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# Minimum Energy Control of Fractional Positive Electrical Circuits with Bounded Inputs

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**Abstract**  Minimum energy control problem for the fractional positive electrical circuits with bounded inputs is formulated and solved. Sufficient conditions for the existence of solution to the problem are established. A procedure for solving of the problem is proposed and illustrated by example of fractional positive electrical circuit.

## 1 Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs [3,17]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

Mathematical fundamentals of the fractional calculus are given in the monographs [27–29]. The positive fractional linear systems have been investigated in [5,6,8,18, 19,21]. Stability of fractional linear continuous-time systems has been investigated in the papers [1,20,21]. The notion of practical stability of positive fractional linear systems has been introduced in [20]. Some recent interesting results in fractional systems theory and its applications can be found in [2,29,30,33,34].

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The minimum energy control problem for standard linear systems has been formulated and solved by J. Klamka in [22–25] and for 2D linear systems with variable coefficients in [22]. The controllability and minimum energy control problem of fractional discrete-time linear systems has been investigated by Klamka in [26]. The minimum energy control of fractional positive continuous-time linear systems has been addressed in [4,11,13] and for descriptor positive discrete-time linear systems in [4,6,10,12]. The minimum energy control problem for positive and positive fractional electrical circuits has been investigated in [13–15].

In this paper, the minimum energy control problem for fractional positive electrical circuits with bounded inputs will be formulated and solved.

The paper is organized as follows. In Sect. 2, the basic definitions and theorems of the fractional positive electrical circuits are recalled and the necessary and sufficient conditions for the reachability of the electrical circuits are given.

The main result of the paper is given in Sect. 3 where minimum energy control problem is formulated, sufficient conditions for its solution are established, and a procedure is proposed. Illustrating example of fractional positive electrical circuit is given in Sect. 4. An extension to the method is presented in Sect. 5. Concluding remarks are given in Sect. 6.

The following notation will be used: $\Re$—the set of real numbers, $\Re^{n \times m}$—the set of $n \times m$ real matrices, $\Re_+^{n \times m}$—the set of $n \times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n \times 1}$, $M_n$—the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), $I_n$—the $n \times n$ identity matrix.

## 2 Preliminaries

The following Caputo definition of the fractional derivative will be used [21]

$$D^\alpha f(t) = \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}}{(t-\tau)^{\alpha+1-n}} \mathrm{d}\tau, \quad n-1 < \alpha \le n \in N = \{1, 2, \ldots\}$$

(2.1)

where $\alpha \in \Re$ is the order of fractional derivative and $f^{(n)}(\tau) = \frac{\mathrm{d}^n f(\tau)}{\mathrm{d}\tau^n}$ and $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \mathrm{d}t$ is the gamma function.

Consider a fractional electrical circuit composed of resistors, coils, condensators and voltage (current) sources. Using the Kirchhoff's laws, we may describe the transient states in the electrical circuit by state equations [7,9,16,17,21,28]

$$D^\alpha x(t) = Ax(t) + Bu(t), \quad 0 < \alpha \le 1 \tag{2.2}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ are the state and input vectors and $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$.

**Theorem 2.1** [21] *The solution of* Eq. (2.2) *is given by*

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau, \quad x(0) = x_0 \tag{2.3}$$

*where*

$$\Phi_0(t) = E_\alpha(At^\alpha) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha + 1)} \tag{2.4}$$

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} \tag{2.5}$$

*and* $E_\alpha(At^\alpha)$ *is the Mittag–Leffler matrix function* [21].

**Definition 2.1** [21] The fractional system (2.2) is called the (internally) positive fractional system if and only if $x(t) \in \Re_+^n, t \geq 0$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m, \quad t \geq 0$.

**Theorem 2.2** [21] *The continuous-time fractional system* (2.2) *is (internally) positive if and only if*

$$A \in M_n, \quad B \in \Re_+^{n\times m}. \tag{2.6}$$

**Definition 2.2** The state $x_f \in \Re_+^n$ of the fractional system (2.2) is called reachable in time $t_f$ if there exist an input $u(t) \in \Re_+^m, \quad t \in [0, \ t_f]$ which steers the state of system (2.2) from zero initial state $x_0 = 0$ to the state $x_f$.

A real square matrix is called monomial if each of its row and each of its column contain only one positive entry and the remaining entries are zero.

