Circuits Syst Signal Process (2018) 37:135–162
https://doi.org/10.1007/s00034-017-0550-5

Asynchronous H_∞ Control of Switched Uncertain Discrete-Time Fuzzy Systems via Basis-Dependent Multiple Lyapunov Functions Approach

Yang Li[1] · **Hongbin Zhang**[1]

Received: 20 May 2016 / Revised: 24 March 2017 / Accepted: 24 March 2017 /
Published online: 4 April 2017

Abstract This paper first investigates robust stability of open-loop switched uncertain discrete-time fuzzy systems (SUDFSs) under mode-dependent average dwell time (MDADT) switching. By a basis-dependent multiple Lyapunov functions (BLFs) approach, which has more flexibility than the multiple quadratic Lyapunov functions approach, computable robust stability conditions are presented in terms of linear matrix inequalities (LMIs). Then, the investigation is extended to robust H_∞ control of closed-loop SUDFSs by using the same approach. The asynchronous state feedback H_∞ controllers which can stabilize the SUDFSs and guarantee weighted H_∞ performance are obtained by solving a set of LMIs. A numerical example and a practical example are provided to show the advantage of the proposed approach.

Keywords H_∞ control · Switched uncertain discrete-time fuzzy systems (SUDFSs) · Basis-dependent multiple Lyapunov functions (BLFs) · Mode-dependent average dwell time (MDADT)

1 Introduction

Switched systems, which are made up of several continuous-time or discrete-time subsystems and the switching rules among them, have attracted a large number of researchers' interest in recent years not only due to its excellent modeling ability

✉ Yang Li
13648065067@163.com

Hongbin Zhang
zhanghb@uestc.edu.cn

1 School of Electronic Engineering, University of Electronic Science and Technology of China, Chengdu 611731, People's Republic of China

for actual system, but also the applicability in relevant results of the existing control theory. It can be applied to many practical systems such as the flight control systems, the network control systems, the automobile engine systems [7,16,19,20,23]. Switched systems' features, especially the stability and stabilization, are researched deeply in recent years, and many important results have been derived [1,4,7,8].

Stability issue has always been important research content of switched systems. According to the property of switching signal, two stability issues have been addressed, i.e., the stability under arbitrary switching and the stability under constrained switching. For the arbitrary switching, we often try to construct a common Lyapunov function (CLF) for all subsystems [7]. But it is difficult to find this function in general. Due to this approach's conservatism, some scholars put forward an improved approach in discrete-time domain by using the switched Lyapunov functions (SLFs) [4]. Despite the arbitrary switching's stability issue, researchers are also interested in constrained switchings. The constrained switching is divided into time-constrained switching and state-constrained switching. We mainly discuss the time-constrained switching signals here. The average dwell time (ADT) and mode-dependent average dwell time (MDADT) logic are put forward to help research the time-constrained switching in [6,28]. For these kinds of switching signals, many research results have been obtained by using the multiple Lyapunov-like functions (MLFs) approach [9,12,16,23,25,26,30–32]. Due to the switching signals' particularity, if the declining rate of Lyapunov function in the system's running time is high enough, then we can allow Lyapunov function rise to some extent at the switching moment. In this paper's research, we also need to adopt this basic thought.

Nowadays, most research results focus on switched linear systems, which is mainly because most research results of non-switched linear systems have been obtained in the past. However, research results about nonlinear switched systems seem to be scattered. Due to the nonlinearity of the system, it is hard to analyze it directly. In this paper, we use the Takagi–Sugeno (T-S) fuzzy model to describe the nonlinear system. We can get the complex nonlinear switched systems' local linear description for each fuzzy rule through T-S fuzzy model [21], and the whole nonlinear system can be expressed by using the "fuzzy blending," and this can help us to investigate nonlinear systems' stability issue by using linear systems' research method. Some stability conditions and H_∞ controllers design methods of non-switched nonlinear systems have been found, e.g., [2,5,10,17,22], but it seems that there exist few researches of switched nonlinear systems' stability and controller design issues. A basis-dependent Lyapunov function [11,29] is used to study the linear discrete-time systems' stability and H_∞ control issue. The basis-dependent Lyapunov functions depend on system's fuzzy model, which lowers the difficulty in solving LMIs, while the single quadratic Lyapunov functions do not depend on it. Zhou et al. [29] had shown that the results of discrete-time nonlinear systems given by basis-dependent Lyapunov functions are less conservative than that given by the single quadratic Lyapunov functions which are basis-independent. Through combining switched systems' switching information and fuzzy model, we apply basis-dependent multiple Lyapunov functions (BLFs) and multiple quadratic Lyapunov functions (QLFs) into SUDFSs' stability analysis and H_∞ control synthesis and compare the two approaches' conservatism.

When dealing with practical problems, perturbation of systems' characteristics or parameters is often unavoidable; therefore, robustness is needed to be considered for a control system. The l_2-gain also attracts more and more researchers' attention due to its function of describing systems' ability of constraining disturbing signal. Usually, in the case of guaranteeing systems' stability we also hope it has good weighted H_∞ performance. We need to find a controller which can guarantee that the l_2-gain from the disturbance to the estimation error is within a prescribed level. Some methodologies have been developed in recent years for the H_∞ controller design, e.g., by using basis-dependent Lyapunov function, an H_∞ control design approach is developed for a class of discrete-time fuzzy systems with uncertainty [29], two sufficient LMI conditions, which guarantee the existence of the H_∞ controllers based on fuzzy observers for the T-S fuzzy systems have been proposed in [11]. However, most H_∞ controllers are designed in the synchronous conditions. This circumstance is too ideal. In practice, it rarely holds since there is always a lag in the controller when the switching happens. Some time should be taken to identify the system modes and apply the matched controller. Asynchronous behaviors' influence on the control of switched linear systems and nonlinear systems has been investigated in [12,13,30–32]. The research results have shown that developing H_∞ controllers in asynchronous conditions is necessary. At present, some papers have investigated the design of discrete-time linear systems' H_∞ controllers in asynchronous conditions. For instance, the conditions of the existence of admissible asynchronous H_∞ controllers are derived in [31]. H_∞ controllers for a class of discrete-time switched linear parameter-varying systems with asynchronous switching are designed in [12]. Recently, there also exist some results on stability analysis and controller design of discrete-time switched nonlinear systems. For instance, stabilization of discrete-time switched nonlinear systems without stable subsystems is investigated in [14]. Robust stability and H_∞ control of discrete-time switched T-S fuzzy systems with time-varying delays are investigated in [15]. However, in most of these papers, the controller is designed in synchronous conditions. The design of discrete-time nonlinear systems' H_∞ controller in asynchronous conditions remains unsolved.

The contributions of the paper focus on two points: first, SUDFSs' stability condition and asynchronous state feedback H_∞ controllers are obtained through BLFs approach. Second, we compare two different approaches' performance to show the BLFs approach's advantage. This paper is organized as follows: definitions and lemmas on stability and l_2-gain of switched systems are shown in Sect. 2. Main results are obtained in Sect. 3. Two numerical examples are presented to verify the feasibility and effectiveness of the proposed techniques in Sect. 4. We conclude the paper at last in Sect. 5.

1.1 Notations

The notation used in this paper is fairly standard. The superscript "T" stands for matrix transposition. The symbol "$*$" in a matrix stands for the transposed elements in the symmetric positions, and $diag\{\cdots\}$ stands for a block-diagonal matrix. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{N}$ represents the set of nonnegative integers, and the notation $\| \cdot \|$ refers to the Euclidean vector norm. $l_2[0, \infty)$ is the space of square summable infinite sequence, and for $w = w(k) \in l_2[0, \infty)$, its norm is given by $\| w \|_2 = \sqrt{\sum_{k=0}^{\infty} | w(k) |^2}$. $\mathbb{C}^1$ denotes the space of continuously differentiable functions, and a function $\alpha : [0, \infty) \to [0, \infty)$ is said to be of class $\mathcal{K}_\infty$ if it is continuous, strictly increasing, unbounded, and $\alpha(0) = 0$. Also, a function $\beta : [0, \infty) \times [0, \infty)$ is said to be of class $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(s, t)$ decreases to 0 as $t \to \infty$ for each fixed $s \geq 0$.

2 System Descriptions and Preliminaries

Let us consider the following switched nonlinear systems described by

$$\begin{cases} x(k+1) = f_{\sigma(k)}(x(k), u(k), w(k)) \\ y(k) = s_{\sigma(t)}(x(t), u(t), w(k)) \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^{n_x}$, $u(k) \in \mathbb{R}^{n_u}$ and $y(k) \in \mathbb{R}^{n_y}$ denote the state vector, input vector and output vector, respectively; $w(k) \in \mathbb{R}^{n_w}$ is the disturbance that belongs to $l_2[0, \infty)$; $f_{\sigma(k)}$ and $s_{\sigma(k)}$ are nonlinear functions; $\sigma(k)$ is defined as a switching signal, which is a piecewise constant function of time and takes its values in the finite set $\mathcal{I} = \{1, 2, \ldots, N\}$, where $N > 1$ is the number of subsystems. For a switching sequence $0 < k_0 < k_1 < \cdots < k_l < k_{l+1} < \cdots$, $\sigma(k)$ is continuous from right everywhere. k_0 is the initial time which can be any positive integer, and we assume that there exists no switching between any two adjacent switching times k_l and k_{l+1}, $l \geq 0$. When $k \in [k_l, k_{l+1})$, $\sigma(k_l) = i$, $i \in \mathcal{I}$, we say the ith subsystem is activated.

The T-S fuzzy model which is described by fuzzy IF-THEN rules [21] is employed here to represent each subsystem of the switched nonlinear systems. The fuzzy model of ith subsystem is described as the following form:
Rule n: IF $v_{i1}(k)$ is N_{i1m} and $\cdots$ and $v_{ig}(k)$ is N_{igm}, THEN

$$\begin{cases} x(k+1) = \mathscr{A}_{im}x(k) + \mathscr{B}_{im}u(k) + E_{im}w(k) \\ y(k) = C_{im}x(k) + D_{im}u(k) + F_{im}w(k) \end{cases} \tag{2}$$

where $v_i(k) = (v_{i1}(k), v_{i2}(k), \ldots, v_{ig}(k))$ are some measurable premise variables and $N_{ipm}(p = 1, 2, \ldots, g)$ are fuzzy sets. $\mathscr{A}_{im}$, $\mathscr{B}_{im}$,C_{im}, D_{im} , E_{im} and F_{im} are real matrices of the mth local model of the ith subsystem; among them matrix $\mathscr{A}_{im}$ and $\mathscr{B}_{im}$ have parametric uncertainties. $\mathscr{A}_{im}$ and $\mathscr{B}_{im}$ satisfy

$$\begin{bmatrix} \mathscr{A}_{im} & \mathscr{B}_{im} \end{bmatrix} = \begin{bmatrix} A_{im} & B_{im} \end{bmatrix} + M_i F(k) \begin{bmatrix} N_{1i} & N_{2i} \end{bmatrix} \tag{3}$$

where M_i, N_{1i} and N_{2i} are known real constant matrices and $F(k)$ is an unknown time-varying matrix function satisfying

$$F(k)^T F(k) \leq I \quad \forall k \in \mathbb{N} \tag{4}$$

Remark 1 Here we let M_i and N_{1i}, N_{2i} unchanged in different fuzzy rules; in other words, M_i and N_{1i}, N_{2i} are given linear matrixes which do not need to be fuzzified. This brings convenience to our research because the derivations will be very complex if M_i and N_{1i}, N_{2i} need to be fuzzified.

