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# Asynchronous $H_\infty$ Control of Switched Systems with Mode-Dependent Average Dwell Time

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**Abstract** This paper is concerned with the problem of $H_\infty$ control for a class of switched systems. Time delays that appear in both the state and the output are considered. In addition, the switching of the controllers experiences a time delay with respect to that of subsystems, which is called "asynchronous switching." By the utilization of the piecewise Lyapunov function technique, sufficient conditions that ensure the exponential stability and a weighted $H_\infty$ performance level for the closed-loop system under a mode-dependent average dwell time (MDADT) scheme is proposed. MDADT means that each subsystem has its own average dwell time (ADT), which is more general than ADT. Two types of MDADT are gained by dividing all the subsystems into two parts. Then, the asynchronous $H_\infty$ dynamical output feedback controller is designed in terms of linear matrix inequalities. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed method.

**Keywords** Asynchronous switching · Mode-dependent average dwell time · Time-varying delay · $H_\infty$ control · Switched system

## 1 Introduction

Switched systems belong to an important class of hybrid systems, represented by a finite number of subsystems and a switching signal orchestrating the switching among

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them. Due to the significance in theoretical development and practical application, the investigation of switched systems has been attracting increasing attention. A considerable number of results have been reported [2,7,8,16]. Since time-delay phenomena exist widely in many engineering systems, which may lower the system performance and even lead to system instability, switched delay systems have been extensively studied [5,13,17,35,38].

Switched systems display complicated dynamical behavior. Switched systems might be stable or unstable for different switching signal. Switched systems might be unstable even if each subsystem is stable [9]. Thus, it is necessary to co-design the switching signal and controller to obtain the performance of the system.

For the switching signal design, an effective method is average dwell time (ADT) [6, 34], which has been widely used to investigate the stability and stabilization problems of switched systems [10,23,25]. Recently, the authors in [32] put forward mode-dependent average dwell time (MDADT) in which each subsystem has its own ADT. It has been proved that MDADT is a more general class of ADT [26,27,33].

For the controller design, the mainly used technique is linear matrix inequalities (LMIs) [1,4]. Various controllers have been designed in [10,14,15,18,22–25,30] without asynchronous switching. However, as stated in [36,37], in actual operation, it takes some time to identify the active subsystem and apply the matched controller, so there inevitably exists asynchronous switching between the subsystems and controllers. The results related to the switched systems under asynchronous switching have been reported. To mention a few, asynchronous $H_\infty$ filtering problem has been investigated in [28,29], asynchronous output feedback control problem has been addressed in [3,20], and asynchronous state feedback control problem has been studied in [11,12,21,31]. In practice, it is often not possible to obtain full information on the state variables to use them for feedback control. This makes it necessary to study the dynamical output feedback (DOF) control problem [20]. To the best of our knowledge, the asynchronous $H_\infty$ DOF control problem of switched systems with time-varying delay, especially based on the MDADT approach, has been rarely studied. The presence of time-varying delay makes the DOF control problem much more complicated. Meanwhile, its presence adds the difficulty for the design of the DOF controller. How to choose the piecewise Lyapunov function technique to establish solvable conditions for the DOF controller is a crucial issue, which has not been resolved. Thus, research in this area should be of both theoretical and practical importance, which motivates us to undertake this work.

In this paper, we are interested in investigating the asynchronous $H_\infty$ DOF control for switched time-varying delay systems. By using the MDADT approach combined with the piecewise Lyapunov function technique, sufficient conditions are proposed to guarantee the exponential stability with a weighted $H_\infty$ performance for the switched closed-loop system. By dividing the subsystems into two parts, two types of MDADT are gained. Moreover, the conditions for solving the DOF controller are given in terms of LMIs. Finally, the simulation result is provided to illustrate the effectiveness of the proposed theory. The contribution of this paper is as follows: (1) The DOF controller under asynchronous switching for switched time-varying delay systems is designed; (2) the weighted $H_\infty$ performance is introduced to study the DOF control problem of switched time-varying delay systems, which has rarely been addressed before;

(3) a more general class of switching signal, i.e., the MDADT switching signal, is considered; (4) two types of smaller MDADT are gained.

The remainder of the article is organized as follows. Preliminaries and problem formulation are introduced in Sect. 2. Section 3 presents the main results. A numerical example is provided in Sect. 4. The conclusions are summarized in Sect. 5.

### 1.1 Notations

$\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space. $\mathbb{R}^{m\times n}$ is the set of all real $m \times n$ matrices. $P > 0$ means that $P$ is a positive definite symmetric matrix. $\lambda_{min}(P)$ ($\lambda_{max}(P)$) is the minimum (maximum) eigenvalue of matrix $P$. $A^{\mathrm{T}}$ denotes the transpose of matrix $A$. * stands for the symmetric terms in matrices. $||\cdot||$ refers to the Euclidean vector norm. $I$ and 0 denote the identity matrix and the zero matrix with appropriate dimension, respectively. diag$\{\cdots\}$ stands for a block diagonal matrix. $L_2[0,\infty)$ is the space of square-integrable vector functions over $[0,\infty)$. $\mathbb{N}$ represents the set of all nonnegative integers.

## 2 Problem Formulation and Preliminaries

Consider a class of switched delay systems

$$\begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + D_{\sigma(t)}x(t-d(t)) + B_{\sigma(t)}u(t) + E_{\sigma(t)}\omega(t), \\ y(t) = C_{\sigma(t)}x(t) + F_{\sigma(t)}x(t-d(t)) + G_{\sigma(t)}\omega(t), \\ z(t) = L_{\sigma(t)}x(t) + U_{\sigma(t)}x(t-d(t)) + H_{\sigma(t)}\omega(t), \\ x(t) = \varphi(t), t \in [-h, 0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $y(t) \in R^{n_y}$ is the measurement output, $z(t) \in R^{n_z}$ is the controller output, $\omega(t) \in R^{n_\omega}$ is the disturbance input which belongs to $L_2[0,\infty)$. $d(t)$ denotes the time-varying delay satisfying $0 \le d(t) \le h$ and $\dot{d}(t) \le h_d < 1$. $\varphi(t)$ is a vector-valued initial function on $[-h, 0]$.$\sigma(t) : [t_0,\infty) \to \mathfrak{M} = \{1, 2, ..., M\}$, called the switching signal, is a piecewise right continuous function. $M$ is the number of subsystems, and $t_0$ is the initial time. For a switching sequence of the subsystems $\Sigma = \{(\sigma(t_o), t_o), (\sigma(t_1), t_1), ..., (\sigma(t_k), t_k), ...|k \in \mathbb{N}\}$, when $t \in [t_k, t_{k+1})$, $\sigma(t) = \sigma(t_k) = p \in \mathfrak{M}$, we say that the $p$th subsystem is active. $A_p$, $D_p$, $B_p$, $E_p$, $C_p$, $F_p$, $G_p$, $L_p$, $U_p$, and $H_p$ are known real constant matrices with appropriate dimensions.

Due to the asynchronous switching between the controllers and subsystems, we consider the dynamical output feedback (DOF) controller as follows:

$$\begin{cases} \dot{x}_c(t) = A_{c,\sigma(t-\Delta_k)}x_c(t) + B_{c,\sigma(t-\Delta_k)}y(t), \quad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \\ u(t) = C_{c,\sigma(t-\Delta_k)}, \quad x_c(0) = 0, \end{cases} \tag{2}$$

where $\Delta_0 = 0$, and $\Delta_k < t_{k+1} - t_k$ represents the delayed period.

Let $\sigma(t_k) = p \in \mathfrak{M}, \sigma(t_{k-1}) = q \in \mathfrak{M}, p \neq q$. Applying the controller (2) to system (1), we obtain the following closed-loop system

$$\begin{cases} \dot{\bar{x}}(t) = \bar{A}_{\tilde{\sigma}}\bar{x}(t) + \bar{D}_{\tilde{\sigma}}\bar{x}(t - d(t)) + \bar{E}_{\tilde{\sigma}}\omega(t), \\ z(t) = \bar{L}_p\bar{x}(t) + \bar{U}_p\bar{x}(t - d(t)) + \bar{H}_p\omega(t), \quad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \\ \bar{x}(t) = \bar{\varphi}(t),\ t \in [-h, 0], \end{cases} \tag{3}$$

where

$$\tilde{\sigma} = \begin{cases} pq, t \in [t_k, t_k + \Delta_k) \\ p, t \in [t_k + \Delta_k, t_{k+1}), \end{cases}$$

$$\bar{x}(t) = \begin{bmatrix} x^{\mathrm{T}}(t) & x_c^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}, \quad \bar{H}_p = \begin{bmatrix} H_p \end{bmatrix}, \quad \bar{L}_p = \begin{bmatrix} L_p & 0 \end{bmatrix}, \quad \bar{U}_p = \begin{bmatrix} U_p & 0 \end{bmatrix},$$

$$\bar{A}_p = \begin{bmatrix} A_p & B_pC_{c,p} \\ B_{c,p}C_p & A_{c,p} \end{bmatrix}, \quad \bar{D}_p = \begin{bmatrix} D_p & 0 \\ B_{c,p}F_p & 0 \end{bmatrix}, \quad \bar{E}_p = \begin{bmatrix} E_p \\ B_{c,p}G_p \end{bmatrix},$$

$$\bar{A}_p = \begin{bmatrix} A_p & B_pC_{c,q} \\ B_{c,q}C_p & A_{c,q} \end{bmatrix}, \quad \bar{D}_{pq} = \begin{bmatrix} D_p & 0 \\ B_{c,q}F_p & 0 \end{bmatrix}, \quad \bar{E}_{pq} = \begin{bmatrix} E_p \\ B_{c,q}G_p \end{bmatrix}.$$

Now, we state the following definitions and lemma for latter development.