**Theorem 2.3** *The positive fractional system* (2.2) *is reachable in time* $t \in [0, \ t_f]$ *if and only if the matrix* $A \in M_n$ *is diagonal and the matrix* $B \in \Re_+^{n\times n}$ *is monomial.*

*Proof Sufficiency* It is well known [17,21] that if $A \in M_n$ is diagonal, then $\Phi(t) \in \Re_+^{n\times n}$ is also diagonal and if $B \in \Re_+^{n\times m}$ is monomial, then $BB^T \in \Re_+^{n\times n}$ is also monomial. In this case, the matrix

$$R_f = \int_0^{t_f} \Phi(\tau)BB^T\Phi^T(\tau)d\tau \in \Re_+^{n\times n} \tag{2.7}$$

is also monomial and $R_f^{-1} \in \Re_+^{n\times n}$. The input

$$u(t) = B^T\Phi^T(t_f - t)R_f^{-1}x_f \tag{2.8}$$

steers the state of the system (2.2) from $x_0 = 0$ to $x_f$ since using (2.3) for $x_0 = 0$ and (2.5) we obtain

$$x(t_f) = \int_0^{t_f} \Phi(t_f - \tau)Bu(\tau)\mathrm{d}\tau = \int_0^{t_f} \Phi(t_f - \tau)BB^{\mathrm{T}}\Phi^{\mathrm{T}}(t_f - \tau)\mathrm{d}\tau R_f^{-1}x_f$$

$$= \int_0^{t_f} \Phi(\tau)BB^{\mathrm{T}}\Phi^{\mathrm{T}}(\tau)\mathrm{d}\tau R_f^{-1}x_f = x_f. \tag{2.9}$$

The proof of necessity is given in [11].

## 3 Problem Formulation and Its Solution

Consider the fractional positive electrical circuit (2.2) with $A \in M_n$ and $B \in \Re_+^{n \times m}$ monomial. If the system is reachable in time $t \in [0, t_f]$, then usually there exist many different inputs $u(t) \in \Re_+^n$ that steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$. Among these inputs, we are looking for input $u(t) \in \Re_+^n$, $t \in [0, t_f]$ satisfying the condition

$$u(t) \le U \in \Re_+^n, \quad t \in [0, t_f] \tag{3.1}$$

that minimizes the performance index

$$I(u) = \int_0^{t_f} u^{\mathrm{T}}(\tau)Qu(\tau)\mathrm{d}\tau \tag{3.2}$$

where $Q \in \Re_+^{n \times n}$ is a symmetric positive defined matrix and $Q^{-1} \in \Re_+^{n \times n}$.

The performance index (3.2) is a measure of the energy used for steering the state of the systems from $x_0 = 0$ to $x_f$.

The minimum energy control problem for the fractional positive electrical circuit (2.2) can be stated as follows.

Given the matrices $A \in M_n$, $B \in \Re_+^{n \times m}$, $\alpha$, $U \in \Re_+^n$ and $Q \in \Re_+^{n \times n}$ of the performance matrix (3.2), $x_f \in \Re_+^n$ and $t > 0$, find an input $u(t) \in \Re_+^n$ for $t \in [0, t_f]$ satisfying (3.1) that steers the state vector of the system from $x_0 = 0$ to $x_f \in \Re_+^n$ and minimizes the performance index (3.2).

To solve the problem, we define the matrix

$$W(t_f) = \int_0^{t_f} \Phi(t_f - \tau)BQ^{-1}B^{\mathrm{T}}\Phi^{\mathrm{T}}(t_f - \tau)\mathrm{d}\tau \tag{3.3}$$

where $\Phi(t)$ is defined by (2.5). From (3.3) and Theorem 2.3, it follows that the matrix (3.3) is monomial if and only if the fractional positive electrical circuit (2.2) is reachable in time $[0, t_f]$. In this case, we may define the input

$$\hat{u}(t) = Q^{-1}B^{\mathrm{T}}\Phi^{\mathrm{T}}(t_f - t)W^{-1}(t_f)x_f \quad \text{for} \quad t \in [0, t_f]. \tag{3.4}$$

Note that the input (3.4) satisfies the condition $u(t) \in \Re_+^n$ for $t \in [0, t_f]$ if

$$Q^{-1} \in \Re_+^{n\times n} \quad \text{and} \quad W^{-1}(t_f) \in \Re_+^{n\times n}. \tag{3.5}$$

**Theorem 3.1** *Let $\bar{u}(t) \in \Re_+^n$ for $t \in [0, t_f]$ be an input satisfying (3.1) that steers the state of the fractional positive electrical circuit (2.2) from $x_0 = 0$ to $x_f \in \Re_+^n$. Then the input (3.4) satisfying (3.1) also steers the state of the electrical circuit from $x_0 = 0$ to $x_f \in \Re_+^n$ and minimizes the performance index (3.2), i.e., $I(\hat{u}) \leq I(\bar{u})$.*

