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Stability Analysis of Delayed Impulsive Systems and Applications

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Abstract This paper investigates exponential stability of nonlinear delayed impulsive systems. The effects of impulses with sufficiently small input delays and arbitrary sizes of input delays are thoroughly examined according to whether the continuous

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dynamics of the systems is stable. Several exponential stability criteria are obtained and are then applied to study constant time-delay systems with linear input impulses. Numerical examples are provided to illustrate the usefulness of the main study results.

Keywords Exponential stability · Time-delays · Average impulsive interval

1 Introduction

Stability of nonlinear dynamical systems has been extensively investigated for the past decades because of its importance in mathematical modeling and applications in nonlinear sciences [7,18,20,30–32]. Impulsive systems, which describe abrupt changes in the state at certain discrete moments, have been widely applied to model practical problems in various fields, such as mechanical systems, control systems, and complex networks [3,8,9,11,27,35]. Existence of impulses creates a huge challenge to investigate the stability of impulsive systems. Therefore, the study of stability of impulsive systems has received considerable attention during the last decade in the dynamical system research community [1,2,4,5,14–17,19,21,22,33,34].

Time-delays are inevitable in the process of information transmission and impulsive sampling. The dynamics of time-delay systems depends on the current moments and is closely related to certain time periods in the past. This scenario has resulted in theoretical difficulties in investigating the stability of time-delay systems. Practical significance and theoretical value of the time-delay impulsive systems have led to a research surge in recent years, and many important results have been obtained [1–5,19,21,29,33]. However, there are only a few results on the study of effects of delayed impulses. Recently, Khadra et al. [13] investigated delay-free autonomous systems with delayed impulses. Using exponential estimation for delay-free systems, the authors obtained the state estimation at impulsive moments and derived a sufficient condition for asymptotic stability. Nonetheless, this method cannot deal with time-delay systems. Subsequently, Ho et al. [12] investigated time-delay neural networks with destabilizing delayed impulses. By imposing certain conditions on delays and impulsive intervals, Ho et al. [12] applied the differential inequality method to attain exponential stability. Chen and Zheng [6] examined nonlinear time-delay systems with destabilizing delayed impulses and stabilizing delayed impulses, and then obtained some interesting exponential stability criteria.

In most existing results concerning stability of impulsive systems [6,10,13,16,28], the magnitude of impulsive intervals is usually assumed to satisfy $\alpha \leq t_k - t_{k-1} \leq \beta$ for all $k \in \mathcal{N}$ and some positive numbers α, β, $\alpha \leq \beta$. This assumption on the randomness of impulsive sampling significantly limits the scope of the obtained results. It is the purpose of this research to remove such a restriction and thereby to obtain more robust results.

Two notable methods are employed to investigate the stability of impulsive systems, namely, the Lyapunov–Razumikhin function method [6,23,26] and the Lyapunov–Krasovskii function method [25]. The current study examines the exponential stability of time-delay impulsive systems with delayed impulses based on the Lyapunov–Razumikhin method. Our contributions can be summarized as follows. (1) A new description concerning impulsive sequences is introduced, which can be used to characterize a wide range of impulses, including those whose interval length $t_k - t_{k-1}$ may be extremely small or very large for some $k \in \mathcal{N}$. The obtained results can be applied to extensive classes of problems. (2) Several modified conditions on the Lyapunov function are proposed, by which substantial understanding of how this function varies with the system state can be attained. Moreover, the obtained results under these conditions are considerably convenient to apply in real problems as shown by the examples in Sect. 6. (3) The study results can be applied to deal with the exponential stability of time-delay systems with synchronizing impulses, inactive impulses, or desynchronizing impulses.

The rest of this paper is organized as follows. Section 2 introduces our model descriptions, notation, and definitions. Section 3 derives the exponential stability criteria of stable time-delay systems with delayed impulses based on our analysis on impulsive differential equations. Section 4 examines the unstable time-delay systems. Section 5 investigates a constant time-delay system with linear input delayed impulses by applying the results obtained in Sects. 3 and 4. Finally, Sect. 6 furnishes several numerical examples to illustrate the effectiveness and usefulness of the obtained criteria.

2 Model Description and Preliminaries

For $\rho \geq 0$, $r > 0$, let $\mathcal{B}(\rho) = \{x \in \mathcal{R}^n | |x| \leq \rho\}$, and $PRC([-r, 0], \mathcal{R}^n) = \{\varphi: [-r, 0] \to \mathcal{R}^n | \varphi$ is piecewise right continuous$\}$, endowed with norm $\|\cdot\|_r$: $\|\varphi\|_r = \sup_{-r\leq\theta\leq 0} |\varphi(\theta)|$. For $x \in PRC([t_0 - r, +\infty), \mathcal{R}^n)$ and $t \geq t_0$, define $x_t \in PRC([-r, 0], \mathcal{R}^n)$ by $x_t(s) = x(t + s)$. Let $D \subset \mathcal{R}^n$ be an open set satisfying $\mathcal{B}(\rho) \subset D$ for some $\rho \geq 0$. Let $f: \mathcal{R}^+ \times PRC([-r, 0], D) \to \mathcal{R}^n$ satisfy $f(t, 0) = 0$. $g_k: D \times D \to \mathcal{R}^n$, $k \in \mathcal{N}$.

We consider the following nonlinear time-delay system with delayed impulses

$$\begin{cases} \dot{x}(t) = f(t, x_t); & t > t_0, t \neq t_k \quad (a) \\ x(t) = g_k(x(t^-), x(t - d_k)^-), & t = t_k, k \in \mathcal{N} \quad (b) \\ x(t_0 + \theta) = \phi(\theta), & \theta \in [-\tau, 0], \quad (c) \end{cases} \tag{1}$$

where $x \in \mathcal{R}^n$ is the system state, $x(t^+)$ and $x(t^-)$ are the right-hand limit and left-hand limit of x at t, respectively, $\{d_k \geq 0, k \in \mathcal{N}\}$ are impulsive delays satisfying $\max_k\{d_k\} = d < \infty$, the nonegative t_0 is the initial time, $\{t_k\}$ is a nonnegative increasing sequence with $t_k \to +\infty$, $\phi \in PRC([-\tau, 0], \mathcal{R}^n)$ is the initial state, and $\tau = \max\{r, d\}$.

Throughout the rest of the paper, we suppose that for any $\phi \in PRC([-\tau, 0], \mathcal{R}^n)$, (1) has a unique solution $x(t) = x(t, t_0, \phi)$, which is right-hand continuous, i.e., $x(t^+) = x(t)$.

