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Analysis and Design of H_∞ Controllers for 2D Singular Systems with Delays

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Abstract The H_∞ control design problem is solved for the class of 2D discrete singular systems with delays. More precisely, the problem addressed is the design of state-feedback controllers such that the acceptability, internal stability and causality of the resulting closed-loop system are guaranteed, while a prescribed H_∞ performance level is simultaneously fulfilled. By establishing a novel version of the bounded real lemma, a linear matrix inequality condition is derived for the existence of these H_∞ controllers. They can then be designed by solving an iterative algorithm based on LMI optimizations. An illustrative example shows the applicability of the algorithm proposed.

Keywords Two-dimensional systems · Systems with delays · H_∞ control · Singular systems

1 Introduction

Two-dimensional (2D) systems appear frequently in practical problems, so they have been extensively studied over the last fewdecades, with many important results

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reported in the literature [5,21]. Of these previous results, the H_∞ filtering problem for 2D linear systems has been studied in [2,7,8,10–12,15,27,28,30–34,41]; for 2D linear parameter-varying systems, the related work can be found in [9,32]; for 2D systems with delays, this filtering problem has been investigated in [27,31]; the stability and stabilization of 2D systems have been solved in [1,17–19,26], while the H_∞ control for 2D nonlinear systems with delays and the nonfragile H_∞ and $l_2 - l_1$ problems were studied in [36]. Nonetheless, as no systematic and general approach to analyze 2D SRM systems exists, there are still many unsolved problems.

This paper focuses on 2D singular systems, which have not been fully studied in the literature, even though they have extensive applications [6,20]. Some fundamental results in 1D singular systems have already been extended to 2D singular systems. For example, the minimum energy control problem for 2D singular models with shift-invariant and shift-varying coefficients was solved in [35,38]; [23] extended the geometric method to the 2D singular case; the input admissibility of 2D singular systems was investigated in [22], whereas [42] proposed an asymptotic stability theory based on the concept of jump modes. It should be emphasized that the admissibility and jump modes play an important role when analyzing the robust stability of 2D singular systems [42]. In fact, the presence of jump modes means that the system is noncausal, and its structural stability will be violated. Hence, in most synthesis studies (such as the robust H_∞ control [37]), the closed-loop system is required to be free of jump modes. The class of 2D singular systems studied here is 2D singular Roesser models, as they are the simplest and most popular 2D singular system models. In appearance, they resemble 1D singular systems, but there is no equivalent to the 1D Kronecker canonical form (which is the basis of many developments of 1D singular systems). This makes 2D singular systems more difficult to study. For example, the problems of robust H_∞ control, model reduction and duality have already been shown to be significantly difficult to solve [37,39,43].

Thus, this paper provides new techniques for analyzing and designing H_∞ controllers for uncertain 2D singular systems with state delays. Given a 2D system described by a Roesser-like model, the focus is on analyzing and designing state-feedback controllers such that the closed-loop 2D system is acceptable, asymptotically stable, causal and has a prescribed H_∞ disturbance attenuation performance. A new version of the bounded real lemma (BRL) is provided for this class of systems, based on slack variables matrices, to provide more flexibility for H_∞ controller design. In the literature, [40] used a particular choice of state-feedback control and imposed the same delays in the horizontal and vertical states, which reduces the achievable H_∞ performance. It must be pointed out that the BRL leads to bilinear matrix inequality (BMI) conditions for state-feedback design, so an iterative algorithm is provided to characterize the existence of admissible state-feedback controllers. Based on this result, the state-feedback design problem is converted into a convex optimization problem, which can be readily solved via standard numerical software.

Notations: For real symmetric matrices X and Y, the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite). $*$ stands for the symmetric terms of a square symmetric matrix. I denotes the identity matrix with appropriate dimension. The superscript ^{T} represents the transpose of a matrix. diag(...) stands for a block-diagonal matrix. For a given matrix $B \in$

$\mathbb{R}^{m\times n}$ (respectively, $B \in \mathbb{R}^{n\times m}$) such that $\text{rank}(B) = r$, we define $B^{\perp} \in \mathbb{R}^{n\times(n-r)}$ (respectively, $B^{\perp} \in \mathbb{R}^{(n-r)\times n}$) as the right (respectively, left) orthogonal complement of B by $BB^{\perp} = 0$ (respectively, $B^{\perp}B = 0$) and $B^{\perp T}B^{\perp} > 0$ (respectively, $B^{\perp}B^{\perp T} > 0$). The Euclidean vector norm is denoted by $\| \,.\, \|$. The l_2 norm of a 2D signal $w(i, j)$ is given by

$$\| w(i,j) \|_2 = \sqrt{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w^T(i,j)w(i,j)}$$

where $w(i, j)$ is said to be in the space $l_2\{[0,\infty),[0,\infty)\}$ or l_2, for simplicity, if $\| w(i,j) \|_2 < \infty$.