The minimal value of the performance index (3.2) is equal to

$$I(\hat{u}) = x_f^{\mathrm{T}}W^{-1}(t_f)x_f. \tag{3.6}$$

*Proof* If the conditions (3.5) are met, then the input (3.4) is well defined and $\hat{u}(t) \in \Re_+^n$ for $t \in [0, t_f]$. We shall show that the input steers the state of the electrical circuit from $x_0 = 0$ to $x_f \in \Re_+^n$. Substitution of (3.4) into (2.3) for $t = t_f$ and $x_0 = 0$ yields

$$x(t_f) = \int_0^{t_f} \Phi(t_f - \tau)B\hat{u}(\tau)\mathrm{d}\tau = \int_0^{t_f} \Phi(t_f - \tau)BQ^{-1}B^{\mathrm{T}}\Phi^{\mathrm{T}}(t_f - \tau)\mathrm{d}\tau\, W_f^{-1}x_f = x_f \tag{3.7a}$$

since (3.3) holds. By assuming the inputs $\bar{u}(t)$ and $\hat{u}(t)$, $t \in [0, t_f]$ steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$, i.e.,

$$x_f = \int_0^{t_f} \Phi(t_f - \tau)B\bar{u}(\tau)\mathrm{d}\tau = \int_0^{t_f} \Phi(t_f - \tau)B\hat{u}(\tau)\mathrm{d}\tau \tag{3.7b}$$

and

$$\int_0^{t_f} \Phi(t_f - \tau)B[\bar{u}(\tau) - \hat{u}(\tau)]\mathrm{d}\tau = 0. \tag{3.7c}$$

By transposition of (3.7c) and postmultiplication by $W^{-1}(t_f)x_f$, we obtain

$$\int_0^{t_f} [\bar{u}(\tau) - \hat{u}(\tau)]^{\mathrm{T}}B^{\mathrm{T}}\Phi^{\mathrm{T}}(t_f - \tau)\mathrm{d}\tau\, W^{-1}(t_f)x_f = 0. \tag{3.8}$$

Substitution of (3.4) into (3.8) yields

$$\int_0^{t_f} [\bar{u}(\tau) - \hat{u}(\tau)]^{\mathrm{T}}B^{\mathrm{T}}\Phi^{\mathrm{T}}(t_f - \tau)\mathrm{d}\tau\, W^{-1}x_f = \int_0^{t_f} [\bar{u}(\tau) - \hat{u}(\tau)]^{\mathrm{T}}Q\hat{u}(\tau)\mathrm{d}\tau = 0. \tag{3.9}$$

Using (3.9), it is easy to verify that

$$\int_0^{t_f} \bar{u}(\tau)^{\mathrm{T}} Q \bar{u}(\tau) \mathrm{d}\tau = \int_0^{t_f} \hat{u}(\tau)^{\mathrm{T}} Q \hat{u}(\tau) \mathrm{d}\tau + \int_0^{t_f} [\bar{u}(\tau) - \hat{u}(\tau)]^{\mathrm{T}} Q [\bar{u}(\tau) - \hat{u}(\tau)] \mathrm{d}\tau.$$

(3.10)

From (3.10), it follows that $I(\hat{u}) < I(\bar{u})$ since the second term in the right-hand side of the inequality is nonnegative. To find the minimal value of the performance index (3.2), we substitute (3.4) into (3.2) and we obtain

$$I(\hat{u}) = \int_0^{t_f} \hat{u}^{\mathrm{T}}(\tau) Q u(\hat{u}) \mathrm{d}\tau = x_f^{\mathrm{T}} W^{-1} \int_0^{t_f} \Phi(t_f - \tau) B Q^{-1} B^{\mathrm{T}} \Phi^{\mathrm{T}}(t_f - \tau) \mathrm{d}\tau W^{-1} x_f$$

$$= x_f^{\mathrm{T}} W^{-1} x_f$$

(3.11)

since (3.3) holds. $\qquad \square$

## 4 Procedure and Example

To find $t \in [0, t_f]$ for which $\hat{u}(t) \in \mathfrak{R}_+^n$ reaches its minimal value using (3.4), we compute the derivative

$$\frac{\mathrm{d}\hat{u}(t)}{\mathrm{d}t} = Q^{-1} B^{\mathrm{T}} \Psi(t) W^{-1}(t_f) x_f, \quad t \in [0, t_f] \tag{4.1}$$

where

$$\Psi(t) = \frac{\mathrm{d}}{\mathrm{d}t} [\Phi^{\mathrm{T}}(t_f - t)]. \tag{4.2}$$

Knowing $\Psi(t)$ and using the equality

$$\Psi(t) W^{-1}(t_f) x_f = 0 \tag{4.3}$$

we can find $t \in [0, t_f]$ for which $\hat{u}(t)$ reaches its maximal value.

