\quad \widetilde{B}=\begin{bmatrix}1&0&0.3\\-0.6&0&0.3\end{bmatrix};\quad \widetilde{C}_1=\begin{bmatrix}0.0298&0.1800&0.2218\\0.1665&0.3139&0.2933\\0.5598&0.1536&0.2398\end{bmatrix};$$

$$\widetilde{C}_2=\begin{bmatrix}0.3001&0.3232&0.0321\\0.3112&0.3515&0.5586\\0.3772&0.2107&0.3361\end{bmatrix};\quad \widetilde{A}=\begin{bmatrix}1&0&-1\\0&0&1.2\end{bmatrix};\quad \widetilde{E}=\begin{bmatrix}0.6&-0.2\\0&0.5\end{bmatrix};$$

$$\widetilde{D}1=\begin{bmatrix}0.6&0.5\\-0.6&0.4\end{bmatrix};\quad \widetilde{H}=\begin{bmatrix}0.09&0.06\\0.06&0.09\end{bmatrix};\quad \sigma=\tau=1;$$
$$\alpha=\widetilde{\alpha}=1.$$

Then,

$$\Gamma_1=\begin{bmatrix}0.1210&0.3159\\0.3508&0.2028\\0.2731&0.3656\\0.2548&0.2324\\0.1049&0.2319\\0.2084&0.2393\end{bmatrix};\Gamma_2=\begin{bmatrix}0.0298&0.1800&0.2218\\0.1665&0.3139&0.2933\\0.5598&0.1536&0.2398\\0.3001&0.3232&0.0321\\0.3112&0.3515&0.5586\\0.3772&0.2107&0.3361\end{bmatrix}.$$

By solving LMIs (3)–(9) for $\epsilon_i > 0(i = 1, 2, 3, 4)$, $\rho_X > 0(X = Q_2, P_2, X_1, X_2, Z_1, Z_2)$ and matrices $X_1 > 0$, $X_2 > 0$, $Z_1 > 0$, $Z_2 > 0$, $P > 0$, $Q > 0$, $P_1 > 0$, $Q_1 > 0$, $P_2 > 0$, $Q_2 > 0$, we obtain

$$P=\begin{bmatrix}0.4654&-0.0064&-0.0340\\-0.0064&0.7287&-0.0290\\-0.0340&-0.0290&0.4735\end{bmatrix};\quad Q=\begin{bmatrix}0.7960&0.0238\\0.0238&0.7093\end{bmatrix};$$

$$P_1=\begin{bmatrix}0.6368&-0.0280&-0.1055\\-0.0280&0.7792&0.2243\\-0.1055&0.2243&0.5100\end{bmatrix};\quad P_2=\begin{bmatrix}0.7973&0\\0&0.7973\end{bmatrix};$$

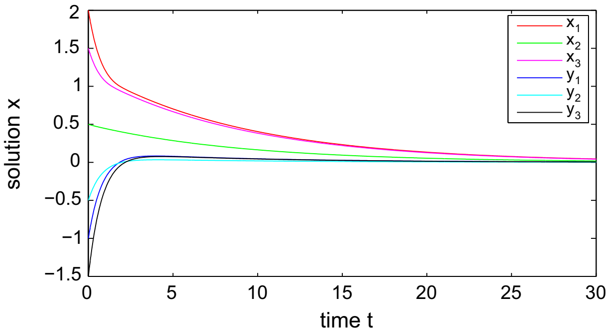

Fig. 1 State of system Example 1

$$Q_1 = \begin{bmatrix} 0.9273 & 0.2330 \\ 0.2330 & 1.1490 \end{bmatrix}; \quad Q_2 = \begin{bmatrix} 1.1561 & 0 & 0 \\ 0 & 1.1561 & 0 \\ 0 & 0 & 1.1561 \end{bmatrix};$$