2 Preliminaries

Consider first a 2D singular Roesser model (2D SRM) of the following form:

$$\begin{aligned} E\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} &= A\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + A_d\begin{bmatrix} x^h(i-d_1,j) \\ x^v(i,j-d_2) \end{bmatrix} + Bu(i,j) + Lw(i,j) \\ z(i,j) &= H\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Gw(i,j) \end{aligned} \tag{2.1}$$

with the boundary conditions

$$\begin{aligned} &\{x^h(\phi,j) = 0\}, \forall j \geq 0, \phi = -d_1, -d_1+1, \ldots, 0; \\ &\{x^v(i,\varphi) = 0\}, \forall i \geq 0, \phi = -d_2, -d_2+1, \ldots, 0; \end{aligned} \tag{2.2}$$

where $x^h(i,j) \in \mathbb{R}^{n_h}$ and $x^v(i,j) \in \mathbb{R}^{n_v}$ are the horizontal and vertical states; $w(i,j) \in \mathbb{R}^q$ is the disturbance (or noise) vector, which belongs to $l_2\{[0,\infty),[0,\infty)\}$; $u(i,j) \in \mathbb{R}^m$ is the input vector; $z(i,j) \in \mathbb{R}^p$ is the controlled output vector; d_1, $d_2 > 0$ are constant delays; A, A_d, B, L, H and G are known constant real matrices with appropriate dimensions; E is singular, satisfying the 2D regular pencil condition: For some finite pairs (z, w), the following holds:

$$\det[EI(z^{d_1+1}, w^{d_2+1}) - AI(z^{d_1}, w^{d_2}) - A_d] = \sum_{k=0}^{\bar{n}_1}\sum_{l=0}^{\bar{n}_2} a_{kl}z^k w^l$$

where $I(z,w) = \text{diag}\{zI_{n_h}, wI_{n_v}\}$, where $a_{\bar{n}_1,0} \neq 0$ and $a_{0,\bar{n}_2} \neq 0$.

When $a_{\bar{n}_1,\bar{n}_2} \neq 0$, system (2.1) is called *acceptable* [21,42]: It has been shown in [42] that nonacceptable systems are ill-posed.

The *jump modes* of the 2D SRM (2.1) and (2.2) can be defined equivalently by the nonzero positive power items ($a_{ij}z^i w^j$, $i > 0$ or $j > 0$) in the Laurent expansion of the matrices $[EI(z,w) - A - A_d I(z^{d_1}, w^{d_2})]^{-1}$, $1 \leq |z| < \infty$, $1 \leq |w| < \infty$ [4]. The lack of jump modes has been shown equivalent to causality. In fact , [42] suggested that if the 2D acceptable SRM (2.1)–(2.2) is causal, then it can be transformed into a standard form via linear transformations, with $E = \mathrm{diag}(E_h, E_v)$, where $E_h \in \mathbb{R}^{n_h \times n_h}$, $E_v \in \mathbb{R}^{n_v \times n_v}$. Therefore, for simplicity and convenience, we use the following assumption:

Assumption 2.1 $E = \mathrm{diag}(E_h, E_v)$, where $E_h \in \mathbb{R}^{n_h \times n_h}$, $E_v \in \mathbb{R}^{n_v \times n_v}$, and $n_h + n_v = n$.

Lemma 2.2 *The 2D SRM system* (2.1) *is acceptable and internally stable if and only if*

$$p(z, w) \neq 0, \ \ for\ 0 <\mid z \mid\leq 1, \quad 0 <\mid w \mid\leq 1, \tag{2.3}$$

or

$$c(z, w) \neq 0,\ for\ (z, w) \in D^2,\ \ D^2 = \{(z, w), 1 \leq\mid z \mid< \infty, 1 \leq\mid w \mid< \infty\}, \tag{2.4}$$

where $p(z, w) = det[E - AI(z, w) - A_d I(z^{d_1+1}, w^{d_2+1})]$ *and* $c(z, w) = det[EI(z, w) - A - A_d I(z^{-d_1}, w^{-d_2})]$.