Note that if the electrical circuit is asymptotically stable $\lim_{t \to \infty} \Phi(t) = 0$, then $\hat{u}(t)$ reaches its maximal value for $t = t_f$ and if it is unstable, then for $t = 0$.

From the above considerations, we have the following procedure for computation of the optimal inputs satisfying the condition (3.1) that steers the state of the system from $x_0 = 0$ to $x_f \in \mathfrak{R}_+^n$ and minimizes the performance index (3.2).
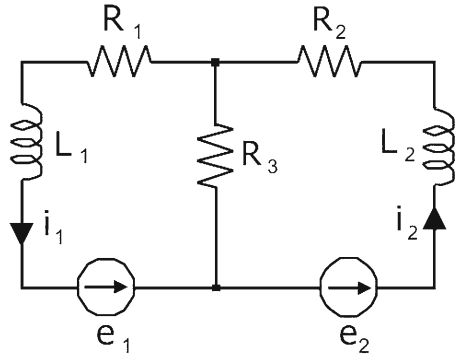
**Procedure 4.1** Step 1 Knowing $A \in M_n$ and using (2.5), compute $\Phi(t)$.
Step 2 Using (3.3), compute the matrix $W_f$ for given $A, B, Q, \alpha$ and some $t_f$.
Step 3 Using (3.4) and (4.3), find $t_f$ for which $\hat{u}(t)$ satisfying (3.1) reaches its maximal value and the desired $\hat{u}(t)$ for given $U \in \mathfrak{R}_+^n$ and $x_f \in \mathfrak{R}_+^n$.
Step 4 Using (3.6), compute the maximal value of the performance index.

**Fig. 1** Electrical circuit



*Example 4.1* Following [17], consider the fractional electrical circuit shown in Fig. 1 with given resistances $R_1$, $R_2$, $R_3$, inductances $L_1$, $L_2$ and source voltages $e_1$, $e_2$.

Using the Kirchhoff's laws, we can write the equations

$$e_1 = R_3(i_1 - i_2) + R_1 i_1 + L_1 \frac{d^\alpha i_1}{dt^\alpha}, \tag{4.4a}$$

$$e_2 = R_3(i_2 - i_1) + R_2 i_2 + L_2 \frac{d^\alpha i_2}{dt^\alpha} \tag{4.4b}$$

where $0 < \alpha < 1$, which can be written in the form

$$\frac{d^\alpha}{dt^\alpha} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{4.5a}$$

where

$$A = \begin{bmatrix} -\frac{R_1+R_3}{L_1} & \frac{R_3}{L_1} \\ \frac{R_3}{L_2} & -\frac{R_2+R_3}{L_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}. \tag{4.5b}$$

The fractional electrical circuit is positive since the matrix $A$ is a Metzler matrix and the matrix $B$ has nonnegative entries.

In [17], it was shown that the electrical circuit is reachable if $R_3 = 0$. In this case, the matrix $A$ has the form

$$A = \begin{bmatrix} -\frac{R_1}{L_1} & 0 \\ 0 & -\frac{R_2}{L_2} \end{bmatrix} \tag{4.6}$$

Consider the fractional positive reachable electrical circuit shown in Fig. 1 for $R_1 = R_2 = 1$, $R_3 = 0$, $L_1 = L_2 = 1$ and $\alpha = 0.7$. Compute the input $\hat{u}(t)$ satisfying the condition

$$\hat{u}(t) = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \le \begin{bmatrix} 20 \\ 20 \end{bmatrix} \quad \text{for} \quad t \in [0, t_f] \tag{4.7}$$

that steers the state of the electrical circuit from zero state to final state $x_f = [1 \ 1]^T$ ($T$ denotes the transpose) and minimizes the performance index (3.1) with

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \tag{4.8}$$

Using the Procedure 4.1, we obtain the following:

Step 1 Taking into account that

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{4.9}$$

and using (2.5), we obtain

$$\Phi(t) = \sum_{k=0}^{\infty} \frac{A^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]} = \sum_{k=0}^{\infty} \begin{bmatrix} (-1)^k & 0 \\ 0 & (-1)^k \end{bmatrix} \frac{t^{(k+1)0.7-1}}{\Gamma[(k+1)0.7]}. \tag{4.10}$$