$$X_1 = \begin{bmatrix} 0.9090 & 0 & -0.1013 \\ 0 & 0.7952 & 0 \\ -0.1013 & 0 & 1.0165 \end{bmatrix}; \quad X_2 = \begin{bmatrix} 2.0404 & -0.0182 \\ -0.0182 & 1.6456 \end{bmatrix};$$

$$Z_1 = \begin{bmatrix} 1.2089 & 0 & 0.0262 \\ 0 & 0.9862 & 0 \\ 0.0262 & 0 & 1.0117 \end{bmatrix}; \quad Z_2 = \begin{bmatrix} 1.4291 & -0.0035 \\ -0.0035 & 1.6198 \end{bmatrix};$$

$$\epsilon_1 = 0.6443; \quad \epsilon_2 = 1.1712; \quad \epsilon_3 = 0.0945; \quad \epsilon_4 = 0.4812.$$

which implies from Theorem 1 that the above delayed stochastic high-order neural network is robustly, globally, asymptotically stable in the mean square with the impulsive control inputs $J(\cdot) = 0$, $\widetilde{J}(\cdot) = 0$. The time trajectories of Example 1 with initial conditions

$$\begin{cases} \phi_1(s) = 2, & s \in [-5, 0]; \\ \phi_2(s) = 1.5, & s \in [-5, 0]; \\ \phi_3(s) = 0.5, & s \in [-5, 0]; \end{cases} \qquad \begin{cases} \varphi_1(s) = -0.5, & s \in [-7, 0]; \\ \varphi_2(s) = -1, & s \in [-7, 0]; \\ \varphi_3(s) = -1.5, & s \in [-7, 0]; \end{cases}$$

and $f_1(x) = \tanh(0.9x)$, $f_2(x) = \tanh(0.78x)$, $f_3(x) = \tanh(0.96x)$; $g_1(x) = \tanh(0.83x)$, $g_2(x) = \tanh(0.88x)$, $g_3(x) = \tanh(0.93x)$ are shown Fig. 1.

Example 2 Consider impulsive neural networks (2) with:

$$D = \begin{bmatrix} 2.9 & 0 & 0 \\ 0 & 3.2 & 0 \\ 0 & 0 & 3 \end{bmatrix}; \quad A = \begin{bmatrix} -1 & 2 \\ 2 & 1 \\ -2 & 1 \end{bmatrix}; \quad E = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{bmatrix};$$

$$C_1 = \begin{bmatrix} 0.1210 & 0.3159 \\ 0.3508 & 0.2028 \end{bmatrix}; \quad C_2 = \begin{bmatrix} 0.2731 & 0.3656 \\ 0.2548 & 0.2324 \end{bmatrix}; \quad C_3 = \begin{bmatrix} 0.1049 & 0.2319 \\ 0.2084 & 0.2393 \end{bmatrix};$$

$$B = \begin{bmatrix} -1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}; \quad K = \begin{bmatrix} 0.9 & 0 \\ 0 & 0.8 \end{bmatrix};$$

$$U = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}; \quad D1 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & -2 & 1 \end{bmatrix};$$

$$H = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.03 & 0 \\ 0 & 0 & 0.03 \end{bmatrix}; \quad \widetilde{D} = \begin{bmatrix} 2.9 & 0 \\ 0 & 2.9 \end{bmatrix};$$

$$\widetilde{B} = \begin{bmatrix} 1 & 0 & 0.3 \\ -0.6 & 0 & 0.3 \end{bmatrix}; \quad \widetilde{C}_1 = \begin{bmatrix} 0.0298 & 0.1800 & 0.2218 \\ 0.1665 & 0.3139 & 0.2933 \\ 0.5598 & 0.1536 & 0.2398 \end{bmatrix};$$

$$\widetilde{C}_2 = \begin{bmatrix} 0.3001 & 0.3232 & 0.0321 \\ 0.3112 & 0.3515 & 0.5586 \\ 0.3772 & 0.2107 & 0.3361 \end{bmatrix}; \quad \widetilde{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1.2 \end{bmatrix}; \quad \widetilde{E} = \begin{bmatrix} 0.6 & -0.2 \\ 0 & 0.5 \end{bmatrix};$$

