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# Improved Results on Delay-Dependent $H_\infty$ Control for Uncertain Systems with Time-Varying Delays

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**Abstract** This paper deals with the problem of delay-dependent robust $H_\infty$ control for uncertain systems with time-varying delays and norm-bounded parameter uncertainties. Firstly, some new delay-dependent stability criteria are proposed by exploiting a new Lyapunov–Krasovskii functional and free-weighting matrices method. Secondly, based on the criteria obtained, a delay-dependent criterion for the existence of a memoryless state feedback $H_\infty$ controller that ensures asymptotic stability and a prescribed $H_\infty$ performance level of the closed-loop system for all admissible uncertainties is proposed in terms of linear matrix inequalities (LMIs). These developed results enjoy much less conservatism than the existing ones due to the introduction of delay segmen-

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tation approach to estimate the upper bound of the derivative of Lyapunov functional without ignoring some useful terms that take into account information of the time-delay. Finally, numerical examples are provided to demonstrate the effectiveness and benefits of the proposed method.

**Keywords** $H_\infty$ control · Delay-dependent criterion · Lyapunov–Krasovskii functional · Uncertain system · Time-delay system

## 1 Introduction

During the past two decades, considerable attention has been devoted by the control community in devising techniques for ascertaining stability of dynamical systems with time-delays; refer [8,15,21,25,41], and the references cited therein. Time-delays are associated in various physical systems like communication systems, air-craft stabilization, nuclear reactors, population dynamics, ship stabilization and electric power systems with lossless transmission lines; these delays are time-varying in nature, and their presence in a system has an adverse impact not only on system performance, but also on its stability. Depending upon whether or not the stability criteria for a time-delay system contains the information of time-delay, the criteria can be classified, respectively, into two categories: namely, the delay-dependent stability criteria and delay-independent stability criteria. Since delay-dependent criteria make use of information on the length of the time-delay, they are less conservative than the delay-independent ones. Hence, researchers have focussed on the delay-dependent stability problem of time-delay systems, and many significant results have been reported in the recent literature [1,3,32,36,45,47,48]. Also, the parameter uncertainties are inherent features of many physical processes and often encountered in engineering systems; so, their presence must be considered in realistic dynamics.

In recent years, the $H_\infty$ control concept was proposed to reduce the effect of the disturbance input on the regulated output within a prescribed level and guarantee that the closed-loop system is stable [4,13,14,33,34,38]. More recently, $H_\infty$ control theory has been applied to an actual building in Tokyo, Japan, using a pair of mass dampers to reduce the bending-torsion motion due to earthquakes [6]. Further, a liquid monopropellant rocket motor with a pressure feeding system has been considered as a numerical design example in [16,17,30]. In [40,46], the problem of controlling the yaw angles of a satellite system with delays has been discussed. This satellite system consisting of two rigid bodies joined by a flexible link was assumed to have the state-space representations. Recently, $H_\infty$ optimal control techniques have been found to be an effective solution to treat robust stabilization and tracking problems. See [5,7,11,12,28,31,42]. An area of particular interest has been control of linear time-delay systems. Fridman and Shaked [5] proposed a descriptor model transformation of time-delay systems and used the bounding techniques from both Park and Moon et al. for $H_\infty$ controller design. Following the technique of Moon et al. [16], Gao and Wang [7] improved the results of [5]. In [12], Lee et al. proposed a delay-dependent robust $H_\infty$ control, which was less conservative than that in [5], for uncertain linear systems with state delay. Xu et al. [42] added null sum terms to Lyapunov functionals derivative and obtained less conservative results than those from previous methods due to the avoidance of using any bounding technology. Jiang and Han [11] concerned with the problem of robust $H_\infty$ control for systems with interval time-varying delay in a range by employing the free-weighting matrix method. In all previous results when estimating the upper bound of the derivative of Lyapunov functional, the derivative of $\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)Z\dot{x}(s)\mathrm{d}sd\theta$ is often estimated as $\tau\dot{x}^T(t)Z\dot{x}(t)-\int_{t-\tau}^{t}\dot{x}^T(s)Z\dot{x}(s)\mathrm{d}s$ and the term $-\int_{t-\tau}^{t-\tau(t)}\dot{x}^T(s)Z\dot{x}(s)\mathrm{d}s$ is neglected, which may lead to considerable conservativeness. In addition, the criteria in [11] are available only to systems with fast time-varying delay. In [28,31], authors used the bounding lemma to deal with the cross-terms. Therefore, there is room for further investigation. It is worth pointing out that, up until now, there have been few results considering delay-dependent robust $H_\infty$ control for uncertain nonlinear systems with time-varying delay and parameter uncertainties, which remains open but challenging.

In this paper, the problem of $H_\infty$ control for uncertain systems with time-varying delay is studied. Our aim is to design a memoryless state feedback control law such that the closed-loop system is robustly asymptotically stable and the effect of the disturbance input on the controlled output is less than a prescribed level for all admissible parameter uncertainties. It is noted that these improved criteria are derived without resorting to any model transformations and bounding techniques for some cross-terms by using Lyapunov–Krasovskii functional method and the free-weighting matrix method. Based on the improved criteria, a delay-dependent condition for the existence of a state feedback controller that ensures asymptotic stability and a prescribed $H_\infty$ performance level of the closed-loop system for all admissible uncertainties is obtained in terms of LMI. Finally, some illustrative examples are provided to show the effectiveness and advantages of the developed method.

**Notations** The following notations will be used throughout this paper. $\mathcal{R}^n$ and $\mathcal{R}^{n\times m}$ denote the $n$-dimensional Euclidean space and the set of all $n\times m$ real matrices, respectively. The notation $X\geq 0$ (respectively, $X>0$), where $X$ is symmetric matrices, means that $X$ is positive semidefinite (respectively, positive definite). The subscript $T$ denotes the transpose of the matrix. $L_2[0,\infty]$ is the space of square integrable vectors on $[0,\infty]$.

## 2 Problem Formulation

Consider a class of linear systems with time-varying delays and parameter uncertainties described by

$$\begin{aligned}\dot{x}(t)&=[A+\Delta A(t)]x(t)+[A_d+\Delta A_d(t)]x(t-\tau(t))+Bu(t)+B_w w(t),\\ z(t)&=Cx(t)+C_d x(t-\tau(t))+Du(t)+D_w w(t),\\ x(t)&=\phi(t),\forall\, t\in[-h,0],\end{aligned}\tag{1}$$

where $x(t)\in R^n$ is the state vector, $u(t)\in R^p$ is the input vector, $w(t)\in\mathcal{L}_2[0,\infty]$ is the exogenous disturbance signal, and $z(t)\in R^r$ is the controlled output. $A, A_d, B, B_w, C, C_d, D, D_w$ are known real constant matrices of appropriate dimensions. $\Delta A(t)$ and $\Delta A_d(t)$ are unknown real matrices of appropriate dimensions representing time-varying parameter uncertainties of system (1) and satisfy

$$\begin{bmatrix}\Delta A(t) & \Delta A_d(t)\end{bmatrix} = EF(t)\begin{bmatrix}G_1 & G_2\end{bmatrix} \tag{2}$$

where $E, G_1, G_2$ are known real constant matrices of appropriate dimensions, and $F(t) \in R^{l\times m}$ is an unknown matrix function with Lebesgue measurable elements satisfying

$$F^T(t)F(t) \leq I, \qquad \forall t \geq 0 \tag{3}$$

$\phi(t)$ is the initial condition of system (1). $\tau(t)$ is a continuous time-varying function satisfying

$$0 \leq \tau(t) \leq h, \quad \dot{\tau}(t) \leq \mu, \tag{4}$$

where $h$ and $\mu$ are known constants.

For a prescribed scalar $\gamma > 0$, we define the performance index as

$$J(t) = \int_0^{\infty} \left[z^T(s)z(s) - \gamma^2 w^T(s)w(s)\right] \mathrm{d}s. \tag{5}$$

We are interested in designing a memoryless state feedback controller

$$u(t) = Kx(t), \tag{6}$$

where $K \in R^{n\times m}$ is a constant matrix. The purpose of this paper is to develop a delay-dependent $H_\infty$ conditions such that, for all admissible uncertainties satisfying (2) and any $\tau(t)$ satisfying (4),

(i) The closed-loop system is asymptotically stable for all admissible uncertainties;
(ii) Under zero initial condition, the closed-loop system satisfies $\| z(t) \|_2 < \gamma \| w(t) \|_2$ for any nonzero $w(t) \in \mathcal{L}_2[0, \infty]$, where $\gamma > 0$ is a prescribed scalar.

The following lemmas are necessary in the proof of the main results.

**Lemma 2.1** ([9]). *For any positive definite matrix $M \in \mathcal{R}^{n\times n}$, scalars $h_2 > h_1 > 0$, vector function $w : [h_1, h_2] \to \mathcal{R}^n$ such that the integrations concerned are well defined, the following inequality holds:*

$$-(h_2 - h_1)\int_{t-h_2}^{t-h_1} w^T(s)Mw(s)ds \leq -\left(\int_{t-h_2}^{t-h_1} w(s)ds\right)^T M\left(\int_{t-h_2}^{t-h_1} w(s)ds\right),$$

$$-\frac{1}{2}(h_2^2 - h_1^2)\int_{-h_2}^{-h_1}\int_{t+\theta}^{t} w^T(s)Mw(s)dsd\theta$$
$$\leq -\left(\int_{-h_2}^{-h_1}\int_{t+\theta}^{t} w(s)dsd\theta\right)^T M\left(\int_{-h_2}^{-h_1}\int_{t+\theta}^{t} w(s)dsd\theta\right).$$

**Lemma 2.2** ([39]). *Given matrices* $Q = Q^T, H, E$ *with appropriate dimensions, then*

$$Q + HF(t)E + E^T F^T(t)H^T < 0$$

*for all* $F(t)$ *satisfying* $F^T(t)F(t) \le I$, *if and only if there exists a scalar* $\lambda > 0$ *such that*

$$Q + \lambda HH^T + \lambda^{-1}E^T E < 0.$$

**Lemma 2.3** (Schur complement [2]). *For a symmetric matrix*

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix},$$

*the following conditions are equivalent:*

*(1)* $S < 0$,
*(2)* $S_{11} < 0$, *and* $S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$,
*(3)* $S_{22} < 0$, *and* $S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

**Lemma 2.4** [44] *For any matrix* $\begin{bmatrix} M & S \\ * & M \end{bmatrix} \ge 0$, *scalars* $\tau > 0, \tau(t) > 0$ *satisfying* $0 < \tau(t) \le \tau$, *vector function* $\dot{x} : [-\tau, 0] \to R^n$ *such that the concerned integrations are well defined, then*

$$-\tau \int_{t-\tau}^{t} \dot{x}^T(\alpha)M\dot{x}(\alpha)d\alpha \le \varpi^T(t)\Omega\varpi(t),$$