Step 2 Using (3.3), (4.8), (4.9) and taking into account that $B = I_2$, we get

$$W(t_f) = \int_0^{t_f} \Phi(t_f - \tau)BQ^{-1}B^T\Phi^T(t_f - \tau)\mathrm{d}\tau = \int_0^{t_f} \Phi(\tau)BQ^{-1}B^T\Phi^T(\tau)\mathrm{d}\tau$$

$$= \frac{1}{2} \int_0^{t_f} \Phi^2(\tau)\mathrm{d}\tau = \frac{1}{2} \int_0^{t_f} \left[ \sum_{k=0}^{\infty} \begin{bmatrix} (-1)^k & 0 \\ 0 & (-1)^k \end{bmatrix} \frac{t^{(k+1)0.7-1}}{\Gamma[(k+1)0.7]} \right]^2 \mathrm{d}\tau$$

$$= \frac{1}{2} \int_0^{t_f} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \begin{bmatrix} (-1)^{k+l} & 0 \\ 0 & (-1)^{k+l} \end{bmatrix} \frac{\tau^{(k+l)0.7-0.6}}{\Gamma[(k+1)0.7]\Gamma[(l+1)0.7]} \mathrm{d}\tau$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \begin{bmatrix} (-1)^{k+l} & 0 \\ 0 & (-1)^{k+l} \end{bmatrix} \frac{1}{\Gamma[(k+1)0.7]\Gamma[(l+1)0.7]} \frac{t_f^{(k+l)0.7+0.4}}{(k+l)0.7+0.4}$$

$$= \begin{bmatrix} 0.3729 & 0 \\ 0 & 0.3729 \end{bmatrix}. \tag{4.11}$$

Step 3 Using (2.7) and (2.8), we obtain

$$\hat{u}(t) = Q^{-1}B^T\Phi^T(t_f - t)W^{-1}(t_f)x_f$$

$$= \frac{1}{2} \left( \sum_{k=0}^{\infty} \begin{bmatrix} (-1)^k & 0 \\ 0 & (-1)^k \end{bmatrix} \frac{(t_f - t)^{(k+1)0.7-1}}{\Gamma[(k+1)0.7]} \right) W^{-1}(t_f) \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{4.12}$$

Note that the electrical circuit is stable. Therefore, $\hat{u}(t)$ reach its maximal value for $t = t_f$ (Fig. 2).
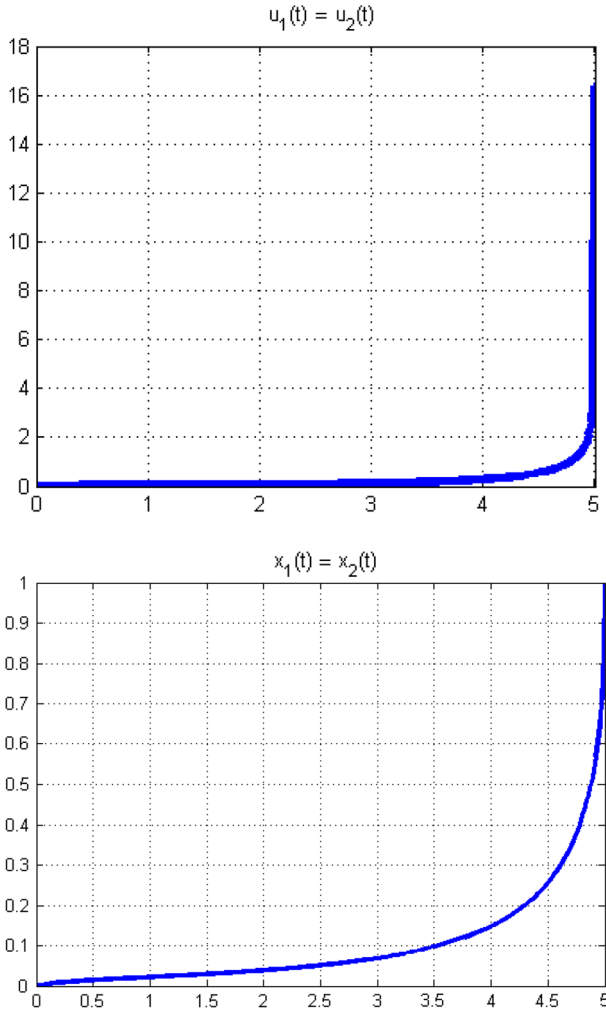
**Fig. 2** Input signal and state variable for $t_f = 5$[s]

Step 4 From (3.6), we have the minimal value of the performance index

$$I(\hat{u}) = x_f^T W^{-1}(t_f) x_f \tag{4.13}$$

where $W(t_f)$ is given by (4.11).