Through using "fuzzy blending," the final output of the ith subsystem is inferred as follows:

$$\begin{cases} x(k+1) = \sum\limits_{m=1}^{r_i} \sum\limits_{n=1}^{r_i} h_{im}h_{in}[\mathscr{A}_{imn}x(k) + E_{im}w(k)] \\ y(k) = \sum\limits_{m=1}^{r_i} \sum\limits_{n=1}^{r_i} h_{im}h_{in}[C_{imn}x(k) + F_{im}w(k)] \end{cases} \tag{5}$$

where $\mathscr{A}_{imn} = \mathscr{A}_{im} + \mathscr{B}_{im}K_{in}$, $C_{imn} = C_{im} + D_{im}K_{in}$, and

$$h_{im} = l_{im} / \sum_{k=1}^{r_i} l_{im}, \quad l_{im} = \prod_{p=1}^{g} N_{ipm}(v_{ip}(k)) \tag{6}$$

in which $N_{ipm}(v_{ip}(k))$ is the grade of the membership function of v_{ip} in N_{ipm}. It is assumed that $l_{im} \geq 0$ for all k and $m = 1, 2, \ldots, r_i$. Therefore, the normalized membership function h_{im} satisfies

$$h_{im} \geq 0, \sum_{m=1}^{r_i} h_{im} = 1 \quad \forall k \in \mathbb{N}. \tag{7}$$

In view of the asynchronous behaviors, asynchronous control problem is considered. The controller $u(k)$ is divided into two parts $\bar{u}(k)$ and $\hat{u}(k)$, where $\hat{u}(k)$ denotes the unmatched controller, and $\bar{u}(k)$ represents the matched controller. $\bar{u}(k)$ and $\hat{u}(k)$ have the following form:

$$\begin{cases} \hat{u}(k) = \sum\limits_{n=1}^{r_i} h_{jn}K_{jn}x(k), \quad k \in [k_l, k_l + T_i) \\ \bar{u}(k) = \sum\limits_{n=1}^{r_i} h_{in}K_{in}x(k), \quad k \in [k_l + T_i, k_{l+1}) \end{cases} \tag{8}$$

where k_l and k_{l+1} are the beginning time and ending time of the activated subsystem $\sigma(k_l)$. When $k \in T[k_l, k_{l+1})$, $\forall l \in \mathbb{N}$, the subsystem $\sigma(k_l) = i (i \in \mathcal{I})$ is activated. The energy function of the system when asynchronous switching occurs is shown in Fig. 1. It can be seen in Fig. 1 that $T[k_l, k_{l+1})$ is divided into $T[k_l, k_l + T_i)$ and $T[k_l + T_i, k_{l+1})$. During $T[k_l, k_l + T_i)$ period, the controller and the system are unmatched; then, the Lyapunov function may increase or decrease; T_i represents the running time of the unmatched controller in $T[k_l, k_l + T_i)$, in other words, the ith subsystem's asynchronous time in $T[k_l, k_l + T_i)$. During $T[k_l + T_i, k_{l+1})$ period, the controller and the system are matched. The Lyapunov function is strictly decreasing with the matched controller. K_{in} and K_{jn} are controller gain matrices which remain to be solved. Substituting (8) into (5), we can obtain the following closed-loop SUDFSs:

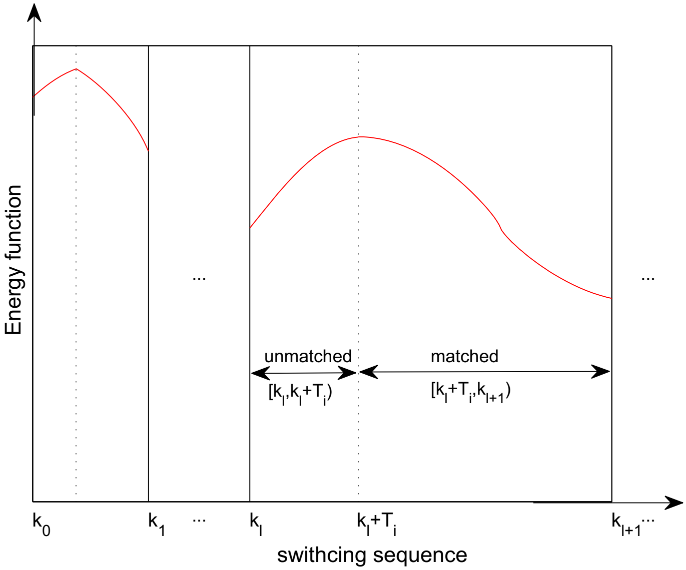

Fig. 1 Energy function for the asynchronous switched system

$$\begin{cases}\begin{cases} x(k+1) = \sum_{m=1}^{r_i}\sum_{n=1}^{r_i} h_{im}h_{jn}\hat{\mathscr{A}}_{imn}x(k) + \sum_{m=1}^{r_i} h_{im}E_{im}w(k) \\ y(k) = \sum_{m=1}^{r_i}\sum_{n=1}^{r_i} h_{im}h_{jn}\hat{C}_{imn}x(k) + \sum_{m=1}^{r_i} h_{im}F_{im}w(k) \end{cases}, \forall k \in [k_l, k_l + T_i) \\ \begin{cases} x(k+1) = \sum_{m=1}^{r_i}\sum_{n=1}^{r_i} h_{im}h_{in}\bar{\mathscr{A}}_{imn}x(k) + \sum_{m=1}^{r_i} h_{im}E_{im}w(k) \\ y(k) = \sum_{m=1}^{r_i}\sum_{n=1}^{r_i} h_{im}h_{in}\bar{C}_{imn}x(k) + F_{im}w(k) \end{cases}, \forall k \in [k_l + T_i, k_{l+1}) \end{cases} \tag{9}$$

where $\hat{\mathscr{A}}_{imn} = \mathscr{A}_{im} + \mathscr{B}_{im}K_{jn}$, $\hat{C}_{imn} = C_{im} + D_{im}K_{jn}$ and $\bar{\mathscr{A}}_{imn} = \mathscr{A}_{im} + \mathscr{B}_{im}K_{in}$, $\bar{C}_{imn} = C_{im} + D_{im}K_{in}$.

Our purpose in this paper is to design an asynchronous state feedback H_∞ controller which can ensure weighted H_∞ performance and GUAS of the SUDFS (9) under MDADT switching condition. The following definitions are first recalled.

Definition 1 [7] System (1) with $w(k) \equiv 0$ and $u(k) \equiv 0$ is globally uniformly asymptotically stable (GUAS) if there exists a class $\mathcal{KL}$ function β such that for all switching signals $\sigma(k)$ and all initial condition $x(k_0)$, the solutions of (1) satisfy the following inequality

$$\|x(k)\| \le \beta(\|x(k_0)\|, k), \quad \forall k \ge k_0. \tag{10}$$

Definition 2 [28] For a switching signal $\sigma(k)$ and any $0 \le k \le K$, let $N_{\sigma(k)i}(k, K)$ be the switching numbers of the ith subsystem activated during the interval $[k, K]$, and $H_i(k, K)$ is the total running time of the ith subsystem in $[k, K]$, $i \in \mathcal{I}$. We say that

$\sigma(k)$ has a MDADT called T_{oai}, if there exist positive N_{0i} (we call mode-dependent chattering bounds) and T_{oai} such that

$$N_{\sigma(k)i}(k, K) \leq N_{0i} + H_i(k, K)/T_{oai}, \quad 0 \leq k \leq K. \tag{11}$$

Remark 2 The MDADT T_{oai} is mentioned here. If the switching signal of system (1) satisfies the MDADT property, its ith ($\forall i \in \mathcal{I}$) subsystem's average running time among the switching process should be larger than T_{oai}. In Fig. 1, we can see whether the subsystems' average running time is long enough to offset the increment caused by the asynchronous phenomenon. The energy function of system (1) will converge to zero.

Definition 3 [33] For $0 < \alpha < 1$ and $\gamma > 0$, system (1) is said to have a weighted l_2-gain no greater than γ, if under zero initial condition, the inequality $\sum_{k=k_0}^{\infty}(1-\alpha)^k y_k^T y_k \leq \gamma^2 \sum_{k=k_0}^{\infty} w_k^T w_k$, $0 < \alpha < 1$ holds for all nonzero $w(k) \in l_2[0, \infty)$.

Remark 3 The definition of non-weighted l_2-gain is different from the weighted one in Definition 3. For $\gamma > 0$, system (1) is said to have a non-weighted l_2-gain no greater than γ, if the inequality $\sum_{k=k_0}^{\infty} y_k^T y_k \leq \gamma^2 \sum_{k=k_0}^{\infty} w_k^T w_k$, holds for all nonzero $w(k) \in l_2[0, \infty)$. The non-weighted one can be seen as a special case that the weighted index is 1. The non-weighted l_2-gain can describe the system's input–output performance better than the weighted one. In [24], via a dwell time-dependent Lyapunov function approach, non-weighted l_2-gain results of uncertain discrete-time switched linear system under minimum dwell time switching are presented in terms of LMIs. However, this approach only applies to analysis of switched systems under minimum dwell time switching. There exist few results on non-weighted l_2-gain analysis of switched system under ADT switching and MDADT switching. In this paper, we mainly investigate the weighted l_2-gain of the SUDFSs under MDADT switching via the BLFs approach.

Lemma 1 [28] *Consider SUDFS (9) with $w(k) \equiv 0$ and $u(k) \equiv 0$, and let $0 < \alpha_i < 1$, $\mu_i > 1$, $i \in \mathcal{I}$ be given constants. If there exist positive definite $\mathbb{C}^1$ function $V_{\sigma(k_l)} : \mathbb{R}^n \to \mathbb{R}$, and class $\mathcal{K}_\infty$ functions κ_{1i} and κ_{2i}, $i \in \mathcal{I}$ satisfying*

$$\kappa_{1i}(\|x_k\|) \leq V_i(x_k) \leq \kappa_{2i}(\|x_k\|) \tag{12}$$

$$V_i(k_l) \leq \mu_i V_j(k_l) \tag{13}$$

and

$$\Delta V_i(k) \leq -\alpha_i V_i(k), \quad k \in [k_l, k_{l+1}). \tag{14}$$

Then, SUDFS (9) is GUAS for any switching signal satisfying

$$T_{oai} \geq T_{oai}^* = \frac{-\ln \mu_i}{\ln(1-\alpha_i)} \tag{15}$$

Remark 4 Lemma 1 is the stability result on discrete-time switched nonlinear system and has been proved in [28]. In [28], based on Lemma 1, some results on stability and stabilization of discrete-time switched linear systems under MDADT switching are given in terms of LMIs, and computable results of discrete-time switched nonlinear systems under MDADT switching are not given. In this paper, we utilize Lemma 1 to analyze stability of SUDFSs (9) and present some computable results in terms of LMIs.

Lemma 2 *Given two matrices* $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times n}$, *and symmetric positive definite matrix* $P \in \mathbb{R}^{m\times m}$, *then*

$$A^T PB + B^T PA \le A^T PA + B^T PB.$$

Proof From $P > 0$, we have

$$(A - B)^T P(A - B) \ge 0,$$

which is equal to

$$A^T PB + B^T PA \le A^T PA + B^T PB.$$

Then, the proof is end. □

Lemma 3 [29] *Suppose* $P_m > 0$, $P_n > 0$, *and* $P_u > 0$. *If*

$$A_m^T P_u A_m - P_m < 0, \quad A_n^T P_u A_n - P_n < 0,$$

then

$$A_m^T P_u A_n + A_n^T P_u A_m - P_m - P_n < 0.$$

Lemma 4 *[12] Consider SUDFS (9), and let* $0 < \alpha_i < 1$, $\beta_i \ge 0$ *and* $\gamma_i > 0$. $\forall i > 0$ *be given constants. Suppose that there exist positive* $\mathbb{C}^1$ *functions* $V_{\sigma(k)} : R^n \to R, \sigma(k) \in \mathcal{I}$, *with* $V_{\sigma(k_0)}(x_{k_0}) \equiv 0$ *such that* $\forall (i, j) \in \mathcal{I} \times \mathcal{I}, i \ne j$, $V_i(x_{k_l}) \le \mu_i V_j(x_{k_l})$ *and* $\forall i \in \mathcal{I}$, *denoting* $\Gamma(k) \triangleq y_k^T y_k - \gamma_i^2 \omega_k^T \omega_k$, *if the following inequality is satisfied*

$$\Delta V_i(x_k) \le \begin{cases} -\alpha_i V_i(k) - \Gamma(k), \forall k \in [k_l, k_l + T_i) \\ \beta_i V_i(k) - \Gamma(k), \forall k \in [k_l + T_i, k_{l+1}) \end{cases}, \tag{16}$$

then SUDFS (9) is GUAS for any switching signal satisfying

$$T_{oai} \ge T_{oai}^* = \frac{T_i(\ln(1-\alpha_i) - \ln(1+\beta_i)) - \ln \mu_i}{\ln(1-\alpha_i)} \tag{17}$$

and the H_∞ *performance no greater than* $\hat{\gamma} = \sqrt{\prod_{i\in\mathcal{I}} (\theta_i^{T_M}\mu_i)^{N_{0i}} \frac{\alpha_{\max}}{\alpha_{\min}} \theta_{\max}^{T_M - 1}}\gamma$, *where* T_i *denotes the increasing interval of the ith subsystem,* $T_M = \max\{T_i\}$, $\theta_{\max} = \max\{\theta_i\} = \max\{(1+\beta_i)/(1-\alpha_i)\}$, $\forall i \in \mathcal{I}$.