Definition 1 For a given impulsive sequence $\{t_k\}$, the trivial solution of (1) is said to be exponentially stable if there exist positive numbers ρ_0, M and λ such that for ϕ with $\|\phi\|_\tau < \rho_0$, the solution $x(t, t_0, \phi)$ of (1) satisfies

$$|x(t, t_0, \phi)| \leq M\|\phi\|_\tau e^{-\lambda(t-t_0)}, \quad t \geq t_0.$$

Moreover, if the trivial solution of (1) is exponentially stable, and ρ_0 can be arbitrarily large, then the trivial solution is said to be globally exponentially stable.

Definition 2 The average impulsive interval of $\zeta = \{t_1, t_2, \ldots\}$ is greater than T_{a_1} if there exists $N_1 \in \mathcal{Z}^+$ such that

$$N_\zeta(T, t) \leq \frac{T - t}{T_{a_1}} + N_1, \ \forall\, T \geq t \geq 0.$$

The average impulsive interval of ζ is less than T_{a_2} if there exists $N_2 \in \mathcal{Z}^+$ such that

$$\frac{T - t}{T_{a_2}} - N_2 \leq N_\zeta(T, t), \ \forall\, T \geq t \geq 0.$$

The average impulsive interval of ζ is T_a if there exists $N_0 \in \mathcal{Z}^+$ such that

$$\frac{T - t}{T_a} - N_0 \leq N_\zeta(T, t) \leq \frac{T - t}{T_a} + N_0, \ \forall\, T \geq t \geq 0,$$

where $N_0, N_i, T_{a_i}, i = 1, 2$, are positive scalars, and $N_\zeta(T, t)$ is the number of impulsive moments of ζ on (t, T).

Remark 1 The concept of "average impulsive interval" was first introduced in [24] to investigate impulsive systems with nonunited distributive impulses.

Definition 3 A function $V: [-\tau, \infty) \times \mathcal{B}(\rho) \to \mathcal{R}^+$ is said to belong to the class ν_0 if

(1) V is continuous in each of the sets $[t_{k-1}, t_k) \times \mathcal{B}(\rho)$. For each $k \in \mathcal{N}$, $\lim_{(t,y)\to(t_k^-,x)} V(t, y) = V(t_k^-, x)$ exists;
(2) $V(t, x)$ is local Lipschitz with respect to $x \in \mathcal{B}(\rho)$, and for all $t \geq t_0$, $V(t, 0) \equiv 0$.

Definition 4 Suppose $V \in \nu_0$. For the solution $x(t) = x(t, t_0, \phi)$, let $V(t) = V(t, x(t))$. Define the upper right-hand derivative of V at $t^* \in [t_0, +\infty)$ with respect to (1) as

$$D^+V(t^*) = \limsup_{s\to 0} \frac{1}{s}[V(t^* + s, x(t^*) + sf(t^*, x_{t^*})) - V(t^*, x(t^*))].$$

3 Stable Time-Delay Systems with Delayed Impulses

This section focuses on exponential stability of (1), where the continuous dynamics (a) in (1) is stable. We make the following assumptions.

$(\mathbf{A_1})$ There exists $L_1 > 0$ such that for each $\varphi \in PRC([-r, 0], \mathcal{B}(\rho))$, $|f(t, \varphi)| \leq L_1\|\varphi\|_r$.

($\mathbf{A_2}$) There exist $L_2, L_3 > 0$ such that for all $k \in \mathcal{N}$ and $x, y_1, y_2 \in \mathcal{B}(\rho)$, $|g_k(x, 0) - x| \leq L_2|x|$, $|g_k(x, y_1) - g_k(x, y_2)| \leq L_3|y_1 - y_2|$.

($\mathbf{A_3}$) The average impulsive interval of $\zeta = \{t_k\}$ is greater than $T_a > 0$, that is, there exists $N_0 \in \mathcal{N}$ such that for all $T \geq t > t_0$, $N_\zeta(T, t) \leq \frac{T-t}{T_a} + N_0$. Thus, there exist at most $l = \lceil \frac{d}{T_a} \rceil + N_0$ impulses on $(t_0, t_0 + d]$, where $\lceil \frac{d}{T_a} \rceil$ means the upper integer of $\frac{d}{T_a}$. Let $\nabla = 1 + L_2 + L_3$, $\varrho = \nabla^l e^{L_1 d}$.

Let the impulsive moments on $(t_0, t_0 + d]$ be $\{t_i\},\ i = 1, 2, \cdots, m_0,\ m_0 \leq l$. We first estimate the solution of (1) on $[t_0 - \tau, t_0 + d]$.

Lemma 1 *Suppose* (A_1)–(A_3) *hold. Then for any* $\phi \in PRC([-\tau, 0], \mathcal{B}(\rho/\varrho))$*, the solution* $x(t, t_0, \phi)$ *of (1) satisfies* $|x(t, t_0, \phi)| \leq \varrho\|\phi\|_\tau$, $t \in [t_0 - \tau, t_0 + d]$.

Proof The proof is standard. We include it for reader's convenience.

Obviously, $|x(t_0+\theta)| = |\phi(\theta)| \leq \|\phi\|_\tau < \rho(\theta \in [-\tau, 0])$. Then for $t \in [t_0-\tau, t_0]$, $|x(t)| < \rho$. We claim $|x(t)| < \rho$ for $t \in [t_0 - \tau, t_1)$. If not, there exists $t^* \in (t_0, t_1)$ such that for $t \in [t_0 - \tau, t^*)$, $|x(t)| < \rho$ and $|x(t^*)| = \rho$.

For $t \in [t_0, t^*]$, $\theta \in [-r, 0]$, without loss of generality, we assume $t + \theta > t_0$. Integrating (a) in (1) from t_0 to $t + \theta$ and using (A_1), we have

$$|x(t+\theta)| = |x(t_0) + \int_{t_0}^{t+\theta} f(s, x_s)ds| \leq |\phi(0)| + \int_{t_0}^{t} |f(s, x_s)|ds$$
$$\leq \|\phi\|_\tau + L_1 \int_{t_0}^{t} \|x_s\|_r ds.$$

Then,

$$\|x_t\|_r \leq \|\phi\|_\tau + L_1 \int_{t_0}^{t} \|x_s\|_r ds, \quad t \in [t_0, t^*].$$

By the Gronwall's inequality, we get

$$\|x_t\|_r \leq \|\phi\|_\tau e^{L_1(t-t_0)}, \quad t \in [t_0, t^*].$$

Therefore,

$$|x(t^*)| \leq \frac{\rho}{\varrho} e^{L_1 d} < \rho,$$

which is a contradiction.