$$\widetilde{D}1 = \begin{bmatrix} 0.6 & 0.5 \\ -0.6 & 0.4 \end{bmatrix}; \quad \widetilde{H} = \begin{bmatrix} 0.09 & 0.06 \\ 0.06 & 0.09 \end{bmatrix}; \quad M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0.2 & 0.8 \\ 0 & 1 & 1 \end{bmatrix};$$

$$\widetilde{M} = \begin{bmatrix} 0.6 & 1 \\ 0.7 & 0.8 \end{bmatrix}; \quad \widetilde{R} = \begin{bmatrix} 3.9 \\ 3 \end{bmatrix}; \quad R = \begin{bmatrix} 3 \\ 3 \\ 3.5 \end{bmatrix}; \quad \sigma = \tau = 1;$$

$$\alpha = \widetilde{\alpha} = 1.$$

Then,

$$\Gamma_1 = \begin{bmatrix} 0.1210 & 0.3159 \\ 0.3508 & 0.2028 \\ 0.2731 & 0.3656 \\ 0.2548 & 0.2324 \\ 0.1049 & 0.2319 \\ 0.2084 & 0.2393 \end{bmatrix}; \quad \Gamma_2 = \begin{bmatrix} 0.0298 & 0.1800 & 0.2218 \\ 0.1665 & 0.3139 & 0.2933 \\ 0.5598 & 0.1536 & 0.2398 \\ 0.3001 & 0.3232 & 0.0321 \\ 0.3112 & 0.3515 & 0.5586 \\ 0.3772 & 0.2107 & 0.3361 \end{bmatrix}.$$

By solving LMIs (3)–(9) for $\epsilon_i > 0 (i = 1, 2, 3, 4)$, $\rho_X > 0 (X = Q_2, P_2, X_1, X_2, Z_1, Z_2)$ and matrices $X_1 > 0$, $X_2 > 0$, $Z_1 > 0$, $Z_2 > 0$, $P > 0$, $Q > 0$, $P_1 > 0$, $Q_1 > 0$, $P_2 > 0$, $Q_2 > 0$, we obtain

$$P = \begin{bmatrix} 0.2257 & -0.0169 & -0.0391 \\ -0.0169 & 0.3624 & 0.0212 \\ -0.0391 & 0.0212 & 0.2539 \end{bmatrix}; \quad Q = \begin{bmatrix} 0.3316 & 0.1228 \\ 0.1228 & 0.5125 \end{bmatrix};$$

$$P_1 = \begin{bmatrix} 0.3605 & -0.0005 & -0.0016 \\ -0.0005 & 0.1667 & 0.0474 \\ -0.0016 & 0.0474 & 0.1246 \end{bmatrix}; \quad P_2 = \begin{bmatrix} 0.3239 & 0 \\ 0 & 0.3239 \end{bmatrix};$$

$$Q_1 = \begin{bmatrix} 0.5007 & 0.0097 \\ 0.0097 & 0.2897 \end{bmatrix}; \quad Q_2 = \begin{bmatrix} 0.4190 & 0 & 0 \\ 0 & 0.4190 & 0 \\ 0 & 0 & 0.4190 \end{bmatrix};$$
$$X_1 = \begin{bmatrix} 0.3861 & 0 & -0.0057 \\ 0 & 0.3290 & 0 \\ -0.0057 & 0 & 0.4381 \end{bmatrix}; \quad X_2 = \begin{bmatrix} 0.7768 & 0.0136 \\ 0.0136 & 0.6089 \end{bmatrix};$$
$$Z_1 = \begin{bmatrix} 0.4788 & 0 & -0.0014 \\ 0 & 0.4079 & 0 \\ -0.0014 & 0 & 0.4400 \end{bmatrix}; \quad Z_2 = \begin{bmatrix} 0.6490 & -0.0223 \\ -0.0223 & 0.6819 \end{bmatrix};$$
$$\epsilon_1 = 0.6443; \quad \epsilon_2 = 1.1712; \quad \epsilon_3 = 0.0945; \quad \epsilon_4 = 0.4812$$