Remark 5 Lemma 4 has been proved in [12] and can apply to analysis of discrete-time switched nonlinear systems under asynchronous switching. In [12], based on Lemma 4, H_∞ control of discrete-time switched linear parameter-varying systems with both MDADT and asynchronous switching is investigated. Some results are given in terms of LMIs. However, there exist no results on H_∞ control of discrete-time nonlinear switched systems with both mode-dependent average dwell time (MDADT) and asynchronous switching. In this paper, we utilize Lemma 4 to analyze the problem of asynchronous H_∞ control of SUDFSs (9) and fill the gap.

Lemma 5 [18] *Given matrices P, M and N of appropriate dimensions and with P symmetrical, then*

$$P + MF(k)N + N^T F^T(k)M^T < 0$$

holds for all F(k) satisfying $F^T(k)F(k) \leq I$ *if and only if for some* $\epsilon > 0$

$$P + \epsilon^{-1}MM^T + \epsilon N^T N < 0.$$

3 Main Results

3.1 Stability Analysis

In this section, we analyze the stability condition of open-loop SUBFSs (9) through BLFs and QLFs approach which have been mentioned before. The normal quadratic multiple Lyapunov function is like this form:

$$V_i(k) = x^T(k)P_i x(k),$$

the basis-dependent multiple Lyapunov function assumes that

$$V_i(k) = x(k)^T \sum_{u=1}^{r_i} h_{iu} P_{iu} x(k),$$

this means:

$$P_i = \sum_{u=1}^{r_i} h_{iu} P_{iu}.$$

Theorem 1 *Consider SUDFS (9) with* $w(k) \equiv 0$ *and* $u(k) \equiv 0$ *and let* $0 < \alpha_i < 1$, $\mu_i > 1$ *be given constants, if there exist matrices* $P_{im} > 0$*, for* $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$, $m = 1, 2, \ldots, r_i$, $n = 1, 2, \ldots, r_i$, $u = 1, 2, \ldots, r_i$ *satisfying*

$$P_{in} \leq \mu_i P_{jm} \tag{18}$$

$$\mathscr{A}_{im}^T P_{iu} \mathscr{A}_{im} - (1 - \alpha_i) P_{im} < 0 \tag{19}$$

then SUDFS (9) is GUAS for any switching signal satisfying (15).

Proof We choose the BLFs which is described by $V_i(k) = x(k)^T \sum_{u=1}^{r_i} h_{iu} P_{iu} x(k)$ to analysis open-loop SUDFS(9)'s stability conditions, due to $\sum_{u=1}^{r_i} h_{iu} P_{iu} > 0$, we surely can find κ_{1i} and κ_{2i} which satisfy (12). Make $P_i = \sum_{u=1}^{r_i} h_{iu} P_{iu}$ and combine (9) we can get

$$\begin{aligned}
&\Delta V_i(k) + \alpha_i V_i(k) \\
&= V_i(k+1) + (\alpha_i - 1) V_i(k) \\
&= x(k+1)^T P_i x(k+1) + (\alpha_i - 1) x(k)^T P_i x(k) \\
&= x(k)^T \left[\sum_{u=1}^{r_i} h_{iu} \left(\sum_{m=1}^{r_i} \sum_{n=1}^{r_i} h_{in} h_{im} (\mathscr{A}_{im}^T P_{iu} \mathscr{A}_{in} - (1-\alpha_i) P_{im}) \right) \right] x(k) \\
&= x(k)^T \left[\sum_{u=1}^{r_i} h_{iu} \left(\sum_{m=1}^{r_i} h_{im}^2 (\mathscr{A}_{im}^T P_{iu} \mathscr{A}_{im} - (1-\alpha_i) P_{im}) + \right.\right. \\
&\quad \sum_{n=1}^{r_i} \sum_{m>n}^{r_i} h_{im} h_{in} (\mathscr{A}_{im}^T P_{iu} \mathscr{A}_{in} + \mathscr{A}_{in}^T P_{iu} \mathscr{A}_{im} \\
&\quad \left.\left. - (1-\alpha_i) P_{in} - (1-\alpha_i) P_{im}) \right) \right] x(k),
\end{aligned}$$

from (21), we have

$$\mathscr{A}_{im}^T P_{iu} \mathscr{A}_{im} - (1-\alpha_i) P_{im} < 0, \quad \mathscr{A}_{in}^T P_{iu} \mathscr{A}_{in} - (1-\alpha_i) P_{in} < 0,$$

from Lemma 3, we can conclude that

$$\mathscr{A}_{im}^T P_{iu} \mathscr{A}_{in} + \mathscr{A}_{in}^T P_{iu} \mathscr{A}_{im} - (1-\alpha_i) P_{in} - (1-\alpha_i) P_{im} < 0,$$

then we can obtain (14). From (18), we can get $\sum_{n=1}^{r_i} h_{in} P_{in} \le \mu_i \sum_{m=1}^{r_i} h_{jm} P_{jm}$, which is equal to $x(k_l)^T \sum_{n=1}^{r_i} h_{in} P_{in} x(k_l) \le \mu_i x(k_l)^T \sum_{m=1}^{r_i} h_{jm} P_{jm} x(k_l)$, then (13) is derived. From Lemma 1, we can know the open-loop SUDFS (9) is GUAS for any switching signal satisfying (15). This completes the proof. □

Theorem 2 *Consider SUDFS (9) with* $w(k) \equiv 0$ *and* $u(k) \equiv 0$ *and let* $0 < \alpha_i < 1$, $\mu_i > 1$ *be given constants, if there exist matrices* $X_{im} > 0$, *matrices* Ω_i, *positive constant* ϵ_i *for* $\forall (i, j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$, $m = 1, 2, \ldots, r_i$, $n = 1, 2, \ldots, r_i$, $u = 1, 2, \ldots, r_i$ *satisfying*

$$X_{jm} \le \mu_i X_{in} \tag{20}$$

$$\Xi_{imu} < 0 \tag{21}$$

where

$$\Xi_{imu} = \begin{bmatrix} \frac{X_{im}}{1-\alpha_i} - \Omega_i^T - \Omega_i & \Omega_i^T A_{im}^T & 0 & \Omega_i^T N_{1i}^T \\ * & -X_{iu} & \frac{M_i}{\epsilon_i} & 0 \\ * & * & -\frac{I}{\epsilon_i} & 0 \\ * & * & * & -\frac{I}{\epsilon_i} \end{bmatrix}.$$

Then, SUDFS (9) is GUAS for any switching signal satisfying (15).

Proof Consider the closed-loop SUDFS (9), applying the Schur complement to (21) yields:

$$\begin{bmatrix} \frac{X_{im}}{1-\alpha_i} - \Omega_i^T - \Omega_i & \Omega_i^T A_{im}^T \\ * & -X_{iu} \end{bmatrix} + \epsilon_i \begin{bmatrix} 0 \\ \frac{M_i}{\epsilon_i} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{M_i}{\epsilon_i} \end{bmatrix}^T + \epsilon_i \begin{bmatrix} \Omega_i^T N_i^T \\ 0 \end{bmatrix} \begin{bmatrix} \Omega_i^T N_i^T \\ 0 \end{bmatrix}^T < 0,$$

which is equal to

$$\begin{bmatrix} \frac{X_{im}}{1-\alpha_i} - \Omega_i^T - \Omega_i & \Omega_i^T A_{im}^T \\ * & -X_{iu} \end{bmatrix} + \epsilon_i^{-1} \begin{bmatrix} 0 \\ M_i \end{bmatrix} \begin{bmatrix} 0 \\ M_i \end{bmatrix}^T + \epsilon_i \begin{bmatrix} \Omega_i^T N_i^T \\ 0 \end{bmatrix} \begin{bmatrix} \Omega_i^T N_i^T \\ 0 \end{bmatrix}^T < 0.$$

From Lemma 5, we can know:

$$\begin{bmatrix} \frac{X_{im}}{1-\alpha_i} - \Omega_i^T - \Omega_i & \Omega_i^T \mathscr{A}_{im}^T \\ * & -X_{iu} \end{bmatrix} < 0$$

which gives $0 < \frac{X_{im}}{1-\alpha_i} < \Omega_i^T + \Omega_i$. This implies that Ω_i is invertible. The inequalities $\left(\frac{X_{im}}{1-\alpha_i} - \Omega_i\right)^T \left(\frac{X_{im}}{1-\alpha_i}\right)^{-1} \left(\frac{X_{im}}{1-\alpha_i} - \Omega_i\right) \geq 0$ imply that $\Omega_i^T (\frac{X_{im}}{1-\alpha_i})^{-1} \Omega_i \geq \Omega_i^T + \Omega_j - \frac{X_{im}}{1+\alpha_i}$, that is

$$\begin{bmatrix} -(1-\alpha_i)\Omega_i^T X_{im}^{-1} \Omega_i & \Omega_i^T \mathscr{A}_{im}^T \\ * & -X_{iu} \end{bmatrix} < 0.$$

Note that Ω_i is invertible. Pre-multiplying diag(Ω_i^{-T},I) and post-multiplying diag(Ω_i^{-1},I) to above inequality give

$$\begin{bmatrix} -(1-\alpha_i)X_{im} & \mathscr{A}_{im}^T \\ * & -X_{iu} \end{bmatrix} < 0.$$

Setting $X_{im}^{-1} = P_{im}$, we have

$$\begin{bmatrix} -(1-\alpha_i)P_{im} & \mathscr{A}_{im}^T \\ * & -P_{iu}^{-1} \end{bmatrix} < 0.$$

It follows from Schur complement equivalence that we can get (19), and (20) is equal to (18):

$$X_{jm} \leq \mu_i X_{in} \Leftrightarrow X_{in}^{-1} \leq \mu_i X_{jm}^{-1} \Leftrightarrow P_{in} \leq \mu_i P_{jm}.$$

From Theorem 1, we can know the open-loop SUDFS (9) is GUAS for any switching signal satisfying (15).

Remark 6 By setting $P_{im} = P_i$, $V_i(k)$ becomes $V_i(k) = x(k)^T P_i x(k)$ which is a multiple quadratic Lyapunov function which is widely used in related works. We get the following Corollary 1 where the QLFs are used. So the LMIs condition in Theorem 2 has less conservatism than those in Corollary 1.

Corollary 1 *Consider SUDFS (9) with $w(k) \equiv 0$ and $u(k) \equiv 0$ and let $0 < \alpha_i < 1$, $\mu_i > 1$ be given constants, if there exist matrices $X_i > 0$, matrices Ω_i, positive constant ϵ_i for $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$, $m = 1, 2, \ldots, r_i$, $n = 1, 2, \ldots, r_i$, satisfying*

$$X_{jn} \leq \mu_i X_{im} \tag{22}$$

$$\Xi_{im} < 0 \tag{23}$$

where

$$\Xi_{im} = \begin{bmatrix} \frac{X_i}{1-\alpha_i} - \Omega_i^T - \Omega_i & \Omega_i^T A_{im}^T & 0 & \Omega_i^T N_{1i}^T \\ * & -X_i & \frac{M_i}{\epsilon_i} & 0 \\ * & * & -\frac{I}{\epsilon_i} & 0 \\ * & * & * & -\frac{I}{\epsilon_i} \end{bmatrix}$$

Then, the open-loop SUDFS (9) is GUAS for any switching signal satisfying (15).

Proof The proof is similar to the proof of Theorem 2 with the function $V_i(k)$ given by $V_i(k) = x(k)^T P_i x(k)$. It is omitted here. □

3.2 Asynchronous H_∞ Controller design

In this section, we are in a position to design the asynchronous H_∞ controller for closed-loop SUDFSs (9). Through the BLFs approach and QLFs approach, we obtain the admissible controller's existing conditions in the following theorem.