*where*

$$\varpi(t) = \left[ x^T(t)\ x^T(t-\tau(t))\ x^T(t-\tau) \right]^T,$$

$$\Omega = \begin{bmatrix} -M & M-S & S \\ * & -2M+S+S^T & -S+M \\ * & * & -M \end{bmatrix}.$$

**Lemma 2.5** [44] *For any positive matrix Z, and for differentiable signal x in* $[\alpha, \beta] \to R^n$, *the following inequality holds:*

$$\int_{\alpha}^{\beta} \dot{x}^T(u)Z\dot{x}(u)du \ge \frac{1}{\beta-\alpha} \begin{bmatrix} x(\beta) \\ x(\alpha) \\ \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} x(u)du \end{bmatrix}^T \hat{\Omega} \begin{bmatrix} x(\beta) \\ x(\alpha) \\ \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta} x(u)du \end{bmatrix},$$

*where*

$$\hat{\Omega} = \begin{bmatrix} 4Z & 2Z & -6Z \\ * & 4Z & -6Z \\ * & * & 12Z \end{bmatrix}.$$

## 3 Main Results

Based on Lyapunov–Krasovskii functional approach, we first investigate a delay-dependent stability condition for the following nominal system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\tau(t)) + B_w w(t), \\ z(t) &= Cx(t) + C_d x(t-\tau(t)) + D_w w(t). \end{aligned} \tag{7}$$

**Theorem 3.1** *For given scalars $h > 0, \gamma > 0$ and $\mu > 0$, system (7) is asymptotically stable and satisfies $\| z(t) \|_2 < \gamma \| w(t) \|_2$ for any nonzero $w(t) \in \mathcal{L}_2[0,\infty)$ under the zero initial condition if there exist symmetric positive matrices $Q = [Q_{ij}]_{5\times 5}$, $Q_i (i = 1, 2, \ldots, 6)$, $\begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix}$, $Z_a = \begin{bmatrix} Z_1 & S \\ * & Z_1 \end{bmatrix}$, $Z_2$ and any matrices $P, M_i, H_i, L_i, N_i (i = 1, 2, \ldots, 14)$, such that the following LMI holds:*

$$\Omega = \begin{bmatrix} \Pi & \Gamma^T \\ * & -I \end{bmatrix} < 0, \tag{8}$$

*where*

$$\begin{aligned}
\Pi &= (\Pi_{ij})_{15\times 15}, \\
\Pi_{11} &= Q_{12} + Q_{12}^T - Q_{14} - Q_{14}^T + X_{11} + \left(\frac{h}{2}\right)^2 [Q_3 + Q_4] \\
&\quad - Z_1 - 4Z_2 + L_1 + L_1^T + \frac{h}{2}(H_1 + H_1^T) \\
&\quad + \frac{h}{2}(M_1 + M_1^T), \quad \Pi_{12} = Z_1 - S + L_2^T - L_1 + N_1 + \frac{h}{2}H_2^T + \frac{h}{2}M_2^T, \\
\Pi_{13} &= -Q_{12} + Q_{13} + Q_{14} - Q_{15} + X_{12} + L_3^T + \frac{h}{2}H_3^T + \frac{h}{2}M_3^T, \\
\Pi_{14} &= -Q_{13} + Q_{15} + S - 2Z_2 + L_4^T - N_1 + \frac{h}{2}H_4^T + \frac{h}{2}M_4^T, \\
\Pi_{15} &= Q_{11} + \frac{h}{2}Q_{14} + \frac{h}{2}Q_{15} + L_5^T + \frac{h}{2}H_5^T + \frac{h}{2}M_5^T + A^T P^T, \\
\Pi_{16} &= L_6^T + \frac{h}{2}H_6^T + \frac{h}{2}M_6^T, \quad \Pi_{17} = L_7^T + \frac{h}{2}H_7^T + \frac{h}{2}M_7^T, \\
\Pi_{18} &= Q_{22}^T - Q_{24}^T + L_8^T + \frac{h}{2}H_8^T - H_1 + \frac{h}{2}M_8^T, \\
\Pi_{19} &= Q_{23} - Q_{34}^T + L_9^T + \frac{h}{2}H_9^T + \frac{h}{2}M_9^T - M_1,
\end{aligned}$$

$$\Pi_{1_{10}} = Q_{24} - Q_{44}^T + L_{10}^T + \frac{h}{2}H_{10}^T - H_1 + \frac{h}{2}M_{10}^T,$$

$$\Pi_{1_{11}} = Q_{25} - Q_{45} + L_{11}^T + \frac{h}{2}H_{11}^T + \frac{h}{2}M_{11}^T - M_1,$$

$$\Pi_{1_{12}} = -L_1 + L_{12}^T + \frac{h}{2}H_{12}^T + \frac{h}{2}M_{12}^T, \quad \Pi_{1_{13}} = L_{13}^T - N_1 + \frac{h}{2}H_{13}^T + \frac{h}{2}M_{13}^T,$$

$$\Pi_{1_{14}} = \frac{6}{h}Z_2 + L_{14}^T + \frac{h}{2}H_{14}^T + \frac{h}{2}M_{14}^T,$$

$$\Pi_{22} = -2Z_1 + S + S^T - L_2 - L_2^T + N_2 + N_2^T,$$

$$\Pi_{23} = -L_3^T + N_3^T, \quad \Pi_{24} = -S + Z_1 - L_4^T + N_4^T - N_2,$$

$$\Pi_{25} = -L_5^T + N_5^T + A_d^T P^T,$$

$$\Pi_{26} = -L_6^T + N_6^T, \quad \Pi_{27} = -L_7^T + N_7^T, \quad \Pi_{28} = -L_8^T + N_8^T - H_2,$$

$$\Pi_{29} = -L_9^T + N_9^T - M_2,$$

$$\Pi_{2_{10}} = -L_{10}^T + N_{10}^T - H_2, \quad \Pi_{2_{11}} = -L_{11}^T + N_{11}^T - M_2,$$

$$\Pi_{2_{12}} = -L_2 - L_{12}^T + N_{12}^T,$$

$$\Pi_{2_{13}} = -L_{13}^T - N_2 + N_{13}^T, \quad \Pi_{2_{14}} = -L_{14}^T + N_{14}^T,$$

$$\Pi_{33} = X_{22} - X_{11}, \quad \Pi_{34} = -X_{12} - N_3,$$

$$\Pi_{38} = -Q_{22}^T + Q_{23}^T + Q_{24}^T - Q_{25}^T - H_3,$$

$$\Pi_{39} = -Q_{23} + Q_{33}^T + Q_{34}^T - Q_{35}^T - M_3,$$

$$\Pi_{3_{10}} = -Q_{24} + Q_{34} + Q_{44}^T - Q_{45}^T - H_3,$$

$$\Pi_{3_{11}} = -Q_{25} + Q_{35} + Q_{45} - Q_{55}^T - M_3, \quad \Pi_{3_{12}} = -L_3,$$

$$\Pi_{3_{13}} = -N_3, \quad \Pi_{44} = -X_{22} - Z_1 - 4Z_2 - N_4 - N_4^T, \quad \Pi_{45} = -N_5^T, \Pi_{46} = -N_6^T,$$

$$\Pi_{47} = -N_7^T, \quad \Pi_{48} = -Q_{23}^T + Q_{25}^T - N_8^T - H_4,$$

$$\Pi_{49} = -Q_{33}^T + Q_{35}^T - N_9^T - M_4,$$

$$\Pi_{4_{10}} = -Q_{34} + Q_{45}^T - N_{10}^T - H_4, \quad \Pi_{4_{11}} = -Q_{35} + Q_{55}^T - N_{11}^T - M_4,$$

$$\Pi_{4_{12}} = -L_4 - N_{12}^T,$$

$$\Pi_{4_{13}} = -N_4 - N_{13}^T, \quad \Pi_{4_{14}} = \frac{6}{h}Z_2 - N_{14}^T,$$

$$\Pi_{55} = \left(\frac{h}{2}\right)^2[Q_1 + Q_2] + \frac{h^4}{64}Q_5 + \frac{9h^4}{64}Q_6 + h^2(Z_1 + Z_2)$$

$$-P - P^T, \quad \Pi_{58} = Q_{12} + \frac{h}{2}Q_{24}^T + \frac{h}{2}Q_{25}^T - H_5,$$

$$\Pi_{59} = Q_{13} + \frac{h}{2}Q_{34}^T + \frac{h}{2}Q_{35}^T - M_5,$$

$$\Pi_{5_{10}} = Q_{14} + \frac{h}{2}Q_{44}^T + \frac{h}{2}Q_{45}^T - H_5,$$

$$\Pi_{5_{11}} = Q_{15} + \frac{h}{2}Q_{45} + \frac{h}{2}Q_{55}^T - M_5, \quad \Pi_{5_{12}} = -L_5,$$

$$\Pi_{5_{13}} = -N_5, \quad \Pi_{5_{15}} = PB_w, \quad \Pi_{66} = -Q_1, \quad \Pi_{68} = -H_6,$$

$$\Pi_{69} = -M_6, \quad \Pi_{6_{10}} = -H_6,$$
$$\Pi_{6_{11}} = -M_6, \quad \Pi_{6_{12}} = -L_6, \quad \Pi_{6_{13}} = -N_6, \quad \Pi_{77} = -Q_2,$$
$$\Pi_{78} = -H_7, \quad \Pi_{79} = -M_7,$$
$$\Pi_{7_{10}} = -H_7, \quad \Pi_{7_{11}} = -M_7, \quad \Pi_{7_{12}} = -L_7 - M_{12}^T,$$
$$\Pi_{7_{13}} = -N_7, \quad \Pi_{88} = -Q_3 - H_8 - H_8^T,$$
$$\Pi_{89} = -H_9^T - M_8, \quad \Pi_{8_{10}} = -H_{10}^T - H_8,$$
$$\Pi_{8_{11}} = -H_{11}^T - M_8, \quad \Pi_{8_{12}} = -L_8 - H_{12}^T,$$
$$\Pi_{8_{13}} = -N_8 - H_{13}^T, \quad \Pi_{8_{14}} = -H_{14}^T, \quad \Pi_{99} = -Q_4 - M_9 - M_9^T,$$
$$\Pi_{9_{10}} = -H_9 - M_{10}^T,$$
$$\Pi_{9_{11}} = -M_{11}^T - M_9, \quad \Pi_{9_{12}} = -L_9, \quad \Pi_{9_{13}} = -N_9 - M_{13}^T,$$
$$\Pi_{9_{14}} = -M_{14}^T, \quad \Pi_{10_{10}} = -Q_5 - H_{10} - H_{10}^T,$$
$$\Pi_{10_{11}} = -H_{11}^T - M_{10}, \quad \Pi_{10_{12}} = -L_{10} - H_{12}^T,$$
$$\Pi_{10_{13}} = -N_{10} - H_{13}^T,$$
$$\Pi_{10_{14}} = -H_{14}^T, \quad \Pi_{11_{11}} = -Q_6 - M_{11} - M_{11}^T, \quad \Pi_{11_{12}} = -L_{11} - M_{12}^T,$$
$$\Pi_{11_{13}} = -N_{11} - M_{13}^T,$$
$$\Pi_{11_{14}} = -M_{14}^T, \quad \Pi_{12_{12}} = -L_{12} - L_{12}^T, \quad \Pi_{12_{13}} = -L_{13}^T - N_{12},$$
$$\Pi_{12_{14}} = -L_{14}^T, \quad \Pi_{13_{13}} = -N_{13} - N_{13}^T,$$
$$\Pi_{13_{14}} = -N_{14}^T, \quad \Pi_{14_{14}} = -\frac{12}{h} Z_2, \quad \Pi_{15_{15}} = -\gamma^2 I,$$
$$\Gamma^T = \begin{bmatrix} C & C_d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_w \end{bmatrix}^T.$$

