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Exponential Stability of Nonlinear Impulsive and Switched Time-Delay Systems with Delayed Impulse Effects

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Abstract The exponential stability problem is considered for a class of nonlinear impulsive and switched time-delay systems with delayed impulse effects by using the method of multiple Lyapunov–Krasovskii functionals. Lyapunov-based sufficient conditions for exponential stability are derived, respectively, for stabilizing delayed impulses and destabilizing delayed impulses. It is shown that even if all the subsystems governing the continuous dynamics without impulse input delays are not exponential stable, if impulsive and switching signal satisfy a dwell-time upper bound condition, stabilizing delayed impulses can stabilize the systems in the exponential stability sense. Moreover, it is also shown that if the magnitude of the delayed impulses is sufficiently small, the exponential stability properties can be derived irrespective of the size of the impulse input delays under some conditions. The opposite situation is also developed. The efficiency of the proposed results is illustrated by two numerical examples.

Keywords Impulsive systems · Switched time-delayed systems · Multiple Lyapunov–Krasovskii functionals · Delayed impulse

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1 Introduction

Impulsive systems have attracted considerable attentions on both the theoretical and practical fronts in recent years because they provide a natural framework for mathematical modeling of many real-world problems where the reactions undergo abrupt changes [2,15]. For example, for many realistic networks, the state of nodes is often subject to instantaneous perturbations and experience abrupt change at certain instants which may be caused by switching phenomena, frequency change or other sudden noise; that is, it exhibits impulsive effects [1,4,9,21,37,39]. It is found that the impulsive systems have important applications in various fields, such as control systems with communication constraints [5,12], sampled-data systems [27,32], networked control systems with scheduling protocol [13,28,29], mechanical systems [3], etc. Impulsive systems can be viewed as a class of hybrid systems that consist of there elements: continuous dynamics, which govern the continuous evolution of the system between impulses; discrete dynamics, which govern the way the system states are changed at impulse times; and a criteria for determining when the states of the system are to reset. During the past two decades, the stability of impulsive systems has been extensively investigated in literatures [1,4,5,20,23,33,34] and references therein. Switched systems are a special kind of hybrid systems that consist of a family of continuous time or discrete-time dynamical systems and a rule called the switching signal (or law) to control the switching between modes [30,46]. The different switching signals differentiate switched systems from general time-varying systems, since the evolution of the former are dependent on not only the systems' initial conditions but also the switching signals. We usually study the stability properties not under a particular switching signal, but rather under various classes of switching signals [11,18,38,41,47,48]. During the past few years, switched systems have been deeply studied due to their potential application in the control of mechanical systems, the automotive industry and complex network control systems [6–8,16,40,42–45]. In the recent years, the study of impulsive switched systems has also received more and more attentions, and a large number of stability criteria of these systems have been reported [4,5,22,31,36].

In general, the impulses are mostly assumed to take the form: $\Delta x(t_k) = x(t_k^+) - x(t_k^-) = B_k x(t_k^-)$, which indicates the state is reset at the switching instants t_k. Due to the phenomena of input delays usually existing in transmission of the impulse information, i.e., for networked control systems, computation time and network-induced delays result in sensor-to-controller delay and controller-to-actuator delay, the corresponding impulsive control law $\Delta x(t_k)$ becomes

$$\Delta x(t) = B_k x((t - d_k)^-), t = t_k, k = 1, 2, \dots, \tag{1.1}$$

A few interesting results on stability have been obtained for nonlinear systems with impulse (1.1). It should be pointed out that the impulses (1.1) for modeling abrupt state changes may have destabilizing effects.

Recently, Chen and Zheng [5] proposed more general impulse form:

$$x(t^+) = C_{0k} x(t^-) + C_{1k} x((t - d_k)^-), \quad t = t_k, k = 1, 2, \dots. \tag{1.2}$$