From the above argument, we also obtain

$$|x(t)| \leq \|x_t\|_r \leq \|\phi\|_\tau e^{L_1(t-t_0)}, \quad t \in [t_0 - \tau, t_1).$$

By (A_2), for $k \in \mathcal{N}$ and $x, y \in \mathcal{B}(\rho)$,

$$|g_k(x, y)| = |g_k(x, y) - g_k(x, 0)| + |g_k(x, 0) - x| + |x| \leq (1 + L_2)|x| + L_3|y|.$$

Hence,

$$\begin{aligned}|x(t_1)| &= |g_k(x(t_1^-), x((t_1-d_1)^-))| \\ &\le (1+L_2)|x(t_1^-)| + L_3|x((t_1-d_1)^-)| \\ &\le (1+L_2+L_3)\|\phi\|_\tau e^{L_1(t_1-t_0)} \\ &= \nabla\|\phi\|_\tau e^{L_1(t_1-t_0)}.\end{aligned}$$

Thus, we have proved that

$$|x(t)| \le \nabla\|\phi\|_\tau e^{L_1(t-t_0)},\ t\in[t_0,t_1].$$

Repeatedly, we obtain

$$|x(t)| \le \nabla^{m_0}\|\phi\|_\tau e^{L_1(t-t_0)} \le \nabla^{l}\|\phi\|_\tau e^{L_1(t-t_0)},\ t\in[t_0,t_{m_0}].$$

Since there are no impulses on $(t_{m_0}, t_0+d]$, we have

$$|x(t)| \le \nabla^{l}\|\phi\|_\tau e^{L_1(t-t_0)} \le \nabla^{l}\|\phi\|_\tau e^{L_1 d} = \varrho\|\phi\|_\tau, t\in[t_0-\tau, t_0+d].$$

□

Theorem 1 *Suppose* (1) *satisfies* (A_1)–(A_3). *There exist* $V\in\nu_0$, *positive scalars* a, b, c, ν, k_1, k_2, *and* $p\ge 1$, *satisfying*

- **(S₁)** *For all* $(t,x)\in[-\tau,\infty)\times\mathcal{B}(\rho)$, $a|x|^p \le V(t,x)\le b|x|^p$.
- **(S₂)** *For all* $t=t_k$ *and* $x, y_1, y_2\in\mathcal{B}(\rho)$ *with* $y_1+y_2\in\mathcal{B}(\rho)$, $V(t, g_k(x,x))\le \nu V(t^-,x)$ *and* $V(t, g_k(x, y_1+y_2))\le k_1V(t, g_k(x,y_1)) + k_2V(t, g_k(0,y_2))$.
- **(S₃)** *For* $t\in[t_0,\infty)$, $t\ne t_k$ *and* $x(\cdot)\in PRC([-\tau,0],\mathcal{B}(\rho))$, $D^+V(t,x(t))\le -cV(t,x(t))$ *whenever* $e^{c\tau}V(t,x(t))\ge V(t+s, x(t+s))$, $s\in[-\tau,0)$.

If there exists $d\ge 0$ *such that*

$$k_1\nu + \frac{b}{a}k_2L_3^p[dL_1 + l(L_2+L_3)]^p < 1, \tag{2}$$

then (1) *is exponentially stable for input delays* $\{d_k\}$ *satisfying* $d_k\le d$, $k\in\mathcal{N}$.

Proof Let λ, $0<\lambda<c$ be such that

$$k_1\nu + \frac{b}{a}k_2L_3^p\left[dL_1e^{\lambda(r+d)/p} + l(L_2+L_3)e^{2\lambda d/p}\right]^p < 1. \tag{3}$$

For any fixed scalar $\delta\in(0, (\sqrt[p]{\frac{b}{a}}(2L_3+1)\nabla\varrho)^{-1}\rho)$, suppose that the solution of (1) subject to $(t_0,\phi)\in\mathcal{R}^+\times PRC([-\tau,0],\mathcal{B}(\delta))$ is $x(t)=x(t,t_0,\phi)$, with maximal existence interval $[t_0-\tau,\bar{T})$. Obviously, Lemma 1 implies $\bar{T}>t_0$. We will show that $\bar{T}=\infty$ and

$$V(t)\le b\varrho^p\|\phi\|_\tau^p e^{-\lambda(t-t_0-d)},\ \ t\in[t_0+d,\bar{T}). \tag{4}$$

For simplicity, we still denote the sequence in $(t_0+d,\infty)\cap(t_0-\tau,\bar{T})$ by $\{t_i\}$, $i = 1, 2, \ldots$ For $t \in [t_n, t^*_{n+1})$, $n \in \mathcal{N}$, define

$$W(s) = e^{\lambda(s-t_0-d)}V(s),\ \ s \in [t_0-\tau, t]. \tag{5}$$

For any fixed $k \in \mathcal{N}$, we first show by mathematical induction that,

$$W(s) \le b\varrho^p \|\phi\|_\tau^p \quad s \in [t_0-\tau, t_k). \tag{6}$$

By Lemma 1,

$$|x(s)| \le \varrho\|\phi\|_\tau, s \in [t_0-\tau, t_0+d]. \tag{7}$$

(7) and (S_1) implies that (6) holds on $[t_0-\tau, t_0+d]$.

We claim that for $t \in [t_0-\tau, t_1)$, (6) is also true. If not, there exists $t^* \in [t_0+d, t_1)$ and $0 < \epsilon < b[(\nabla + L_3)^p - 1]$ such that

$$W(t^*) = (b+\epsilon)\varrho^p\|\phi\|_\tau^p,\ \ D^+W(t^*) \ge 0, \tag{8}$$

and for $s \in [t_0-\tau, t^*)$,

$$W(s) < W(t^*). \tag{9}$$

Combing (8), (9) and (S_1), for $s \in (t_0+d, t^*)$, we have

$$|x(s)| \le \sqrt[p]{\frac{b+\epsilon}{a}}\varrho\|\phi\|_\tau < \rho.$$

For $s \in [t^*-\tau, t^*)$, (9) implies

$$V(t^*) > e^{-\lambda(t^*-s)}V(s) \ge e^{-\lambda\tau}V(s) \ge e^{-c\tau}V(s).$$

Thus from (S_2), we obtain $D^+V(t^*) \le -cV(t^*)$. It follows that

$$D^+W(t^*) \le -(c-\lambda)e^{\lambda(t^*-t_0-d)}V(t^*) < 0,$$

which is a contradiction to (8). Hence (6) holds on $[t_0-\tau, t_1)$.