*Proof* Construct a Lyapunov functional candidate as follows:

$$V(x(t)) = V_1(x(t)) + V_2(x(t)) + V_3(x(t)) + V_4(x(t)) + V_5(x(t)) + V_6(x(t)), \quad (9)$$

where

$$V_1(x(t)) = \xi^T(t) Q \xi(t),$$
$$V_2(x(t)) = \int_{t-\frac{h}{2}}^{t} \begin{bmatrix} x(s) \\ x(s-\frac{h}{2}) \end{bmatrix}^T \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix} \begin{bmatrix} x(s) \\ x(s-\frac{h}{2}) \end{bmatrix} \mathrm{d}s,$$
$$V_3(x(t)) = \frac{h}{2} \int_{-\frac{h}{2}}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) Q_1 \dot{x}(s) \mathrm{d}s \mathrm{d}\theta + \frac{h}{2} \int_{-h}^{-\frac{h}{2}} \int_{t+\theta}^{t} \dot{x}^T(s) Q_2 \dot{x}(s) \mathrm{d}s \mathrm{d}\theta,$$
$$V_4(x(t)) = \frac{h}{2} \int_{-\frac{h}{2}}^{0} \int_{t+\theta}^{t} x^T(s) Q_3 x(s) \mathrm{d}s \mathrm{d}\theta + \frac{h}{2} \int_{-h}^{-\frac{h}{2}} \int_{t+\theta}^{t} x^T(s) Q_4 x(s) \mathrm{d}s \mathrm{d}\theta,$$
$$V_5(x(t)) = \frac{h^2}{8} \int_{-\frac{h}{2}}^{0} \int_{\theta}^{0} \int_{t+\lambda}^{t} \dot{x}^T(s) Q_5 \dot{x}(s) \mathrm{d}s \mathrm{d}\lambda \mathrm{d}\theta$$
$$+ \frac{3h^2}{8} \int_{-h}^{-\frac{h}{2}} \int_{\theta}^{0} \int_{t+\lambda}^{t} \dot{x}^T(s) Q_6 \dot{x}(s) \mathrm{d}s \mathrm{d}\lambda \mathrm{d}\theta,$$

$$V_6(x(t)) = h\int_{-h}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)(Z_1+Z_2)\dot{x}(s)\mathrm{d}s\mathrm{d}\theta,$$

where

$$\xi^T(t) = \Big[x^T(t) \quad \int_{t-\frac{h}{2}}^{t} x^T(s)\mathrm{d}s \quad \int_{t-h}^{t-\frac{h}{2}} x^T(s)\mathrm{d}s \quad \int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)\mathrm{d}s\mathrm{d}\theta \\ \int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t}\dot{x}^T(s)\mathrm{d}s\mathrm{d}\theta\Big].$$

The time derivative of $V(x_t)$ along the trajectory of system (7) is given by

$$\dot{V}(x(t)) = \dot{V}_1(x(t)) + \dot{V}_2(x(t)) + \dot{V}_3(x(t)) + \dot{V}_4(x(t)) + \dot{V}_5(x(t)) + \dot{V}_6(x(t)),$$

where

$$\begin{aligned}
\dot{V}_1(x(t)) &= 2\xi^T(t)Q\dot{\xi}(t) \\
&= 2\begin{bmatrix} x(t) \\ \int_{t-\frac{h}{2}}^{t} x(s)\mathrm{d}s \\ \int_{t-h}^{t-\frac{h}{2}} x(s)\mathrm{d}s \\ \int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta \\ \int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & Q_{15} \\ Q_{12}^T & Q_{22} & Q_{23} & Q_{24} & Q_{25} \\ Q_{13}^T & Q_{23}^T & Q_{33} & Q_{34} & Q_{35} \\ Q_{14}^T & Q_{24}^T & Q_{34}^T & Q_{44} & Q_{45} \\ Q_{15}^T & Q_{25}^T & Q_{35}^T & Q_{45}^T & Q_{55} \end{bmatrix} \\
&\quad\times \begin{bmatrix} \dot{x}(t) \\ x(t) - x(t-\frac{h}{2}) \\ x(t-\frac{h}{2}) - x(t-h) \\ \frac{h}{2}\dot{x}(t) - x(t) + x(t-\frac{h}{2}) \\ \frac{h}{2}\dot{x}(t) - x(t-\frac{h}{2}) + x(t-h) \end{bmatrix} \\
&= 2x^T(t)Q_{11}\dot{x}(t) + 2\dot{x}^T(t)Q_{12}\int_{t-\frac{h}{2}}^{t} x(s)\mathrm{d}s + 2\dot{x}^T(t)Q_{13}\int_{t-h}^{t-\frac{h}{2}} x(s)\mathrm{d}s \\
&\quad + 2\dot{x}^T(t)Q_{14}\int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta + 2\dot{x}^T(t)Q_{15}\int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta \\
&\quad + x^T(t)[Q_{12}+Q_{12}^T]x(t) + 2x^T(t)Q_{22}^T\int_{t-\frac{h}{2}}^{t} x(s)\mathrm{d}s + 2x^T(t)Q_{23}\int_{t-h}^{t-\frac{h}{2}} x(s)\mathrm{d}s \\
&\quad + 2x^T(t)Q_{24}\int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta + 2x^T(t)Q_{25}\int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta \\
&\quad - 2x^T(t)Q_{12}x\left(t-\frac{h}{2}\right) - 2x^T\left(t-\frac{h}{2}\right)Q_{22}^T\int_{t-\frac{h}{2}}^{t} x(s)\mathrm{d}s \\
&\quad - 2x^T\left(t-\frac{h}{2}\right)Q_{23}\int_{t-h}^{t-\frac{h}{2}} x(s)\mathrm{d}s
\end{aligned}$$

$$-2x^T\left(t-\frac{h}{2}\right)Q_{24}\int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta-2x^T\left(t-\frac{h}{2}\right)Q_{25}\int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta$$

$$+2x^T(t)Q_{13}x\left(t-\frac{h}{2}\right)+2x^T\left(t-\frac{h}{2}\right)Q_{23}^T\int_{t-\frac{h}{2}}^{t}x(s)\mathrm{d}s$$

$$+2x^T\left(t-\frac{h}{2}\right)Q_{33}^T\int_{t-h}^{t-\frac{h}{2}}x(s)\mathrm{d}s$$

$$+2x^T\left(t-\frac{h}{2}\right)Q_{34}\int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta+2x^T\left(t-\frac{h}{2}\right)Q_{35}\int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta$$

$$-2x^T(t)Q_{13}x(t-h)-2x^T(t-h)Q_{23}^T\int_{t-\frac{h}{2}}^{t}x(s)\mathrm{d}s$$

$$-2x^T(t-h)Q_{33}^T\int_{t-h}^{t-\frac{h}{2}}x(s)\mathrm{d}s$$

$$-2x^T(t-h)Q_{34}\int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta-2x^T(t-h)Q_{35}\int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta$$

$$+2\frac{h}{2}x^T(t)Q_{14}\dot{x}(t)+2\frac{h}{2}\dot{x}^T(t)Q_{24}^T\int_{t-\frac{h}{2}}^{t}x(s)\mathrm{d}s+2\frac{h}{2}\dot{x}^T(t)Q_{34}^T\int_{t-h}^{t-\frac{h}{2}}x(s)\mathrm{d}s$$

$$+2\frac{h}{2}\dot{x}^T(t)Q_{44}^T\int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta+2\frac{h}{2}\dot{x}^T(t)Q_{45}\int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta$$

$$+x^T(t)[-Q_{14}-Q_{14}^T]x(t)-2x^T(t)Q_{24}^T\int_{t-\frac{h}{2}}^{t}x(s)\mathrm{d}s-2x^T(t)Q_{34}^T\int_{t-h}^{t-\frac{h}{2}}x(s)\mathrm{d}s$$

$$-2x^T(t)Q_{44}^T\int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s d\theta-2x^T(t)Q_{45}\int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta$$

$$+2x^T(t)Q_{14}x\left(t-\frac{h}{2}\right)+2x^T\left(t-\frac{h}{2}\right)Q_{24}^T\int_{t-\frac{h}{2}}^{t}x(s)\mathrm{d}s$$

$$+2x^T\left(t-\frac{h}{2}\right)Q_{34}^T\int_{t-h}^{t-\frac{h}{2}}x(s)\mathrm{d}s$$

$$+2x^T\left(t-\frac{h}{2}\right)Q_{44}^T\int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta+2x^T\left(t-\frac{h}{2}\right)Q_{45}\int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta$$

$$+2\frac{h}{2}x^T(t)Q_{15}\dot{x}(t)+2\frac{h}{2}\dot{x}^T(t)Q_{25}^T\int_{t-\frac{h}{2}}^{t}x(s)\mathrm{d}s+2\frac{h}{2}\dot{x}^T(t)Q_{35}^T\int_{t-h}^{t-\frac{h}{2}}x(s)\mathrm{d}s$$

$$+2\frac{h}{2}\dot{x}^T(t)Q_{45}^T\int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta+2\frac{h}{2}\dot{x}^T(t)Q_{55}^T\int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta$$

$$-2x^T(t)Q_{15}x\left(t-\frac{h}{2}\right)-2x^T\left(t-\frac{h}{2}\right)Q_{25}^T\int_{t-\frac{h}{2}}^{t}x(s)\mathrm{d}s$$