## 5 Extension to Fractional Positive Electrical Circuits with Different Orders

Consider an electrical circuit composed of resistors, $n$ capacitors and $m$ voltage (current) sources. Using the Kirchhoff's laws, we may describe the transient states in the electrical circuit by the fractional differential equation

$$\frac{\mathrm{d}^{\alpha} x(t)}{\mathrm{d}t^{\alpha}} = Ax(t) + Bu(t), \quad 0 < \alpha < 1, \tag{5.1}$$

where $x(t) \in \Re^{n}$, $u(t) \in \Re^{m}$, $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$. The components of the state vector $x(t)$ and input vector $u(t)$ are the voltages on the capacitors and source voltages, respectively. Similarly, using the Kirchhoff's laws, we may describe the transient states in the electrical circuit composed of resistances, inductances and voltage (current) sources by the fractional differential equation

$$\frac{\mathrm{d}^{\beta} x(t)}{\mathrm{d}t^{\beta}} = Ax(t) + Bu(t), \quad 0 < \beta < 1, \tag{5.2}$$

where $x(t) \in \Re^{n}$, $u(t) \in \Re^{m}$, $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$. In this case, the components of the state vector $x(t)$ are the currents in the coils.

Now let us consider electrical circuit composed of resistors, capacitors, coils and voltage (current) sources. As the state variables (the components of the state vector $x(t)$), we choose the voltages on the capacitors and the currents in the coils. Using Eqs. (5.1), (5.2) and Kirchhoff's laws, we may write for the fractional linear circuits in the transient states the state equation

$$\begin{bmatrix} \frac{\mathrm{d}^{\alpha} x_C}{\mathrm{d}t^{\alpha}} \\ \frac{\mathrm{d}^{\beta} x_L}{\mathrm{d}t^{\beta}} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_C \\ x_L \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \quad 0 < \alpha, \beta < 1, \tag{5.3}$$

where the components of $x_C \in \Re^{n_1}$ are voltages on the capacitors, the components of $x_L \in \Re^{n_2}$ are currents in the coils and the components of $u \in \Re^{m}$ are the source voltages and

$$A_{ij} \in \Re^{n_i \times n_j}, \quad B_i \in \Re^{n_i \times m}, \quad i, j = 1, 2. \tag{5.4}$$

**Theorem 5.1** *The solution of* Eq. (5.3) *for* $0 < \alpha < 1; 0 < \beta < 1$ *with initial conditions*

$$x_C(0) = x_{10} \quad and \quad x_L(0) = x_{20} \tag{5.5}$$

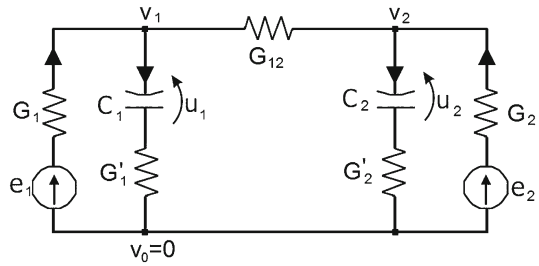*has the form*

$$x(t) = \Phi_0(t)x_0 + \int_0^t [\Phi_1(t-\tau)B_{10} + \Phi_2(t-\tau)B_{01}] u(\tau)\mathrm{d}\tau, \tag{5.6a}$$

*where*

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad x_0 = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \quad B_{10} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad B_{01} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix},$$

**Fig. 3** Fractional electrical circuit



$$T_{kl} = \begin{cases} I_n & for \quad k = l = 0 \\ \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix} & for \quad k = 1, l = 0 \\ \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} & for \quad k = 0, l = 1 \\ T_{10}T_{k-1,l} + T_{01}T_{k,l-1} & for \quad k + l > 0 \end{cases}$$

$$\Phi_0(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T_{kl} \frac{t^{k\alpha+l\beta}}{\Gamma(k\alpha + l\beta + 1)},$$

$$\Phi_1(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T_{kl} \frac{t^{(k+1)\alpha+l\beta-1}}{\Gamma[(k + 1)\alpha + l\beta]},$$

$$\Phi_2(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} T_{kl} \frac{t^{k\alpha+(l+1)\beta-1}}{\Gamma[k\alpha + (l + 1)\beta]}. \tag{5.6b}$$

*Proof is given in [18,21].*

The extension of Theorem 5.1 to systems consisting of $n$ subsystems with different fractional orders is given in [18,21].