**Definition 1** [32] For a switching signal $\sigma(t)$ and any $T > t \geq 0$, let $N_{\sigma p}(t, T)$ be the switching numbers that the $p$th subsystem is activated over the interval $[t, T)$ and $T_p(t, T)$ denote the total running time of the $p$th subsystem over the interval $[t, T)$, $p \in \mathfrak{M}$. We say that $\sigma(t)$ has a mode-dependent average dwell time (MDADT) $\tau_p$ if there exist positive numbers $N_{0p}$ (we call $N_{0p}$ the mode-dependent chatter bounds) and $\tau_p$ such that

$$N_{\sigma p}(t, T) \leq N_{0p} + \frac{T_p(t, T)}{\tau_p}. \tag{4}$$

**Definition 2** [17] The equilibrium $\bar{x} = 0$ of closed-loop system (3) with $w(t) = 0$ is globally uniformly exponentially stable (GUES) under certain switching signal $\sigma(t)$ and initial condition $\bar{x}(t_0)$, if there exist constants $\delta > 0$ and $\eta > 0$ such that the solution of the system satisfies

$$||\bar{x}(t)|| \leq \delta \mathrm{e}^{-\eta(t-t_o)} \, ||\bar{x}(t_0)||_{c^1}, \forall t \geq t_0, \tag{5}$$

where $||\bar{x}(t_0)||_{c^1} = \sup_{-h \leq \theta \leq 0} \left\{ ||\bar{x}(t_0 + \theta)||, \left|\left|\dot{\bar{x}}(t_0 + \theta)\right|\right| \right\}$.

**Definition 3** For the given constants $\alpha_p > 0$ and $\gamma > 0$, system (3) is said to be GUES with a weighted $H_\infty$ performance $\gamma$, if the following conditions are satisfied:

(1) System (3) is exponentially stable with $w(t) = 0$;
(2) Under zero initial condition, i.e., $\bar{\varphi}(t) = 0, t \in [-h, 0]$, it holds for any nonzero $w(t) \in L_2[0, \infty)$ that

$$\int_{t_0}^{\infty} \exp\left\{ -\sum_{p=1}^{M} \left[ \alpha_p T_p(t_0, t) \right] \right\} z^{\mathrm{T}}(t) z(t) \mathrm{d}t \leq \gamma^2 \int_{t_0}^{\infty} w^{\mathrm{T}}(t) w(t) \mathrm{d}t. \tag{6}$$

*Remark 1* The standard $H_\infty$ performance, which has been commonly adopted for non-switched systems, cannot be achieved in general for switched systems with an ADT switching. Thus, weighed $H_\infty$ performance with the weighted term $e^{-\alpha t}$ is used in [23,34]. In this paper, since the MDADT switching technique is used, the weighted term is replaced by $\exp\{-\sum_{p=1}^{M}[\alpha_P T_p(t_0, t)]\}$. It can be seen that when $\alpha_p = \alpha, \forall p \in \mathfrak{M}$, Definition 3 is turned into that in [23,34]. Thus, Definition 3 can be viewed as an extension of that in [23,34].

**Lemma 1** [38] *Let $x(t) \in \mathbb{R}^n$ be a vector-valued function with first-order continuous-derivative entries. Then, the following integral inequality holds for any matrices $N_1, N_2 \in \mathbb{R}^{n\times n}$ and $X = X^{\mathrm{T}} > 0$, and a scalar function $0 \le d(t) \le h$ :*

$$-\int_{t-d(t)}^{t} \dot{x}^{\mathrm{T}}(s)X\dot{x}(s)ds \le \zeta^{\mathrm{T}}(t)\begin{bmatrix} N_1^{\mathrm{T}} + N_1 & -N_1^{\mathrm{T}} + N_2 \\ * & -N_2^{\mathrm{T}} + N_2 \end{bmatrix}\zeta(t) + h\zeta^{\mathrm{T}}(t)\begin{bmatrix} N_1^{\mathrm{T}} \\ N_2^{\mathrm{T}} \end{bmatrix} X^{-1}\begin{bmatrix} N_1 & N_2 \end{bmatrix}\zeta(t), \tag{7}$$

*where $\zeta(t) = [x^{\mathrm{T}}(t)\; x^{\mathrm{T}}(t - d(t))]^{\mathrm{T}}$.*

**Lemma 2** [1] (Schur complement) *For a given symmetric matrix with the partition*

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix},$$

*where $W_{11}$ and $W_{22}$ is a square matrix and $W_{12}^{\mathrm{T}} = W_{21}$, the following three conditions are equivalent*

(1) $W < 0$;
(2) $W_{11} < 0$ *and* $W_{22} - W_{12}^{\mathrm{T}} W_{11}^{-1} W_{12} < 0$;
(3) $W_{22} < 0$ *and* $W_{11} - W_{12} W_{22}^{-1} W_{12}^{\mathrm{T}} < 0$.

## 3 Main Results

### 3.1 Stability and $H_\infty$ Performance Analysis

In this section, we focus on the stability and $H_\infty$ performance of the closed-loop system (3) with asynchronous behaviors. For concise notation, let $T_\downarrow(0, t)$ ($T_\uparrow(0, t)$) represent the total periods that the controllers and the subsystems are matched (unmatched) during $[0, t)$. Let $T_{\downarrow p}(0, t)$ ($T_{\uparrow p}(0, t)$) denote the total running time of the $p$th subsystem controlled by the matched (unmatched) controller during $[0, t)$.

The following theorem presents a sufficient condition of exponential stability for the system (3) with $w(t) = 0$.

**Theorem 1** *For the switched system (3) with $w(t) = 0$, let $\alpha_p > 0$, $\beta_P$, $\mu_p \ge 1$ and $\hat{\mu}_p \ge 1$, $p \in \mathfrak{M}$ be given constants, if there exist matrices $P_p > 0$, $Q_p > 0$, $S_p > 0$, $P_{pq} > 0$, $Q_{pq} > 0$ and $S_{pq} > 0$, such that $\forall (p, q) \in \mathfrak{M} \times \mathfrak{M}$, $p \ne q$,*

$$\begin{bmatrix} \Gamma_{11}^{p} & \Gamma_{12}^{p} & h\bar{A}_p^{\mathrm{T}}K^{\mathrm{T}} \\ * & \Gamma_{22}^{p} & h\bar{D}_p^{\mathrm{T}}K^{\mathrm{T}} \\ * & * & -hS_p^{-1} \end{bmatrix} < 0, \tag{8}$$

$$\begin{bmatrix} \Gamma_{11}^{pq} & \Gamma_{12}^{pq} & h\bar{A}_{pq}^{\mathrm{T}}K^{\mathrm{T}} \\ * & \Gamma_{22}^{pq} & h\bar{D}_{pq}^{\mathrm{T}}K^{\mathrm{T}} \\ * & * & -hS_{pq}^{-1} \end{bmatrix} < 0, \tag{9}$$

$$\begin{aligned} &P_p \le \mu_p P_{pq}, Q_p \le \mu_p Q_{pq}, S_p \le \mu_p S_{pq}, \\ &P_{pq} \le \hat{\mu}_p P_p, Q_{pq} \le \hat{\mu}_p Q_p, S_{pq} \le \hat{\mu}_p S_q, \end{aligned} \tag{10}$$

*where*

$$\begin{aligned} \Gamma_{11}^{p} &= \bar{A}_p^{\mathrm{T}} P_p + P_p \bar{A}_p + \alpha_p P_p + Q_p, \\ \Gamma_{12}^{p} &= P_p \bar{D}_p + e^{-\alpha_p h} K^{\mathrm{T}} K, \\ \Gamma_{22}^{p} &= -(1-h_d)e^{-\alpha_p h} Q_p - 2e^{-\alpha_p h} K^{\mathrm{T}} K + he^{-\alpha_p h} K^{\mathrm{T}} S_p^{-1} K, \\ \Gamma_{11}^{pq} &= \bar{A}_{pq}^{\mathrm{T}} P_{pq} + P_{pq} \bar{A}_{pq} - \beta_p P_{pq} + Q_{pq}, \\ \Gamma_{12}^{pq} &= P_{pq} \bar{D}_{pq} + K^{\mathrm{T}} K, \\ \Gamma_{22}^{pq} &= -(1-h_d) Q_{pq} - 2K^{\mathrm{T}} K + hK^{\mathrm{T}} S_{pq}^{-1} K. \end{aligned}$$

*Then, the closed-loop system* (3) *is GUES for any switching signal* $\sigma(t)$ *with the following MDADT*

$$\begin{aligned} \tau_p \ge \tau_p^* &= \frac{ln(\mu_p \hat{\mu}_p \tilde{\mu}_p)}{\alpha_p}, \beta_p + \alpha_p \le 0, \\ \tau_p \ge \tau_p^* &= \frac{ln(\mu_p \hat{\mu}_p \tilde{\mu}_p) + (\alpha_p + \beta_p)\Delta_{pM}}{\alpha_p}, \beta_p + \alpha_p > 0, \end{aligned} \tag{11}$$

*where* $\tilde{\mu}_p = \max_{q \in \mathfrak{M}, q \neq p} \{\mu_{qp}\}$, $\mu_{qp} = e^{\alpha_q + \beta_p}$, $\Delta_{pM} = \max T_{\upharpoonright p}(t_k, t_{k+1}), \forall k \in \mathbb{N}$.

*Proof* According to the value of $\alpha_p + \beta_p$, we divide all the subsystems into two parts: if $\alpha_p + \beta_p \le 0$, the subsystem belongs to set $\Psi_1 = \{1, \ldots, l\}$; otherwise, it belongs to set $\Psi_2 = \{l+1, \ldots, M\}$.