$$
\begin{aligned}
&-2x^T\left(t-\frac{h}{2}\right)Q_{35}^T\int_{t-h}^{t-\frac{h}{2}}x(s)\mathrm{d}s\\
&-2x^T\left(t-\frac{h}{2}\right)Q_{45}^T\int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\\
&-2x^T\left(t-\frac{h}{2}\right)Q_{55}^T\int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\\
&+2x^T(t)Q_{15}x(t-h)+2x^T(t-h)Q_{25}^T\int_{t-\frac{h}{2}}^{t}x(s)\mathrm{d}s\\
&+2x^T(t-h)Q_{35}^T\int_{t-h}^{t-\frac{h}{2}}x(s)\mathrm{d}s\\
&+2x^T(t-h)Q_{45}^T\int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta+2x^T(t-h)Q_{55}^T\int_{-h}^{-\frac{h}{2}}\\
&\times\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta,
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
\dot{V}_2(x(t))=&\begin{bmatrix}x(t)\\x(t-\frac{h}{2})\end{bmatrix}^T\begin{bmatrix}X_{11}&X_{12}\\ *&X_{22}\end{bmatrix}\begin{bmatrix}x(t)\\x(t-\frac{h}{2})\end{bmatrix}\\
&-\begin{bmatrix}x(t-\frac{h}{2})\\x(t-h)\end{bmatrix}^T\begin{bmatrix}X_{11}&X_{12}\\ *&X_{22}\end{bmatrix}\begin{bmatrix}x(t-\frac{h}{2})\\x(t-h)\end{bmatrix},
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\dot{V}_3(x(t))=&\left(\frac{h}{2}\right)^2\dot{x}^T(t)[Q_1+Q_2]\dot{x}(t)-\frac{h}{2}\int_{t-\frac{h}{2}}^{t}\dot{x}^T(s)Q_1\dot{x}(s)\mathrm{d}s\\
&-\frac{h}{2}\int_{t-h}^{t-\frac{h}{2}}\dot{x}^T(s)Q_2\dot{x}(s)\mathrm{d}s,
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\dot{V}_4(x(t))=&\left(\frac{h}{2}\right)^2x^T(t)[Q_3+Q_4]x(t)-\frac{h}{2}\int_{t-\frac{h}{2}}^{t}x^T(s)Q_3x(s)\mathrm{d}s\\
&-\frac{h}{2}\int_{t-h}^{t-\frac{h}{2}}x^T(s)Q_4x(s)\mathrm{d}s,
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\dot{V}_5(x(t))=&\dot{x}^T(t)\left[\frac{h^4}{64}Q_5+\frac{9h^4}{64}Q_6\right]\dot{x}(t)-\frac{h^2}{8}\int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)Q_5\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\\
&-\frac{3h^2}{8}\int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t}\dot{x}^T(s)Q_6\dot{x}(s)\mathrm{d}s\mathrm{d}\theta,
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
\dot{V}_6(x(t))=&h^2\dot{x}^T(t)[Z_1+Z_2]\dot{x}(t)-h\int_{t-h}^{t}\dot{x}^T(s)Z_1\dot{x}(s)\mathrm{d}s\\
&-h\int_{t-h}^{t}\dot{x}^T(s)Z_2\dot{x}(s)\mathrm{d}s.
\end{aligned}
\tag{15}
$$

It follows from Lemma 2.1 that

$$-\frac{h}{2}\int_{t-\frac{h}{2}}^{t}\dot{x}^T(s)Q_1\dot{x}(s)\mathrm{d}s \le -\left(\int_{t-\frac{h}{2}}^{t}\dot{x}(s)\mathrm{d}s\right)^T Q_1\left(\int_{t-\frac{h}{2}}^{t}\dot{x}(s)\mathrm{d}s\right), \tag{16}$$

$$-\frac{h}{2}\int_{t-h}^{t-\frac{h}{2}}\dot{x}^T(s)Q_2\dot{x}(s)\mathrm{d}s \le -\left(\int_{t-h}^{t-\frac{h}{2}}\dot{x}(s)\mathrm{d}s\right)^T Q_2\left(\int_{t-h}^{t-\frac{h}{2}}\dot{x}(s)\mathrm{d}s\right), \tag{17}$$

$$-\frac{h}{2}\int_{t-\frac{h}{2}}^{t}x^T(s)Q_3x(s)\mathrm{d}s \le -\left(\int_{t-\frac{h}{2}}^{t}x(s)\mathrm{d}s\right)^T Q_3\left(\int_{t-\frac{h}{2}}^{t}x(s)\mathrm{d}s\right), \tag{18}$$

$$-\frac{h}{2}\int_{t-h}^{t-\frac{h}{2}}x^T(s)Q_4x(s)\mathrm{d}s \le -\left(\int_{t-h}^{t-\frac{h}{2}}x(s)\mathrm{d}s\right)^T Q_4\left(\int_{t-h}^{t-\frac{h}{2}}x(s)\mathrm{d}s\right), \tag{19}$$

$$-\frac{h^2}{8}\int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)Q_5\dot{x}(s)\mathrm{d}s\mathrm{d}\theta \le -\left(\int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\right)^T Q_5\left(\int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\right), \tag{20}$$

$$-\frac{3h^2}{8}\int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t}\dot{x}^T(s)Q_6\dot{x}(s)\mathrm{d}s\mathrm{d}\theta \le -\left(\int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\right)^T Q_6\left(\int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t}\dot{x}(s)\mathrm{d}s\mathrm{d}\theta\right). \tag{21}$$

By Lemma 2.4, we get

$$\begin{aligned}&-h\int_{t-h}^{t}\dot{x}^T(s)Z_1\dot{x}(s)\mathrm{d}s\\ &\le \begin{bmatrix} x(t)\\ x(t-\tau(t))\\ x(t-h)\end{bmatrix}^T \begin{bmatrix} -Z_1 & Z_1-S & S\\ * & -2Z_1+S+S^T & -S+Z_1\\ * & * & -Z_1\end{bmatrix}\begin{bmatrix} x(t)\\ x(t-\tau(t))\\ x(t-h)\end{bmatrix}.\end{aligned} \tag{22}$$

By Lemma 2.5, we get

$$\begin{aligned}&-h\int_{t-h}^{t}\dot{x}^T(s)Z_2\dot{x}(s)\mathrm{d}s\\ &\le -\begin{bmatrix} x(t)\\ x(t-h)\\ \frac{1}{h}\int_{t-h}^{t}x(s)\mathrm{d}s\end{bmatrix}^T \begin{bmatrix} 4Z_2 & 2Z_2 & -6Z_2\\ * & 4Z_2 & -6Z_2\\ * & * & 12Z_2\end{bmatrix}\begin{bmatrix} x(t)\\ x(t-h)\\ \frac{1}{h}\int_{t-h}^{t}x(s)ds\end{bmatrix}.\end{aligned} \tag{23}$$

From the Leibniz–Newton formula, the following equations are true for any matrices $M, N, L, H, P$ with appropriate dimensions:

$$0 = 2\zeta^T(t)L\left[x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t} \dot{x}(s)\mathrm{d}s\right], \tag{24}$$

$$0 = 2\zeta^T(t)N\left[x(t-\tau(t)) - x(t-h) - \int_{t-h}^{t-\tau(t)} \dot{x}(s)\mathrm{d}s\right] = 0, \tag{25}$$

$$0 = 2\zeta^T(t)H\left[\frac{h}{2}x(t) - \int_{t-\frac{h}{2}}^{t} x(s)\mathrm{d}s - \int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t} \dot{x}(s)\mathrm{d}s\mathrm{d}\theta\right] = 0, \tag{26}$$

$$0 = 2\zeta^T(t)M\left[\frac{h}{2}x(t) - \int_{t-h}^{t-\frac{h}{2}} x(s)\mathrm{d}s - \int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t} \dot{x}(s)\mathrm{d}s\mathrm{d}\theta\right] = 0, \tag{27}$$

$$0 = 2\dot{x}^T(t)P\left[-\dot{x}(t) + Ax(t) + A_d x(t-\tau(t)) + B_w w(t)\right] = 0, \tag{28}$$

where

$$\begin{aligned}\zeta^T(t) = \Big[ & x^T(t) \quad x^T(t-\tau(t)) \quad x^T\left(t-\frac{h}{2}\right) \quad x^T(t-h) \quad \dot{x}^T(t) \quad \int_{t-\frac{h}{2}}^{t} \dot{x}^T(s)\mathrm{d}s \\ & \int_{t-h}^{t-\frac{h}{2}} \dot{x}^T(s)\mathrm{d}s \int_{t-\frac{h}{2}}^{t} x^T(s)\mathrm{d}s \quad \int_{t-h}^{t-\frac{h}{2}} x^T(s)\mathrm{d}s \quad \int_{-\frac{h}{2}}^{0}\int_{t+\theta}^{t} \dot{x}^T(s)\mathrm{d}s\mathrm{d}\theta \\ & \int_{-h}^{-\frac{h}{2}}\int_{t+\theta}^{t} \dot{x}^T(s)\mathrm{d}s\mathrm{d}\theta \\ & \int_{t-\tau(t)}^{t} \dot{x}^T(s)\mathrm{d}s \quad \int_{t-h}^{t-\tau(t)} \dot{x}^T(s)\mathrm{d}s \quad \int_{t-h}^{t} x^T(s)\mathrm{d}s \quad w^T(t)\Big].\end{aligned} \tag{29}$$

Combining from (10) to (28) together, we have

$$\dot{V}(x(t)) \leq \zeta(t)\Omega_1\zeta(t) - z^T(t)z(t) + \gamma^2 w^T(t)w(t), \tag{30}$$

where $\Omega_1 = \Pi + \Gamma^T\Gamma^T$. $\Pi$ and $\Gamma$ are defined in Theorem 3.1. □

Based on (8) and by Schur complement, (30) implies that

$$\dot{V}(x(t)) \leq -z^T(t)z(t) + \gamma^2 w^T(t)w(t). \tag{31}$$

Integrating both sides of (31) from $t_0$ to $t$, we obtain

$$V(t) - V(t_0) \leq -\int_{t_0}^{t} z^T(s)z(s)\mathrm{d}s + \int_{t_0}^{t} \gamma^2 w^T(s)w(s)\mathrm{d}s. \tag{32}$$

Then, letting $t \to \infty$ and under zero initial condition, we obtain from (32) that

$$\int_{t_0}^{\infty} z^T(s)z(s)\mathrm{d}s \leq \int_{t_0}^{\infty} \gamma^2 w^T(s)w(s)\mathrm{d}s.$$

Therefore, $J(t) < 0$, and hence $\| z(t) \|_2 \leq \gamma \| w(t) \|_2$ is satisfied for any nonzero $w(t) \in \mathcal{L}_2[0, \infty)$.