Theorem 3 *Consider closed-loop SUDFS (9), and let $0 < \alpha_i < 1$, $\beta_i > 0$, $\gamma > 0$ and $\mu_i \geq 1$ be given constants. If there exist matrices $P_{im} > 0$ for $\forall (i,j) \in \mathcal{I} \times \mathcal{I}$, $i \neq j$, $m = 1, 2, \ldots, r_i$, $n = 1, 2, \ldots, r_i$, $s = 1, 2, \ldots, r_i$ such that*

$$P_{im} \leq \mu_i P_{jn} \tag{24}$$

$$\Delta_{imn}^T \begin{bmatrix} P_{is} & 0 \\ 0 & I \end{bmatrix} \Delta_{imn} - \begin{bmatrix} (1-\alpha_i) P_{im} & 0 \\ 0 & \gamma^2 I \end{bmatrix} < 0 \tag{25}$$

$$\Psi_{imn}^T \begin{bmatrix} P_{is} & 0 \\ 0 & I \end{bmatrix} \Psi_{imn} - \begin{bmatrix} (1+\beta_i) P_{im} & 0 \\ 0 & \gamma^2 I \end{bmatrix} < 0 \tag{26}$$

where

$$\Delta_{imn} = (\Phi_{imn} + \Phi_{inm})/2, \Phi_{imn} = \begin{bmatrix} \bar{\mathscr{A}}_{imn} & E_{im} \\ \bar{C}_{imn} & F_{im} \end{bmatrix}, \Psi_{imn} = \begin{bmatrix} \hat{\mathscr{A}}_{imn} & E_{im} \\ \hat{C}_{imn} & F_{im} \end{bmatrix},$$

then closed-loop SUDFS (9) is GUAS for any switching signal satisfying (20) and has a weighted H_∞ performance index no greater than $\hat{\gamma} = \sqrt{\prod_{i\in\mathcal{I}} (\theta_i^{T_M}\mu_i)^{N_{0i}} \frac{\alpha_{\max}}{\alpha_{\min}} \theta_{\max}^{T_M-1}}\gamma$.

Proof We choose the BLFs which has been defined in Theorem 1 to analyze stability conditions and H_∞ performance of closed-loop SUDFS (9) here. Let $\xi_k = \begin{bmatrix} x_k^T & \omega_k^T \end{bmatrix}^T$. First we consider the synchronous switching situation, considering $k \in T[k_l + T_i, k_{l+1})$, from SUDFS (9) we have that:

$$
\begin{aligned}
&\Delta V_k + \alpha_i V_k + y_k^T y_k - \gamma^2 \omega_k^T \omega_k \\
&= \xi_k^T \sum_{s=1}^{r_i} h_{is} \sum_{n=1}^{r_i}\sum_{m=1}^{r_i}\sum_{p=1}^{r_i}\sum_{q=1}^{r_i} h_{im}h_{in}h_{ip}h_{iq} \\
&\quad \times \left(\begin{bmatrix} \bar{\mathscr{A}}_{imn}^T \\ E_{im}^T \end{bmatrix} P_{is} \begin{bmatrix} \bar{\mathscr{A}}_{ipq} & E_{ip} \end{bmatrix} + \begin{bmatrix} \bar{C}_{imn}^T \\ F_{im}^T \end{bmatrix} \begin{bmatrix} \bar{C}_{ipq} & F_{ip} \end{bmatrix} - \begin{bmatrix} (1-\alpha_i)P_{im} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right) \xi_k \\
&= \xi_k^T \sum_{s=1}^{r_i} h_{is} \sum_{n=1}^{r_i}\sum_{m=1}^{r_i}\sum_{p=1}^{r_i}\sum_{q=1}^{r_i} h_{im}h_{in}h_{ip}h_{iq} \\
&\quad \times \left(\begin{bmatrix} \bar{\mathscr{A}}_{imn}^T & \bar{C}_{imn}^T \\ E_{im}^T & F_{im}^T \end{bmatrix} \begin{bmatrix} P_{is} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{\mathscr{A}}_{ipq} & E_{ip} \\ \bar{C}_{ipq} & F_{ip} \end{bmatrix} - \begin{bmatrix} (1-\alpha_i)P_{im} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right) \xi_k.
\end{aligned}
$$

Let $\begin{bmatrix} \bar{\mathscr{A}}_{imn}^T & \bar{C}_{imn}^T \\ E_{im}^T & F_{im}^T \end{bmatrix} = \Phi_{imn}^T$, $\begin{bmatrix} \bar{\mathscr{A}}_{ipq} & E_{ip} \\ \bar{C}_{ipq} & F_{ip} \end{bmatrix} = \Phi_{ipq}$, the above equation can be rewritten as

$$
\begin{aligned}
&\Delta V_k + \alpha_i V_k + y_k^T y_k - \gamma^2 \omega_k^T \omega_k \\
&= \xi_k^T \sum_{s=1}^{r_i} h_{is} \sum_{n=1}^{r_i}\sum_{m=1}^{r_i}\sum_{p=1}^{r_i}\sum_{q=1}^{r_i} h_{im}h_{in}h_{ip}h_{iq} \\
&\quad \times \left(\Phi_{imn}^T \begin{bmatrix} P_{is} & 0 \\ 0 & I \end{bmatrix} \Phi_{ipq} - \begin{bmatrix} (1-\alpha_i)P_{im} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right) \xi_k \\
&= \xi_k^T \sum_{s=1}^{r_i} h_{is} \sum_{n=1}^{r_i}\sum_{m=1}^{r_i}\sum_{p=1}^{r_i}\sum_{q=1}^{r_i} \frac{1}{4} h_{im}h_{in}h_{ip}h_{iq} \\
&\quad \times \left((\Phi_{imn} + \Phi_{inm})^T \begin{bmatrix} P_{is} & 0 \\ 0 & I \end{bmatrix} (\Phi_{ipq} + \Phi_{iqp}) - 4 \begin{bmatrix} (1-\alpha_i)P_{im} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right) \xi_k \\
&= \xi_k^T \sum_{s=1}^{r_i} h_{is} \sum_{n=1}^{r_i}\sum_{m=1}^{r_i}\sum_{p=1}^{r_i}\sum_{q=1}^{r_i} \frac{1}{8} h_{im}h_{in}h_{ip}h_{iq} \\
&\quad \times \left((\Phi_{imn} + \Phi_{inm})^T \begin{bmatrix} P_{is} & 0 \\ 0 & I \end{bmatrix} (\Phi_{ipq} + \Phi_{iqp}) \right. \\
&\quad \left. - 8 \begin{bmatrix} (1-\alpha_i)P_{im} & 0 \\ 0 & \gamma^2 I \end{bmatrix} + (\Phi_{ipq} + \Phi_{iqp})^T \begin{bmatrix} P_{is} & 0 \\ 0 & I \end{bmatrix} (\Phi_{imn} + \Phi_{inm}) \right) \xi_k.
\end{aligned}
$$

From Lemma 2, we can get the following inequality:

$$
\begin{aligned}
&\Delta V_k + \alpha_i V_k + y_k^T y_k - \gamma^2 \omega_k^T \omega_k \\
&\leq \xi_k^T \sum_{s=1}^{r_i} h_{is} \sum_{n=1}^{r_i} \sum_{m=1}^{r_i} \sum_{p=1}^{r_i} \sum_{q=1}^{r_i} \frac{1}{8} h_{im} h_{in} h_{ip} h_{iq} \\
&\quad \times \Bigg((\Phi_{imn} + \Phi_{inm})^T \begin{bmatrix} P_{is} & 0 \\ 0 & I \end{bmatrix} (\Phi_{imn} + \Phi_{inm}) \\
&\quad -8 \begin{bmatrix} (1-\alpha_i) P_{im} & 0 \\ 0 & \gamma^2 I \end{bmatrix} + (\Phi_{ipq} + \Phi_{iqp})^T \begin{bmatrix} P_{is} & 0 \\ 0 & I \end{bmatrix} (\Phi_{ipq} + \Phi_{iqp}) \Bigg) \xi_k \\
&= \xi_k^T \sum_{s=1}^{r_i} h_{is} \sum_{n=1}^{r_i} \sum_{m=1}^{r_i} \frac{1}{4} h_{im} h_{in} \\
&\quad \times \left((\Phi_{imn} + \Phi_{inm})^T \begin{bmatrix} P_{is} & 0 \\ 0 & I \end{bmatrix} (\Phi_{imn} + \Phi_{inm}) - 4 \begin{bmatrix} (1-\alpha_i) P_{im} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right) \xi_k \\
&= \xi_k^T \sum_{s=1}^{r_i} h_{is} \sum_{n=1}^{r_i} \sum_{m=1}^{r_i} h_{im} h_{in} \\
&\quad \times \left(\Delta_{imn}^T \begin{bmatrix} P_{is} & 0 \\ 0 & I \end{bmatrix} \Delta_{inm} - \begin{bmatrix} (1-\alpha_i) P_{im} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right) \xi_k .
\end{aligned}
$$

From (25), we can know

$$
\Delta V_i(x_k) \leq -\alpha_i V_i(k) - \Gamma(k), \quad \forall k \in T[k_l + T_i, k_{l+1}). \tag{27}
$$

Then, we consider the asynchronous switching situation. That is $k \in T[k_l, k_l + T_i)$, supposing the ith subsystem is activated and the former one is the jth subsystem, by the same procedure, from (9) we can know:

$$
\begin{aligned}
&\Delta V_k - \beta_i V_k + y_k^T y_k - \gamma^2 \omega_k^T \omega_k \\
&= \xi_k^T \sum_{s=1}^{r_i} h_{is} \sum_{n=1}^{r_i} \sum_{m=1}^{r_i} \sum_{p=1}^{r_i} \sum_{q=1}^{r_i} h_{im} h_{jn} h_{ip} h_{jq} \\
&\quad \times \left(\begin{bmatrix} \hat{\mathscr{A}}_{imn}^T \\ E_{im}^T \end{bmatrix} P_{is} \begin{bmatrix} \hat{\mathscr{A}}_{ipq} & E_{ip} \end{bmatrix} + \begin{bmatrix} \hat{C}_{imn}^T \\ F_{im}^T \end{bmatrix} \begin{bmatrix} \hat{C}_{ipq} & F_{ip} \end{bmatrix} - \begin{bmatrix} (1+\beta_i) P_{im} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right) \xi_k \\
&= \xi_k^T \sum_{s=1}^{r_i} h_{is} \sum_{n=1}^{r_i} \sum_{m=1}^{r_i} \sum_{p=1}^{r_i} \sum_{q=1}^{r_i} h_{im} h_{jn} h_{ip} h_{jq} \\
&\quad \times \left(\begin{bmatrix} \hat{\mathscr{A}}_{imn}^T & \hat{C}_{imn}^T \\ E_{im}^T & F_{im}^T \end{bmatrix} \begin{bmatrix} P_{is} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{\mathscr{A}}_{ipq} & E_{ip} \\ \hat{C}_{ipq} & F_{ip} \end{bmatrix} - \begin{bmatrix} (1+\beta_i) P_{im} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right) \xi_k .
\end{aligned}
$$

Let $\begin{bmatrix} \hat{\mathscr{A}}_{imn}^T & \hat{C}_{imn}^T \\ E_{im}^T & F_{im}^T \end{bmatrix} = \Psi_{imn}^T$, $\begin{bmatrix} \hat{\mathscr{A}}_{ipq} & E_{ip} \\ \hat{C}_{ipq} & F_{ip} \end{bmatrix} = \Psi_{ipq}$, the above equation can be rewritten as

$$\begin{aligned}
&\Delta V_k - \beta_i V_k + y_k^T y_k - \gamma^2 \omega_k^T \omega_k \\
&= \xi_k^T \sum_{s=1}^{r_i} h_{is} \sum_{n=1}^{r_i} \sum_{m=1}^{r_i} \sum_{p=1}^{r_i} \sum_{q=1}^{r_i} h_{im} h_{jn} h_{ip} h_{jq} \\
&\quad \times \left(\Psi_{imn}^T \begin{bmatrix} P_{is} & 0 \\ 0 & I \end{bmatrix} \Psi_{ipq} - \begin{bmatrix} (1+\beta_i) P_{im} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right) \xi_k \\
&= \xi_k^T \sum_{s=1}^{r_i} h_{is} \sum_{n=1}^{r_i} \sum_{m=1}^{r_i} \sum_{p=1}^{r_i} \sum_{q=1}^{r_i} \frac{1}{2} h_{im} h_{jn} h_{ip} h_{jq} \\
&\quad \times \left(\Psi_{imn}^T \begin{bmatrix} P_{is} & 0 \\ 0 & I \end{bmatrix} \Psi_{ipq} + \Psi_{ipq}^T \begin{bmatrix} P_{is} & 0 \\ 0 & I \end{bmatrix} \Psi_{imn} - 2 \begin{bmatrix} (1+\beta_i) P_{im} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right) \xi_k.
\end{aligned}$$