For $\forall t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$, let $t_0 = 0$, and define $K = [I \ 0]$. Choose a piecewise Lyapunov function of the following form:

$$\begin{aligned} V(t) = \bar{x}^{\mathrm{T}}(t) P_{\tilde{\sigma}} \bar{x}(t) + \int_{t-d(t)}^{t} \mathrm{e}^{\kappa(t-s)} \bar{x}^{\mathrm{T}}(s) Q_{\tilde{\sigma}} \bar{x}(s) \mathrm{d}s \\ + \int_{-h}^{0} \int_{t+\theta}^{t} \mathrm{e}^{\kappa(t-s)} \dot{\bar{x}}^{\mathrm{T}}(s) K^{\mathrm{T}} S_{\tilde{\sigma}} K \dot{\bar{x}}(s) \mathrm{d}s \mathrm{d}\theta, \end{aligned} \tag{12}$$

where

$$\kappa = \begin{cases} \beta_p, & t \in [t_k, t_k + \Delta_k) \\ -\alpha_p, & t \in [t_k + \Delta_k, t_{k+1}). \end{cases}$$

Taking the derivation of the Lyapunov function, we have

$$\begin{aligned} \dot{V}(t) \le\ & \kappa V(t) - \kappa \bar{x}^{\mathrm{T}}(t) P_{\tilde{\sigma}} \bar{x}(t) + 2\dot{\bar{x}}^{\mathrm{T}}(t) P_{\tilde{\sigma}} \bar{x}(t) \\ & + \bar{x}^{\mathrm{T}}(t) Q_{\tilde{\sigma}} \bar{x}(t) + h\dot{\bar{x}}^{\mathrm{T}}(t) K^{\mathrm{T}} S_{\tilde{\sigma}} K \dot{\bar{x}}(t) \mathrm{d}s \\ & - (1 - h_d) v \bar{x}^{\mathrm{T}}(t - d(t)) Q_{\tilde{\sigma}} \bar{x}(t - d(t)) \\ & - v \int_{t-d(t)}^{t} \dot{\bar{x}}^{\mathrm{T}}(s) K^{\mathrm{T}} S_{\tilde{\sigma}} K \dot{\bar{x}}(s) \mathrm{d}s, \end{aligned} \tag{13}$$

where

$$v = \begin{cases} 1, & t \in [t_k, t_k + \Delta_k) \\ \mathrm{e}^{-\alpha_p h}, & t \in [t_k + \Delta_k, t_{k+1}). \end{cases}$$

Define $\xi(t) = [\bar{x}^{\mathrm{T}}(t)\ \bar{x}^{\mathrm{T}}(t - d(t))]^{\mathrm{T}}$, it follows from Lemma 1 with $N_1 = 0$, $N_2 = I$:

$$\begin{aligned} & -v \int_{t-d(t)}^{t} \dot{\bar{x}}^{\mathrm{T}}(s) K^{\mathrm{T}} S_{\tilde{\sigma}} K \dot{\bar{x}}(s) \mathrm{d}s \\ & \le v \left\{ \xi^{\mathrm{T}}(t) \begin{bmatrix} 0 & K^{\mathrm{T}} K \\ * & -2K^{\mathrm{T}} K \end{bmatrix} \xi(t) + h \bar{x}^{\mathrm{T}}(t - d(t)) K^{\mathrm{T}} S_{\tilde{\sigma}}^{-1} K \bar{x}(t - d(t)) \right\}. \end{aligned} \tag{14}$$

From (12)–(14), it yields that

$$\dot{V}(t) \le \begin{cases} \beta_p V(t) + \xi^{\mathrm{T}}(t)(\Gamma^{pq} + d\Theta_{pq} S_{pq} \Theta_{pq}^{\mathrm{T}})\xi(t), & t \in [t_k, t_k + \Delta_k) \\ -\alpha_p V(t) + \xi^{\mathrm{T}}(t)(\Gamma^{p} + d\Theta_{p} S_{p} \Theta_{p}^{\mathrm{T}})\xi(t), & t \in [t_k + \Delta_k, t_{k+1}), \end{cases} \tag{15}$$

where

$$\Gamma^p = \begin{bmatrix} \Gamma_{11}^p & \Gamma_{12}^p \\ * & \Gamma_{22}^p \end{bmatrix}, \Gamma^{pq} = \begin{bmatrix} \Gamma_{11}^{pq} & \Gamma_{12}^{pq} \\ * & \Gamma_{22}^{pq} \end{bmatrix}, \Theta_p = \begin{bmatrix} \bar{A}_p^{\mathrm{T}} & K^{\mathrm{T}} \\ \bar{D}_p^{\mathrm{T}} & K^{\mathrm{T}} \end{bmatrix}, \Theta_{pq} = \begin{bmatrix} \bar{A}_{pq}^{\mathrm{T}} & K^{\mathrm{T}} \\ \bar{D}_{pq}^{\mathrm{T}} & K^{\mathrm{T}} \end{bmatrix}.$$

By Schur complement Lemma, (8) and (9) imply

$$\dot{V}(t) \le \begin{cases} \beta_p V(t), & t \in [t_k, t_k + \Delta_k) \\ -\alpha_p V(t), & t \in [t_k + \Delta_k, t_{k+1}) \end{cases}, \tag{16}$$

which gives that

$$V(t) \le \begin{cases} \mathrm{e}^{\beta_p (t - t_k)} V(t_k), & t \in [t_k, t_k + \Delta_k) \\ \mathrm{e}^{-\alpha_p (t - t_k - \Delta_k)} V(t_k + \Delta_k), & t \in [t_k + \Delta_k, t_{k+1}). \end{cases} \tag{17}$$

Using (10) and (12), we get

$$\begin{aligned}&V(t_k) \le \hat{\mu}_p \tilde{\mu}_p V(t_k^-),\\&V(t_k+\Delta_k) \le \mu_p V((t_k+\Delta_k)^-).\end{aligned} \tag{18}$$

For $\forall t \in [t_k, t_{k+1})$, combining (17) and (18) yields

$$\begin{aligned}V(t) &\le \mu_{\sigma(t_k)} \mathrm{e}^{\beta_{\sigma(t_k)} T_\uparrow(t_k,t) - \alpha_{\sigma(t_k)} T_\downarrow(t_k,t)} V(t_k)\\
&\le \mu_{\sigma(t_k)} \hat{\mu}_{\sigma(t_k)} \tilde{\mu}_{\sigma(t_k)} \mathrm{e}^{\beta_{\sigma(t_k)} T_\uparrow(t_k,t) - \alpha_{\sigma(t_k)} T_\downarrow(t_k,t)} V(t_k^-)\\
&\le \mu_{\sigma(t_{k-1})} \mu_{\sigma(t_k)} \hat{\mu}_{\sigma(t_k)} \tilde{\mu}_{\sigma(t_k)} \mathrm{e}^{\beta_{\sigma(t_k)} T_\uparrow(t_k,t) - \alpha_{\sigma(t_k)} T_\downarrow(t_k,t)}\\
&\quad \times \mathrm{e}^{\beta_{\sigma(t_{k-1})} T_\uparrow(t_{k-1},t_k) - \alpha_{\sigma(t_{k-1})} T_\downarrow(t_{k-1},t_k)} V(t_{k-1})\\
&\le \prod_{i=k-1}^{k} (\mu_{\sigma(t_i)} \hat{\mu}_{\sigma(t_i)} \tilde{\mu}_{\sigma(t_i)}) \mathrm{e}^{\beta_{\sigma(t_k)} T_\uparrow(t_k,t) - \alpha_{\sigma(t_k)} T_\downarrow(t_k,t)}\\
&\quad \times \mathrm{e}^{\beta_{\sigma(t_{k-1})} T_\uparrow(t_{k-1},t_k) - \alpha_{\sigma(t_{k-1})} T_\downarrow(t_{k-1},t_k)} V(t_{k-1}^-)\\
&\le \cdots\\
&\le \prod_{i=1}^{k} (\mu_{\sigma(t_i)} \hat{\mu}_{\sigma(t_i)} \tilde{\mu}_{\sigma(t_i)}) \mathrm{e}^{\beta_{\sigma(t_k)} T_\uparrow(t_k,t) - \alpha_{\sigma(t_k)} T_\downarrow(t_k,t)}\\
&\quad \times \mathrm{e}^{\sum_{i=1}^{k} \left[\beta_{\sigma(t_{i-1})} T_\uparrow(t_{i-1},t_i) - \alpha_{\sigma(t_{i-1})} T_\downarrow(t_{i-1},t_i)\right]} V(t_0)\\
&= \exp\left\{\sum_{p=1}^{M} \left[\beta_p T_{\uparrow p}(0,t) - \alpha_p T_{\downarrow p}(0,t)\right]\right\} \prod_{p=1}^{M} (\mu_p \hat{\mu}_p \tilde{\mu}_p)^{N_{\sigma p(0,t)}} V(t_0)\\
&= \Omega_1 \Omega_2 V(t_0),\end{aligned} \tag{19}$$

where

$$\begin{aligned}\Omega_1 &= \exp\left\{\sum_{p=1}^{l} \left[\beta_p T_{\uparrow p}(0,t) - \alpha_p T_{\downarrow p}(0,t)\right]\right\} \prod_{p=1}^{l} (\mu_p \hat{\mu}_p \tilde{\mu}_p)^{N_{\sigma p(0,t)}},\\
\Omega_2 &= \exp\left\{\sum_{p=l+1}^{M} \left[\beta_p T_{\uparrow p}(0,t) - \alpha_p T_{\downarrow p}(0,t)\right]\right\} \prod_{p=l+1}^{M} (\mu_p \hat{\mu}_p \tilde{\mu}_p)^{N_{\sigma p(0,t)}}.\end{aligned}$$

For $\Omega_1$ noticing that $\beta_p \le -\alpha_p$, together with Definition 1, we get

$$\begin{aligned}\Omega_1 &\le \exp\left\{\sum_{p=1}^{l} \left[-\alpha_p T_p(0,t)\right]\right\} \prod_{p=1}^{l} (\mu_p \hat{\mu}_p \tilde{\mu}_p)^{N_{\sigma p(0,t)}}\\
&\le \exp\left\{\sum_{p=1}^{l} \left[N_{0p} \ln(\mu_p \hat{\mu}_p \tilde{\mu}_p) + T_p(0,t) \left(\frac{\ln(\mu_p \hat{\mu}_p \tilde{\mu}_p)}{\tau_p} - \alpha_p\right)\right]\right\}.\end{aligned} \tag{20}$$