Now we assume that for $s \in [t_0-\tau, t_m)$, $m = 1, 2, \ldots, k-1$, (6) is true. So

$$W\left(t_m^-\right) \le b\varrho^p\|\phi\|_\tau^p, \tag{10}$$

we will show $W(t_m) \le b\varrho^p\|\phi\|_\tau^p$.

Since $N_\zeta(t_m, t_m-d_m) \le \frac{d_m}{T_a} + N_0 \le \frac{d}{T_a} + N_0$, there exist at most $l = \lceil\frac{d}{T_a}\rceil + N_0$ impulses on (t_m-d_m, t_m), which are assumed to be $t_{m_1}, t_{m_2}, \ldots, t_{m_{l_0}}$, $l_0 \le l$.

From (10) and (S_1),

$$|x(s)| \leq \sqrt[p]{\frac{b}{a}}\varrho\|\phi\|_{\tau}e^{-\lambda(s-t_0-d)/p} < \rho, \quad s \in [t_0-\tau, t_m). \tag{11}$$

By (11), (A_1) and (A_2),

$$\begin{aligned}\Delta_m &= \left|x\left(t_m^-\right) - x\left(t_m - d_m\right)^-\right| = \left|\int_{t_m-d_m}^{t_m} \dot{x}(s)ds + \sum_{i=1}^{l_0}\left(x\left(t_{m_i}\right) - x\left(t_{m_i}^-\right)\right)\right| \\ &\leq \left|\int_{t_m-d_m}^{t_m} |f(s,x_s)|\,ds\right| + \sum_{i=1}^{l_0}\left|g_{m_i}\left(x\left(t_{m_i}^-\right), x\left(\left(t_{m_i}-d_{m_i}\right)^-\right)\right) - x\left(t_{m_i}^-\right)\right| \\ &\leq L_1\left|\int_{t_m-d_m}^{t_m}\|x_s\|_r ds\right| + \sum_{i=1}^{l_0}\left[L_2\left|x\left(t_{m_i}^-\right)\right| + L_3\left|x\left(t_{m_i}-d_{m_i}\right)^-\right|\right] \\ &\leq \left[L_1 de^{\lambda(r+d)/p} + l(L_2+L_3)e^{2\lambda d/p}\right]\left(\frac{b}{a}\right)^{1/p}\varrho\|\phi\|_{\tau}e^{-\lambda(t_m-t_0-d)/p}.\end{aligned} \tag{12}$$

It is easy to check by virtue of (11) and (A_2) that

$$\begin{aligned}&|g_m\left(x\left(t_m^-\right), x\left(t_m^-\right)\right)| < \rho, \\ &|g_m\left(x\left(t_m^-\right), x((t_m-d_m)^-)\right)| < \rho, \\ &|g_m(0,\Delta_m)| < \rho.\end{aligned} \tag{13}$$

Combing (3), (10), (12), (S_1) and (S_2), we obtain

$$\begin{aligned}V(t_m) &= V(t_m, x(t_m)) = V\left(t_m, g_m\left(x\left(t_m^-\right), x((t_m-d_m)^-)\right)\right) \\ &= V\left(t_m, g_m\left(x\left(t_m^-\right), x\left(t_m^-\right)+\Delta_m\right)\right) \\ &\leq k_1 V\left(t_m, g_m\left(x\left(t_m^-\right), x\left(t_m^-\right)\right)\right) + k_2 V(t_m, g_m(0,\Delta_m)) \\ &\leq k_1\nu V\left(t_m^-\right) + k_2 b|g_m(0,\Delta_m)|^p \\ &\leq \left\{k_1\nu + \frac{b}{a}k_2 L_3^p\left[dL_1 e^{\lambda(r+d)/p}\right.\right. \\ &\quad \left.\left.+ l(L_2+L_3)e^{2\lambda d/p}\right]^p\right\} b\varrho^p\|\phi\|_{\tau}^p e^{-\lambda(t_m-t_0-d)}. \\ &\leq b\varrho^p\|\phi\|_{\tau}^p e^{-\lambda(t_m-t_0-d)}.\end{aligned} \tag{14}$$

Therefore,

$$W(t_m) \leq b\varrho^p\|\phi\|_{\tau}^p. \tag{15}$$

In summary, we have proved, for $s \in [t_0-\tau, t_m]$,

$$W(s) \leq b\varrho^p\|\phi\|_{\tau}^p. \tag{16}$$

As we did for proving (16) on $[t_0 - \tau, t_1)$, we can prove (16) holds on $[t_m, t_{m+1})$ by contradiction. By the mathematical induction method, for any $k \in \mathcal{N}$, (16) holds on $[t_0 - \tau, t_k)$. Similar to the argument of (15), we can establish $W(t_k) \leq b\varrho^p \|\phi\|_\tau^p$. Again using the contradiction method, (16) holds on $[t_k, t_{k+1}^*)$. By the continuation theorem in [1], we know $\bar{T} = +\infty$, and (1) is exponentially stable for delays d_k with $d_k \leq d, k \in \mathcal{N}$. □

Remark 2 (S_3) means the continuous dynamics (a) in (1) is stable. In order to realize exponential stability of (1), one should remove unfavorable influences from impulsive delays. In the existing results, impulsive intervals were generally assumed to be bounded from below, which implies the interval of any two adjacent impulses cannot be too small. Then the impulsive strategy of continuous sampling is not applied to realize stability. In this paper, the bounded assumption from below on impulsive intervals is replaced by (A_3), which allows the lengths of impulsive intervals to be arbitrary small. See the numerical examples for details.

Remark 3 (S_3) implies that, if the function $V(t, x(t))$ exponentially increases at rate c in $[-\tau, 0]$ before the time t, it should exponentially decrease in some neighborhood of t at the same rate c so that the system realizes exponential stability.

Remark 4 Theorem 1 is suitable to investigate robustly exponential stability for time-delay systems with delayed impulses. Namely, for sufficiently small impulse input delays, under the conditions of Theorem 1, the system (1) is still exponentially stable.

Remark 5 For applications of Theorem 1, one should first determine the delay d by the inequality (2), and then chooses appropriate impulses with upper bound d.

In the next theorem, we will remove the limits to the sizes of the impulse input delays, and investigate exponential stability of (1) with any bounded impulse input delays.

Theorem 2 *Suppose* (1) *satisfies* (A_1)–(A_3). *There exist* $V \in \nu_0$, *positive scalars* a, b, c, *and* $p \geq 1$, *such that* (S_1) *and* (S_3) *hold. Moreover,*

$(S_2^{'})$ *There exist positive scalars* M_1, M_2, $M_1 + M_2 < 1$ *such that for all* $t = t_k, x, y \in \mathcal{B}(\rho)$,

$$V(t, g_k(x, y)) \leq M_1 V(t^-, x) + M_2 V((t - d_k)^-, y),$$

then (1) *is exponentially stable for any bounded input delays* $\{d_k, k \in \mathcal{N}\}$.