Proof 2.3 Similar to that of Lemma 5 in [40], so it is omitted.

Lemma 2.4 *The acceptable 2D SRM system* (2.1) *is causal if and only if*

$$deg(s^{(n_h d_1 + n_v d_2)}\ \ det[sE - A - s^{-(d_1+d_2)} A_d]) = (n_h d_1 + n_v d_2) + rankE \tag{2.5}$$

$$rankE = rankE_h + rankE_v \tag{2.6}$$

Proof 2.5 The proof is similar to that of Lemma 6 in [40], so it is omitted. □

A state-feedback controller of the following form is used:

$$u(i, j) = K \begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix}. \tag{2.7}$$

If $A + BK$ is denoted as A_c, this controller gives the following closed-loop system:

$$\begin{aligned} E \begin{bmatrix} x^h(i+1, j) \\ x^v(i, j+1) \end{bmatrix} &= A_c \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + A_d \begin{bmatrix} x^h(i-d_1, j) \\ x^v(i, j-d_2) \end{bmatrix} + Lw(i, j) \\ z(i, j) &= H \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix} + Gw(i, j). \end{aligned} \tag{2.8}$$

When (2.8) is regular, its transfer function is given by

$$G(z, w) = H[EI(z, w) - A_c - A_d I(z^{-d_1}, w^{-d_2})]^{-1} L + G,$$

so its H_∞ norm is, by definition

$$\|G(z, w)\|_\infty = \sup_{z, w \in [0, 2\pi]} \sigma_{\max}[G(e^{jz}, e^{jw})]$$

where $\sigma_{\max}$ denotes the maximum singular value of a matrix.

Remark 2.6 By using the 2D Parseval's theorem [24], it is not difficult to show that, under zero boundary conditions and with the internal stability of (2.8), the condition $\|G(z, w)\|_\infty < \gamma$ is equivalent to

$$\sup_{0\neq w(i,j)\in\ell_2} \frac{\|z(i, j)\|_2}{\|w(i, j)\|_2} < \gamma.$$

Then, the H_∞ state-feedback control problem addressed in this paper can be expressed as follows: Given the 2D SRM system (2.1), design a suitable state-feedback controller (2.7) such that the following two requirements are satisfied:

1. The closed-loop system (2.8) with $w(i, j) \equiv 0$ is acceptable, internally stable and jump-mode free [42].
2. Under zero boundary conditions, the H_∞ performance $\|G(z, w)\|_\infty < \gamma$ is guaranteed for all nonzero $w(i, j) \in l_2$.

We conclude this section by introducing several lemmas, which will be used in the proof of our main results.

Lemma 2.7 *The 2D SRM delayed system* (2.8) *with* $w(i, j) \equiv 0$ *is acceptable, internally stable and causal if there exist a symmetric matrix* $P = diag(P_h, P_v) \in \mathbb{R}^{(n_h+n_v)\times(n_h+n_v)}$ *and a matrix* $Q = diag(Q_h, Q_v) > 0 \in \mathbb{R}^{(n_h+n_v)\times(n_h+n_v)}$ *such that*

$$E^T PE \geq 0 \tag{2.9}$$

$$\begin{bmatrix} A_c^T PA_c - E^T PE + Q & A_c^T PA_d \\ * & A_d^T PA_d - Q \end{bmatrix} < 0. \tag{2.10}$$

Proof 2.8 We prove Lemma 2.7 by contradiction: Suppose that the conditions 2.9–2.10 are satisfied, but the system 2.8 is unstable. Then, there exists $(z, w) \in D^2$ such that

$$\det[EI(z, w) - A_c - A_d I(z^{-d_1}, w^{-d_2})] = 0. \tag{2.11}$$

Hence, there exists a vector $\upsilon \neq 0$ such that

$$EI(z, w)\upsilon = [A_c + A_d I(z^{-d_1}, w^{-d_2})]\upsilon \tag{2.12}$$

It is easy to see that

$$\begin{aligned} &\upsilon^*[I(z^*, w^*)E^T PEI(z, w) - E^T PE]\upsilon \\ &= \upsilon^*[A_c^T PA_c - E^T PE \\ &\quad + A_c^T PA_d I(z^{-d_1}, w^{-d_2}) \\ &\quad + I(z^{*-d_1}, w^{*-d_2})A_d^T PA_c \\ &\quad + I(z^{*-d_1}, w^{*-d_2})A_d^T PA_d I(z^{-d_1}, w^{-d_2})]\upsilon \end{aligned} \tag{2.13}$$