For $\Omega_2$, noticing that $T_{\uparrow p}(0,t) \le \Delta_{pM} N_{\sigma p}(0,t)$, together with Definition 1, we have

$$\begin{aligned}\Omega_2 &\le \exp\left\{\sum_{p=l+1}^{M}\left[-\alpha_p T_p(0,t) + (\alpha_p+\beta_p)\Delta_{pM} N_{\sigma p}(0,t)\right]\right\}\prod_{p=l+1}^{M}(\mu_p\hat{\mu}_p\tilde{\mu}_p)^{N_{\sigma p(0,t)}} \\ &\le \exp\left\{\sum_{p=l+1}^{M}\left[N_{0p}\ln(\mu_p\hat{\mu}_p\tilde{\mu}_p) + (\alpha_p+\beta_p)\Delta_{pM}\right]\right\} \\ &\quad\times\exp\left\{\sum_{p=l+1}^{M}\left[\left(\frac{\ln(\mu_p\hat{\mu}_p\tilde{\mu}_p)+(\alpha_p+\beta_p)\Delta_{pM}}{\tau_p}-\alpha_p\right)T_p(0,t)\right]\right\}.\end{aligned} \tag{21}$$

Define

$$\pi_1 = \min_{p,q\in\mathfrak{M},p\neq q}\{\lambda_{\min}(P_p),\lambda_{\min}(P_{pq})\},$$
$$\pi_2 = \max_{p\in\mathfrak{M}}\{\lambda_{\max}(P_p)\} + h\max_{p\in\mathfrak{M}}\{\lambda_{\max}(Q_p)\} + \frac{h^2}{2}\max_{p\in\mathfrak{M}}\{\lambda_{\max}(S_p)\}.$$

Set

$$\begin{aligned}\delta &= \sqrt{\frac{\pi_2}{\pi_1}}\exp\left\{\frac{1}{2}\sum_{p=1}^{l}\left[N_{0p}\ln(\mu_p\hat{\mu}_p\tilde{\mu}_p)\right]\right. \\ &\quad\left.+\frac{1}{2}\sum_{p=l+1}^{M}\left[N_{0p}\left(\ln(\mu_p\hat{\mu}_p\tilde{\mu}_p)+(\alpha_p+\beta_p)\Delta_{pM}\right)\right]\right\}, \\ \eta &= -\frac{1}{2}\max\left\{\max_{p\in\Psi_1}\left\{\frac{\ln(\mu_p\hat{\mu}_p\tilde{\mu}_p)}{\tau_p}-\alpha_p\right\},\right. \\ &\quad\left.\max_{p\in\Psi_2}\left\{\frac{\ln(\mu_p\hat{\mu}_p\tilde{\mu}_p)+(\alpha_p+\beta_p)\Delta_{pM}}{\tau_p}-\alpha_p\right\}\right\}.\end{aligned}$$

Then, from (11) and (19)–(21), we can obtain

$$\|\bar{x}(t)\| \le \delta e^{-\eta(t-t_0)}\|\bar{x}(t_0)\|_{c^1}. \tag{22}$$

By Definition 2, we can conclude that the closed-loop system (3) with $w(t) = 0$ is GUES for any switching signal with MDADT (11). This completes the proof. □

*Remark 2* To facilitate the latter design of the DOF controller, in Theorem 1, a matrix $K = [I\ 0]$ is added into the third term of the piecewise Lyapunov function (12).

*Remark 3* A unique feature of the approaches in this paper is the utilization of MDADT. Different from the ADT approach adopted in [12,18–20], where the parameters are mode-independent, and the ADT for all the subsystems are required to be larger than a common constant $\tau_a$, the parameters selected in this paper are mode-dependent, and we only require the ADT among the intervals associated with the $p$th subsystem to be larger than $\tau_p$, where the intervals are not adjacent.

*Remark 4* Different form most existing results on asynchronous control problem [3, 11, 12, 20, 28, 29, 31], in which $\alpha$ and $\beta$ are positive, in this paper, $\beta_p$, $p \in \mathfrak{M}$ can be negative. Based on the value of $\alpha_p + \beta_p$, $p \in \mathfrak{M}$, we get two types of MDADT (11). It can be seen that for the same parameters $\alpha$ and $\mu$ if only $\beta_p < 0$, $p \in \mathfrak{M}$ exist, the MDADT (11) is smaller than that in [3, 11, 12, 20, 28, 29, 31].

Now, we are in a position to give the weighted $H_\infty$ performance analysis for the system (3).

**Theorem 2** *For the switched system* (3)*, let* $\gamma > 0$*,* $\alpha_p > 0$*,* $\beta_p$*,* $\mu_p \geq 1$ *and* $\hat{\mu}_p \geq 1$*,* $p \in \mathfrak{M}$ *be given constants, if there exist matrices* $P_p > 0$*,* $Q_p > 0$*,* $S_p > 0$*,* $P_{pq} > 0$*,* $Q_{pq} > 0$ *and* $S_{pq} > 0$*,* $\forall (p,q) \in \mathfrak{M} \times \mathfrak{M}$*,* $p \neq q$*, such that* (10) *and the following inequalities hold*

$$\begin{bmatrix} \Gamma_{11}^{p} & \Gamma_{12}^{p} & P_p\bar{E}_p & \bar{L}_p^{\mathrm{T}} & h\bar{A}_p^{\mathrm{T}}K^{\mathrm{T}} \\ * & \Gamma_{22}^{p} & 0 & \bar{U}_p^{\mathrm{T}} & h\bar{D}_p^{\mathrm{T}}K^{\mathrm{T}} \\ * & * & -\gamma^2 I & \bar{H}_p^{\mathrm{T}} & h\bar{E}_p^{\mathrm{T}}K^{\mathrm{T}} \\ * & * & * & -I & 0 \\ * & * & * & * & -hS_p^{-1} \end{bmatrix} < 0, \tag{23}$$

$$\begin{bmatrix} \Gamma_{11}^{pq} & \Gamma_{12}^{pq} & P_{pq}\bar{E}_{pq} & \bar{L}_p^{\mathrm{T}} & h\bar{A}_{pq}^{\mathrm{T}}K^{\mathrm{T}} \\ * & \Gamma_{22}^{pq} & 0 & \bar{U}_p^{\mathrm{T}} & h\bar{D}_{pq}^{\mathrm{T}}K^{\mathrm{T}} \\ * & * & -\gamma^2 I & \bar{H}_p^{\mathrm{T}} & h\bar{E}_{pq}^{\mathrm{T}}K^{\mathrm{T}} \\ * & * & * & -I & 0 \\ * & * & * & * & -hS_{pq}^{-1} \end{bmatrix} < 0, \tag{24}$$

*then the closed-loop system* (3) *is GUES with a weighted* $H_\infty$ *performance level* $\tilde{\gamma}$ *for any switching signal* $\sigma(t)$ *with MDADT satisfying* (11)*, where* $\tilde{\gamma} = \gamma\sqrt{\rho}$ *and* $\rho = \exp\{\sum_{p=1}^{l}[N_{0p}\, ln(\mu_p\hat{\mu}_p\tilde{\mu}_p)] + \sum_{p=l+1}^{M}[((\alpha_p + \beta_p)\Delta_{pM} + ln(\mu_p\hat{\mu}_p\tilde{\mu}_p))N_{0p}]\}$.

*Proof* (8) and (9) can be concluded from (23) and (24). By Theorem 1, the exponential stability of the system (3) with $w(t) = 0$ is guaranteed.

Next, we will show the weighted $H_\infty$ performance of the system.

Constructing the Lyapunov function (12) and using the same method in Theorem 1, it gives

$$\dot{V}(t) \leq \begin{cases} \beta_p V(t) - \Upsilon(t), & t \in [t_k, t_k + \Delta_k) \\ -\alpha_p V(t) - \Upsilon(t), & t \in [t_k + \Delta_k, t_{k+1}) \end{cases} \tag{25}$$

where $\Upsilon(t) = z^{\mathrm{T}}(t)z(t) - \gamma^2 w^T(t)w(t)$.

Integrating both sides of (25), it holds that

$$V(t) \le \begin{cases} \mathrm{e}^{\beta_p(t-t_k)}V(t_k) - \int_{t_k}^{t} \mathrm{e}^{\beta_p(t-s)}\Upsilon(s)\mathrm{d}s, & t \in [t_k, t_k+\Delta_k) \\ \mathrm{e}^{-\alpha_p(t-t_k-\Delta_k)}V(t_k+\Delta_k) \\ \qquad - \int_{t_k+\Delta_k}^{t} \mathrm{e}^{-\alpha_p(t-s)}\Upsilon(s)\mathrm{d}s, & t \in [t_k+\Delta_k, t_{k+1}). \end{cases} \tag{26}$$