From *Lemma* 2, we can get the following inequality:

$$\begin{aligned}
&\Delta V_k - \beta_i V_k + y_k^T y_k - \gamma^2 \omega_k^T \omega_k \\
&\leq \xi_k^T \sum_{s=1}^{r_i} h_{is} \sum_{n=1}^{r_i} \sum_{m=1}^{r_i} \sum_{p=1}^{r_i} \sum_{q=1}^{r_i} \frac{1}{2} h_{im} h_{jn} h_{ip} h_{jq} \\
&\quad \times \left(\Psi_{imn}^T \begin{bmatrix} P_{is} & 0 \\ 0 & I \end{bmatrix} \Psi_{imn} + \Psi_{ipq}^T \begin{bmatrix} P_{is} & 0 \\ 0 & I \end{bmatrix} \Psi_{ipq} - 2 \begin{bmatrix} (1+\beta_i) P_{im} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right) \xi_k \\
&= \xi_k^T \sum_{s=1}^{r_i} h_{is} \sum_{n=1}^{r_i} \sum_{m=1}^{r_i} h_{im} h_{jn} \\
&\quad \times \left(\Psi_{imn}^T \begin{bmatrix} P_{is} & 0 \\ 0 & I \end{bmatrix} \Psi_{imn} - \begin{bmatrix} (1+\beta_i) P_{im} & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right) \xi_k.
\end{aligned}$$

From (26), we can know

$$\Delta V_i(x_k) \leq \beta_i V_i(k) - \Gamma(k), \quad \forall k \in T[k_l, k_l + T_i). \tag{28}$$

By combining (27) and (28), we have (16), and from (24), we can get

$$V_i(k_l) \leq \mu_i V_j(k_l^-), \tag{29}$$

then from Lemma 4, we can know SUDFS (9) is GUAS for any switching signal satisfying (17) and has a weighted H_∞ performance index no greater than $\hat{\gamma}$. So SUDFS (9) is GUAS under asynchronous switching conditions. The proof is ended. □

Theorem 4 *Consider closed-loop SUDFS (9), and let* $0 < \alpha_i < 1$, $\beta_i > 0$, $\mu_i \geq 1$ *be the given constants. If there exist matrices* $X_{im} > 0$, *matrices* Ω_i *and* Y_{im}, *and positive* *constant ϵ_i such that for* $\forall (i, j) \in \mathcal{I} \times \mathcal{I}, i \neq j, m = 1, 2, \ldots, r_i, n = 1, 2, \ldots, r_i, s = 1, 2, \ldots, r_i$ *satisfying*

$$X_{jn} \leq \mu_i X_{im} \tag{30}$$

$$\varXi_{imns} < 0 \tag{31}$$

$$\varXi_{imjns} < 0 \tag{32}$$

where

$$\varXi_{imns} \triangleq \begin{bmatrix} \bar{Q}_{im} & 0 & \bar{M}_{imn} & \bar{N}_{imn} & 0 & \bar{P}_{imn} \\ * & -\gamma^2 I & \frac{E_{im}^T + E_{in}^T}{2} & \frac{F_{im}^T + F_{in}^T}{2} & 0 & 0 \\ * & * & -X_{is} & 0 & \frac{M_i}{\epsilon_i} & 0 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -\frac{I}{\epsilon_i} & 0 \\ * & * & * & * & * & -\frac{I}{\epsilon_i} \end{bmatrix},$$

$$\varXi_{imjns} \triangleq \begin{bmatrix} \hat{Q}imj & 0 & \hat{M}_{imjn} & \hat{N}_{imjn} & 0 & \hat{P}_{inj} \\ * & -\gamma^2 I & E_{im}^T & F_{im}^T & 0 & 0 \\ * & * & -X_{is} & 0 & \frac{M_i}{\epsilon_i} & 0 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -\frac{I}{\epsilon_i} & 0 \\ * & * & * & * & * & -\frac{I}{\epsilon_i} \end{bmatrix},$$

and

$$\begin{aligned}
\bar{Q}_{im} &= \frac{X_{im}}{1 - \alpha_i} - \Omega_i^T - \Omega_i, \\
\bar{P}_{imn} &= \frac{2\Omega_i^T N_{1i}^T + (Y_{im} + Y_{in})^T N_{2i}^T}{2}, \\
\bar{M}_{imn} &= \frac{\Omega_i^T (A_{im} + A_{in})^T + Y_{im}^T B_{in}^T + Y_{in}^T B_{im}^T}{2}, \\
\bar{N}_{imn} &= \frac{\Omega_i^T (C_{im} + C_{in})^T + Y_{im}^T D_{in}^T + Y_{in}^T D_{im}^T}{2}, \\
\hat{Q}_{imj} &= \frac{X_{im}}{1 + \beta_i} - \Omega_j^T - \Omega_j, \\
\hat{P}_{inj} &= \Omega_j^T N_{1i}^T + Y_{jn}^T N_{2i}^T, \\
\hat{M}_{imjn} &= \Omega_j^T A_{im}^T + Y_{jn}^T B_{im}^T, \\
\hat{N}_{imjn} &= \Omega_j^T C_{im}^T + Y_{jn}^T D_{im}^T,
\end{aligned}$$

then closed-loop SUDFS (9) is GUAS for any switching signal satisfying (17) and has a weighted H_∞ performance index no greater than $\hat{\gamma} = \sqrt{\prod_{i \in \mathcal{I}} (\theta_i^{T_M} \mu_i)^{N_{0i}} \frac{\alpha_{\max}}{\alpha_{\min}} \theta_{\max}^{T_M - 1}} \gamma$

with the asynchronous state feedback controller gains given by

$$\begin{cases} K_{jn} = Y_{jn}\Omega_j^{-1}, k \in [k_l, \bar{k}_l) \\ K_{in} = Y_{in}\Omega_i^{-1}, k \in [\bar{k}_l, k_{l+1}) \end{cases} \tag{33}$$

Proof Consider the closed-loop SUDFS(9), applying the Schur complement to (31) yields:

$$\begin{bmatrix} \bar{Q}_{im} & 0 & \bar{M}_{imn} & \bar{N}_{imn} \\ * & -\gamma^2 I & \frac{E_{im}^T+E_{in}^T}{2} & \frac{F_{im}^T+F_{in}^T}{2} \\ * & * & -X_{is} & 0 \\ * & * & * & -I \end{bmatrix} + \epsilon_i \begin{bmatrix} 0 \\ 0 \\ \frac{M_i}{\epsilon_i} \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{M_i}{\epsilon_i} \\ 0 \end{bmatrix}^T + \epsilon_i \begin{bmatrix} \bar{P}_{imn} \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \bar{P}_{imn} \\ 0 \\ 0 \\ 0 \end{bmatrix}^T < 0,$$

which is equal to

$$\begin{bmatrix} \bar{Q}_{im} & 0 & \bar{M}_{imn} & \bar{N}_{imn} \\ * & -\gamma^2 I & \frac{E_{im}^T+E_{in}^T}{2} & \frac{F_{im}^T+F_{in}^T}{2} \\ * & * & -X_{is} & 0 \\ * & * & * & -I \end{bmatrix} + \epsilon_i^{-1} \begin{bmatrix} 0 \\ 0 \\ M_i \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ M_i \\ 0 \end{bmatrix}^T + \epsilon_i \begin{bmatrix} \bar{P}_{imn} \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \bar{P}_{imn} \\ 0 \\ 0 \\ 0 \end{bmatrix}^T < 0,$$

from Lemma 5, we can know:

$$\begin{bmatrix} \frac{X_{im}}{1-\alpha_i} - \Omega_i^T - \Omega_i & 0 & \frac{\Omega_i^T(\mathscr{A}_{im}+\mathscr{A}_{in})^T + Y_{im}^T\mathscr{B}_{in}^T + Y_{in}^T\mathscr{B}_{im}^T}{2} & \bar{N}_{imn} \\ * & -\gamma^2 I & \frac{E_{im}^T+E_{in}^T}{2} & \frac{F_{im}^T+F_{in}^T}{2} \\ * & * & -X_{is} & 0 \\ * & * & * & -I \end{bmatrix} < 0,$$

which gives $0 < \frac{X_{im}}{1-\alpha_i} < \Omega_i^T + \Omega_i$, which implies that Ω_i is invertible. The inequalities $(\frac{X_{im}}{1-\alpha_i} - \Omega_i)^T (\frac{X_{im}}{1-\alpha_i})^{-1} (\frac{X_{im}}{1-\alpha_i} - \Omega_i) \geq 0$ imply that $\Omega_i^T (\frac{X_{im}}{1-\alpha_i})^{-1} \Omega_i \geq \Omega_i^T + \Omega_j - \frac{X_{im}}{1+\alpha_i}$, that is

$$\begin{bmatrix} -\Omega_i^T(\frac{X_{im}}{1-\alpha_i})^{-1}\Omega_i & 0 & \frac{\Omega_i^T(\mathscr{A}_{im}+\mathscr{A}_{in})^T + Y_{im}^T\mathscr{B}_{in}^T + Y_{in}^T\mathscr{B}_{im}^T}{2} & \bar{N}_{imn} \\ * & -\gamma^2 I & \frac{E_{im}^T+E_{in}^T}{2} & \frac{F_{im}^T+F_{in}^T}{2} \\ * & * & -X_{is} & 0 \\ * & * & * & -I \end{bmatrix} < 0,$$

Note that Ω_i is invertible. Pre-multiplying diag(Ω_i^{-T},I,I,I) and post-multiplying diag(Ω_i^{-1},I,I,I) to above inequality give

$$\begin{bmatrix} -(1-\alpha_i)X_{im}^{-1} & 0 & \frac{\bar{\mathscr{A}}_{imn}+\bar{\mathscr{A}}_{inm}}{2} & \frac{\bar{C}_{imn}+\bar{C}_{inm}}{2} \\ * & -\gamma^2 I & \frac{E_{im}^T+E_{in}^T}{2} & \frac{F_{im}^T+F_{in}^T}{2} \\ * & * & -X_{is} & 0 \\ * & * & * & -I \end{bmatrix} < 0,$$

Setting $X_{im}^{-1} = P_{im}$, we have

$$\begin{bmatrix} -(1-\alpha_i)P_{im} & 0 & \frac{\bar{\mathscr{A}}_{imn}+\bar{\mathscr{A}}_{inm}}{2} & \frac{\bar{C}_{imn}+\bar{C}_{inm}}{2} \\ * & -\gamma^2 I & \frac{E_{im}^T+E_{in}^T}{2} & \frac{F_{im}^T+F_{in}^T}{2} \\ * & * & -P_{is}^{-1} & 0 \\ * & * & * & -I \end{bmatrix} < 0,$$

It follows from Schur complement equivalence that we can get (25); similarly, (26) can be obtained by (32), and (28) is equal to (24):

$$X_{jm} \le \mu_i X_{in} \Leftrightarrow X_{in}^{-1} \le \mu_i X_{jm}^{-1} \Leftrightarrow P_{in} \le \mu_i P_{jm}.$$

Then, we can know the SUDFS (9) is GUAS with $\omega(k) = 0$ and has a weighted H_∞ performance index no greater than $\hat{\gamma}$ for any switching signals satisfying (17); the proof is ended. □

Remark 7 Theorem 4 can also apply to relevant problems for SUDFSs (9) without asynchronous behaviors, i.e., $T_M = 0$. For this case, if (40) in Theorem 4 is removed, we can easily conclude the SUDFS (9) is GUAS and has a weighted H_∞ performance index no greater than $\bar{\gamma} = \sqrt{\prod_{i\in\mathcal{I}} \mu_i^{N_{0i}} \frac{\alpha_{\max}}{\alpha_{\min}}}\gamma$ for any switching signal satisfying (15).

Remark 8 By setting $P_{im} = P_i$, we get the following Corollary 2 where the QLFs are used. So the LMIs condition in Theorem 4 has less conservatism than those in Corollary 2.