By applying the Schur complement formula to (2.10), we obtain

$$A_c^T PA_c - E^T PE + Q + A_c^T PA_d (Q - A_d^T PA_d)^{-1} A_d^T PA_c < 0. \tag{2.14}$$

It follows from $Q > 0$ and $(z, w) \in D^2$ that

$$\begin{aligned}(I(z^{*-d_1}, w^{*-d_2})(Q - A_d^T PA_d)I(z^{-d_1}, w^{-d_2}) - A_c^T PA_d I(z^{-d_1}, w^{-d_2}))\\ \times(I(z^{*-d_1}, w^{*-d_2})(Q - A_d^T PA_d)I(z^{-d_1}, w^{-d_2}))^{-1}\\ \times(I(z^{*-d_1}, w^{*-d_2})(Q - A_d^T PA_d)I(z^{-d_1}, w^{-d_2}) - I(z^{*-d_1}, w^{*-d_2})A_d^T PA_c) \geq 0\end{aligned} \tag{2.15}$$

which then implies

$$\begin{aligned}A_c^T PA_d I(z^{-d_1}, w^{-d_2}) + I(z^{*-d_1}, w^{*-d_2})A_d^T PA_c\\ +I(z^{*-d_1}, w^{*-d_2})A_d^T PA_d I(z^{-d_1}, w^{-d_2})\\ \leq Q + A_c^T PA_d (Q - A_d^T PA_d)^{-1} A_d^T PA_c.\end{aligned} \tag{2.16}$$

This, together with (2.14) and $\upsilon \neq 0$, means that the right-hand side of (2.13) is negative. On the other hand, $(z, w) \in D^2$ and $E^T PE \geq 0$ implies that $\mathrm{diag}(E_h^T P_h E_h, E_v^T P_v E_v) \geq 0$: Therefore, the left-hand side of (2.13) is nonnegative, leading to a contradiction, which completes the proof. □

We now introduce the following Lyapunov–Krasovskii functional:

$$V(i, j) \triangleq V_1(i, j) + V_2(i, j) \tag{2.17}$$

$$V_1(i, j) \triangleq x^{hT}(i, j)E_h^T P_h E_h x^h(i, j) + \sum_{i-d_1}^{i} x^{hT}(i, j)Q_h x^h(i, j) \tag{2.18}$$

$$V_2(i, j) \triangleq x^{vT}(i, j)E_v^T P_v E_v x^v(i, j) + \sum_{j-d_2}^{j} x^{vT}(i, j)Q_v x^v(i, j) \tag{2.19}$$

with the associated unidirectional variation of $V(i, j)$ in (2.8) defined as in [14]:

$$\Delta V(i, j) \triangleq \Delta V_1(i, j) + \Delta V_2(i, j) \tag{2.20}$$

where $\Delta V_1(i, j) \triangleq V_1(i + 1, j) - V_1(i, j)$, $\Delta V_2(i, j) \triangleq V_2(i, j + 1) - V_2(i, j)$, $E_h^T P_h E_h \geq 0$, $E_v^T P_v E_v \geq 0$, $Q_h > 0$ and $Q_v > 0$. Then, we have the following result:

Lemma 2.9 *The 2D SRM delayed system* (2.8) *is acceptable, internally stable and causal if*

$$\Delta V(i, j) < 0 \tag{2.21}$$

Proof 2.10 By calculation,

$$\Delta V(i,j) = \xi(i,j)\begin{bmatrix} A_c^T PA_c - E^T PE + Q & A_c^T PA_d \\ * & A_d^T PA_d - Q \end{bmatrix}\xi(i,j) \tag{2.22}$$

where $\xi(i,j) = [x^{hT}(i,j) \quad x^{vT}(i,j) \quad x^{hT}(i-d_1,j) \quad x^{vT}(i,j-d_2)]^T$.
Now, for any $\xi(i,j) \neq 0$, $\Delta V(i,j) < 0$ requires that
$\begin{bmatrix} A_c^T P A_c - E^T P E + Q & A_c^T PA_d \\ * & A_d^T PA_d - Q \end{bmatrix} < 0$, so the proof is completed, by using Lemma 2.7. □