For $\forall t \in [t_k, t_{k+1})$, it follows from (18) and (26) that

$$\begin{aligned} V(t) &\le \mu_{\sigma(t_k)}\mathrm{e}^{\beta_{\sigma(t_k)}T_\uparrow(t_k,t)-\alpha_{\sigma(t_k)}T_\downarrow(t_k,t)}V(t_k) \\ &\quad - \int_{t_k}^{t} \mathrm{e}^{\beta_{\sigma(t_k)}T_\uparrow(s,t)-\alpha_{\sigma(t_k)}T_\downarrow(s,t)}\Upsilon(s)\mathrm{d}s \\ &\le \mu_{\sigma(t_k)}\hat{\mu}_{\sigma(t_k)}\tilde{\mu}_{\sigma(t_k)}\mathrm{e}^{\beta_{\sigma(t_k)}T_\uparrow(t_k,t)-\alpha_{\sigma(t_k)}T_\downarrow(t_k,t)}V(t_k^-) \\ &\quad - \int_{t_k}^{t} \mathrm{e}^{\beta_{\sigma(t_k)}T_\uparrow(s,t)-\alpha_{\sigma(t_k)}T_\downarrow(s,t)}\Upsilon(s)\mathrm{d}s \\ &\le \mu_{\sigma(t_{k-1})}\mu_{\sigma(t_k)}\hat{\mu}_{\sigma(t_k)}\tilde{\mu}_{\sigma(t_k)}\mathrm{e}^{\beta_{\sigma(t_k)}T_\uparrow(t_k,t)-\alpha_{\sigma(t_k)}T_\downarrow(t_k,t)} \\ &\quad \times \mathrm{e}^{\beta_{\sigma(t_{k-1})}T_\uparrow(t_{k-1},t_k)-\alpha_{\sigma(t_{k-1})}T_\downarrow(t_{k-1},t_k)}V(t_{k-1}) \\ &\quad - \mu_{\sigma(t_k)}\hat{\mu}_{\sigma(t_k)}\tilde{\mu}_{\sigma(t_k)} \int_{t_{k-1}}^{t} \mathrm{e}^{\beta_{\sigma(t_{k-1})}T_\uparrow(s,t_k)-\alpha_{\sigma(t_{k-1})}T_\downarrow(s,t_k)} \\ &\quad \times \mathrm{e}^{\beta_{\sigma(t_k)}T_\uparrow(t_k,t)-\alpha_{\sigma(t_k)}T_\downarrow(t_k,t)}\Upsilon(s)\mathrm{d}s \\ &\quad - \int_{t_k}^{t} \mathrm{e}^{\beta_{\sigma(t_k)}T_\uparrow(s,t)-\alpha_{\sigma(t_k)}T_\downarrow(s,t)}\Upsilon(s)\mathrm{d}s \\ &\le \prod_{i=k-1}^{k}(\mu_{\sigma(t_i)}\hat{\mu}_{\sigma(t_i)}\tilde{\mu}_{\sigma(t_i)})\mathrm{e}^{\beta_{\sigma(t_k)}T_\uparrow(t_k,t)-\alpha_{\sigma(t_k)}T_\downarrow(t_k,t)} \\ &\quad \times \mathrm{e}^{\beta_{\sigma(t_{k-1})}T_\uparrow(t_{k-1},t_k)-\alpha_{\sigma(t_{k-1})}T_\downarrow(t_{k-1},t_k)}V(t_{k-1}^-) \\ &\quad - \mu_{\sigma(t_k)}\hat{\mu}_{\sigma(t_k)}\tilde{\mu}_{\sigma(t_k)} \int_{t_{k-1}}^{t} \mathrm{e}^{\beta_{\sigma(t_{k-1})}T_\uparrow(s,t_k)-\alpha_{\sigma(t_{k-1})}T_\downarrow(s,t_k)} \\ &\quad \times \mathrm{e}^{\beta_{\sigma(t_k)}T_\uparrow(t_k,t)-\alpha_{\sigma(t_k)}T_\downarrow(t_k,t)}\Upsilon(s)\mathrm{d}s \\ &\quad - \int_{t_k}^{t} \mathrm{e}^{\beta_{\sigma(t_k)}T_\uparrow(s,t)-\alpha_{\sigma(t_k)}T_\downarrow(s,t)}\Upsilon(s)\mathrm{d}s \\ &\le \cdots \\ &\le \exp\left\{\sum_{p=1}^{M}\left[\beta_p T_{\uparrow p}(0,t) - \alpha_p T_{\downarrow p}(0,t)\right]\right\}\prod_{p=1}^{M}(\mu_p\hat{\mu}_p\tilde{\mu}_p)^{N_{\sigma p(0,t)}}V(t_0) \\ &\quad - \int_{t_0}^{t} \exp\left\{\sum_{p=1}^{M}\left[\beta_p T_{\uparrow p}(s,t) - \alpha_p T_{\downarrow p}(s,t)\right]\right\} \\ &\quad \times \prod_{p=1}^{M}(\mu_p\hat{\mu}_p\tilde{\mu}_p)^{N_{\sigma p(s,t)}}\Upsilon(s)\mathrm{d}s. \end{aligned}$$

Therefore, under the zero initial condition, we have

$$\int_{t_0}^{t} \exp\left\{\sum_{p=1}^{M}\left[\beta_p T_{\uparrow p}(s,t) - \alpha_p T_{\downarrow p}(s,t)\right]\right\} \prod_{p=1}^{M} (\mu_p \hat{\mu}_p \tilde{\mu}_p)^{N_{\sigma p}(s,t)} \Upsilon(s) \mathrm{d}s \leq 0. \tag{27}$$

That is

$$\begin{aligned} \int_{t_0}^{t} \exp\Bigg\{ & \sum_{p=1}^{l}\left[-\alpha_p T_p(s,t) + f(s,t)\right] \\ & + \sum_{p=l+1}^{M}\left[-\alpha_p T_p(s,t) + f(s,t)\right]\Bigg\} \Upsilon(s)\mathrm{d}s \leq 0, \end{aligned} \tag{28}$$

where $f(s,t) = (\alpha_p + \beta_p) T_{\uparrow p}(s,t) + N_{\sigma p}(s,t)\ln(\mu_p \hat{\mu}_p \tilde{\mu}_p)$.

Multiplying $\exp\left\{-\sum_{p=1}^{M} f(t_0,t)\right\}$ on both sides of (28) yields

$$\int_{t_0}^{t} \exp\{\Phi_1 + \Phi_2\} z^{\mathrm{T}}(s) z(s) \mathrm{d}s \leq \gamma^2 \int_{t_0}^{t} \exp\{\Phi_1 + \Phi_2\} w^{\mathrm{T}}(s) w(s) \mathrm{d}s, \tag{29}$$

where $\Phi_1 = \sum_{p=1}^{l}[-\alpha_p T_p(s,t) - f(t_0,s)]$, $\Phi_2 = \sum_{p=l+1}^{M}[-\alpha_p T_p(s,t) - f(t_0,s)]$.

For $\Phi_1$, from Definition 1 and (11), and noticing that $-(\alpha_p + \beta_p) \geq 0$, we get

$$\begin{aligned} \Phi_1 & \geq \sum_{p=1}^{l}\left[-\alpha_p T_p(s,t) - N_{\sigma p}(t_0,s)\ln(\mu_p \hat{\mu}_p \tilde{\mu}_p)\right] \\ & \geq \sum_{p=1}^{l}\left[-\alpha_p T_p(s,t) - N_{0p}\ln(\mu_p \hat{\mu}_p \tilde{\mu}_p) - \frac{\ln(\mu_p \hat{\mu}_p \tilde{\mu}_p) T_p(t_0,s)}{\tau_p}\right] \\ & \geq \sum_{p=1}^{l}\left[-\alpha_p T_p(t_0,t) - N_{0p}\ln(\mu_p \hat{\mu}_p \tilde{\mu}_p)\right]. \end{aligned} \tag{30}$$

For $\Phi_2$, from Definition 1 and (11), and noticing that $T_{\uparrow p}(t_0,s) \leq \Delta_{pM} N_{\sigma p}(t_0,s)$, we have

$$\begin{aligned} \Phi_2 & \geq \sum_{p=l+1}^{M}\left[-\alpha_p T_p(s,t) - ((\alpha_p + \beta_p)\Delta_{pM} + \ln(\mu_p \hat{\mu}_p \tilde{\mu}_p)) N_{\sigma p}(t_0,s)\right] \\ & \geq \sum_{p=l+1}^{M}\left[-\alpha_p T_p(s,t) - ((\alpha_p + \beta_p)\Delta_{pM} + \ln(\mu_p \hat{\mu}_p \tilde{\mu}_p))\left(N_{0p} + \frac{T_p(t_0,s)}{\tau_p}\right)\right] \\ & \geq \sum_{p=l+1}^{M}\left[-\alpha_p T_p(t_0,t) - ((\alpha_p + \beta_p)\Delta_{pM} + \ln(\mu_p \hat{\mu}_p \tilde{\mu}_p)) N_{0p}\right]. \end{aligned} \tag{31}$$

Combining (30) and (31), and noticing that $-f(t_o, s) \leq 0$, we obtain

$$\exp\left\{\sum_{p=1}^{M}[-\alpha_p T_p(t_0,t)] - \sum_{p=1}^{l}[N_{0p}\ln(\mu_p\hat{\mu}_p\tilde{\mu}_p)] - \sum_{p=l+1}^{M}[((\alpha_p+\beta_p)\Delta_{pM} + \ln(\mu_p\hat{\mu}_p\tilde{\mu}_p))N_{0p}]\right\} \leq \exp\{\Phi_1 + \Phi_2\} \leq \exp\left\{\sum_{p=1}^{M}[-\alpha_p T_p(s,t)]\right\}. \tag{32}$$

From (29) and (32), it follows that

$$\int_{t_o}^{t}\exp\left\{\sum_{p=1}^{M}\left[-\alpha_p T_p(t_0,t)\right]\right\} z^{\mathrm{T}}(s)z(s)\mathrm{d}s \leq \gamma^2\rho\int_{t_o}^{t}\exp\left\{\sum_{p=1}^{M}\left[-\alpha_p T_p(s,t)\right]\right\} w^{\mathrm{T}}(s)w(s)\mathrm{d}s. \tag{33}$$

where $\rho = \exp\{\Sigma_{p=1}^{l}[N_{0p}\ln(\mu_p\hat{\mu}_p\tilde{\mu}_p)] + \Sigma_{p=l+1}^{M}[((\alpha_p + \beta_p)\Delta_{pM} + \ln(\mu_p\hat{\mu}_p\tilde{\mu}_p))N_{0p}]\}$.

Integrating both sides of (33) from $t = t_0$ to $\infty$ yields

$$\int_{t_0}^{\infty}\exp\left\{-\sum_{p=1}^{M}\left[\alpha_p T_p(t_0,s)\right]\right\} z^{\mathrm{T}}(s)z(s)\mathrm{d}s \leq \tilde{\gamma}^2\int_{t_0}^{\infty} w^{\mathrm{T}}(s)w(s)\mathrm{d}s, \tag{34}$$

where $\tilde{\gamma} = \gamma\sqrt{\rho}$.

This means that system (3) achieves a weighted $H_\infty$ performance level $\tilde{\gamma}$.

The proof is completed. □

## 3.2 Controller Design

In this section, based on the proposed weighted $H_\infty$ performance condition, we will give the design method of the DOF controller for the system (1).