*Remark 1* To reduce the conservatism, Lemma 2.1 is used to deal with the derivative of $\dot{V}_5(x(t))$, i.e., $-\frac{h^2}{8} \int_{-\frac{h}{2}}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) Q_5 \dot{x}(s) \mathrm{d}s\mathrm{d}\theta$ is bounded with $-\left( \int_{-\frac{h}{2}}^{0} \int_{t+\theta}^{t} \dot{x}(s)\mathrm{d}s\mathrm{d}\theta \right)^T Q_5 \left( \int_{-\frac{h}{2}}^{0} \int_{t+\theta}^{t} \dot{x}(s)\mathrm{d}s\mathrm{d}\theta \right)$. Similarly in the derivative of $\dot{V}_5(x(t))$, the term $-\frac{3h^2}{8} \int_{-h}^{-\frac{h}{2}} \int_{t+\theta}^{t} \dot{x}^T(s) Q_6 \dot{x}(s)\mathrm{d}s\mathrm{d}\theta$ is bounded with $-\left( \int_{-h}^{-\frac{h}{2}} \int_{t+\theta}^{t} \dot{x}(s)\mathrm{d}s\mathrm{d}\theta \right)^T Q_6 \left( \int_{-h}^{-\frac{h}{2}} \int_{t+\theta}^{t} \dot{x}(s)\mathrm{d}s\mathrm{d}\theta \right)$, which yields less conservative results in the proof of Theorem 3.1.

*Remark 2* By dividing the discrete delay interval $[-h, 0]$ into $[-h, -\frac{h}{2}]$ and $[-\frac{h}{2}, 0]$, then different functionals were chosen on each subintervals. $\frac{h}{2} \int_{-\frac{h}{2}}^{0} \int_{t+\theta}^{t} \dot{x}^T(s) Q_1 \dot{x}(s) \mathrm{d}s\mathrm{d}\theta$ was on the subinterval $[-\frac{h}{2}, 0]$, *and* $\frac{h}{2} \int_{-h}^{-\frac{h}{2}} \int_{t+\theta}^{t} \dot{x}^T(s) Q_2 \dot{x}(s)\mathrm{d}s\mathrm{d}\theta$ was on the subinterval $[-h, -\frac{h}{2}]$. We can see that this division provides extra freedom for discrete delay terms and reduces the conservatism. It was similar to division for other integral terms.

**Theorem 3.2** *For given scalars $h > 0, \gamma > 0$ and $\mu > 0$, system (1) without controlled output is asymptotically stable and satisfies $\| z(t) \|_2 < \gamma \| w(t) \|_2$ for any nonzero $w(t) \in \mathcal{L}_2[0, \infty)$ under the zero initial condition if there exist scalar $\lambda > 0$, symmetric positive matrices $Q = [Q_{ij}]_{5\times5}$, $Q_i (i = 1, 2, \ldots, 6)$, $\begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix}$, $Z_a = \begin{bmatrix} Z_1 & S \\ * & Z_1 \end{bmatrix}$, $Z_2$ and any matrices $P, M_i, H_i, L_i, N_i (i = 1, 2, \ldots, 14)$, such that the following LMI holds:*

$$\begin{bmatrix} \Pi & \Gamma^T & \Gamma_d \\ * & -I & 0 \\ * & * & -\lambda I \end{bmatrix} < 0, \tag{33}$$

*where*

$$\begin{aligned} \Pi &= (\Pi_{ij})_{15\times15}, \\ \Pi_{11} &= Q_{12} + Q_{12}^T - Q_{14} - Q_{14}^T + X_{11} + \left(\frac{h}{2}\right)^2 [Q_3 + Q_4] \\ &\quad - Z_1 - 4Z_2 + L_1 + L_1^T + \frac{h}{2}(H_1 + H_1^T) \\ &\quad + \frac{h}{2}(M_1 + M_1^T) + \lambda G_1^T G_1, \end{aligned}$$

$$\Pi_{12} = Z_1 - S + L_2^T - L_1 + N_1 + \frac{h}{2}H_2^T + \frac{h}{2}M_2^T + \lambda G_1^T G_2,$$
$$\Pi_{22} = -2Z_1 + S + S^T - L_2 - L_2^T + N_2 + N_2^T + \lambda G_2^T G_2.$$

*and other terms are the same as in Theorem 3.1.*

*Proof* Replace $A$, $A_d$ in (8) with $A + \Delta A(t)$, $A_d + \Delta A_d(t)$, respectively. Then, the uncertain system (1) is equivalent to the following condition

$$\Omega + \Gamma_d F(t)\Gamma_e + \Gamma_e^T F^T(t)\Gamma_d^T < 0, \tag{34}$$

where

$$\Gamma_d = \begin{bmatrix} 0\,0\,0\,0\; PE\; 0\,0\,0\,0\,0\,0\,0\,0\,0 \end{bmatrix}^T,$$
$$\Gamma_e = \begin{bmatrix} G_1\; G_2\; 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \end{bmatrix}.$$

By Lemma 2.2, there is a necessary and sufficient condition to satisfy the inequality (34) and there exists a scalar $\lambda > 0$, such that

$$\Omega + \lambda^{-1}\Gamma_d\Gamma_d^T + \lambda\Gamma_e^T\Gamma_e < 0. \tag{35}$$

Now, by applying Schur complement (35) is equivalent to (33). This completes the proof of Theorem 3.2. □

*Remark 3* If $B = B_w = 0$ and without controlled output, system (1) is simplified as

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + [A_d + \Delta A_d(t)]x(t - \tau(t)),$$
$$x(t) = \phi(t), \forall\, t \in [-h, 0]. \tag{36}$$

**Corollary 3.3** *For given scalars $h > 0$ and $\mu > 0$, system (36) is asymptotically stable, if there exist scalar $\lambda > 0$, symmetric positive matrices $Q = [Q_{ij}]_{5\times 5}$, $Q_i(i = 1, 2, \ldots, 6)$, $\begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix}$, $Z_a = \begin{bmatrix} Z_1 & S \\ * & Z_1 \end{bmatrix}$, $Z_2$ and any matrices $P, M_i, H_i, L_i, N_i(i = 1, 2, \ldots, 14)$, such that the following LMI holds:*

$$\begin{bmatrix} \Pi & \Gamma_d \\ * & -\lambda I \end{bmatrix} < 0, \tag{37}$$

*where $\Pi = (\Pi_{ij})_{14\times 14}$.*

*Proof* By taking $C = C_d = B_w = D_w = 0$ in the proof of Theorem 3.2, we can obtain (37) for the stability of system (36) and the proof is omitted here. □

*Remark 4* Different from [28,31], the double-integral term in the Lyapunov functional is divided into two parts, i.e., $h\int_{-h}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)Z_1\dot{x}(s) + h\int_{-h}^{0}\int_{t+\theta}^{t}\dot{x}^T(s)Z_2\dot{x}(s)$. As a result, there are two integral terms, $h\int_{t-h}^{t}\dot{x}^T(s)Z_1\dot{x}(s)$ and $h\int_{t-h}^{t}\dot{x}^T(s)Z_2\dot{x}(s)$, contained in $\dot{V}(x(t))$. Then, two different integral inequalities are adopted to deal with these two integral terms, respectively, which may lead to less conservative results.

## 4 State Feedback $H_\infty$ Control

Using the stability criteria derived in the last section, we now design a feedback controller gain $K$ to make system (1) robustly asymptotically stable with the norm bound $\gamma$.

**Theorem 4.1** *For given scalars $h > 0, \gamma > 0$ and $\mu > 0$, system (1) is asymptotically stable and satisfies $\| z(t) \|_2 < \gamma \| w(t) \|_2$ for any nonzero $w(t) \in \mathcal{L}_2[0, \infty)$ under the zero initial condition if there exist scalar $\lambda > 0$, symmetric positive matrices $\bar{Q} = [\bar{Q}_{ij}]_{5\times5}, \bar{Q}_i (i = 1, 2, \ldots, 6), \begin{bmatrix} \bar{X}_{11} & \bar{X}_{12} \\ * & \bar{X}_{22} \end{bmatrix}, \bar{Z}_a = \begin{bmatrix} \bar{Z}_1 & \bar{S} \\ * & \bar{Z}_1 \end{bmatrix}, \bar{Z}_2$ and any matrices $\bar{P}, \bar{M}_i, \bar{H}_i, \bar{L}_i, \bar{N}_i (i = 1, 2, \ldots, 14), L > 0, Y$ such that the following LMI holds:*

$$\begin{bmatrix} \bar{\Pi} & \bar{\Gamma}^T & \bar{\Gamma}_d \\ * & -I & 0 \\ * & * & -\lambda I \end{bmatrix} < 0, \tag{38}$$

*where*

$$\bar{\Pi} = (\bar{\Pi}_{ij})_{15\times15},$$

$$\bar{\Pi}_{11} = \bar{Q}_{12} + \bar{Q}_{12}^T - \bar{Q}_{14} - \bar{Q}_{14}^T + \bar{X}_{11} + \left(\frac{h}{2}\right)^2 [\bar{Q}_3 + \bar{Q}_4] - \bar{Z}_1 - 4\bar{Z}_2 + \bar{L}_1 + \bar{L}_1^T + \frac{h}{2}(\bar{H}_1 + \bar{H}_1^T) + \frac{h}{2}(\bar{M}_1 + \bar{M}_1^T), \quad \bar{\Pi}_{12} = \bar{Z}_1 - \bar{S} + \bar{L}_2^T - \bar{L}_1 + \bar{N}_1 + \frac{h}{2}\bar{H}_2^T + \frac{h}{2}\bar{M}_2^T,$$

$$\bar{\Pi}_{13} = -\bar{Q}_{12} + \bar{Q}_{13} + \bar{Q}_{14} - \bar{Q}_{15} + \bar{X}_{12} + \bar{L}_3^T + \frac{h}{2}\bar{H}_3^T + \frac{h}{2}\bar{M}_3^T,$$

$$\bar{\Pi}_{14} = -\bar{Q}_{13} + \bar{Q}_{15} + \bar{S} - 2\bar{Z}_2 + \bar{L}_4^T - \bar{N}_1 + \frac{h}{2}\bar{H}_4^T + \frac{h}{2}\bar{M}_4^T,$$

$$\bar{\Pi}_{15} = \bar{Q}_{11} + \frac{h}{2}\bar{Q}_{14} + \frac{h}{2}\bar{Q}_{15} + \bar{L}_5^T + \frac{h}{2}\bar{H}_5^T + \frac{h}{2}\bar{M}_5^T + XA^T + Y^T B^T,$$

$$\bar{\Pi}_{16} = \bar{L}_6^T + \frac{h}{2}\bar{H}_6^T + \frac{h}{2}\bar{M}_6^T, \quad \bar{\Pi}_{17} = \bar{L}_7^T + \frac{h}{2}\bar{H}_7^T + \frac{h}{2}\bar{M}_7^T,$$