Proof The method is similar to the proof of Theorem 1.

For any bounded impulse input delays $\{d_k, k \in \mathcal{N}\}$, let $d = \sup\{d_k, k \in \mathcal{N}\}$. By $(S_2^{'})$, we can take $0 < \lambda < c$ satisfying $M_1 + M_2 e^{\lambda d} < 1$. Under the assumptions of Theorem 2, it is easy to check (16) holds on $[t_0 - \tau, t_1)$. Assuming that for $s \in [t_0 - \tau, t_m)$, $m = 1, 2, \ldots, k - 1$, (16) is true, we only need show

$$W(t_m) \leq b\varrho^p \|\phi\|_\tau^p.$$

This can be derived by virtue of $(S_2^{'})$ as follows.

$$
\begin{aligned}
W(t_m) &= e^{\lambda(t_m-t_0-d)}V(t_m) = e^{\lambda(t_m-t_0-d)}V\left(t_m, g_m\left(x\left(t_m^-\right), x((t_m-d_m)^-)\right)\right) \\
&\le e^{\lambda(t_m-t_0-d)}\left[M_1 V\left(t_m^-\right) + M_2 V((t_m-d_m)^-)\right] \\
&\le M_1 W\left(t_m^-\right) + M_2 e^{\lambda d} W((t_m-d_m)^-) \\
&\le \left(M_1 + M_2 e^{\lambda d}\right) b\varrho^p \|\phi\|_\tau^p \\
&\le b\varrho^p \|\phi\|_\tau^p.
\end{aligned}
$$

□

Remark 6 Since Theorem 2 concerns arbitrary bounded impulsive delays, it is very difficult to realize exponentially stability of the whole system only by virtue of the stable continuous system. So (S_2') is adopted to eliminate the negative effects from these impulse delays.

4 Unstable Time-Delay Systems with Delayed Impulses

In (S_3) of Theorems 1 and 2, condition $c > 0$ implies the continuous system (a) in (1) is stable. If $c \le 0$, (a) is unstable, and then stability of the whole system greatly depends on effects from the input impulses.

Theorem 3 *Suppose that* (1) *satisfies* (A_1), (A_2), *and that there exist* $V \in \nu_0$, *positive scalars* a, b, ν, k_1, k_2 *and* $p \ge 1$ *such that* (S_1) *and* (S_2) *hold. In addition, the following conditions are satisfied.*

$(A_3^{'})$ *The average impulsive interval of* $\zeta = \{t_k\}$ *is* $T_a > 0$, *that is, there exists* $N_0 \in \mathcal{N}$ *such that for all* $T \ge t > t_0$, $\frac{T-t}{T_a} - N_0 \le N_\zeta(T, t) \le \frac{T-t}{T_a} + N_0$. l, ∇ *and* ϱ *are defined as* (A_3).

$(S_3^{'})$ *Let* $\Delta = N_0 T_a$. *There exist* $c \le 0, d > 0$, *such that for all* $t \in [t_0, \infty)$, $t \ne t_k$ *and* $s \in [-\tau, 0)$, $D^+V(t) \le -cV(t)$ *whenever*

$$
\eta_0 e^{-c\tau} V(t) \ge V(t+s)
$$

and

$$
e^{-c\Delta} < \eta_0, \tag{17}
$$

where $\eta_0 = [k_1\nu + \frac{b}{a}k_2 L_3^p(dL_1 + l(L_2 + L_3))^p]^{-1}$, *then* (1) *is exponentially stable for input delays* $\{d_k\}$ *with* $d_k \le d$, $k \in \mathcal{N}$.

Proof It is easy to see that, if (17) is true, then there exist $0 < \lambda < -c, \sigma > 0$ such that

$$
e^{(-c+\lambda)\Delta} < \sigma < \left[k_1\nu + \frac{b}{a}k_2 L_3^p\left(dL_1 e^{\lambda(r+d)/p} + l(L_2 + L_3)e^{2\lambda d/p}\right)^p\right]^{-1}. \tag{18}
$$

Let $\eta = [k_1\nu + \frac{b}{a}k_2L_3^p(dL_1e^{\lambda(r+d)/p} + l(L_2 + L_3)e^{2\lambda d/p})^p]^{-1}$. Take $\delta \in (0, (\sqrt[p]{\frac{b}{a}}(2L_3+1)\triangledown\varrho)^{-1}\rho)$. Suppose $x(t) = x(t, t_0, \phi)$ is the solution of (1) subject to $(t_0, \phi) \in \mathcal{R}^+ \times PRC\ ([-\tau, 0], \mathcal{B}(\delta))$, with maximal existence interval $[t_0 - \tau, \bar{T})$, where $\bar{T} > t_0$ is a positive number. We will show $\bar{T} = \infty$ and

$$V(t) = V(t, x(t)) \le b\sigma\varrho^p\|\phi\|_\tau^p e^{-\lambda(t-t_0-d)}, \quad t \in [t_0 + d, \bar{T}). \tag{19}$$

We still denote the impulsive sequence on $(t_0 + d, \infty) \cap (t_0 - \tau, \bar{T})$ by $\{t_i\}$, $i = 1, 2, \ldots$ For $t \in [t_k, t_{k+1})$, define

$$W(s) = e^{\lambda(s-t_0-d)}V(s), \ s \in [t_0 - \tau, t]. \tag{20}$$

Similar to the proof of Theorem 1, we will use the mathematical induction method to establish that for any fixed $k \ge 1$,

$$W(s) \le b\sigma\varrho^p\|\phi\|_\tau^p, \quad s \in [t_0 - \tau, t_k). \tag{21}$$

By (S_1) and Lemma 1, we have

$$W(s) \le b\varrho^p\|\phi\|_\tau^p, \ \ s \in [t_0 - \tau, t_0 + d].$$

Then (21) holds on $[t_0 - \tau, t_0 + d]$.

We claim that (21) is also true for $t \in [t_0 - \tau, t_1)$. Otherwise, there exists $s \in [t_0 + d, t_1)$ such that

$$W(s) > b\sigma\varrho^p\|\phi\|_\tau^p.$$

Let $t^* = \inf\{t | t \in (t_0 + d, t_1);\ W(t) > b\sigma\varrho^p\|\phi\|_\tau^p\}$. Then $t^* \in (t_0 + d, t_1)$ and

$$W(t^*) = b\sigma\varrho^p\|\phi\|_\tau^p. \tag{*}$$

Let $\bar{t} = \sup\{t | t \in (t_0 - \tau, t^*);\ W(t) \le b\varrho^p\|\phi\|_\tau^p\}$. Then $\bar{t} \in [t_0 + d, t^*)$ and $W(\bar{t}) = b\varrho^p\|\phi\|_\tau^p$.