Corollary 2 *Consider SUDFS (9), and let* $0 < \alpha_i < 1$, $\beta_i > 0$, $\mu_i \ge 1$ *be the given constant. If there exist matrices* $X_i > 0$, *matrices* Ω_i *and* Y_{im} *such that* $\forall(i, j) \in \mathcal{I} \times \mathcal{I}, i \ne j$, $m = 1, 2, \ldots, r_i$, $n = 1, 2, \ldots, r_i$ *satisfying*

$$\begin{aligned} &X_j \le \mu_i X_i \\ &\Xi_{imn} < 0 \\ &\Xi_{imjn} < 0 \end{aligned}$$

where

$$\Xi_{imns} \triangleq \begin{bmatrix} \bar{Q}_i & 0 & \bar{M}_{imn} & \bar{N}_{imn} & 0 & \bar{P}_{imn} \\ * & -\gamma^2 I & \frac{E_{im}^T+E_{in}^T}{2} & \frac{F_{im}^T+F_{in}^T}{2} & 0 & 0 \\ * & * & -X_i & 0 & \frac{M_i}{\epsilon_i} & 0 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -\frac{I}{\epsilon_i} & 0 \\ * & * & * & * & * & -\frac{I}{\epsilon_i} \end{bmatrix},$$

$$\Xi_{imjns} \triangleq \begin{bmatrix} \hat{Q}_{ij} & 0 & \hat{M}_{imjn} & \hat{N}_{imjn} & 0 & \hat{P}_{inj} \\ * & -\gamma^2 I & E_{im}^T & F_{im}^T & 0 & 0 \\ * & * & -X_i & 0 & \frac{M_i}{\epsilon_i} & 0 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -\frac{I}{\epsilon_i} & 0 \\ * & * & * & * & * & -\frac{I}{\epsilon_i} \end{bmatrix},$$

and

$$\begin{aligned}
\bar{Q}_i &= \frac{X_i}{1-\alpha_i} - \Omega_i^T - \Omega_i, \\
\bar{P}_{imn} &= \frac{2\Omega_i^T N_{1i}^T + (Y_{im}+Y_{in})^T N_{2i}^T}{2}, \\
\bar{M}_{imn} &= \frac{\Omega_i^T (A_{im}+A_{in})^T + Y_{im}^T B_{in}^T + Y_{in}^T B_{im}^T}{2}, \\
\bar{N}_{imn} &= \frac{\Omega_i^T (C_{im}+C_{in})^T + Y_{im}^T D_{in}^T + Y_{in}^T D_{im}^T}{2}, \\
\hat{Q}_{ij} &= \frac{X_i}{1+\beta_i} - \Omega_j^T - \Omega_j, \\
\hat{P}_{inj} &= \Omega_j^T N_{1i}^T + Y_{jn}^T N_{2i}^T, \\
\hat{M}_{imjn} &= \Omega_j^T A_{im}^T + Y_{jn}^T B_{im}^T, \\
\hat{N}_{imjn} &= \Omega_j^T C_{im}^T + Y_{jn}^T D_{im}^T.
\end{aligned}$$

Then, the SUDFS (9) is GUAS for any switching signal satisfying (20) and has a weighted H_∞ performance $\hat{\gamma} = \sqrt{\prod_{i\in\mathcal{I}} (\theta_i^{T_M}\mu_i)^{N_{0i}} \frac{\alpha_{\max}}{\alpha_{\min}} \theta_{\max}^{T_M-1}}\gamma$ when the asynchronous state feedback controller gains are given by

$$\begin{cases} K_{jn} = Y_{jn}\Omega_j^{-1}, k \in [k_l, k_l+T_i) \\ K_{in} = Y_{in}\Omega_i^{-1}, k \in [k_l+T_i, k_{l+1}) \end{cases} \tag{34}$$

Proof The proof is similar to the proof of Theorem 3 and Theorem 4 with the function $V_i(k)$ given by $V_i(k) = x(k)^T P_i x(k)$. It is omitted here. □

4 Numerical Examples

Example 1 First, a numerical example is presented to show the effectiveness of our results. Consider the following closed-loop SUDFS consisting of two subsystems, and each subsystem has two fuzzy rules, where

$$A_{11} = \begin{bmatrix} 0.5 & -2 \\ 1 & 3 \end{bmatrix}, A_{12} = \begin{bmatrix} 0.3 & -1.5 \\ 1 & 2 \end{bmatrix}, A_{21} = \begin{bmatrix} 0.1 & 1.2 \\ 1 & -0.05 \end{bmatrix}, A_{22} = \begin{bmatrix} 0.5 & 1.3 \\ 1 & -0.15 \end{bmatrix},$$
$$B_{11} = \begin{bmatrix} 0.25 & -2 \\ -1 & 3 \end{bmatrix}, B_{12} = \begin{bmatrix} 0.2 & -2 \\ -1 & 5 \end{bmatrix}, B_{21} = \begin{bmatrix} 0.05 & -1.5 \\ -1 & -0.5 \end{bmatrix}, B_{22} = \begin{bmatrix} 0.06 & -1.5 \\ -1 & -0.4 \end{bmatrix},$$
$$C_{11} = \begin{bmatrix} 0.3 \; 0.5 \end{bmatrix}, C_{12} = \begin{bmatrix} 0.2 \; 0.2 \end{bmatrix}, C_{21} = \begin{bmatrix} 0.2 \; 0.2 \end{bmatrix}, C_{22} = \begin{bmatrix} 0.2 \; 0.2 \end{bmatrix},$$
$$D_{11} = \begin{bmatrix} 0.25 \; 0.25 \end{bmatrix}, D_{12} = \begin{bmatrix} 0.25 \; 0.25 \end{bmatrix}, D_{21} = \begin{bmatrix} 0.2 \; 0.2 \end{bmatrix}, D_{22} = \begin{bmatrix} 0.18 \; 0.22 \end{bmatrix},$$
$$E_{11} = \begin{bmatrix} 0.35 \\ 0.22 \end{bmatrix}, E_{12} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, E_{21} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, E_{22} = \begin{bmatrix} 0.12 \\ 0.08 \end{bmatrix},$$
$$F_{11} = 0.01, F_{12} = 0.05, F_{21} = 0.01, F22 = 0.01,$$
$$M_1 = \begin{bmatrix} 0.01 \; 0.08 \\ 0.02 \; 0.03 \end{bmatrix}, M_2 = \begin{bmatrix} 0.02 \; 0.03 \\ 0.5 \;\; 0.01 \end{bmatrix},$$
$$N_{11} = \begin{bmatrix} 0.02 \; 0.05 \\ 0.08 \; 0.01 \end{bmatrix}, N_{21} = \begin{bmatrix} 0.03 & 0.4 \\ 0.1 & 0.015 \end{bmatrix}, N_{12} = \begin{bmatrix} 0.01 & 0.15 \\ 0.02 & 0.025 \end{bmatrix}, N_{22} = \begin{bmatrix} 0.01 & 0.2 \\ 0.02 & 0.01 \end{bmatrix}.$$

The fuzzy membership functions are taken as $h_{11} = \cos^2(x_1+0.5)$, $h_{12} = 1-h_{11}$, $h_{21} = \sin^2(x_2+0.5)$, $h_{22} = 1-h_{21}$. According to Lemma 4, the initial conditions are set to be $x(k_0) = \begin{bmatrix} 0 \; 0 \end{bmatrix}^T$, and the disturbance input is assumed to be $\omega(k) = e^{-0.15k}$. Our purpose here is to use our results to design a state feedback controller such that the closed-loop SUDFS is GUAS and the H_∞ performance is achieved. We choose $N_{0i} = 0$, $\mu_1 = 1.1$, $\mu_2 = 1.2$, $\alpha_1 = 0.18$, $\alpha_2 = 0.17$, $\beta_1 = 0.05$, $\beta_2 = 0.09$ and assume that the controlled signal is synchronously with the switched signal, which means $T_M = 0$. From Theorem 4, we can get the synchronous state feedback controllers under the condition of minimizing γ

$$K_{11} = \begin{bmatrix} 0.6574 & 0.0374 \\ 0.0140 & -0.9205 \end{bmatrix}, K_{12} = \begin{bmatrix} 0.5757 & -0.7264 \\ -0.0108 & -0.5930 \end{bmatrix},$$
$$K_{21} = \begin{bmatrix} 0.8081 & -0.4152 \\ 0.0869 & 0.7235 \end{bmatrix}, K_{22} = \begin{bmatrix} 0.7081 & -0.4707 \\ 0.3339 & 0.7903 \end{bmatrix}.$$

At the same time, we can obtain $\bar{\gamma}_{\min} = 0.8799$, $T^*_{oa1} = 0.4803$, $T^*_{oa2} = 0.9785$. We assume that $T_{oa1} = 5 > 0.4803$, $T_{oa2} = 4 > 0.9785$, $T_M = 0$, the state response curve of the closed-loop synchronous switched system is shown in Fig. 2, from which we can know the state response converges to zero rapidly, and the weighted H_∞ performance is 0.3481 which is below $\bar{\gamma}_{\min}$. This shows the synchronous controller works well in synchronous conditions.

But in reality, the synchronous condition is quite ideal. It inevitably takes some time to identify the system modes and apply the matched controller; thus, the asyn-

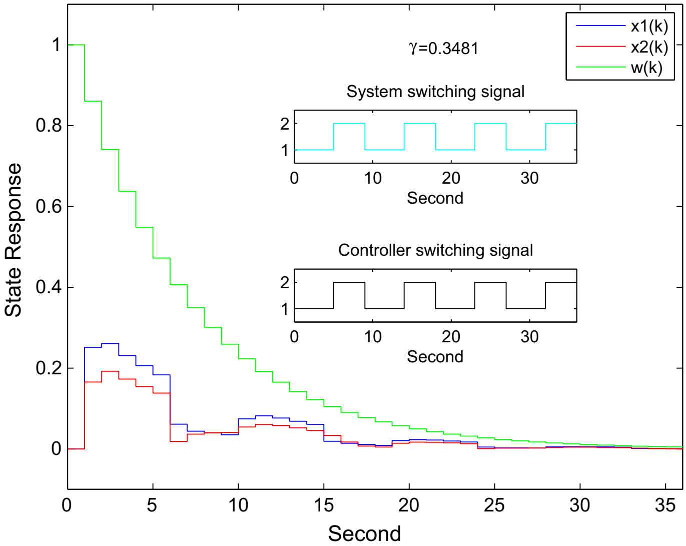

Fig. 2 State response of the closed-loop synchronous SUDFS with controller designed by synchronous conditions, $T_{oa1} = 5$, $T_{oa2} = 4$

chronous switching between the system and the controller generally exists. We assume that the controlled signal is asynchronous with the switched signal; subsystem 1's asynchronous time is $T_1 = 3$, subsystem 2's asynchronous time is $T_2 = 2$, obviously $T_M = 3$. Through Theorem 4, we can obtain $\hat{\gamma}_{\min} = 2.7896$, $T^*_{oa1} = 4.2178$, $T^*_{oa2} = 3.9035$ and the asynchronous controller

$$K_{11} = \begin{bmatrix} 1.1710 & 0.0941 \\ 0.0581 & -0.7210 \end{bmatrix}, K_{12} = \begin{bmatrix} 0.9238 & 0.0956 \\ -0.0287 & -0.5125 \end{bmatrix},$$
$$K_{21} = \begin{bmatrix} 0.9613 & 0.8267 \\ -0.0091 & -0.4261 \end{bmatrix}, K_{22} = \begin{bmatrix} 0.9358 & 0.7839 \\ -0.0164 & -0.4377 \end{bmatrix}.$$

We adopt two switching sequences which satisfy $T_{oa1} = 5 > 4.2178$, $T_{oa2} = 4 > 3.9035$ and $T_{oa1} = 6 > 4.2178$, $T_{oa2} = 5 > 3.9035$, respectively. First, we apply the synchronous controller to the given system. The state response curve under the two switching sequences is shown in Figs. 3 and 4, from which we can see in the case that $T_{oa1} = 5 > 4.2178$, $T_{oa2} = 4 > 3.9035$, the synchronous controller cannot stabilize the given system, and in the case that $T_{oa1} = 6 > 4.2178$, $T_{oa2} = 5 > 3.9035$, the synchronous controller can stabilize the given system but cannot guarantee the weighted H_∞ performance.