$$\bar{\Pi}_{18} = \bar{Q}_{22}^T - \bar{Q}_{24}^T + \bar{L}_8^T + \frac{h}{2}\bar{H}_8^T - \bar{H}_1 + \frac{h}{2}\bar{M}_8^T,$$

$$\bar{\Pi}_{19} = \bar{Q}_{23} - \bar{Q}_{34}^T + \bar{L}_9^T + \frac{h}{2}\bar{H}_9^T + \frac{h}{2}\bar{M}_9^T - \bar{M}_1,$$

$$\bar{\Pi}_{1_{10}} = \bar{Q}_{24} - \bar{Q}_{44}^T + \bar{L}_{10}^T + \frac{h}{2}\bar{H}_{10}^T - \bar{H}_1 + \frac{h}{2}\bar{M}_{10}^T,$$

$$\bar{\Pi}_{1_{11}} = \bar{Q}_{25} - \bar{Q}_{45} + \bar{L}_{11}^T + \frac{h}{2}\bar{H}_{11}^T + \frac{h}{2}\bar{M}_{11}^T - \bar{M}_1,$$

$$\bar{\Pi}_{1_{12}} = -\bar{L}_1 + \bar{L}_{12}^T + \frac{h}{2}\bar{H}_{12}^T + \frac{h}{2}\bar{M}_{12}^T, \quad \bar{\Pi}_{1_{13}} = \bar{L}_{13}^T - \bar{N}_1 + \frac{h}{2}\bar{H}_{13}^T + \frac{h}{2}\bar{M}_{13}^T,$$

$$\bar{\Pi}_{1_{14}} = \frac{6}{h}\bar{Z}_2 + \bar{L}_{14}^T + \frac{h}{2}\bar{H}_{14}^T + \frac{h}{2}\bar{M}_{14}^T,$$
$$\bar{\Pi}_{22} = -2\bar{Z}_1 + \bar{S} + \bar{S}^T - \bar{L}_2 - \bar{L}_2^T + \bar{N}_2 + \bar{N}_2^T,$$
$$\bar{\Pi}_{23} = -\bar{L}_3^T + \bar{N}_3^T, \quad \bar{\Pi}_{24} = -\bar{S} + \bar{Z}_1 - \bar{L}_4^T + \bar{N}_4^T - \bar{N}_2,$$
$$\bar{\Pi}_{25} = -\bar{L}_5^T + \bar{N}_5^T + XA_d^T,$$
$$\bar{\Pi}_{26} = -\bar{L}_6^T + \bar{N}_6^T, \quad \bar{\Pi}_{27} = -\bar{L}_7^T + \bar{N}_7^T, \quad \bar{\Pi}_{28} = -\bar{L}_8^T + \bar{N}_8^T - \bar{H}_2,$$
$$\bar{\Pi}_{29} = -\bar{L}_9^T + \bar{N}_9^T - \bar{M}_2,$$
$$\bar{\Pi}_{2_{10}} = -\bar{L}_{10}^T + \bar{N}_{10}^T - \bar{H}_2, \quad \bar{\Pi}_{2_{11}} = -\bar{L}_{11}^T + \bar{N}_{11}^T - \bar{M}_2,$$
$$\bar{\Pi}_{2_{12}} = -\bar{L}_2 - \bar{L}_{12}^T + \bar{N}_{12}^T,$$
$$\bar{\Pi}_{2_{13}} = -\bar{L}_{13}^T - \bar{N}_2 + \bar{N}_{13}^T, \quad \bar{\Pi}_{2_{14}} = -\bar{L}_{14}^T + \bar{N}_{14}^T,$$
$$\bar{\Pi}_{33} = \bar{X}_{22} - \bar{X}_{11}, \quad \bar{\Pi}_{34} = -\bar{X}_{12} - \bar{N}_3,$$
$$\bar{\Pi}_{38} = -\bar{Q}_{22}^T + \bar{Q}_{23}^T + \bar{Q}_{24}^T - \bar{Q}_{25}^T - \bar{H}_3,$$
$$\bar{\Pi}_{39} = -\bar{Q}_{23} + \bar{Q}_{33}^T + \bar{Q}_{34}^T - \bar{Q}_{35}^T - \bar{M}_3,$$
$$\bar{\Pi}_{3_{10}} = -\bar{Q}_{24} + \bar{Q}_{34} + \bar{Q}_{44}^T - \bar{Q}_{45}^T - \bar{H}_3,$$
$$\bar{\Pi}_{3_{11}} = -\bar{Q}_{25} + \bar{Q}_{35} + \bar{Q}_{45} - \bar{Q}_{55}^T - \bar{M}_3, \quad \bar{\Pi}_{3_{12}} = -\bar{L}_3,$$
$$\bar{\Pi}_{3_{13}} = -\bar{N}_3, \quad \bar{\Pi}_{44} = -\bar{X}_{22} - \bar{Z}_1 - 4\bar{Z}_2 - \bar{N}_4 - \bar{N}_4^T,$$
$$\bar{\Pi}_{45} = -\bar{N}_5^T, \bar{\Pi}_{46} = -\bar{N}_6^T,$$
$$\bar{\Pi}_{47} = -\bar{N}_7^T, \quad \bar{\Pi}_{48} = -\bar{Q}_{23}^T + \bar{Q}_{25}^T - \bar{N}_8^T - \bar{H}_4,$$
$$\bar{\Pi}_{49} = -\bar{Q}_{33}^T + \bar{Q}_{35}^T - \bar{N}_9^T - \bar{M}_4,$$
$$\bar{\Pi}_{4_{10}} = -\bar{Q}_{34} + \bar{Q}_{45}^T - \bar{N}_{10}^T - \bar{H}_4, \quad \bar{\Pi}_{4_{11}} = -\bar{Q}_{35} + \bar{Q}_{55}^T - \bar{N}_{11}^T - \bar{M}_4,$$
$$\bar{\Pi}_{4_{12}} = -\bar{L}_4 - \bar{N}_{12}^T,$$
$$\bar{\Pi}_{4_{13}} = -\bar{N}_4 - \bar{N}_{13}^T, \quad \bar{\Pi}_{4_{14}} = \frac{6}{h}\bar{Z}_2 - \bar{N}_{14}^T,$$
$$\bar{\Pi}_{55} = \left(\frac{h}{2}\right)^2\left[\bar{Q}_1 + \bar{Q}_2\right] + \frac{h^4}{64}\bar{Q}_5 + \frac{9h^4}{64}\bar{Q}_6 + h^2(\bar{Z}_1 + \bar{Z}_2) - X - X^T,$$
$$\bar{\Pi}_{58} = \bar{Q}_{12} + \frac{h}{2}\bar{Q}_{24}^T + \frac{h}{2}\bar{Q}_{25}^T - \bar{H}_5, \quad \bar{\Pi}_{59} = \bar{Q}_{13} + \frac{h}{2}\bar{Q}_{34}^T + \frac{h}{2}\bar{Q}_{35}^T - \bar{M}_5,$$
$$\bar{\Pi}_{5_{10}} = \bar{Q}_{14} + \frac{h}{2}\bar{Q}_{44}^T + \frac{h}{2}\bar{Q}_{45}^T - \bar{H}_5,$$
$$\bar{\Pi}_{5_{11}} = \bar{Q}_{15} + \frac{h}{2}\bar{Q}_{45} + \frac{h}{2}\bar{Q}_{55}^T - \bar{M}_5, \quad \bar{\Pi}_{5_{12}} = -\bar{L}_5,$$
$$\bar{\Pi}_{5_{13}} = -\bar{N}_5, \quad \bar{\Pi}_{5_{15}} = B_w, \quad \bar{\Pi}_{66} = -\bar{Q}_1, \quad \bar{\Pi}_{68} = -\bar{H}_6,$$
$$\bar{\Pi}_{69} = -\bar{M}_6, \quad \bar{\Pi}_{6_{10}} = -\bar{H}_6,$$
$$\bar{\Pi}_{6_{11}} = -\bar{M}_6, \quad \bar{\Pi}_{6_{12}} = -\bar{L}_6, \quad \bar{\Pi}_{6_{13}} = -\bar{N}_6, \quad \bar{\Pi}_{77} = -\bar{Q}_2,$$
$$\bar{\Pi}_{78} = -\bar{H}_7, \quad \bar{\Pi}_{79} = -\bar{M}_7,$$
$$\bar{\Pi}_{7_{10}} = -\bar{H}_7, \quad \bar{\Pi}_{7_{11}} = -\bar{M}_7, \quad \bar{\Pi}_{7_{12}} = -\bar{L}_7 - \bar{M}_{12}^T,$$
$$\bar{\Pi}_{7_{13}} = -\bar{N}_7, \quad \bar{\Pi}_{88} = -\bar{Q}_3 - \bar{H}_8 - \bar{H}_8^T,$$
$$\bar{\Pi}_{89} = -\bar{H}_9^T - \bar{M}_8, \quad \bar{\Pi}_{8_{10}} = -\bar{H}_{10}^T - \bar{H}_8,$$

$$\begin{aligned}
\bar{\Pi}_{8_{11}} &= -\bar{H}_{11}^T - \bar{M}_8, \quad \bar{\Pi}_{8_{12}} = -\bar{L}_8 - \bar{H}_{12}^T, \\
\bar{\Pi}_{8_{13}} &= -\bar{N}_8 - \bar{H}_{13}^T, \quad \bar{\Pi}_{8_{14}} = -\bar{H}_{14}^T, \quad \bar{\Pi}_{99} = -\bar{Q}_4 - \bar{M}_9 - \bar{M}_9^T, \\
\bar{\Pi}_{9_{10}} &= -\bar{H}_9 - \bar{M}_{10}^T, \\
\bar{\Pi}_{9_{11}} &= -\bar{M}_{11}^T - \bar{M}_9, \quad \bar{\Pi}_{9_{12}} = -\bar{L}_9, \quad \bar{\Pi}_{9_{13}} = -\bar{N}_9 - \bar{M}_{13}^T, \quad \bar{\Pi}_{9_{14}} = -\bar{M}_{14}^T, \\
\bar{\Pi}_{10_{10}} &= -\bar{Q}_5 - \bar{H}_{10} - \bar{H}_{10}^T, \\
\bar{\Pi}_{10_{11}} &= -\bar{H}_{11}^T - \bar{M}_{10}, \quad \bar{\Pi}_{10_{12}} = -\bar{L}_{10} - \bar{H}_{12}^T, \quad \bar{\Pi}_{10_{13}} = -\bar{N}_{10} - \bar{H}_{13}^T, \\
\bar{\Pi}_{10_{14}} &= -\bar{H}_{14}^T, \quad \bar{\Pi}_{11_{11}} = -\bar{Q}_6 - \bar{M}_{11} - \bar{M}_{11}^T, \quad \bar{\Pi}_{11_{12}} = -\bar{L}_{11} - \bar{M}_{12}^T, \\
\bar{\Pi}_{11_{13}} &= -\bar{N}_{11} - \bar{M}_{13}^T, \\
\bar{\Pi}_{11_{14}} &= -\bar{M}_{14}^T, \quad \bar{\Pi}_{12_{12}} = -\bar{L}_{12} - \bar{L}_{12}^T, \quad \bar{\Pi}_{12_{13}} = -\bar{L}_{13}^T - \bar{N}_{12}, \\
\bar{\Pi}_{12_{14}} &= -\bar{L}_{14}^T, \quad \bar{\Pi}_{13_{13}} = -\bar{N}_{13} \\
&\qquad -\bar{N}_{13}^T, \quad \bar{\Pi}_{13_{14}} = -\bar{N}_{14}^T, \quad \bar{\Pi}_{14_{14}} = -\frac{12}{h}\bar{Z}_2, \quad \bar{\Pi}_{15_{15}} = -\gamma^2 I, \\
\bar{\Gamma}^T &= \left[ CX \; C_d X \; 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \; D_w X \right]^T, \\
\bar{\Gamma}_d &= \left[ 0\,0\,0\,0 \; EX \; 0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \right]^T,
\end{aligned}$$