For $s \in [\bar{t}, t^*)$,

$$W(s) \ge b\varrho^p\|\phi\|_\tau^p = \frac{1}{\sigma} \cdot b\sigma\varrho^p\|\phi\|_\tau^p \ge \frac{1}{\sigma}W(s + \theta), \ \theta \in [-r, 0],$$

by which and (18), for $\theta \in [-r, 0]$,

$$\begin{aligned} V(s) &\geq \frac{1}{\sigma} e^{\lambda\theta} V(s+\theta) \geq \frac{1}{\sigma} e^{-\lambda\tau} V(s+\theta) \\ &\geq \frac{1}{\sigma} e^{c\tau} V(s+\theta) \geq \frac{1}{\eta} e^{c\tau} V(s+\theta) \\ &\geq \frac{1}{\eta_0} e^{c\tau} V(s+\theta). \end{aligned}$$

That is, $\eta_0 e^{-c\tau} V(t) \geq V(t+s)$. For $s \in [\bar{t}, t^*)$, using $W(s) \leq b\sigma\varrho^p \|\phi\|_\tau^p$ and (S_1), we obtain

$$|x(s)| \leq \sqrt[p]{\frac{b\sigma}{a}} \varrho \|\phi\|_\tau < \rho.$$

Then by $(S_3^{'})$,

$$D^+ V(s) \leq -cV(s), \quad s \in [\bar{t}, t^*).$$

Integrating the above formula from $\bar{t}$ to t^*, we have

$$\begin{aligned} V(t^*) &\leq e^{-c(t^*-\bar{t})} V(\bar{t}) = e^{-c(t^*-\bar{t})} e^{-\lambda(\bar{t}-t_0-d)} b\varrho^p \|\phi\|_\tau^p \\ &= e^{(-c+\lambda)(t^*-\bar{t})} e^{-\lambda(t^*-t_0-d)} b\varrho^p \|\phi\|_\tau^p. \end{aligned} \tag{22}$$

Since there are no impulses in $(\bar{t}, t^*)$, $N_\zeta(t^*, \bar{t}) = 0$. By the definition of average impulsive interval, $t^* - \bar{t} \leq N_0 T_a = \Delta$.

By (22), we have

$$V(t^*) \leq e^{(-c+\lambda)\triangle} e^{-\lambda(t^*-t_0-d)} b\varrho^p \|\phi\|_\tau^p < \sigma e^{-\lambda(t^*-t_0-d)} b\varrho^p \|\phi\|_\tau^p, \tag{23}$$

which implies that $W(t^*) < b\sigma\varrho^p \|\phi\|_\tau^p$, a contradiction to $(*)$.

Now, assuming for $m \in \mathcal{N}$, $1 \leq m \leq k-1$,

$$W(s) \leq b\sigma\varrho^p \|\phi\|_\tau^p, \; s \in [t_0 - \tau, t_m), \tag{24}$$

we will show that

$$W(s) \leq b\sigma\varrho^p \|\phi\|_\tau^p, \; s \in [t_m, t_{m+1}). \tag{25}$$

By (24) and (S_1),

$$|x(s)| \leq \sqrt[p]{\frac{b\sigma}{a}} \varrho \|\phi\|_\tau e^{-\lambda(s-t_0-d)/p} < \rho, \quad s \in [t_0 - \tau, t_m). \tag{26}$$

Since $N_\zeta(t_m, t_m - d_m) \le \frac{d_m}{T_a} + N_0 \le \frac{d}{T_a} + N_0$, there are at most $l = [\frac{d}{T_a}] + N_0$ impulses on $(t_m - d_m, t_m)$. As in the proof of Theorem 1, we can establish

$$\begin{aligned}|\Delta_m| &= |x\left(t_m^-\right) - x(t_m - d_m)^-| \\ &\le \left[L_1 d e^{\lambda(r+d)/p} + l(L_2 + L_3)e^{2\lambda d/p}\right] \sqrt[p]{\frac{b\sigma}{a}} \varrho \|\phi\|_\tau e^{-\lambda(t_m - t_0 - d)/p}.\end{aligned}$$

By (26) and (A_2), and by the choice of δ, we can easily check that

$$\begin{aligned}&|g_m\left(x\left(t_m^-\right), x\left(t_m^-\right)\right)| < \rho, \\ &|g_m\left(x\left(t_m^-\right), x((t_m - d_m)^-)\right)| < \rho, \\ &|g_m(0, \Delta_m)| < \rho.\end{aligned} \tag{27}$$

Then using the same argument as (14), we obtain

$$\begin{aligned}V(t_m) \le &\left\{k_1\nu + \frac{b}{a}k_2 L_3^p \left[L_1 d e^{\lambda(r+d)/p}\right.\right. \\ &\left.\left.+ l(L_2 + L_3)e^{2\lambda d/p}\right]^p\right\} b\sigma\varrho^p\|\phi\|_\tau^p e^{-\lambda(t_m - t_0 - d)} \\ \le\ & b\varrho^p\|\phi\|_\tau^p e^{-\lambda(t_m - t_0 - d)}.\end{aligned}$$

That is

$$W(t_m) \le b\varrho^p\|\phi\|_\tau^p.$$

We can prove that (21) holds on $[t_m, t_{m+1})$ by the proof we used for $[t_0 - \tau, t_1)$. Using mathematical induction method again, we can show that (21) holds on $[t_0 - \tau, t_k)$ for any $k \in \mathcal{N}$. We omit the details. Similar to the above argument, we obtain $W(t_k) \le b\varrho^p\|\phi\|_\tau^p$. Again using the contradiction method, we can show (21) holds on $[t_k, t_{k+1})$. By the Continuation Theorem in [1], we obtain $\bar{T} = +\infty$, and (1) is exponentially stable for delays $\{d_k\}$ satisfying $d_k \le d, k \in \mathcal{N}$. □

Remark 7 By comparing with the existing results [6], we remark that our conditions are simpler and more convenient to apply since there are fewer parameters need to be checked.

Similar to Theorem 2, we will also investigate stability for unstable time-delay systems with any bounded delayed input impulses.