Then, we adopt the asynchronous switching controller instead of the synchronous one, as is shown in Figs. 5 and 6 that the state response of the closed-loop asynchronous switched system which utilizes the asynchronous controller converges to zero. The weighted H_∞ performance index is 0.3897 and 0.3840, which is much smaller than $\hat{\gamma}_{\min} = 2.7896$. The simulated results show that the controller designed by asyn-

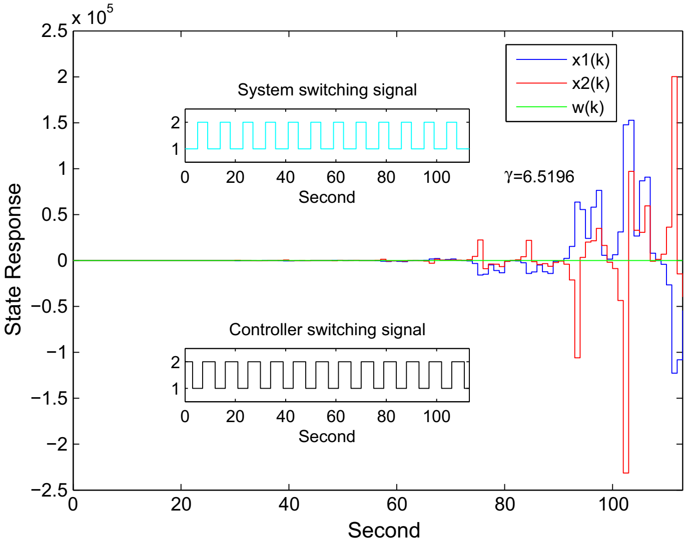

Fig. 3 State response of the closed-loop asynchronous SUDFS with controller designed by synchronous conditions, $T_{oa1} = 5$, $T_{oa2} = 4$

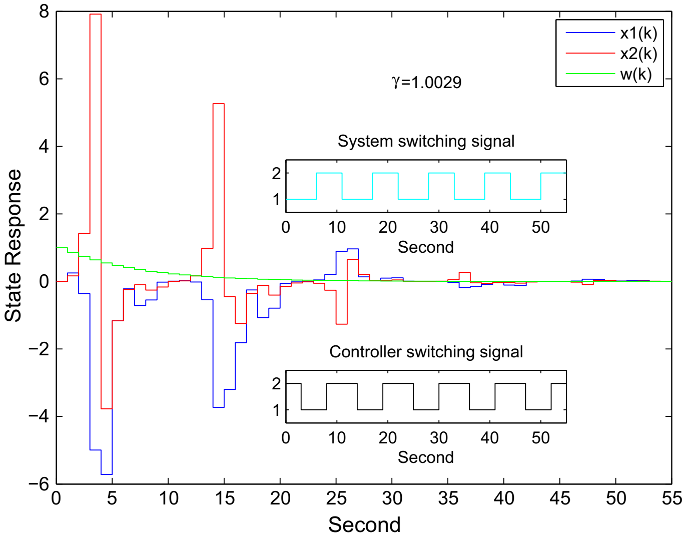

Fig. 4 State response of the closed-loop asynchronous SUDFS with controller designed by synchronous conditions, $T_{oa1} = 6$, $T_{oa2} = 5$

chronous conditions can stabilize the given system and guarantee good weighted H_∞ performance.

In Table 1, we change μ_1, μ_2, α_1 and α_2's values and let other parameters remain the same. For different T^*_{cai} sets, we compute the minimal value of $\hat{\gamma}_{\min}$ by BLFs approach and QLFs approach. In Table 1, we can see $\hat{\gamma}_{\min}$ is decreasing with μ_i increasing. This

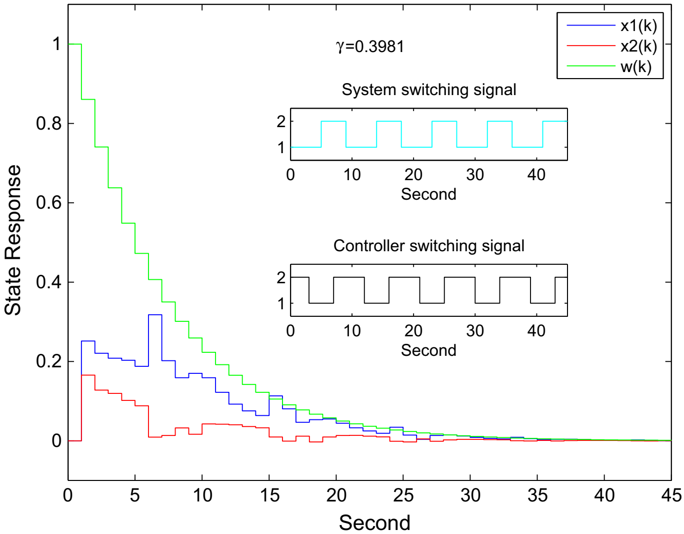

Fig. 5 State response of the closed-loop asynchronous SUDFS with controller designed by asynchronous conditions, $T_{oa1} = 5$, $T_{oa2} = 4$

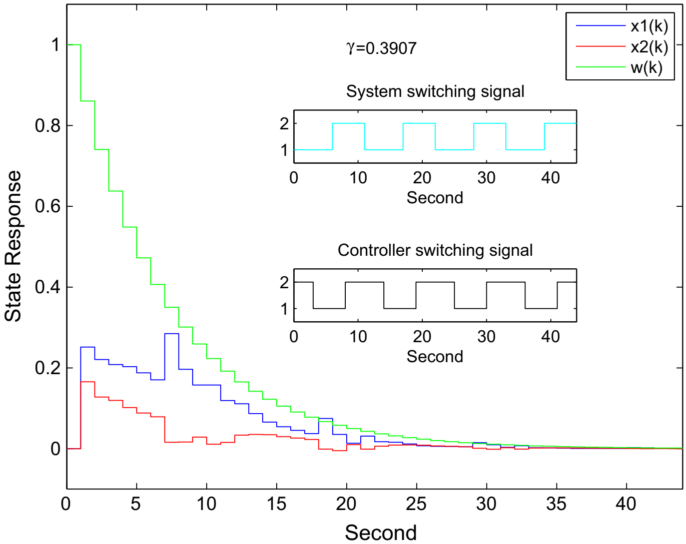

Fig. 6 State response of the closed-loop asynchronous SUDFS with controller designed by asynchronous conditions, $T_{oa1} = 6$, $T_{oa2} = 5$

is because a smaller μ_i will bring a smaller T^*_{oai} (faster switching), which will result in a worse system performance, i.e., a larger $\hat{\gamma}_{\min}$. However, it is also seen that although a smaller α_i brings a larger T^*_{oai}, $\hat{\gamma}_{\min}$ still may increase. This is because the change in α_i also influences the value of $\hat{\gamma}$'s weighted index $\sqrt{\frac{\alpha_{\max}}{\alpha_{\min}}\theta_{\max}^{T_M-1}}$ ($N_{0i} = 0$), and

Table 1 The minimal $\hat{\gamma}_{\min}$ of BLFs method and QLFs method for different T^*_{cai} sets

T^*_{ca1} values	4.2178	5.0596	5.0596	5.1936	5.1936
(μ_1 values)	(1.1)	(1.3)	(1.3)	(1.3)	(1.3)
(α_1 values)	(0.18)	(0.18)	(0.18)	(0.17)	(0.17)
T^*_{ca2} values	3.9035	3.9035	4.7308	4.7308	4.9184
(μ_2 values)	(1.2)	(1.2)	(1.4)	(1.4)	(1.4)
(α_2 values)	(0.17)	(0.17)	(0.17)	(0.17)	(0.16)
The $\hat{\gamma}_{\min}$ of BLFs method	2.7896	2.6885	2.5610	2.4524	2.4699
The $\hat{\gamma}_{\min}$ of QLFs method	2.7994	2.7083	2.5868	2.4763	2.4938

the change in μ_i does not influence the value of weighted index. Although we can obtain a smaller γ by letting α_i decrease, $\hat{\gamma}$ still may increase due to the change in weighted index. The same case will occur when β_i changes; for brevity, the simulated results for different β_i are omitted. We also found that the calculated $\hat{\gamma}_{\min}$ of BLFs method is smaller than the calculated $\hat{\gamma}_{\min}$ of QLFs method in the same condition. This proves that BLFs method has more flexibility than the QLFs method in designing H_∞ controller and improves the switched system's weighted H_∞ performance.

Example 2 A practical example is also presented to show the effectiveness of our approach. Let us consider a continuous stirred tank reactor (CSTR) where an exothermic, irreversible reaction of the form $A \to B$ happens. There exist two different feeding streams to feed the reactor, and these two feeding streams are selected by a selector [27]. Source 1 feeds pure species A at the flow rate $F_1 = 50$ L/min, concentration $C_{A_1} = 1.5$ mol/L and temperature $T_{A1} = 350$ K, and source 2 feeds pure species A at the flow rate F_2 200 L/min, concentration $C_{A_2} = 0.75$ mol/L and temperature $T_{A_2} = 350$ K, i.e., the reactor has two modes corresponding to the feeding stream. For each mode, the system model for the chemical process is given as follows [3,27]:

$$\begin{aligned} \dot{C}_A &= \tfrac{F_\sigma}{V}(C_{A_\sigma} - C_A) - k_0 e^{-E/RT_R} C_A \\ \dot{T}_R &= \tfrac{F_\sigma}{V}(T_{A_\sigma} - T_R) + \tfrac{(-\Delta H)}{\rho c_p} k_0 e^{-E/RT_R} C_A + \tfrac{Q_\sigma}{\rho c_p V} \end{aligned} \tag{35}$$

It is clear that the system (35) is a switched nonlinear system, where C_A denotes the concentration of the species A; T_R denotes the temperature of the reactor; Q_σ is the heat removed from the reactor; V is the volume of the reactor; k_0, E, ΔH are the pre-exponential constant, the activation energy and enthalpy of the reaction, respectively; c_p, ρ are the heat capacity and fluid density in the reactor, and $\sigma(t) \in \{1, 2\}$ is the switching signal which is a discrete variable. All process parameters have been given in [27]. By substituting all the process parameters into Eq. (35), we can get the following two subsystems:

Subsystem 1: ($\sigma = 1$)

$$\dot{C}_A = -0.0334 C_A - 1.2 \times 10^9 e^{-10000 T_R} C_A + 0.026386,$$

$$\dot{T_R} = -0.0334T_R + 2.4 \times 10^{11} e^{-10000T_R} C_A + 11.77684 + \frac{Q_\sigma}{23.9}$$

Subsystem 2: ($\sigma = 2$)

$$\dot{C_A} = -0.0167C_A - 1.2 \times 10^9 e^{-10000T_R} C_A + 0.0167,$$
$$\dot{T_R} = -0.0167T_R + 2.4 \times 10^{11} e^{-10000T_R} C_A + 5.177 + \frac{Q_\sigma}{23.9}$$

When $Q_\sigma = 0$, the switched nonlinear systems' two steady states: $(C_A, T_R)_1 = (0.57, 395.3)$ and $(C_A, T_R)_2 = (0.57, 395.3)$, can be obtained.

It has been shown in [27] that these subsystems are all unstable. Here we assume that the asynchronous phenomenon exists, and the asynchronous H_∞ control problem of the system (35) is considered. Similar to [14], for the convenience of research, we discretize the system (35) by Euler approximation; then, the system (35) is transformed into a discrete-time switched nonlinear system. The discrete-time switched nonlinear system can be represented by the following T-S fuzzy models:

Subsystem $\sigma(t)$

Rule 1: IF the concentration of the species A is $M_\sigma(x_1)$ (i.e., $x_1(t)$ is 0.57). THEN

$$\delta x(k+1) = (A_{\sigma 1} + M_\sigma F(k) N_{1\sigma})\delta x(k) + B_{\sigma 1}\delta u(k) + E_{\sigma 1}\omega(k),$$
$$\delta y(k) = C_{\sigma 1}\delta x(k) + D_{\sigma 1}\delta u(k) + F_{\sigma 1}\omega(k),$$

Rule 2: IF the concentration of the species A is $M_\sigma(x_1)$ (i.e., $x_1(t)$ is 0.738). THEN

$$\delta x(k+1) = (A_{\sigma 2} + M_\sigma F(k) N_{1\sigma})\delta x(k) + B_{\sigma 2}\delta u(k) + E_{\sigma 2}\omega(k),$$
$$\delta y(k) = C_{\sigma 2}\delta x(k) + D_{\sigma 2}\delta u(k) + F_{\sigma 2}\omega(k),$$

where $\sigma \in \{1, 2\}$ is the subsystem subscript, $x(k) = [C_A \quad T_R]^T$, $\delta x(k) = x(k) - x_d$, x_d is a stationary point of the discrete-time switched nonlinear system, $\delta u(k)$ is the asynchronous controller which need to be designed, and $\omega(k)$ is the disturbance input created by external disturbance. Due to change in environment, the system matrices have some uncertainties. The value of system matrices are given as follows:

$$A_{11} = \begin{bmatrix} 0.9995 & 0.0000 \\ 0.0248 & 1.0000 \end{bmatrix}, A_{12} = \begin{bmatrix} 0.9649 & 0.0000 \\ 6.9536 & 0.9999 \end{bmatrix}, A_{21} = \begin{bmatrix} 0.9997 & 0.0000 \\ 0.0248 & 1.0000 \end{bmatrix},$$
$$A_{22} = \begin{bmatrix} 0.9650 & 0.0000 \\ 6.9366 & 0.9999 \end{bmatrix}, M_1 = \begin{bmatrix} 0.01 & 0.08 \\ 0.02 & 0.03 \end{bmatrix}, N_{11} = \begin{bmatrix} 0.02 & 0.05 \\ 0.08 & 0.01 \end{bmatrix},$$
$$N_{12} = \begin{bmatrix} 0.01 & 0.15 \\ 0.02 & 0.025 \end{bmatrix}, B_{11} = B_{12} = B_{21} = B_{22} = \begin{bmatrix} 0 & 0.003 \\ -0.05 & 0 \end{bmatrix},$$
$$E_{11} = E_{12} = E_{21} = E_{22} = \begin{bmatrix} 0.06 \\ 0.6 \end{bmatrix}, C_{11} = C_{12} = C_{21} = C_{22} = \begin{bmatrix} 0 & 1 \end{bmatrix},$$
$$D_{11} = D_{12} = D_{21} = D_{21} = 0, F_{11} = F_{12} = F_{21} = F_{22} = 0,$$

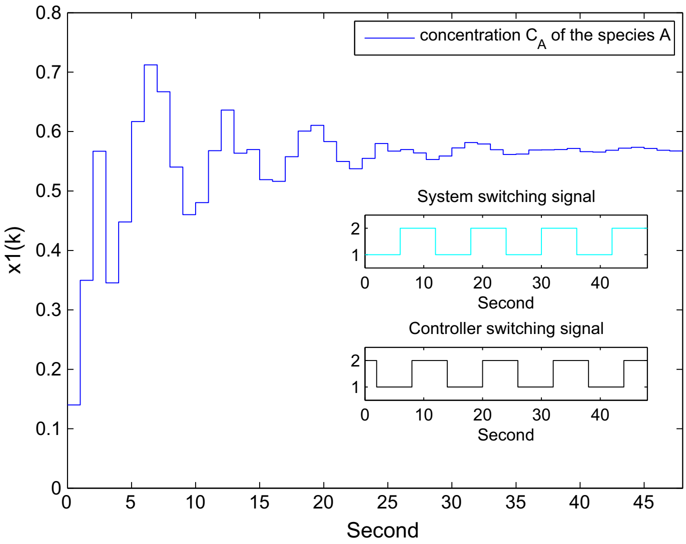

Fig. 7 State trajectory of $x_1(k)$

From above, we can see the temperature of the reactor T_R can be obtained directly by system output y(k). The given system can be seen as a SUDFS with asynchronous controller. The membership functions for Rules 1 and 2 for each individual system are taken as those of [3].