*Proof* Assume that a proportional feedback controller is employed, i.e., $u(t) = Kx(t)$. Replacing $A$ and $C$ with $A + BK$ and $C + DK$ in (33), respectively, and then pre- and post-multiplying them with diag $(\underbrace{L \ \cdots \ L}_{14} \quad I \quad I \quad I)$ and its transpose, setting $P = L^{-1}$, $\bar{Q}_{ij} = LQ_{ij}L^T (i, j = 1, 2, \ldots, 5)$, $\bar{Q}_i = LQ_iL^T (i = 1, 2, \ldots, 6)$, $\bar{X}_{11} = LX_{11}L^T$, $\bar{X}_{12} = LX_{12}L^T$, $\bar{X}_{22} = LX_{22}L^T$, $\bar{Z}_1 = LZ_1L^T$, $\bar{S} = LSL^T$, $\bar{Z}_2 = LZ_2L^T$, $\bar{M}_i = LM_iL^T$, $\bar{H}_i = LH_iL^T$, $\bar{L}_i = LL_iL^T$, $\bar{N}_i = LN_iL^T (i = 1, 2, \ldots, 14)$ and $Y = KL^T$, the proof can be completed. □

## 5 Numerical Examples

In this section, some numerical examples are provided to demonstrate that the proposed methods in this paper are effective and are improvements over some existing methods.

*Example 1* Consider the following system with time-varying delays:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + A_d x(t - \tau(t)) + B_w w(t), \\
z(t) &= Cx(t) + C_d x(t - \tau(t)).
\end{aligned}$$

where

$$A = \begin{bmatrix} -0.6238 & -1.0132 \\ 2.0116 & -0.2106 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.5011 & -0.7871 \\ -0.3002 & 0.5231 \end{bmatrix}, \quad C = \begin{bmatrix} 0.2134 & -0.0191 \\ 0.1119 & -0.1665 \end{bmatrix},$$
$$B_w = \begin{bmatrix} -0.4326 & 0.1253 \\ -1.6656 & 0.2877 \end{bmatrix}, \quad C_d = \begin{bmatrix} 0.0816 & 0.1290 \\ 0.0712 & 0.0669 \end{bmatrix}, \quad D_w = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

**Table 1** Maximum upper bound of $h$ for different values of $\gamma$ (Example 1)

| $\gamma$ | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 |
|---|---|---|---|---|---|---|
| [42] | 0.4203 | 0.4779 | 0.5146 | 0.5401 | 0.5589 | 0.5733 |
| [31] | 0.4734 | 0.5237 | 0.5545 | 0.5754 | 0.5904 | 0.6018 |
| [28] | 0.9094 | 0.9302 | 0.9446 | 0.9553 | 0.9635 | 0.9700 |
| Theorem 3.1 | 0.9425 | 0.9814 | 1.0354 | 1.0569 | 1.0742 | 1.0833 |

**Table 2** Maximum upper bound of $h$ for different values of $\mu$ with $\gamma = 4$ (Example 1)

| $\mu$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.7 | 0.9 |
|---|---|---|---|---|---|---|---|
| [42] | 0.5258 | 0.4863 | 0.4377 | 0.3755 | 0.2917 | 0.2686 | - |
| [31] | 0.5562 | 0.5153 | 0.4651 | 0.4009 | 0.3826 | 0.3826 | - |
| [28] | 0.9311 | 0.9149 | 0.9078 | 0.9040 | 0.9017 | 0.8992 | 0.8985 |
| Theorem 3.1 | 1.0525 | 1.0126 | 0.9846 | 0.9354 | 0.9165 | 0.9021 | 0.9013 |

To compare our results with those in [28,31,42], we use Theorem 3.1 with $\mu = 0$, and Table 1 gives the comparison results on the upper bound of the time-delay for given $\gamma$ among [28,31,42] and Theorem 3.1. Through the comparison in Table 1, it is clear that the method proposed in this paper gives less conservative results than those in [28,31,42].

For $\mu > 0$, Table 2 shows the comparison results between [28,31,42] and Theorem 3.1 in this paper for given $\gamma = 4$. It is seen from Table 2 that the stability criterion proposed in this paper is much less conservative than that in [28,31,42].

*Example 2* Consider the following uncertain system with parametric coefficients as follows:

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + [A_d + \Delta A_d(t)]x(t - \tau(t)) + B_w w(t),$$
$$z(t) = Cx(t) + C_d x(t - \tau(t)) + D_w w(t),$$

with

$$A = \begin{bmatrix} -3.0242 & 2.7527 \\ 0.8104 & -4.3988 \end{bmatrix}, \quad A_d = \begin{bmatrix} 2.8409 & -1.2355 \\ -9.8952 & -0.1443 \end{bmatrix}, \quad B_w = \begin{bmatrix} -0.9043 & 0.4325 \\ -0.7774 & 0.1846 \end{bmatrix},$$
$$C = \begin{bmatrix} -0.9647 & -1.6555 \\ 0.8245 & -0.8378 \end{bmatrix}, \quad C_d = \begin{bmatrix} 1.2723 & 0.2718 \\ 0.4810 & -0.2368 \end{bmatrix}, \quad D_w = \begin{bmatrix} 0.1352 & -1.0236 \\ -0.0125 & 0.3368 \end{bmatrix},$$
$$E = \begin{bmatrix} 1.5 \\ 0.8 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0.2 & 0.3 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}.$$

We assume $\mu = 0$; then by solving LMI (33) in Theorem 3.2, Tables 3 and 4 tabulate the maximum allowable upper delay bound $h$ for a prescribed $\gamma$ and the minimum

**Table 3** Maximum upper bound of $h$ for different values of $\gamma$ (Example 2)

| $\gamma$ | 3.0 | 4.0 | 6.0 |
|---|---|---|---|
| [42] | 0.1677 | 0.2406 | 0.3024 |
| [18] | 0.2243 | 0.2711 | 0.3141 |
| [28] | 0.4743 | 0.5086 | 0.5384 |
| Theorem 3.2 | 0.5132 | 0.5574 | 0.5871 |

**Table 4** Minimum allowable $\gamma$ for given $h$. (Example 2)

| $h$ | 0.1 | 0.2 | 0.3 |
|---|---|---|---|
| [42] | 2.5209 | 3.3495 | 5.8767 |
| [18] | 2.4699 | 2.7626 | 5.1596 |
| [28] | 1.1009 | 1.1646 | 1.3861 |
| Theorem 3.2 | 0.7983 | 1.0548 | 1.2574 |

**Table 5** Maximum upper bound of $h$ for given $\mu$ (Example 3)

| $\mu$ | 0.5 | 0.9 |
|---|---|---|
| [43] | 0.9428 | 0.8189 |
| [35] Theorem 2 | 0.9428 | 0.8189 |
| [35] Theorem 1 | 0.9561 | 0.8919 |
| [28] | 0.9682 | 0.9103 |
| Corollary 3.3 | 0.9954 | 0.9324 |

allowable $\gamma$ for a prescribed delay bound $h$, respectively. It can be seen that these comparisons show that Theorem 3.2 for delay systems with uncertainties in this paper is less conservative than those in [18,28,42].

*Example 3* Consider a time-delay system (36) with the parameters as follows:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 1.6 & 0 \\ 0 & 0.05 \end{bmatrix},$$
$$G_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}.$$

For different $\mu$, maximum delay bounds obtained by Corollary 3.3 are listed and compared with the results of the existing works in Table 5. From Table 5, it can be shown that our results for this example also give larger upper bound of time-delay than the ones in [28,35,43].

*Example 4* Consider a time-delay system (1) with the following parameters:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix},$$
$$C_d = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = 0.1.$$

**Table 6** Comparison results for Example 4 with $\gamma = 0.1287$

| | [12] | [42] | [37] | [31] | Theorem 4.1. |
|---|---|---|---|---|---|
| $h$ | 1.25 | 1.38 | 1.4075 | 1.4142 | 1.6254 |
| Gain matrix K | [0.6407 −89.1149] | Not given | Not given | [−0.4396 −9.4901] | [−0.0058 −2.3516] |

**Table 7** Comparison results for Example 4 with $h = 0.999$

| | $\gamma_{min}$ | Feedback gain |
|---|---|---|
| [37] | 0.6331 | [0 −4.8400] |
| [5] | 0.1287 | $[0\ -1.0285 \times 10^6]$ |
| [12] | 0.1015 | [3.6828 −827.0898] |
| [31] | 0.00025 | [0 −10.0000] |
| Theorem 4.1. | 0.00005 | [0 −4.3568] |

**Table 8** Comparison results for Example 4.

| | [29] | [37] | [31] | Theorem 4.1 | [12] | [42] | [31] | Theorem 4.1. |
|---|---|---|---|---|---|---|---|---|
| $h$ | 0.2 | 0.2 | 0.2 | 0.2 | 0.8 | 0.9 | 1.03 | 1.21 |
| $\gamma_{min}$ | 0.66 | 0.1093 | 0.0071 | 0.0025 | 0.05 | 0.03 | 0.03 | 0.02 |

We consider this example in the following two cases.

Case 1: $\Delta A = \Delta A_d = 0$.
For $\gamma = 0.1287$, $\mu = 0$, by solving LMI (38), Table 6 shows the comparison results between [12,31,37,42] and Theorem 4.1.
For $h = 0.999$, $\mu = 0$, the comparison results between [5,12,31,37] and Theorem 4.1 are listed in Table 7.
Case 2: $\| \Delta A \| \leq 0.2$, $\| \Delta A_d \| \leq 0.2$,
Choose $G_1$, $G_2$ and $E$ as

$$G_1 = G_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix},$$

Also, for different $h$ and $\gamma_{min}$, Table 8 shows the comparison results between [12,29, 31,37,42] and Theorem 4.1. From Tables 6, 7, 8 one can seen that our method proposed in this paper gives less conservative than the methods in [31] and the state feedback gain obtained in this paper is smaller than the corresponding ones in [5,12,31].