Theorem 4 *Suppose that* (1) *satisfies* (A_1), (A_2) *and* $(A_3^{'})$, *that there exist* $V \in \nu_0$, *positive scalars* a, b *and* $p \ge 1$, *such that* (S_1) *holds, and that*
(S_2'') *There exist positive scalars* M_1, M_2 *such that*

$$V(t, g_k(x, y)) \le M_1 V(t^-, x) + M_2 V((t - d_k)^-, y)\ for\ all\ t = t_k, x, y \in \mathcal{B}(\rho).$$

(S_3'') *There exist* $\gamma > 0$ *and* $c \leq 0$ *such that, for all* $t \in [t_0, +\infty), t \neq t_k$ *and* $s \in [-\tau, 0)$,

$$D^+V(t) \leq -cV(t) \text{ whenever } \gamma e^{-c\tau} V(t) \geq V(t+s) \tag{28}$$

and

$$e^{-cN_0T_a} < \gamma < (M_1 + M_2)^{-1}.$$

Then (1) *is exponentially stable for any bounded impulse input delays* $\{d_k, k \in \mathcal{N}\}$.

Proof For any bounded impulse input delays $\{d_k, k \in \mathcal{N}\}$, let $d = \sup\{d_k, k \in \mathcal{N}\}$. Take $0 < \lambda < -c$ and $\sigma > 0$ such that

$$M_1 + M_2 e^{\lambda d} \leq \gamma^{-1}, \quad e^{(-c+\lambda)N_0T_a} < \sigma < \gamma.$$

Under the assumptions of Theorem 4, it is easy to check (21) holds on $[t_0 - \tau, t_1)$. Assuming that (21) is true for $[t_0 - \tau, t_m)$, $m = 1, 2, \ldots, k-1$, we only need show

$$W(t_m) \leq b\varrho^p \|\phi\|_\tau^p.$$

This can be obtained by (S_2'') as follows.

$$\begin{aligned}
W(t_m) &= e^{\lambda(t_m - t_0 - d)} V(t_m) \\
&\leq e^{\lambda(t_m - t_0 - d)} \left[M_1 V\left(t_m^-\right) + M_2 V((t_m - d_m)^-)\right] \\
&\leq M_1 W\left(t_m^-\right) + M_2 e^{\lambda d} W((t_m - d_m)^-) \\
&\leq \left(M_1 + M_2 e^{\lambda d}\right) b\sigma\varrho^p \|\phi\|_\tau^p \\
&\leq \frac{\sigma}{\gamma} b\varrho^p \|\phi\|_\tau^p < b\varrho^p \|\phi\|_\tau^p.
\end{aligned}$$

□

5 Applications of Theorems

In this section, we apply the above results to investigate the following time-delay impulsive system

$$\begin{cases} \dot{x}(t) = Ax(t) + \Phi(t, x(t-r)), & t > t_0, t \neq t_i \\ x(t) = \mu x(t^-) + B_i x((t - d_i)^-), & t = t_i, i \in \mathcal{N} \\ x(t) = \phi(t - t_0), & t_0 - \tau \leq t \leq t_0, \end{cases} \tag{29}$$

where A is an $n \times n$ constant matrix, $r \geq 0$, impulsive sequence $\{t_i\}$ satisfies $t_0 < t_1 < t_2 < \cdots < t_n < \cdots, \to +\infty$, d_i is the input delay, $\phi(t) \in C^1([t_0 - \tau, 0])$, $\tau = \max\{r, d\}$, and B_i, $i = 1, 2, \ldots,$ are $n \times n$ matrices.

For a matrix B, let $\|B\| = \sqrt{\lambda_{\max}(B^T B)}$, where $\lambda_{\max}(B^T B)$ is the most maximum eigenvalue of $B^T B$.

In [1], the authors investigated equi-attraction of (29) without considering time-delays [i.e., $r = 0$ in (29)]. Here we consider exponential stability of (29). We make the following assumptions.

($\mathbf{C_0}$) There exist M_0, M_1, L_0 such that $\|B_i + \mu I\| \le M_0$, $\|B_i\| \le M_1$, $i = 1, 2, \ldots$ $|\Phi(t, x)| \le L_0|x|$ for $(t, x) \in \mathcal{R}^+ \times \mathcal{R}^n$.

($\mathbf{C_1}$) The average impulsive interval of the impulsive sequence $\{t_i\}$ is T_a. That is there exist $N_0 \in \mathcal{N}$, $T_a > 0$ such that $\frac{T-t}{T_a} - N_0 \le \mathcal{N}(T, t) \le \frac{T-t}{T_a} + N_0$.

($\mathbf{C_2}$) $\bar{\lambda} = \lambda_{\max}(A) < -L_0$.

By [1], (C_0) guarantees that (29) has a solution for any $(t_0, \phi) \in \mathcal{R}^+ \times C([-\tau, 0], \mathcal{R}^n)$, which is denoted by $x(t) = x(t, t_0, \phi)$. Suppose $x(t)$ to be right-hand continuous on its existence interval.

Corollary 1 *Suppose that (29) satisfies $(C_0), (C_1), (C_2)$. Then the following statements hold.*

(a) *If there exists $d \ge 0$ such that*

$$M_0 + M_1 \left[d(\|A\| + L_0) + \left(\left\lceil \frac{d}{T_a} \right\rceil + N_0 \right) (|\mu - 1| + M_1) \right] < 1, \tag{30}$$

then for any input delays $d_k \le d$, (29) is exponentially stable.

(b) *If*

$$|\mu| + M_1 < 1, \tag{31}$$

then for any bounded input delays d_k, (29) is exponentially stable.

Proof We use Theorems 1 and 2 to establish statements (a) and (b).

Let $f(t, x_t) = Ax + \Phi(t, x(t - r))$, $g_i(x, y) = \mu x + B_i y$. Then for $L_1 = \|A\| + L_0$, $L_2 = |\mu - 1|$ and $L_3 = M_1$, $(A_1), (A_2)$ are satisfied. (C_1) is as same as (A_3).

Take $V(t, x) = V(x) = |x|$. Obviously, $V(x) \in \nu_0$ and satisfies (S_1) for $a = b = p = 1$.

It is easy to see (S_2) holds for $\nu = M_0$, $k_1 = k_2 = 1$.