which implies from Theorem 2 that the above neural network is robustly, globally, asymptotically stable in the mean square by designing a state feedback memory control law and a impulsive feedback control law of the form

$$\begin{aligned} u(t) &= [0.11510.3782 - 0.9017]x(t) + [9.5251 - 5.65521.9394]x(t - \sigma(t)), \\ \widetilde{u}(t) &= [-13.706910.9838]y(t) + [3.2767 - 2.4000]y(t - \tau(t)), \\ J(t_k^-) &= \begin{bmatrix} -0.2118 & -0.0994 & 0.3143 \\ -0.1684 & 0.2283 & -0.4115 \\ 0.3271 & -0.2291 & -0.2115 \end{bmatrix} x(t_k^-), \\ \widetilde{J}(t_k^-) &= \begin{bmatrix} 2.4283 & -2.0264 \\ -2.8833 & 1.7573 \end{bmatrix} y(t_k^-) \end{aligned} \tag{18}$$

The time trajectories of Example 2 with initial conditions

$$\begin{cases} \phi_1(s) = 2, & s \in [-2.5, 0]; \\ \phi_2(s) = 1.5, & s \in [-2.5, 0]; \\ \phi_3(s) = 0.5, & s \in [-2.5, 0]; \end{cases} \quad \begin{cases} \varphi_1(s) = -0.5, & s \in [-3, 0]; \\ \varphi_2(s) = -1, & s \in [-3, 0]; \\ \varphi_3(s) = -1.5, & s \in [-3, 0]; \end{cases}$$

and $f_i(x) = \tanh(x)(i = 1, 2, 3)$; $g_j(x) = \tanh(x)(j = 1, 2, 3)$ are shown in Figs. 2 [without impulsive controller (18)] and 3 [with impulsive controller (18)].

5 Conclusion and Future Works

In this article, we have investigated some high-order bidirectional neural networks with time-varying delays and impulsive, established a new globally asymptotical stability and designed a state feedback memory controller to robustly stabilize this network. Due to the development of the fractional-order calculus and its applications [23,24], some results on fractional-order neural networks have been obtained [8,14]. To the best of our knowledge, the model of the high-order fractional-order BAM neural networks has not been investigated until now. The corresponding results will appear in the near future.

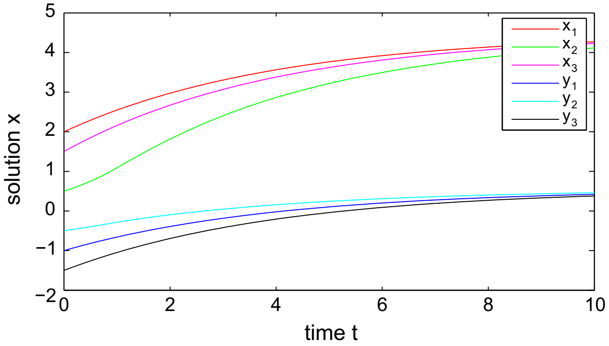

Fig. 2 State of system Example 2 without impulsive control

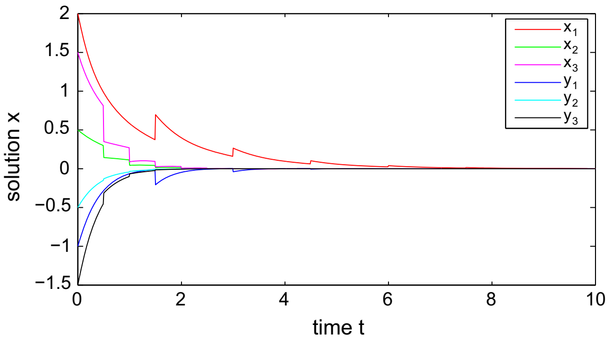

Fig. 3 State of system Example 2 with impulsive controller (18)

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