## 6 Conclusion

This paper has investigated the problem of delay-dependent robust $H_\infty$ control design for time-varying delay system. Based on the Lyapunov–Krasovskii functional and free-weighting matrices approach, sufficient conditions have been obtained as a set of linear matrix inequalities. Numerical examples have been provided to show the less conservativeness of the proposed method. By utilizing the proposed idea of this paper, future works will focus on stabilization for various dynamic systems with time-delays such as Markovian jumping complex networks [19,20], robust nonfragile control [15, 22], exponential dissipativity system [23,24], $H_\infty$ filtering system [27], polytopic systems [10] and networked control systems [26].

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## References

1. S. Arik, An improved global stability result for delayed cellular neural networks. IEEE Trans. Circuits Syst. **I**(49), 1211–1214 (2002)
2. S. Boyd, L. Ghaoui, E. Feron, V. Balakrishnam, *Linear Matrix Inequalities in Systems and Control* (SIAM, Philadelphia, 1994)
3. X. Chen, Q. Song, Global stability of complex-valued neural networks with both leakage time delay and discrete time delay on time scales. Neurocomputing **121**, 254–264 (2013)
4. B. Chen, X. Liu, C. Lin, K. Liu, Robust $H_\infty$ control of Takagi–Sugeno fuzzy systems with state and input time delays. Fuzzy Sets Syst. **160**, 403–422 (2009)
5. E. Fridman, U. Shaked, A descriptor system approach to $H_\infty$ control of linear time-delay systems. IEEE Trans. Automat. Control **47**, 253–270 (2002)
6. T. Fujinami, Y. Saito, M. Morishita, Y. Koike, K. Tanida, A hybrid mass damper system controlled by $H_\infty$ control theory for reducing bending-torsion vibration of an actual building. Earthq. Eng. Struct. Dyn. **30**, 1639–1653 (2001)
7. H. Gao, C. Wang, Comments and further results on “a descriptor system approach to $H_\infty$ control of linear time-delay systems”. IEEE Trans. Automat. Control **48**, 520–525 (2003)
8. K. Gu, S.I. Niculescu, Survey on recent results in the stability and control of time-delay systems. J. Dyn. Syst. Meas. Control **125**(2), 158–165 (2003)
9. K. Gu, An integral inequality in the stability problem of time delay systems. in *Proceedings of the 39th IEEE Conference on Decision Control*, 2000, pp. 2805–2810
10. L.V. Hien, T.D. Tran, H.M. Trinh, New $H_\infty$ control design for polytopic systems with mixed time-varying delays in state and input. Int. J. Innov. Comput. Inf. Control **11**(1), 105–121 (2015)
11. X. Jiang, Q.L. Han, On $H_\infty$ control for linear systems with interval time-varying delay. Automatica **41**, 2099–2106 (2005)
12. Y.S. Lee, Y.S. Moon, W.H. Kwon, P.G. Park, Delay-dependent robust $H_\infty$ control for uncertain systems with a state-delay. Automatica **40**, 65–72 (2004)
13. C.H. Lien, K.W. Yu, C.T. Huang, P.Y. Chou, L.Y. Chung, J.D. Chen, Robust $H_\infty$ control for uncertain T–S fuzzy time-delay systems with sampled-data input and nonlinear perturbations. Nonlinear Anal. Hybrid Syst. **4**, 550–556 (2010)
14. H. Li, B. Chen, Q. Zhou, C. Lin, A delay-dependent approach to robust $H_\infty$ control for uncertain stochastic systems with state and input delays. Circuits Syst. Signal Process. **28**, 169–183 (2009)
15. F. Li, P. Shi, L. Wu, X. Zhang, Fuzzy-model-based $\mathcal{D}$-stability and non-fragile control for discrete-time descriptor systems with multiple delays. IEEE Trans. Fuzzy Syst. **22**(4), 1019–1025 (2014)
16. Y.S. Moon, P. Park, W.H. Kwon, Y.S. Lee, Delay-dependent robust stabilization of uncertain state-delayed systems. Int. J. Control **74**, 1447–1455 (2001)
17. M.N.A. Parlakci, I.B. Kucukdemiral, Robust delay-dependent $H_\infty$ control of time-delay systems with state and input delays. Int. J. Robust Nonlinear Control **21**, 974–1007 (2011)
18. C. Peng, Y.C. Tian, Delay-dependent robust $H_\infty$ control for uncertain systems with time-varying delay. Inform. Sci. **179**, 3187–3197 (2009)

19. R. Rakkiyappan, B. Kaviarasan, F.A. Rihan, S. Lakshmanan, Synchronization of singular Markovian jumping complex networks with additive time-varying delays via pinning control. J. Frankl. Inst. **352**, 3178–3195 (2015)
20. R. Rakkiyappan, S. Lakshmanan, R. Sivasamy, C.P. Lim, Leakage-delay-dependent stability analysis of Markovian jumping linear systems with time-varying delays and nonlinear perturbations. Appl. Math. Model. **40**, 5026–5043 (2016)
21. J.P. Richard, Time-delay systems: an overview of some recent advances and open problems. Automatica **39**(10), 1667–1694 (2003)
22. K. Sivaranjani, R. Rakkiyappan, S. Lakshmanan, C.P. Lim, Robust non-fragile control for offshore steel jacket platform with nonlinear perturbations. Nonlinear Dyn. **81**(4), 2043–2057 (2015)
23. Q. Song, Stochastic dissipativity analysis on discrete-time neural networks with time-varying delays. Neurocomputing **74**(5), 838–845 (2011)
24. Q. Song, Exponential stability of recurrent neural networks with both time-varying delays and general activation functions via LMI approach. Neurocomputing **71**, 2823–2830 (2011)
25. Q. Song, Exponential stability of recurrent neural networks with both time-varying delays and general activation functions via LMI approach. Neurocomputing **71**(13–5), 2823–2830 (2008)
26. H. Shao, G. Miao, Z. Zhang, State feedback control design for a networked control model of systems with two additive time-varying delays. Int. J. Innov. Comput. Inf. Control **11**(4), 1457–1469 (2015)
27. P. Shi, X. Luan, F. Liu, $H_\infty$ filtering for discrete-time systems with stochastic incomplete measurement and mixed delays. IEEE Trans. Ind. Electron. **59**(6), 2732–2739 (2012)
28. M. Syed Ali, R. Saravanakumar, Novel delay-dependent robust $H_\infty$ control of uncertain systems with distributed time-varying delays. Appl. Math. Comput. **249**, 510–520 (2014)
29. C.E. de Souza, X. Li, Delay-dependent robust $H_\infty$ control of uncertain linear state-delayed systems. Automatica **35**, 1313–1321 (1999)
30. N.T. Thanh, V.N. Phat, Decentralized $H_\infty$ control for large-scale interconnected nonlinear time-delay systems via LMI approach. J. Process Control **22**, 1325–1339 (2012)
31. E. Tian, D. Yue, Y. Zhang, On improved delay-dependent robust $H_\infty$ control for systems with interval time-varying delay. J. Frankl. Inst. **348**, 555–567 (2011)
32. J. Tian, W. Xiong, F. Xu, Improved delay-partitioning method to stability analysis for neural networks with discrete and distributed time-varying delays. Appl. Math. Comput. **233**, 152–164 (2014)
33. C. Wang, Y. Shen, Robust $H_\infty$ control for stochastic systems with nonlinearity, uncertainty and time-varying delay. Comput. Math. Appl. **63**, 985–998 (2012)
34. C. Wang, Y. Shen, Delay-dependent non-fragile robust stabilization and $H_\infty$ control of uncertain stochastic systems with time-varying delay and nonlinearity. J. Frankl. Inst. **348**, 2174–2190 (2011)
35. C. Wang, Y. Shen, Improved delay-dependent robust stability criteria for uncertain time delay systems. Appl. Math. Comput. **218**, 2880–2888 (2011)
36. Z. Wang, X. Liao, S. Guo, G. Liu, Stability analysis of genetic regulatory network with time delays and parameter uncertainties. IET Control Theory Appl. **4**, 2018–2028 (2010)
37. J. Wu, T.W. Chen, L. Wang, Delay-dependent robust stability and $H_\infty$ control for jump linear systems with delays. Syst. Control Lett. **55**, 939–948 (2006)
38. J. Xia, S. Xu, Y. Zou, Robust reliable H-infinity control for nonlinear uncertain stochastic time-delay systems with Markovian jumping parameters. J. Control Theory Appl. **6**(4), 410–414 (2008)
39. L.H. Xie, Output feedback $H_\infty$ control of systems with parameter uncertainty. Int. J. Control **63**, 741–750 (1996)
40. W. Xie, Improved delay-independent $H_\infty$ performance analysis and memoryless state feedback for linear delay systems with polytopic uncertainties. Int. J. Control Autom. Syst. **6**, 263–268 (2011)
41. S. Xu, J. Lam, A survey of linear matrix inequality techniques in stability analysis of delay systems. Int. J. Syst. Sci. **39**(12), 1095–1113 (2008)
42. S. Xu, J. Lam, Y. Zou, New results on delay-dependent robust $H_\infty$ control for systems with time-varying delays. Automatica **42**, 343–348 (2006)
43. H. Yan, H. Zhang, M.Q. Meng, Delay-range-dependent robust $H_\infty$ control for uncertain systems with interval time-varying delays. Neurocomputing **73**, 1235–1243 (2010)
44. H.B. Zeng, J.H. Park, J.W. Xia, S.P. Xiao, Improved delay-dependent stability criteria for T–S fuzzy systems with time-varying delay. Appl. Math. Comput. **235**, 492–501 (2014)
45. Z. Zhang, T. Zhang, S. Huang, P. Xiao, New global exponential stability result to a general Cohen–Grossberg neural networks with multiple delays. Nonlinear Dyn. **67**, 2419–2432 (2012)

46. X. Zhu, G. Yang, T. Li, C. Lin, L. Guo, LMI stability criterion with less variables for time-delay systems. Int. J. Control Autom. Syst. **7**, 530–535 (2009)
47. Q. Zhu, J. Cao, Exponential stability of stochastic neural networks with both Markovian jump parameters and mixed time delays. IEEE Trans. Syst. Man Cybern. B **41**(2), 341–353 (2011)
48. Q. Zhu, J. Cao, Stability analysis of Markovian jump stochastic BAM neural networks with impulse control and mixed time delays. IEEE Trans. Neural Netw. **23**, 467–479 (2012)