Calculating yields

$$\begin{aligned} D^+V(t) &= \frac{(x^T, Ax(t)) + (x^T(t), \Phi(t, x(t - r)))}{|x|} \\ &\le \bar{\lambda}|x| + L_0|x(t - r)|. \end{aligned}$$

Take c satisfying $L_0 e^{\tau t} + t + \bar{\lambda} = 0$, then if $e^{c\tau} V(t) \ge V(t - r)$, i.e., $|x(t - r)| \le e^{c\tau}|x(t)|$, we have

$$D^+V(t) \le \bar{\lambda}|x(t)| + L_0|x(t - r)| \le \left(\bar{\lambda} + L_0 e^{c\tau}\right)|x(t)| = -cV(t).$$

Moreover, according to the choice of c, we see $c > 0$ if $\bar{\lambda} < -L_0$. Hence (S_3) holds. Equation (30) implies (2), and (31) implies $(S_2^{'})$. Therefore, all conditions in Theorem 1 and 2 are satisfied and statements (a) and (b) are established. □

Corollary 2 *Suppose (29) satisfies (C_0) and (C_1). Then the following statements hold.*

(a) $\eta_0 = \{M_0 + M_1[d(\|A\| + L_0) + (\lceil \frac{d}{T_a} \rceil + N_0)(|\mu - 1| + M_1)]\}^{-1}$ *and* $\bar{\lambda} = \lambda_{\max}(A)$ *satisfy*

$$- L_0\eta_0 \leq \bar{\lambda} \leq -\frac{1 + \ln(\tau L_0 \eta_0)}{\tau}. \tag{32}$$

Let c be a solution of $L_0\eta_0 e^{-\tau t} + t + \bar{\lambda} = 0$. If there exists $d \geq 0$ such that

$$e^{-cN_0T_a} < \eta_0, \tag{33}$$

then for any input delays $d_k \leq d$, (29) is exponentially stable.

(b) *If there exist $\gamma > 0$ and $c \leq 0$ satisfying*

$$L_0\gamma e^{-c\tau} + c + \bar{\lambda} = 0 \tag{34}$$

and

$$e^{-cN_0T_a} < \gamma < (|\mu| + M_1)^{-1}, \tag{35}$$

then for any bounded input delays $\{d_k\}$, (29) is exponentially stable.

Proof (32) ensures equation $L_0\eta_0 e^{-\tau t} + t + \bar{\lambda} = 0$ has a nonnegative solution c. For the functional $V(t, x) = V(x) = |x|$, if $\eta_0 e^{-c\tau} V(t) \geq V(t - r)$, we have

$$D^+V(t) \leq \bar{\lambda}|x(t)| + L_0|x(t - r)| \leq \left(\bar{\lambda} + L_0\eta_0 e^{-c\tau}\right)|x(t)| = -cV(t).$$

Therefore, $(S_3^{'})$ holds. Theorem 3 establishes statement (a).

The statement (b) follows since $(S_2^{''})$ and $(S_3^{''})$ in Theorem 4 are satisfied by (34) and (35), respectively. □

6 Numerical Examples

Example 1 We consider

$$\begin{cases} \dot{x}_1(t) = ax_1(t) + \frac{1}{4}x_1(t - r)\sin(x_2(t - r)) & t \neq t_k \\ \dot{x}_2(t) = ax_2(t) + \frac{1}{4}x_2(t - r)\cos(x_1(t - r)) & t \neq t_k \\ x_1(t^+) = \mu x_1(t^-) + bx_1(t - d_k)^-) & t = t_k \\ x_2(t^+) = \mu x_2(t^-) + bx_2(t - d_k)^-) & t = t_k \end{cases} \tag{36}$$

where a, b, μ are constants. $r, d_k \geq 0, k \in \mathcal{N}$.

Suppose that the impulsive sequence $\zeta = \{\epsilon, 2\epsilon, \ldots, (N_0-1)\epsilon, N_0T_a, N_0T_a + \epsilon, N_0T_a + 2\epsilon, \ldots, N_0T_a + (N_0-1)\epsilon, 2N_0T_a, \ldots\}$, where $N_0 \in \mathcal{Z}^+$ and $T_a > 0$.

Obviously, $\inf_{k\in\mathcal{N}^+}\{t_k - t_{k-1}\} = \epsilon$, $\sup_{k\in\mathcal{N}^+}\{t_k - t_{k-1}\} = N_0(T_a - \epsilon) + \epsilon$. Therefore, if ϵ is arbitrarily small, so is the minimum interval length of ζ. While if N_0 is very large, so is the maximum interval length of ζ.

We take $T_a = 0.5, N_0 = 2, r = 1, d_k = d = \frac{1}{4}, k \in \mathcal{N}$.

Let $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $B_i = B = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$, $\Phi(t, x) = \begin{bmatrix} \frac{1}{4}x_1 \sin x_2 \\ \frac{1}{4}x_2 \cos x_1 \end{bmatrix}$. Then $\|A\| = |a|, M_0 = |b + \mu|, M_1 = |b|, \bar{\lambda} = a$. Take $a = -\frac{1}{2}, L_0 = \frac{1}{4}$.

1. For $b = -\frac{1}{16}, \mu = 1$, or $b = -\frac{2}{16}, \mu = \frac{33}{32}$, (30) holds. Therefore, for any d_k satisfying $0 \leq d_k \leq \frac{1}{4}$, (36) is exponentially stable by(a) of Corollary 1.
2. Let $b = \frac{1}{16}, \mu = \frac{13}{16}$, then $|\mu| + M_1 < 1$, i.e., (31) holds. Then, for any bounded input delays $d_k, k \in \mathcal{N}$, (36) is exponentially stable by (b) of Corollary 1.

Example 2 For (36), we take $a = -\frac{1}{4}$. Suppose that the values of T_a, N_0, L_0, d_k, r are the same as Example 1. Then $\tau = \max\{r, d\} = 1$.

1. Let $b = -\frac{1}{16}, \mu = \frac{15}{16}$. Then $\eta_0 = \frac{128}{115}$ by a simple computation. It follows that (32) holds. So equation $L_0\eta_0 e^{-\tau t} + t + \lambda = 0$, i.e., $\frac{32}{115}e^{-t} + t - \frac{1}{4} = 0$ has a negative value $c \in (-0.04, -0.03)$ satisfying (33).
2. Take $\gamma = \eta_0 = \frac{128}{115}, \mu = b = \frac{1}{4}$. It is easy to check that (34) and (35) are satisfied.

Corollary 2 establishes the exponential stability of (36).

7 Conclusions

In this paper, we investigated exponential stability of nonlinear time-delay impulsive systems and obtained some sufficient conditions to realize exponential stability of the systems. The main results are divided into two parts according to whether the continuous dynamics is stable. Some applications and numerical examples are supplied to illustrate the validity of the main results. Some open problems are worth to explore. For example, the condition $M_1 + M_2 < 1$ in Theorem 2 seems to be somewhat strong. If this condition can be removed or weakened, the results will be more meaningful. We plan to study this issue for our future research.

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