*Example 5.1* Consider the fractional electrical circuit shown in Fig. 3 [19,31,32] with given source voltages $e_1, e_2$, ultracapacitor $C_1 = 1$ of the fractional order $\alpha = 0.7$, ultracapacitor $C_2 = 2$ of the fractional order $\beta = 0.6$, conductances $G_1 = 4, G'_1 = 4, G_2 = 3, G'_2 = 6, G_{12} = 0$ and $N = n_1 + n_2 = 2$.

Using the Kirchhoff's laws, we can write the equations

$$C_1 \frac{d^\alpha u_1}{dt^\alpha} = G'_1(v_1 - u_1),$$

$$C_2 \frac{d^\beta u_2}{dt^\beta} = G'_2(v_2 - u_2) \tag{5.7}$$

and

$$\begin{bmatrix} (G_1 + G'_1) & 0 \\ 0 & (G_2 + G'_2) \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} G'_1 & 0 \\ 0 & G'_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{5.8}$$

From (5.8), we obtain

$$
\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} (G_1 + G_1') & 0 \\ 0 & (G_2 + G_2') \end{bmatrix}^{-1} \begin{bmatrix} G_1' & 0 \\ 0 & G_2' \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}
$$
$$
+ \begin{bmatrix} (G_1 + G_1') & 0 \\ 0 & (G_2 + G_2') \end{bmatrix}^{-1} \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}. \tag{5.9}
$$

Substitution of (5.9) into

$$
\begin{bmatrix} \frac{d^\alpha u_1}{dt^\alpha} \\ \frac{d^\beta u_2}{dt^\beta} \end{bmatrix} = \begin{bmatrix} -\frac{G_1'}{C_1} & 0 \\ 0 & -\frac{G_2'}{C_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} \frac{G_1'}{C_1} & 0 \\ 0 & \frac{G_2'}{C_2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{5.10}
$$

we obtain

$$
\begin{bmatrix} \frac{d^\alpha u_1}{dt^\alpha} \\ \frac{d^\beta u_2}{dt^\beta} \end{bmatrix} = A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \tag{5.11}
$$

where

$$
A = \begin{bmatrix} \frac{G_1'}{C_1}\left(\frac{G_1'}{(G_1+G_1')} - 1\right) & 0 \\ 0 & \frac{G_2'}{C_2}\left(\frac{G_2'}{(G_2+G_2')} - 1\right) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},
$$
$$
B = \begin{bmatrix} \frac{G_1' G_1}{C_1(G_1+G_1')} & 0 \\ 0 & \frac{G_2' G_2}{C_2(G_2+G_2')} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}. \tag{5.12}
$$

From (5.12), it follows that $A$ is a diagonal Metzler matrix and the matrix $B$ is mono-mial matrix with positive diagonal entries. Therefore, the fractional electrical circuit is positive for all values of the conductances and capacitances.

Find the optimal input (source voltage) $\hat{e}(t) \in \mathfrak{R}_+^2$, $t \in [0, t_f]$ satisfying the condition

$$
\hat{e}(t) = \begin{bmatrix} \hat{e}_1(t) \\ \hat{e}_2(t) \end{bmatrix} < \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{for} \quad t \in [0, t_f] \tag{5.13}
$$

for the performance index (3.2) with $Q = \text{diag}[2, 2]$ which steers the system from initial state (voltage drop on capacitances) $u_0 = [0\ 0]^T$ to the finite state $u_f = [2\ 3]^T$ and minimizes the performance index (3.2) with (5.12).

Using (5.6) and (5.12), we obtain

$$
\Phi_1(t) B_{10} + \Phi_2(t) B_{01} = \begin{bmatrix} \Phi_{11}^1(t) & 0 \\ 0 & \Phi_{22}^2(t) \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = M(t), \tag{5.14}
$$

where

$$\Phi_{11}^1(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{t^{(k+1)0.7+l0.6-1}}{\Gamma\,[(k+1)0.7+l0.6]}(-2)^k,$$

$$\Phi_{22}^2(t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{t^{k0.7+(l+1)0.6-1}}{\Gamma\,[k0.7+(l+1)0.6]}(-1)^l. \qquad (5.15)$$

From (5.6a), (5.12) and (3.3), we have

$$W(t_f) = \int_0^{t_f} M(t_f-\tau)Q^{-1}M^{\mathrm{T}}(t_f-\tau)\mathrm{d}\tau$$

$$= \int_0^{t_f} \begin{bmatrix} \Phi_{11}^1(\tau) & 0 \\ 0 & \Phi_{22}^2(\tau) \end{bmatrix}\begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} \Phi_{11}^1(\tau) & 0 \\ 0 & \Phi_{22}^2(\tau) \end{bmatrix}^{\mathrm{T}}\mathrm{d}\tau. \quad (5.16)$$

Note that the electrical circuit is stable. Therefore, $\hat{e}(t)$ reach its maximal value for $t = t_f$.