Lemma 2.11 *Given a scalar $\gamma > 0$, the 2D SRM with delay (2.8) is acceptable, internally stable, causal and satisfies $\|G(z,w)\|_\infty < \gamma$ if there exist a symmetric matrix $P = diag(P_h, P_v) \in \mathbb{R}^{n_h \times n_v}$ and a matrix $Q = diag(Q_h, Q_v) > 0 \in \mathbb{R}^{(n_h+n_v)\times(n_h+n_v)}$ such that the following LMI holds:*

$$E^T P E \geq 0 \tag{2.23}$$

$$\begin{bmatrix} A_c^T P A_c - E^T P E + Q & A_c^T P A_d & A_c^T PL & H^T \\ * & A_d^T PA_d - Q & A_d^T PL & 0 \\ * & * & -\gamma^2 I + L^T PL & G^T \\ * & * & * & -I \end{bmatrix} < 0 \tag{2.24}$$

Proof 2.12 We shall show that $\|e(i,j)\|_2 < \gamma\|w(i,j)\|_2$ under zero boundary conditions for any nonzero $w(i,j) \in l_2$. For this, consider the following associated performance index:

$$\daleth = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} z^T(i,j)z(i,j) - \gamma^2 w^T(i,j)w(i,j)$$

Inspired by [2,29] and [12], we have that

$$\begin{aligned}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta V(i,j) &\triangleq \sum_{i=0}^{\infty}\Delta V_1(i,j) + \sum_{j=0}^{\infty}\Delta V_2(i,j) \\ &= V_1(\infty,j) - V_1(0,j) + V_2(i,\infty) - V_2(i,0).\end{aligned} \tag{2.25}$$

Under zero boundary conditions, $V_1(0,j) = V_2(i,0) = 0$, $V_1(\infty,j) \geq 0$ and $V_2(i,\infty) \geq 0$. Thus, from (2.25), we can deduce that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\Delta V(i,j) = V_1(\infty,j) + V_2(i,\infty) \geq 0.$$

Then,

$$\daleth \leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} z^T(i,j)z(i,j) - \gamma^2 w^T(i,j)w(i,j) + \Delta V(i,j)$$
$$\leq \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \psi^T(i,j)\Xi\psi(i,j)$$

where

$$\psi(i,j) = [x^{hT}(i,j) \quad x^{vT}(i,j) \quad x^{hT}(i-d_1,j) \quad x^{vT}(i,j-d_2) \quad w^T(i,j)]^T$$

and

$$\Xi = \begin{bmatrix} A_c^T PA_c - E^T PE + Q + H^T H & A_c^T PA_d & A_c^T PL + H^T G \\ * & A_d^T PA_d - Q & A_d^T P B \\ * & * & -\gamma^2 I + B^T PB + G^T G \end{bmatrix} \tag{2.26}$$

If $\Xi < 0$, then the LMI (2.24) is negative, so by the Schur complement (2.8), it is acceptable, internally stable, causal and satisfies $\|G(z,w)\|_\infty < \gamma$, which completes the proof. □

Lemma 2.13 *Let $\xi \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times n}$ with rank$(B) < n$ and $B^\perp$ such that $BB^\perp = 0$. Then, the following conditions are equivalent:*

(i) $\xi^T Q\xi < 0, \forall \xi \neq 0 : B\xi = 0$
(ii) $B^{\perp^T} QB^\perp < 0$
(iii) $\exists \mu \in \Re : Q - \mu B^T B < 0$
(iv) $\exists \chi \in \Re^{n\times m} : Q + \chi B + B^T \chi^T < 0$

3 Analysis of H_∞ Performance

We present now a novel bounded real lemma for 2D SRM with delays:

Theorem 3.1 *The 2D SRM with delay (2.1) with $u(i,j) \equiv 0$ is acceptable, internally stable, causal and satisfies $\|G(z,w)\|_\infty < \gamma$ if there exist a symmetric matrix $P = diag(P_h, P_v) \in \mathbb{R}^{(n_h+n_v)\times(n_h+n_v)}$, a matrix $Q = diag(Q_h, Q_v) > 0 \in \mathbb{R}^{(n_h+n_v)\times(n_h+n_v)}$ and matrices $S \in \mathbb{R}^{(n_h+n_v)\times(n_h+n_v)}$, $R \in \mathbb{R}^{(n_h+n_v)\times(n_h+n_v)}$, $M \in \mathbb{R}^{(n_h+n_v)\times(n_h+n_v)}$, $N \in \mathbb{R}^{q\times(n_h+n_v)}$ and $F \in \mathbb{R}^{p\times((n_h+n_v))}$ such that the following LMIs hold:*

$$E^T PE \geq 0 \tag{3.1}$$

$$\begin{bmatrix} \Theta_{11} & SA_d + A^T R^T & SL + A^T N^T & -S + A^T M^T & H^T + A^T F^T \\ * & \Theta_{22} & RL + A_d^T N^T & -R + A_d^T M^T & A_d^T F^T \\ * & * & \Theta_{33} & -N + L^T M^T & G^T + L^T F^T \\ * & * & * & -M - M^T + P & -F^T \\ * & * & * & * & -I_p \end{bmatrix} < 0 \tag{3.2}$$

where

$$\Theta_{11} = -E^T P E + SA + A^T S^T + Q$$
$$\Theta_{22} = -Q + A_d^T R^T + RA_d$$
$$\Theta_{33} = NL + L^T N^T - \gamma^2 I_q$$

Proof 3.2 The LMI condition (4.2) is obtained by considering $\chi = \begin{bmatrix} S & R & N & M & F \end{bmatrix}^T$, $\mathfrak{B} = \begin{bmatrix} A & A_d & L & -I & 0 \end{bmatrix}$, and

$$Q = \begin{bmatrix} -E^T P E + Q & 0 & 0 & 0 & H^T \\ * & -Q & 0 & 0 & 0 \\ * & * & -\gamma^2 I & 0 & G^T \\ * & * & * & P & 0 \\ * & * & * & * & -I \end{bmatrix}$$

in condition (iv) of Lemma 2.11, with

$$\mathfrak{B}^{\perp} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ A & A_d & B & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and then, by calculation and using the Schur complement and condition (ii) of Lemma 2.11, we can obtain the equivalence between $\mathfrak{B}^{\perp T} Q \mathfrak{B}^{\perp} < 0$ and LMIs (2.24), so (2.24) is equivalent to (4.2). Thus, Theorem 3.1 is equivalent to Lemma 2.9, completing the proof. □

Remark 3.3 By means of Finsler's lemma, in the proof of Theorem 3.1, the slack variables S, R, N, M and F are introduced to decouple the products $A_c^T P A_c$ in Lemma 2.11. This feature facilitates the design of the state-feedback controller.

Remark 3.4 The introduction of the slack variables S, R, N, M and F in Theorem 3.1 makes it possible to reduce the conservatism, as the system matrices are decoupled from the matrix variable P. Thus, our results provide additional flexibility for the analysis and design of state-feedback controllers, at the cost of increased computational complexity.

Remark 3.5 The proposed design method is general in the sense that nonsingular 2D Roesser models can also be treated using Theorem 3.1, fixing $E = I$.

4 State-Feedback H_∞ Controller Design

In the previous section, the H_∞ control analysis problem was studied for 2D SRM with delays. In this section, the design of controllers is solved using the following result:

Theorem 4.1 *The closed-loop 2D SRM with delayrm (2.8) is acceptable, internally stable and causal with prescribed* H_∞ *performance level* $\gamma > 0$ *if there exist a symmetric matrix* $P = diag(P_h, P_v) \in \mathbb{R}^{(n_h+n_v)\times(n_h+n_v)}$, *a matrix* $Q = diag(Q_h, Q_v) > 0 \in \mathbb{R}^{(n_h+n_v)\times(n_h+n_v)}$ *and matrices* $K \in \mathbb{R}^{m\times(n_h+n_v)}$, $S \in \mathbb{R}^{(n_h+n_v)\times(n_h+n_v)}$, $R \in \mathbb{R}^{(n_h+n_v)\times(n_h+n_v)}$, $M \in \mathbb{R}^{(n_h+n_v)\times(n_h+n_v)}$, $N \in \mathbb{R}^{q\times(n_h+n_v)}$ *and* $F \in \mathbb{R}^{p\times((n_h+n_v))}$ *such that the following inequalities hold:*