**Theorem 3** *For the switched system (1), let $\gamma > 0$, $\alpha_p > 0$, $\beta_p$, $\varepsilon_p > 0$, $\mu_p \geq 1$ and $\hat{\mu}_p \geq 1$, $p \in \mathfrak{M}$ be given constants, if there exist matrices $\mathscr{A}_{c,p}$, $\mathscr{B}_{c,p}$, $\mathscr{C}_{c,p}$, $\mathscr{P}_{1p} > 0$, $\mathscr{X}_{1p} > 0$, $\mathscr{L}_p > 0$, $\mathscr{Q}_p > 0$, $\mathscr{I}_p > 0$, $P_{pq} > 0$, $Q_{pq} > 0$ and $S_{pq} > 0$, such that $\forall(p,q) \in \mathfrak{M}\times\mathfrak{M}$, $p \neq q$,*

$$\begin{bmatrix} \mathscr{X}_{1p} & I \\ I & \mathscr{P}_{1p} \end{bmatrix} > 0, \tag{35}$$

$$\begin{bmatrix} \Xi_{11}^{p} & \Xi_{12}^{p} & \Xi_{13}^{p} & 0 & E_p & \mathscr{X}_{1p}L_p^{\mathrm{T}} & \Xi_{17}^{p} & \varepsilon_p\mathscr{X}_{1p} \\ * & \Xi_{22}^{p} & \Xi_{23}^{p} & 0 & \Xi_{25}^{p} & L_p^{\mathrm{T}} & hA_p^{\mathrm{T}} & 0 \\ * & * & \Xi_{33}^{p} & 0 & 0 & U_p^{\mathrm{T}} & hD_p^{\mathrm{T}} & 0 \\ * & * & * & \Xi_{44}^{p} & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & H_p^{\mathrm{T}} & hE_p^{\mathrm{T}} & 0 \\ * & * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & * & -h\mathscr{I}_p & 0 \\ * & * & * & * & * & * & * & -\varepsilon_p I \end{bmatrix} < 0, \tag{36}$$

$$\begin{bmatrix} \Gamma_{11}^{pq} & \Gamma_{12}^{pq} & P_{pq}\bar{E}_{pq} & \bar{L}_p^{\mathrm{T}} & h\bar{A}_{pq}^{\mathrm{T}}K^{\mathrm{T}} \\ * & \Gamma_{22}^{pq} & 0 & \bar{U}_p^{\mathrm{T}} & h\bar{D}_{pq}^{\mathrm{T}}K^{\mathrm{T}} \\ * & * & -\gamma^2 I & \bar{H}_p^{\mathrm{T}} & h\bar{E}_{pq}^{\mathrm{T}}K^{\mathrm{T}} \\ * & * & * & -I & 0 \\ * & * & * & * & -hS_{pq}^{-1} \end{bmatrix} < 0, \tag{37}$$

$$\begin{aligned} &\mathscr{Y}_p\mathscr{J}_p^{-1} \le \mu_p P_{pq},\ diag\{\varepsilon_p I, \mathscr{Q}_p\} \le \mu_p Q_{pq}, \mathscr{I}_p^{-1} \le \mu_p S_{pq}, \\ &P_{pq} \le \hat{\mu}_p\mathscr{Y}_q\mathscr{J}_q^{-1}, Q_{pq} \le \hat{\mu}_p diag\{\varepsilon_q I, \mathscr{Q}_q\}, S_{pq} \le \hat{\mu}_p\mathscr{I}_q^{-1}, \end{aligned} \tag{38}$$

*where*

$$\begin{aligned} &\Xi_{11}^{p} = A_p\mathscr{X}_{1p} + \mathscr{X}_{1p}A_p^{\mathrm{T}} + B_p\mathscr{C}_{c,p} + \mathscr{C}_{c,p}^{\mathrm{T}}B_p^{\mathrm{T}} + \alpha_p\mathscr{X}_{1p} + \mathscr{L}_p, \\ &\Xi_{12}^{p} = A_p + \mathscr{A}_{c,p}^{\mathrm{T}} + \alpha_p I + \varepsilon_p\mathscr{X}_{1p},\ \Xi_{13}^{p} = D_p + e^{-\alpha_p h}\mathscr{X}_{1p}, \\ &\Xi_{17}^{p} = h\mathscr{X}_{1p}A_p^{\mathrm{T}} + h\mathscr{C}_{cp}^{\mathrm{T}}B_p^{\mathrm{T}}, \Xi_{25}^{p} = \mathscr{P}_{1p}E_p + \mathscr{B}_{cp}G_p, \\ &\Xi_{22}^{p} = \mathscr{P}_{1p}A_p + A_p^{\mathrm{T}}\mathscr{P}_{1p} + \mathscr{B}_{c,p}C_p + C_p^{\mathrm{T}}\mathscr{B}_{c,p}^{\mathrm{T}} + \alpha_p\mathscr{P}_{1p}\varepsilon_p I, \\ &\Xi_{23}^{p} = \mathscr{P}_{1p}D_p + \mathscr{B}_{c,p}F_p + e^{-\alpha_p h}I,\ \Xi_{44}^{p} = -(1-h_d)e^{-\alpha_p h}\mathscr{Q}_p, \\ &\Xi_{33}^{p} = -(1-h_d)e^{-\alpha_p h}\varepsilon_p I - 2e^{-\alpha_p h}I + he^{-\alpha_p h}\mathscr{I}_p. \end{aligned}$$

*Then, the closed-loop system* (3) *is GUES with a weighted* $H_\infty$ *performance level* $\tilde{\gamma}$ *for any switching signal* $\sigma(t)$ *with MDADT satisfying* (11)*, where* $\tilde{\gamma} = \gamma\sqrt{\rho}$ *and* $\rho = \exp\{\Sigma_{p=1}^{l}[N_{0p}\, In(\mu_p\hat{\mu}_p\tilde{\mu}_p)] + \Sigma_{p=l+1}^{M}[((\alpha_p+\beta_p)\Delta_{pM} + In(\mu_p\hat{\mu}_p\tilde{\mu}_p))N_{0p}]\}$.

*Moreover, the controller gains are given by*

$$\begin{aligned} A_{c,p} &= \mathscr{P}_{2p}^{-1}[\mathscr{A}_{c,p} - \mathscr{P}_{1p}A_p\mathscr{X}_{1p} - \mathscr{B}_{c,p}C_p\mathscr{X}_{1p} - \mathscr{P}_{1p}B_p\mathscr{C}_{c,p}]\mathscr{X}_{2p}^{-\mathrm{T}}, \\ B_{c,p} &= \mathscr{P}_{2p}^{-1}\mathscr{B}_{c,p}, \\ C_{c,p} &= \mathscr{C}_{c,p}\mathscr{X}_{2p}^{-\mathrm{T}}. \end{aligned} \tag{39}$$

*Proof* Partition $P_p$ and its inverse as

$$P_p = \begin{bmatrix} \mathscr{P}_{1p} & \mathscr{P}_{2p} \\ \mathscr{P}_{2p}^{\mathrm{T}} & \mathscr{P}_{3p} \end{bmatrix}, \ P_p^{-1} = \begin{bmatrix} \mathscr{X}_{1p} & \mathscr{X}_{2p} \\ \mathscr{X}_{2p}^{\mathrm{T}} & \mathscr{X}_{3p} \end{bmatrix}, \tag{40}$$

where $\mathscr{P}_{3p} > 0$, $\mathscr{X}_{3p} > 0$, and $\mathscr{P}_{2p}$, $\mathscr{X}_{2p}$ are invertible matrices.

Define the following matrices

$$\mathscr{J}_p = \begin{bmatrix} \mathscr{X}_{1p} & I \\ \mathscr{X}_{2p}^{\mathrm{T}} & 0 \end{bmatrix}, \ \mathscr{Y}_p = \begin{bmatrix} I & \mathscr{P}_{1p} \\ 0 & \mathscr{P}_{2p}^{\mathrm{T}}, \end{bmatrix}, \ Q_p = \begin{bmatrix} \varepsilon_p I & 0 \\ 0 & \mathscr{Q}_p \end{bmatrix}. \tag{41}$$

By computation, we can get

$$\mathscr{P}_{1p}\mathscr{X}_{1p} + \mathscr{P}_{2p}\mathscr{X}_{2p}^{\mathrm{T}} = I, P_p\mathscr{J}_p = \mathscr{Y}_p. \tag{42}$$

Multiplying diag $\{\mathscr{J}_p^{\mathrm{T}}, I, I, I, I\}$ by pre- and post-(23), we can obtain

$$\begin{bmatrix} \tilde{\Gamma}_{11}^p & \tilde{\Gamma}_{12}^p & \mathscr{J}_p^{\mathrm{T}} P_p \bar{E}_p & \mathscr{J}_p^{\mathrm{T}} \bar{L}_p^{\mathrm{T}} & h\mathscr{J}_p^{\mathrm{T}} \bar{A}_p^{\mathrm{T}} K^{\mathrm{T}} \\ * & \Gamma_{22}^p & 0 & U_p^{\mathrm{T}} & h\bar{D}_p^{\mathrm{T}} K^{\mathrm{T}} \\ * & * & -\gamma^2 I & \bar{H}_p^{\mathrm{T}} & h\bar{E}_p^{\mathrm{T}} K^{\mathrm{T}} \\ * & * & * & -I & 0 \\ * & * & * & * & -hS_p^{-1} \end{bmatrix} < 0, \tag{43}$$

where

$$\tilde{\Gamma}_{11}^p = \mathscr{J}_p^{\mathrm{T}}(\bar{A}_p^{\mathrm{T}} P_p + P_p\bar{A}_p + \alpha_p P_p + Q_p)\mathscr{J}_p,$$
$$\tilde{\Gamma}_{12}^p = \mathscr{J}_p^{\mathrm{T}}(P_p\bar{D}_p + \mathrm{e}^{-\alpha_p h} K^{\mathrm{T}} K).$$

Define the following matrices:

$$\mathscr{A}_{c,p} = \mathscr{P}_{1p}A_p\mathscr{X}_{1p} + \mathscr{P}_{2p}B_{c,p}C_p\mathscr{X}_{1p} + \mathscr{P}_{1p}B_pC_{c,p}\mathscr{X}_{2p}^{\mathrm{T}} + \mathscr{P}_{2p}A_{c,p}\mathscr{X}_{2p}^{\mathrm{T}},$$
$$\mathscr{B}_{c,p} = \mathscr{P}_{2p}B_{c,p}, \mathscr{C}_{c,p} = C_{c,p}\mathscr{X}_{2p}^{\mathrm{T}}, \mathscr{L}_p = \mathscr{X}_{2p}\mathscr{Q}_p\mathscr{X}_{2p}^{\mathrm{T}}, \mathscr{L}_p = S_p^{-1}. \tag{44}$$