Now using (3.4) and (5.16), we obtain

$$\hat{e}(t) = Q^{-1}M^{\mathrm{T}}(t_f-t)W^{-1}(t_f)u_f$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{bmatrix} \Phi_{11}^1(t_f-t) & 0 \\ 0 & \Phi_{22}^2(t_f-t) \end{bmatrix}^{\mathrm{T}}W^{-1}(t_f)\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \Phi_{11}^1(t_f-t) & 0 \\ 0 & 0.5\Phi_{22}^2(t_f-t) \end{bmatrix}^{\mathrm{T}}\left[\int_0^{t_f}\begin{bmatrix} 8\Phi_{11}^1(\tau)[\Phi_{11}^1(\tau)]^{\mathrm{T}} & 0 \\ 0 & 2\Phi_{22}^2(\tau)[\Phi_{22}^2(\tau)]^{\mathrm{T}} \end{bmatrix}\mathrm{d}\tau\right]^{-1}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0.25[\Phi_{11}^1(t_f-t)]^{\mathrm{T}}\int_0^{t_f}[\Phi_{11}^1(\tau)[\Phi_{11}^1(\tau)]^{\mathrm{T}}]^{-1}\mathrm{d}\tau \\ 1.5[\Phi_{22}^2(t_f-t)]^{\mathrm{T}}\int_0^{t_f}[\Phi_{22}^2(\tau)[\Phi_{22}^2(\tau)]^{\mathrm{T}}]^{-1}\mathrm{d}\tau \end{bmatrix}, \qquad (5.17)$$

The minimal value of $t_f$ satisfying the condition (3.1) can be found from the inequality

$$\begin{bmatrix} 0.25[\Phi_{11}^1(t_f-t)]^{\mathrm{T}}\int_0^{t_f}[\Phi_{11}^1(\tau)[\Phi_{11}^1(\tau)]^{\mathrm{T}}]^{-1}\mathrm{d}\tau \\ 1.5[\Phi_{22}^2(t_f-t)]^{\mathrm{T}}\int_0^{t_f}[\Phi_{22}^2(\tau)[\Phi_{22}^2(\tau)]^{\mathrm{T}}]^{-1}\mathrm{d}\tau \end{bmatrix} < \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{for } t \in [0, t_f]$$

$$(5.18)$$

and the minimal value of the performance index (3.2) is equal to

$$I(\hat{e}) = u_f^{\mathrm{T}} W^{-1} u_f = [\,2\ 3\,] \left[ \int\limits_0^{t_f} \begin{bmatrix} 8\Phi_{11}^1(\tau)[\Phi_{11}^1(\tau)]^{\mathrm{T}} & 0 \\ 0 & 2\Phi_{22}^2(\tau)[\Phi_{22}^2(\tau)]^{\mathrm{T}} \end{bmatrix} \mathrm{d}\tau \right]^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= 0.5 \int\limits_0^{t_f} [\Phi_{11}^1(\tau)[\Phi_{11}^1(\tau)]^{\mathrm{T}}]^{-1}\mathrm{d}\tau + 3 \int\limits_0^{t_f} [\Phi_{22}^2(\tau)[\Phi_{22}^2(\tau)]^{\mathrm{T}}]^{-1}\mathrm{d}\tau. \qquad (5.19)$$

## 6 Concluding Remarks

Necessary and sufficient conditions for the reachability of the fractional positive electrical circuits have been established (Theorem 2.3). The minimum energy control problem for the fractional positive electrical circuits with bounded inputs has been formulated and solved. Sufficient conditions for the existence of a solution to the problem have been given (Theorem 3.1), and a procedure for computation of optimal input and the minimal value of performance index has been proposed. The effectiveness of the procedure has been demonstrated on the example of fractional positive electrical circuit. The presented method can be extended to positive discrete-time linear systems and to fractional positive discrete-time linear systems with bounded inputs.

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