$$E^T PE \geq 0 \tag{4.1}$$

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} & \Lambda_{15} \\ * & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} & \Lambda_{24} \\ * & * & \Lambda_{33} & \Lambda_{34} & \Lambda_{35} \\ * & * & * & \Lambda_{44} & -F^T \\ * & * & * & * & -I_p \end{bmatrix} < 0 \tag{4.2}$$

where

$$\begin{aligned}
&\Lambda_{11} = -E^T PE + S(A + BK) + (A + BK)^T S^T + Q \\
&\Lambda_{12} = SA_d + (A + BK)^T R^T, \ \Lambda_{13} = SL + (A + BK)^T N^T \\
&\Lambda_{14} = -S + (A + BK)^T M^T, \ \Lambda_{15} = H^T + (A + BK)^T F^T \\
&\Lambda_{22} = -Q + A_d^T R^T + RA_d, \ \Lambda_{23} = RL + (A + BK)_d^T N^T \\
&\Lambda_{24} = -R + (A + BK)_d^T M^T, \ \Lambda_{25} = (A + BK)_d^T F^T \\
&\Lambda_{33} = NL + L^T N^T - \gamma^2 I_q, \ \Lambda_{34} = -N + B^T M^T \\
&\Lambda_{35} = G^T + L^T F^T, \ \Lambda_{44} = -M - M^T + P
\end{aligned}$$

Corollary 1 *The minimum* γ *that fulfills the conditions of Theorem 4.1 can be obtained using the following optimization problem:*

$$min\, \gamma \text{ s.t. (4.1) and (4.2)} \tag{4.3}$$

Theorem 4.1 is presented in terms of bilinear matrix inequalities (BMI) thanks to the slack variables S, R, N, M and F. The advantages of this approach come from the fact that such variables can be used as variables when searching for the best performance of the closed-loop system. Thus, a lower H_∞ guaranteed cost could be obtained by searching using the variables S, R, N, M and F. Nevertheless, by fixing S, R, N, M and F, the conditions of Theorem 4.1 reduce to LMIs: In this case, Corollary 1 becomes a convex optimization problem that can be handled by semi-definite programming (SDP) algorithms. In order to solve Corollary 1 within the BMI framework, many methods in the literature can be applied, such as the *path-following method* [16], which is based on linearizing the BMIs, and the *alternating SDP method* [3,13,25], which is based on fixing some variables and searching for others in such a way that, at each step, a convex optimization problem is solved. Although in both cases there is no guarantee of convergence, these methods are easy to implement and provide good results. In this paper, we use the methodology proposed in [25], as presented in Table 1.

Remark 4.2 Although other methods could be applied in the solution of the BMI problem (4.2), the algorithm in Table 1 is proposed. It is based on minimizing γ by

Table 1 Algorithm [25]

Initialize matrices S, R, N, M, F and scalars γ_T and γ_K

While $\left|\frac{(\sqrt{\gamma_T}-\sqrt{\gamma_K})}{\sqrt{\gamma_T}}\right| > \epsilon$ AND (Maximum number of iterations not reached) **do**

Solve conditions of Theorem 4.1 with S, R, N, M and F, minimizing $\sqrt{\gamma}$, $\sqrt{\gamma_T} \leftarrow \sqrt{\gamma}$; then calculate matrix K

Solve Theorem 4.1 with K obtained in the previous step, minimizing $\sqrt{\gamma}$, $\sqrt{\gamma_K} \leftarrow \sqrt{\gamma}$; then calculate matrices S, R, N, M and F

End While

alternatively using K as variable (fixing S, R, N, M and F) and using S, R, N, M and F as variables (fixing K): At each step, a convex optimization problem in terms of LMI conditions is solved. This idea has been called alternating semi-definite programming or the Gauss–Seidel method [13].

5 Illustrative Example

An example is now studied to show the effectiveness of the proposed method to design controllers for 2D SRM systems with delays.