From (40), we get

$$\mathscr{J}_p^{\mathrm{T}} P_p\bar{A}_p\mathscr{J}_p = \begin{bmatrix} A_p\mathscr{X}_{1p} + B_p\mathscr{C}_{c,p} & A_p \\ \mathscr{A}_{c,p} & \mathscr{P}_{1p}A_p + \mathscr{B}_{c,p}C_p \end{bmatrix},$$
$$\mathscr{J}_p^{\mathrm{T}} Q_p\mathscr{J}_p = \begin{bmatrix} \varepsilon_p\mathscr{X}_{1p}\mathscr{X}_{1p} + \mathscr{L}_p & \varepsilon_p\mathscr{X}_{1p} \\ \varepsilon_p\mathscr{X}_{1p} & \varepsilon_p I \end{bmatrix},$$
$$\mathscr{J}_p^{\mathrm{T}} P_p\mathscr{J}_p = \begin{bmatrix} \mathscr{X}_{1p} & I \\ I & \mathscr{P}_{1p} \end{bmatrix}, \ \mathscr{J}_p^{\mathrm{T}} P_p\bar{D}_p = \begin{bmatrix} D_p & 0 \\ \mathscr{P}_{1p}D_p + \mathscr{B}_{c,p}F_p & 0 \end{bmatrix},$$

$$\mathscr{J}_p^{\mathrm{T}} K^{\mathrm{T}} K = \begin{bmatrix} \mathscr{X}_{1p} & 0 \\ I & 0 \end{bmatrix}, \quad \mathscr{J}_p^{\mathrm{T}} P_p \bar{E}_p = \begin{bmatrix} E_p \\ \mathscr{P}_{1p} E_p + \mathscr{B}_{c,p} G_p \end{bmatrix},$$
$$\mathscr{J}_p^{\mathrm{T}} \bar{L}_p^{\mathrm{T}} = \begin{bmatrix} \mathscr{X}_{1p} L_p^{\mathrm{T}} \\ L_p^{\mathrm{T}} \end{bmatrix}, \quad \mathscr{J}_p^{\mathrm{T}} \bar{A}_p^{\mathrm{T}} K^{\mathrm{T}} = \begin{bmatrix} \mathscr{X}_{1p} A_p^{\mathrm{T}} + \mathscr{C}_{c,p}^{\mathrm{T}} B_p^{\mathrm{T}} \\ A_p^{\mathrm{T}} \end{bmatrix}. \tag{45}$$

Substituting (45) into (43) and applying Schur complement Lemma, we can obtain (36).

Thus, (36) is equivalent to (23). Notice that (37) is equivalent to (24), and (38) is equivalent to (10). The proof is completed. □

*Remark 5* The asynchronous DOF control problem was also studied in [20] for a class of switched delay systems based on the ADT approach. However, the state delay is time invariant, and the switching delay only involves in partial controller gain matrices. The advantages of the result in this paper are that the state delay considered is time varying, and the switching delay appears in all the controller gain matrices. On the other hand, not only the stability but also the $H_\infty$ performance for the switched system is studied, especially based on the MDADT approach, which brings more flexibility to find the feasible controller.

Notice that the inequality conditions in Theorem 3 are mutually dependent, and we present the following computational algorithm to obtain the DOF controller and the MDADT.

**Algorithm 1**

**Step 1** $\forall p \in \mathfrak{M}$, given constants $\alpha_p$ and $\varepsilon_p$, solve (35) and (36) to obtain $\mathscr{A}_{c,p}, \mathscr{B}_{c,p}, \mathscr{C}_{c,p}, \mathscr{P}_{1p}, \mathscr{X}_{1p}, \mathscr{L}_p, \mathscr{Q}_p$ and $\mathscr{I}_p$.
**Step 2** Compute the invertible matrices $\mathscr{X}_{2p}$ satisfying $\mathscr{L}_p = \mathscr{X}_{2p}\mathscr{Q}_p\mathscr{X}_{2p}^{\mathrm{T}}$ by the function fsolve $(\cdots)$ in MATLAB. Then $\mathscr{P}_{2p}$ can be obtained from $\mathscr{P}_{1p}\mathscr{X}_{1p} + \mathscr{P}_{2p}\mathscr{X}_{2p}^{\mathrm{T}} = I$.
**Step 3** Compute the matrices $\mathscr{Y}_p$ and $\mathscr{J}_p$ by (41).
**Step 4** According to (39), the controller matrices $A_{c,p}$, $B_{c,p}$ and $C_{c,p}$ can be obtained.
**Step 5** Upon substituting the matrices obtained from Step 1–Step 4 to (37) and (38), they can be transformed into LMIs with respect to $P_{pq}$, $Q_{pq}$ and $S_{pq}$.
**Step 6** Solve (37) and (38) for the given constants $\beta_p$, $\mu_p$ and $\hat{\mu}_p$.
**Step 7** Use $\tilde{\mu}_p = \max_{q\in\mathfrak{M}, q\neq p} \{\mu_{qp}\}$ with $\mu_{qp} = \mathrm{e}^{\alpha_q+\beta_p}$ to obtain $\tilde{\mu}_p$.
**Step 8** Substitute $\alpha_p$, $\beta_p$, $\hat{\mu}_p$, and $\tilde{\mu}_p$ to (11) to obtain $\tau_p^*$.

*Remark 6* It can be seen that a smaller $\alpha_p$ will be favorable to the feasibility of (36) and a larger $\beta_p$ will be favorable to the feasibility of (37). In view of this, for the choice of $\alpha_p$ in Algorithm 1, for the first time, we can choose a larger $\alpha_p$, if (36) is unfeasible, we can decrease $\alpha_p$ appropriately. Repeat this until (36) is feasible. For the choice of $\beta_p$ in Algorithm 1, for the first time, we can choose a $\beta_p < -\alpha_p$, if (37) is unfeasible, we can increase $\beta_p$ appropriately. Repeat this until (37) is feasible.

## 4 Example

In this section, we present a numerical example to demonstrate the effectiveness of the proposed method. Consider system (1) consisting of three subsystems,

Subsystem 1:

$$A_1 = \begin{bmatrix} -1.9 & 0 & 1.1 \\ 0 & -0.9 & 0.3 \\ -0.2 & 0.1 & 0.3 \end{bmatrix}, \; D_1 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0.1 & 0 & 0.1 \\ 0 & 0.1 & 0.2 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1.1 \\ 0.4 \\ 2 \end{bmatrix},$$
$$E_1 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.5 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 2.2 & 3 & 3 \end{bmatrix}, \; F_1 = \begin{bmatrix} 0 & 0.1 & 0 \end{bmatrix}, \; G_1 = \begin{bmatrix} 0.8 \end{bmatrix},$$
$$L_1 = \begin{bmatrix} 0 & 0.5 & 0.5 \end{bmatrix}, \; U_1 = \begin{bmatrix} 0.2 & 0 & 0.4 \end{bmatrix}, \; H_1 = \begin{bmatrix} 0.3 \end{bmatrix}.$$

Subsystem 2:

$$A_2 = \begin{bmatrix} -1.8 & 0 & 0.1 \\ 0.1 & -2.4 & 0 \\ 0 & 0.1 & 0.1 \end{bmatrix}, \; D_2 = \begin{bmatrix} 0.2 & 0 & 0.1 \\ 0.1 & 0.1 & 0.1 \\ 0 & 0.1 & 0.2 \end{bmatrix}, \; B_2 = \begin{bmatrix} 0.2 \\ 0.5 \\ 2 \end{bmatrix},$$
$$E_2 = \begin{bmatrix} 0.4 \\ 0 \\ 0.6 \end{bmatrix},$$
$$C_2 = \begin{bmatrix} 0.6 & 1 & 2 \end{bmatrix}, \; F_2 = \begin{bmatrix} 0.1 & 0.1 & 1 \end{bmatrix}, \; G_2 = \begin{bmatrix} 0.3 \end{bmatrix},$$
$$L_2 = \begin{bmatrix} 0.2 & 1.3 & 0 \end{bmatrix}, \; U_2 = \begin{bmatrix} 0.1 & 0.4 & 0 \end{bmatrix}, \; H_2 = \begin{bmatrix} 0.2 \end{bmatrix}.$$

Subsystem 3:

$$A_3 = \begin{bmatrix} -2.2 & 0 & 0.3 \\ 0 & -1.9 & 0.3 \\ 0.2 & 0 & 0.4 \end{bmatrix}, \; D_3 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0.1 & 0 & 0.1 \\ 0 & 0.1 & 0.2 \end{bmatrix}, \; B_3 = \begin{bmatrix} 0.4 \\ -0.4 \\ 3 \end{bmatrix},$$
$$E_3 = \begin{bmatrix} 0 \\ 0.1 \\ 0.6 \end{bmatrix},$$
$$C_3 = \begin{bmatrix} 1.2 & 4 & 2 \end{bmatrix}, \; F_3 = \begin{bmatrix} 0.1 & 0.1 & 0 \end{bmatrix}, \; G_3 = \begin{bmatrix} 0.6 \end{bmatrix},$$
$$L_3 = \begin{bmatrix} 0.4 & 0 & 0 \end{bmatrix}, \; U_3 = \begin{bmatrix} 0.3 & 0.1 & 0 \end{bmatrix}, \; H_3 = \begin{bmatrix} 0.1 \end{bmatrix},$$

Considering $d(t) = 0.9 + 0.1\sin(t)$, we can get that $h = 1$, $h_d = 0.1$. Taking $\alpha_1 = 1.3$, $\alpha_2 = 1.2$, $\alpha_3 = 1.4$, $\varepsilon_1 = 0.5$, $\varepsilon_2 = 1$, $\varepsilon_3 = 0.1$, and $\gamma = 1$. Following Step 1–Step 4 of Algorithm 1, we can obtain the DOF controller gains