We set $N_{0i} = 0$, $\mu_1 = 1.2$, $\mu_2 = 1.2$, $\alpha_1 = 0.12$, $\alpha_2 = 0.12$, $\beta_1 = 0.14$, $\beta_2 = 0.14$ in Theorem 4, and assume that subsystem 1's asynchronous time is $T_1 = 3$, and subsystem 2's asynchronous time is $T_2 = 2$; then, $T_M = 3$. By using the MATLAB LMI toolbox and minimizing the value of γ, we can obtain $\hat{\gamma}_{\min} = 1.0163$, $T^*_{oa1} = 5.4762$, $T^*_{oa2} = 5.4762$, and the controller gain matrices are given as follows

$$K_{11} = \begin{bmatrix} 79.3614 & 15.7267 \\ -297.8623 & -6.1146 \end{bmatrix}, K_{12} = \begin{bmatrix} 100.5547 & 16.3494 \\ -277.5581 & -6.4698 \end{bmatrix},$$
$$K_{21} = \begin{bmatrix} 85.8395 & 16.3758 \\ -290.7458 & -5.7520 \end{bmatrix}, K_{22} = \begin{bmatrix} 109.1907 & 17.1505 \\ -244.3731 & -5.8429 \end{bmatrix}.$$

We assume that the initial states are $x(k_0) = [0.14 \quad 404.9]^T$ and the disturbance input is $\omega(k) = 5cos(k)e^{-0.1k}$; the system's switching signal satisfies $T_{oa1} = 6 > 5.4762$, $T_{oa2} = 6 > 5.4762$. Simulation results of the trajectory of the SUDFS are presented in Figs. 7 and 8, from which we can see although the asynchronous phenomenon and external disturbance exist, the states converge to the stationary point of the discrete-time switched nonlinear system gradually, which shows the designed asynchronous controller stabilizes the SUDFS and guarantees good H_∞ performance.

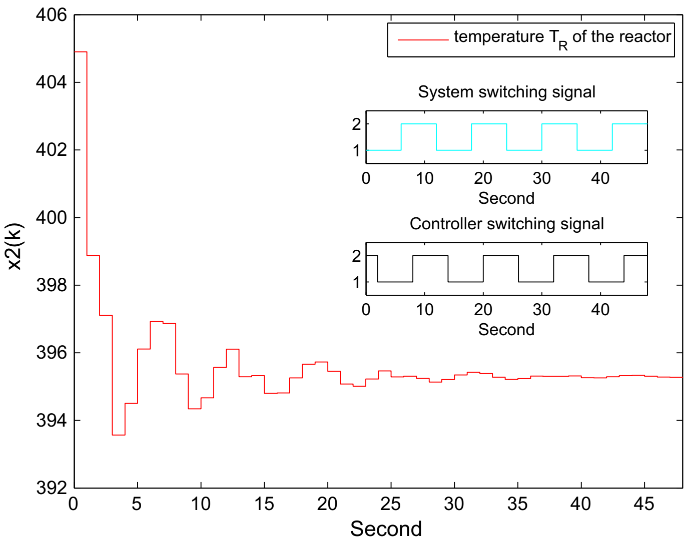

Fig. 8 State trajectory of $x_2(k)$

5 Conclusion

In this paper, we use the BLFs approach to solve asynchronous SUDFSs' stability issues and H_∞ controller problems. We obtain open-loop SUDFSs' robust stability conditions, and asynchronous state feedback H_∞ controller which can ensure the stability of asynchronous closed-loop SUDFSs with a weighted H_∞ performance is designed by solving LMIs. At last, both numerical and practical examples are given to show the importance of considering switched systems' asynchronous behavior and demonstrate the effectiveness of BLFs approach. The research results show that the asynchronous behavior may influence the system's performance or even cause the system's instability. In future work, via the BLFs approach, filter design problem of asynchronous SUDFSs will be analyzed. Characteristics of asynchronous SUDFSs with impulsive dynamical behaviors and time-varying delays will also be investigated.

References

1. M.S. Branicky, Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. IEEE Trans. Autom. Control. **43**(4), 475–482 (1998)
2. D. Choi, P. Park, H_∞ state-feedback controller design for discrete-time fuzzy sytems using fuzzy weighting-dependent Lyapunov functions. IEEE Trans. Fuzzy Syst. **11**, 271–278 (2003)
3. J.S. Chiou, C. Wang, C. Cheng, C. Wang, Analysis and synthesis of switched nonlinear systems using the T-S fuzzy model. Appl. Math. Model. **34**(6), 1467–1481 (2010)
4. J. Daafouz, R. Riedinger, C. Iung, Stability analysis and control synthesis for switched systems: a switched Lyapunov function approach. IEEE Trans. Autom. Control. **47**(4), 1883–1887 (2002)
5. G. Feng, S.G. Cao, N.W. Rees, C.K. Chak, Design of fuzzy control systems with guaranteed stability. Fuzzy Sets Syst. **85**(1), 1–10 (1995)
6. J.P. Hespanha, A.S. Morse, Stability of switched systems with average dwell-time, in *Proc. IEEE Conf. Decision and Control* (1999), pp. 2655-2660
7. D. Liberzon, *Switching in Systems and Control* (Birkhauser, Berlin, 2003)

8. D. Liberzon, A.S. Morse, Basic problems in stability and design of switched systems. IEEE Control Syst. Mag. **19**(5), 59–70 (1999)
9. J. Lian, F. Zhang, P. Shi, Sliding mode control of uncertain stochastic hybrid delay systems with average dwell time. Circuits Syst. Signal Process. **31**(2), 539–553 (2012)
10. D.H. Lee, J.B. Park, Y.H. Joo, A new fuzzy Lyapunov function for relaxed stability condition of continuous-time Takagi-Sugeno fuzzy systems. IEEE Trans. Fuzzy Syst. **19**(4), 785–791 (2011)
11. X. Liu, Q. Zhang, New approaches to H_∞ controller designs based on fuzzy observers for T-S fuzzy systems via LMI. Automatica **39**(5), 1571–1582 (2003)
12. Q. Lu, L. Zhang, H.R. Karimi, Y. Shi, H_∞ control for asynchronously switched linear parameter-varying systems with mode-dependent average dwell time. IET Control Theory Appl. **54**(7), 673–683 (2013)
13. Y. Mao, H. Zhang, S. Xu, The exponential stability and asynchronous stabilization of a class of switched nonlinear system via the T-S fuzzy model. IEEE Trans. Fuzzy Syst. **22**(4), 817–828 (2014)
14. Y. Mao, H. Zhang, Z. Zhang, Finite-time stabilization of discrete-time switched nonlinear systems without stable subsystems via switching signals design. IEEE Trans. Fuzzy Syst. (2016). doi:10.1109/TFUZZ.2016.2554139
15. Y. Mao, H. Zhang, Exponential stability and robust H_∞ control of a class of discrete-time switched non-linear systems with time-varying delays via T-S fuzzy model. Int. J. Syst. Sci. **45**(5), 1112–1127 (2014)
16. O. Ou, Y. Mao, H. Zhang, L. Zhang, Robust H_∞ control of a class of switching nonlinear systems with time-varying delay via T-S fuzzy model. Circuits Syst. Signal Process. **33**, 1411–1437 (2014)
17. H. Ohtake, K. Tanaka, H.O. Wang, Switching fuzzy controller design based on switching lyapunov function for a class of nonlinear systems. IEEE Trans. Syst. Man Cybern. **36**(1), 13–23 (2006)
18. I.R. Petersen, A stabilization algorithm for a class of uncertain linear systems. Syst. Control Lett. **8**, 351–357 (1987)
19. R. Sakthivel, M. Joby, P. Shi, K. Mathiyalagan, Robust reliable sampled-data control for switched systems with application to flight control. Int. J. Syst. Sci. **29**(5), 1–11 (2015)
20. S. Tong, L. Zhang, Y. Li, Observed-based adaptive fuzzy decentralized tracking control for switched uncertain nonlinear large-scale systems with dead zones. IEEE Trans. Syst. Man Cybern. **46**(1), 37–47 (2016)
21. T. Takagi, M. Sugeno, Fuzzy identification of systems and its application to modeling and control. IEEE Trans. Syst. Man Cybern. **SMC–15**(1), 116–132 (1985)
22. K. Tanaka, T. Ikeda, H.O. Wang, Fuzzy regulator and fuzzy observer: relaxed stability conditions and LMI-based designs. IEEE Trans. Fuzzy Syst. **6**(2), 250–265 (1998)
23. G. Wang, R. Xie, H. Zhang, G. Yu, C. Dang, Robust exponential H_∞ filtering for discrete-time switched fuzzy systems with time-varying delay. Circuits Syst. Signal Process. **35**(1), 117–138 (2016)
24. W. Xiang, J. Xiao, Convex sufficient conditions on asymptotic stability and l_2 gain performance for uncertain discrete-time switched linear systems. IET Control Theory Appl. **8**(3), 211–218 (2014)
25. D. Xie, H. Zhang, H. Zhang, B. Wang, Exponential stability of switched systems with unstable subsystems: a mode-dependent average dwell time approach. Circuits Syst. Signal Process. **32**(6), 3093–3105 (2013)
26. Y. Yin, X. Zhao, X. Zheng, New stability and stabilization conditions of switched systems with mode-dependent average dwell time. Circuits Syst. Signal Process. **36**(1), 1–17 (2016)
27. M.B. Yazdi, M.R. Jahed-Motlagh, S.A. Attia, J. Raisch, Modal exact linearization of a class of second-order switched nonlinear systems. Nonlinear Anal. Real World Appl. **11**(4), 2243–2252 (2010)
28. X. Zhao, L. Zhang, P. Shi, M. Liu, Stability and stabilization of switched linear systems with mode-dependent average dwell time. IEEE Trans. Autom. Control. **57**(7), 1809–1815 (2012)
29. S. Zhou, G. Feng, J. Lam, S. Xu, Robust H_∞ control for discrete-time fuzzy systems via basis-dependent Lyapunov functions. Inf. Sci. **174**, 197–217 (2004)
30. L. Zhang, H. Gao, Asynchronously switched control of switched systems with average dwell time. Automatica **46**(5), 953–958 (2010)
31. L. Zhang, P. Shi, Stability, l_2-gain and asynchronous H_∞ control of discrete-time switched systems with average dwell time. IEEE Trans. Autom. Control. **54**(9), 2193–2200 (2009)
32. H. Zhang, Y. Mao, Y. Gao, Exponential stability and asynchronous stabilization of switched systems with stable and unstable subsystems. Asian J. Control **15**(5), 1426–1433 (2013)
33. G. Zhai, B. Hu, K. Yasuda, A. Michel, Disturbance attenuation properties of time-controlled switched systems. J. Frankl. Inst. **338**(7), 765–779 (2001)