Consider a thermal processes in chemical reactors, heat exchangers and pipe furnaces, which can be described by the partial differential equation [40]:

$$\frac{\partial T(x,t)}{\partial x} = -\frac{\partial T(x,t)}{\partial t} - T(x,t) - T(x,t-d_2) + u(t) \tag{5.1}$$

where $T(x,t)$ is the temperature at $x(space) \in [0, xf]$ and $t(time) \in [0, \infty]$. Assuming that the disturbance input is given by $w(i, j)$, then the partial differential equation above can be transformed into the following 2D SRM: (see [40] for more details)

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} 0 & a_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x^h(i-d_1,j) \\ x^v(i,j-d_2) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u(i,j) + \begin{bmatrix} l \\ 0 \end{bmatrix} w(i,j). \tag{5.2}$$

It is easy to check that the given system can be converted to

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \begin{bmatrix} a_1 & 0 \\ -a_1 & 1 \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} 0 & a_2 \\ 0 & -a_2 \end{bmatrix} \begin{bmatrix} x^h(i-d_1,j) \\ x^v(i,j-d_2) \end{bmatrix} + \begin{bmatrix} b \\ -b \end{bmatrix} u(i,j) + \begin{bmatrix} l \\ -l \end{bmatrix} w(i,j) \tag{5.3}$$

where $a_1 = \frac{\Delta t}{\Delta x + \Delta t + \Delta x \Delta t}$, $a_2 = \frac{\Delta x - \Delta x \Delta t}{\Delta x + \Delta t + \Delta x \Delta t}$, $b = \frac{\Delta x \Delta t}{\Delta x + \Delta t + \Delta x \Delta t}$, and the measured output is given by

Table 2 Minimum γ obtained

	Corollary 1	[40]
γ_{min}	0.1505	0.5

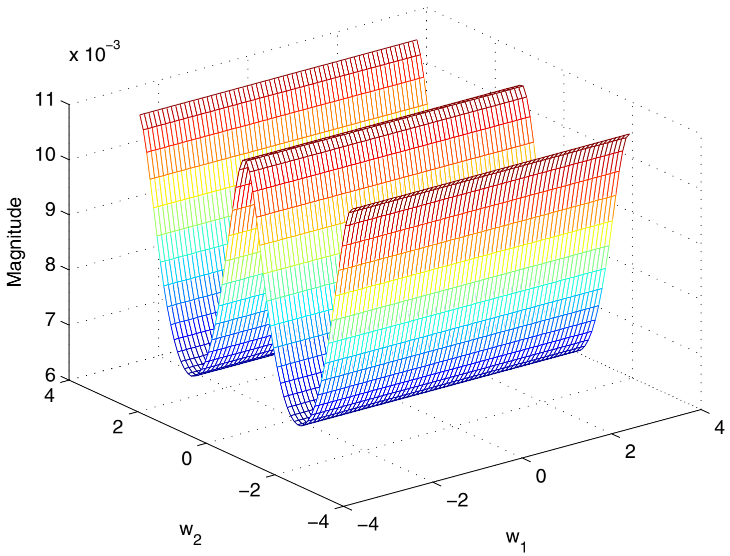

Fig. 1 Frequency response of the closed-loop system with the desired controller

$$z(i, j) = C \begin{bmatrix} x^h(i, j) \\ x^v(i, j) \end{bmatrix}. \tag{5.4}$$

Let $\Delta x = 0.1$, $\Delta t = 0.1$, $l = 0.1$, $d_1 = 1$, $d_2 = 2$, $C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$. We now compare Corollary 1 with the results proposed in [40]: The minimum γ obtained with each algorithm is presented in Table 2, from which it can be seen that our results have less conservatism than those in [40], thanks to the degrees of freedom given by the slack variables.

The obtained state-feedback H_∞ control is given as follows:

$$K = \begin{bmatrix} -10.0000 & -15.0961 \end{bmatrix}$$

With this controller, the closed-loop frequency response is presented in Fig. 1: It can be seen that the maximum value of $\|G(z, w)\|_\infty$ is effectively below the specified level of attenuation $\gamma = 0.1505$.

6 Conclusions

The analysis and design of H_∞ controllers for 2D singular systems with delays have been addressed in this paper, providing a design condition that is clearly less conservative than previous results in the literature. For this, extra-variables were used to derive

BMI conditions that may be explored in the search for a better H_∞ performance. The state-feedback design is accomplished by means of an optimization problem, solved in terms of LMIs by using an iterative algorithm. An example illustrated the applicability and advantages of the proposed method.

The main results in this paper may be further extended to related problems such as H_∞ output-feedback for these systems or time-varying delays.

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