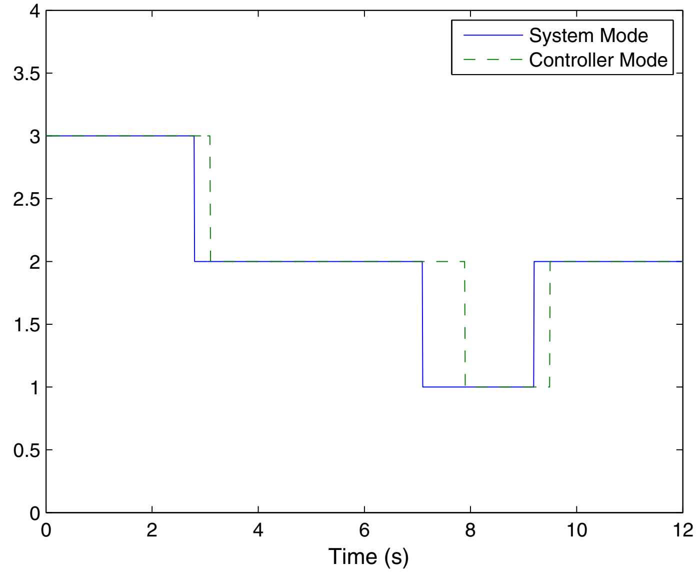


**Fig. 1** Switching signals

$$A_{c,1}=\begin{bmatrix}-4.9484 & -7.2571 & -16.2287\\ -9.1137 & -32.8127 & -70.5270\\ -17.2043 & -59.1784 & -138.7735\end{bmatrix},\ B_{c,1}=\begin{bmatrix}1.4895\\ 7.2964\\ 14.2323\end{bmatrix},$$
$$C_{c,1}=\begin{bmatrix}0.3397 & -1.5057 & -4.0909\end{bmatrix},$$
$$A_{c,2}=\begin{bmatrix}-1.0910 & -0.8166 & 4.8126\\ 1.5152 & -8.2070 & 8.7754\\ -4.7626 & 10.8616 & -48.9274\end{bmatrix},\ B_{c,2}=\begin{bmatrix}-0.5375\\ -1.1477\\ 4.6752\end{bmatrix},$$
$$C_{c,2}=\begin{bmatrix}0.1740 & 1.2542 & -3.8081\end{bmatrix},$$
$$A_{c,3}=\begin{bmatrix}-8.5193 & -4.9855 & -3.7357\\ -5.0012 & -108.5742 & -8.8421\\ -11.8495 & -240.5768 & -24.4987\end{bmatrix},\ B_{c,3}=\begin{bmatrix}0.3924\\ 6.9366\\ 15.5646\end{bmatrix},$$
$$C_{c,3}=\begin{bmatrix}0.1387 & -1.9609 & -2.4129\end{bmatrix}.$$

Then, choosing $\beta_1=-1.33$, $\mu_1=2$, $\hat{\mu}_1=6.5$, $\beta_2=-0.7$, $\mu_2=8.7$, $\hat{\mu}_2=8.5$, $\beta_3=-1.44$, $\mu_3=7.5$ and $\hat{\mu}_3=7$, and following Step 5 and Step 6 of Algorithm 1, we can seek the feasible solutions $P_{pq}$, $Q_{pq}$ and $S_{pq}$ of (37) and (38). Following Step 7 of Algorithm 1, we can get $\tilde{\mu}_1=1.0725$, $\tilde{\mu}_2=2.0138$ and $\tilde{\mu}_3=0.8694$. Assume that $\Delta_{1M}=0.8$, $\Delta_{2M}=0.3$ and $\Delta_{3M}=0.2$, following Step 8 of Algorithm 1, we can obtain $\tau_1^*=2.0268$, $\tau_2^*=4.2945$ and $\tau_3^*=2.7292$.

*Remark 7* Although the matrix inequalities (35)–(38) are coupled. According to Algorithm 1, we can firstly solve (35) and (36) to gain $\mathscr{A}_{c,p}$ $\mathscr{B}_{c,p}$, $\mathscr{C}_{c,p}$, $\mathscr{P}_{1p}$, $\mathscr{X}_{1p}$, $\mathscr{L}_p$, $\mathscr{Q}_p$ and $\mathscr{I}_p$, and compute the matrices $\mathscr{X}_{2p}$, $\mathscr{P}_{2p}$, $\mathscr{Y}_p$ and $\mathscr{Y}_p$ by Step 2 and Step 3. Then, we solve (37) and (38) by substituting the matrices obtained into (37) and

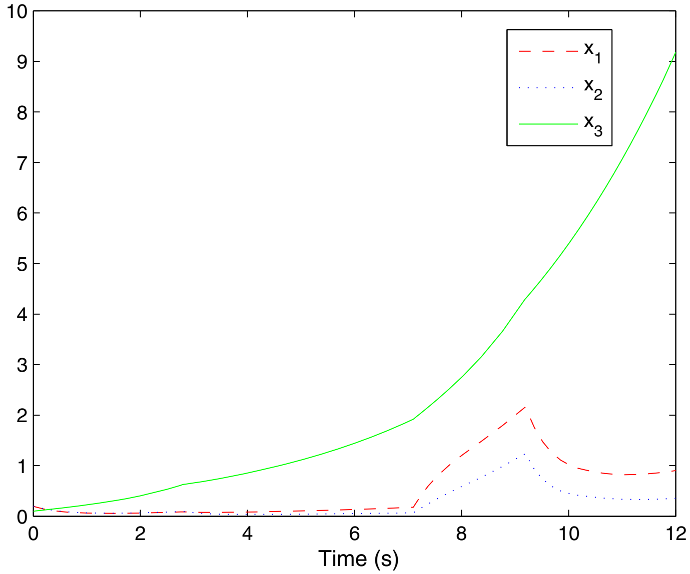


**Fig. 2** States of the open-loop system

(38). By adjusting the parameters $\beta_p$, $\mu_p$ and $\hat{\mu}_p$ appropriately, we seek the feasible solutions $P_{pq}$, $Q_{pq}$ and $S_{pq}$ such that (37) and (38) hold.

In the simulation, we choose the initial condition being $\varphi(t) = [0.2\ 0.2\ 0.1]^{\mathrm{T}}$, $\tau_1 = 2.1$, $\tau_2 = 4.3$ and $\tau_3 = 2.8$. Figure 1 describes the switching signals of system and controller. The states of the open-loop system are shown in Fig. 2.

Figure 3 detects the states of the closed-loop system. The states of the DOF controller are given in Fig. 4. It can be seen that the open-loop system is unstable, and the closed-loop system is exponentially stable, which indicates that the designed controller in (39) under the admissible switching signals is effective despite asynchronous switching. Let $N_{0p} = 0$, $p = 1, 2, 3$, according to Theorem 3, the resulting closed-loop system is exponentially stable with a weighted $H_\infty$ performance $\tilde{\gamma} = 1.0779$.

*Remark 8* The parameters $\alpha_p$, $\beta_p$, $\varepsilon_p$, $\mu_p$ and $\hat{\mu}_p$ in this paper are mode dependent. When we solve (35)–(38), we can adjust any of them to ensure the feasibility. Thus, it will be more feasible in practice to design a MDADT switching than a ADT switching [12,18–20].

## 5 Conclusion

In this paper, the asynchronous $H_\infty$ control problem for switched time-varying delay systems with MDADT has been studied. By adopting MDADT approach and the piecewise Lyapunov function technique, the exponential stability and weighted $H_\infty$ performance results for switched systems are proposed. Based on the value of $\alpha_p + \beta_p$,

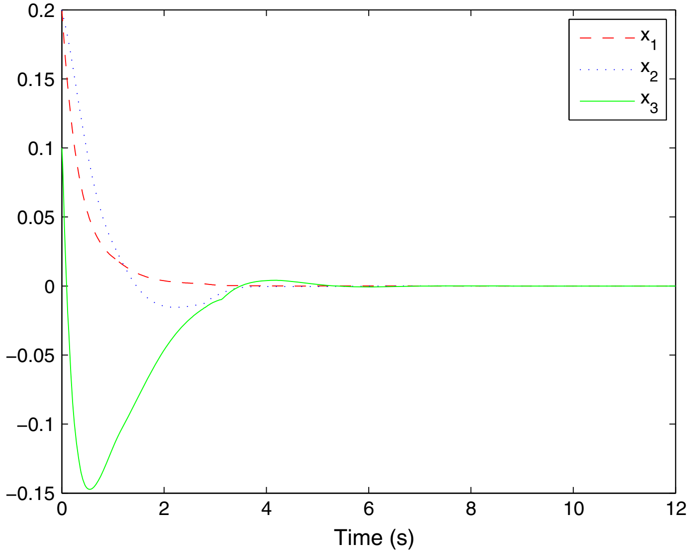


**Fig. 3** States of the closed-loop system

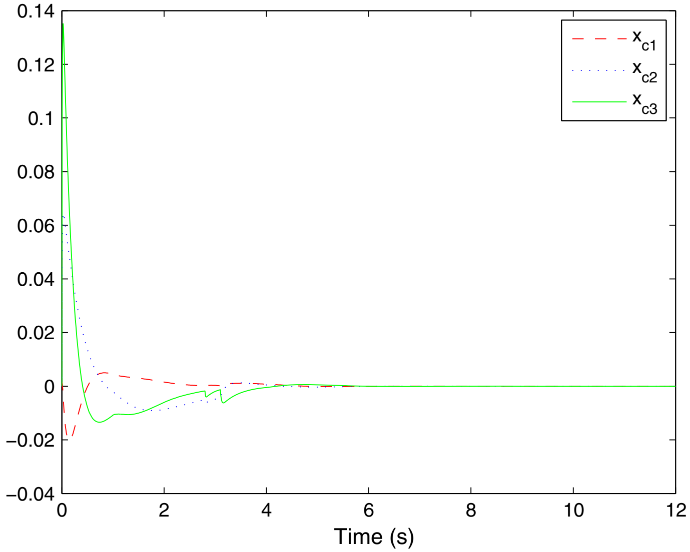


**Fig. 4** States of the controller

$p \in \mathfrak{M}$, two types of MDADT are obtained. Moreover, the corresponding solvability conditions for desired DOF controller are established, and the computational algorithm for the design of the DOF controller and MDADT is presented. Finally, an example is given to illustrate the effectiveness of the proposed design method. In fact, the main approaches utilized in this work can be used to deal with the problem of asynchronous finite-time DOF control of switched systems, which could be our future work.

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