Although a great number of results on stability of impulsive time-delay systems have been reported, there have been very few available results on stability of impulsive systems with delayed impulses. In [11], the problem of asymptotic stability for a class of delay-free autonomous systems with impulses of (1.1) is investigated. By using exponential estimates of delay-free systems to obtain difference inequalities of the system state at impulse times, a sufficient stability condition is derived involving the sizes of impulse input delays. It seems difficult to apply the same method to state-delayed impulsive systems. For a class of state-delayed systems with stabilizing impulse input delay (1.2) and unstable systems matrices, the stability problem was dealt with in [21]. By virtue of the method of variation of parameters and system matrices, several stability criteria are established. However, those results in [25] require the condition $\sup_k(|C_{0k}| + |C_{1k}|) < 1$ and thus cannot be used to tackle the time-delay system with impulses (1.1). Next, Ho et al. [13] studied the exponential stability for a class of delayed neural networks with destabilizing impulse delays by using the method of differential inequality. It is noted that those results in [13] impose some restriction on delays. In order to reject the conservatism of above-mentioned methods, an impulsive system approach was introduced for sampled-data systems in [27] and [32]. And a less conservative stability result was given. The key technique is to construct a novel Lyapunov functions with discontinuity at the impulse times. Recently, by using Lyapunov–Razumikhin functions, some new results are derived for a class of impulsive delay systems with general delayed impulses (1.2) [28]. It should be pointed out that those results in [28] contain no information of state delay. Based on the above consideration, an interesting question is under what conditions the less conservative stability condition can be established and how the Lyapunov functions can be constructed for impulsive switched system with delayed impulse.

On the other hand, in contrast with the Lyapunov–Razumikhin method presented in [19,24,26,28,35], it is well known that the Lyapunov–Krasovskii functionals method are sometimes more general that in the sense the former can be considered as a special case [14,17]. Such an approach is usually more difficult than the Lyapunov–Razumikhin technique. The reason is that, in general, we cannot expect an impulse that occurs at a discrete time to bring the value of a functional down instantaneously, whereas, in the Lyapunov–Razumikhin method, the value of a function can subside simultaneously as the impulse occurs [10]. To the best of our knowledge, the problem of exponential stability for state-delayed systems with switching and impulses input delays based on Lyapunov–Krasovskii technique has not been fully investigated until now, which motivates the present study.

In this paper, we pay close attention to the problem of exponential stability of impulsive and switching time-delay systems with more general delayed impulses by using the method of multiple Lyapunov–Krasovskii functionals. The main contribution of this paper can be listed as follows: (1) the more general system is studied which include the system in [5] as a special case; (2) the more general Lyapunov–Krasovskii functional technique is utilized; (3) the result developed in this paper is less conservative than that in the literature [5]. Especially, when all the subsystems are stable and impulses are not stable, not only the magnitude of impulses input delays but also the size of state delays is taken into account. When applying our results to a class of impulsive and switching time-delay systems, the new stability criteria are established. The efficiency of the proposed results is illustrated by three numerical examples.

2 Preliminaries

Let N denote the set of nonnegative integers, R^+ the set of nonnegative real numbers, and R^n the n-dimensional real Euclidean space. $|\cdot|$ denotes the Euclidean norm for vectors or the spectral norm for matrices. For $r > 0$, let $PC([-r, 0], R^n)$ denote the class of functions from $[-r, 0]$ to R^n satisfying the following: (i) it has at most a finite number of jump discontinuities on $(-r, 0]$, i.e., points at which the function has finite-valued but different left-hand and right-hand limits; (ii) it is continuous from the right at all points in $[-r, 0)$. For function $\phi : [-r, 0] \to R^n$, a norm is defined as $\|\phi\|_r = \sup_{-r\leq\theta\leq 0} \|\phi(\theta)\|$. Given $x \in PC([-r, \infty], R^n)$ and for each $t \in R^+$, define $x_t, x_{t^-} \in PC([-r, 0], R^n)$ by $x_t(s) = x(t + s)$ for $-r \leq s \leq 0$ and $x_{t^-}(s) = x(t + s)$ for $-r \leq s < 0$, respectively. For a given scalar $\rho \geq 0$, let $B(\rho) = \{x \in R^n : |x| \leq \rho\}$.

Let N_c be an arbitrary index set. Consider the following nonlinear switched time-delay system with delayed impulses

$$x'(t) = f_{i_k}(t, x_t), \quad t > t_0, t \neq t_k, i_k \in N_c, \tag{2.1a}$$

$$x(t) = g_k(x(t^-), x(t - d_k)^-), \quad t = t_k, \tag{2.1b}$$

$$x(t_0 + \theta) = \phi(\theta), \qquad \theta \in [-r, 0], \tag{2.1c}$$

where $x(t) \in R^n$ is the systems state, $x'(t)$ the right-hand derivative of $x(t)$, $x(t^+)$ and $x(t^-)$ denote the limit from the right and the left at point t, respectively, $\{t_k : k \in N\} \subset R^+$ a strictly increasing sequence and $\lim_{k\to\infty} t_k = \infty$. $\{d_k \geq 0 : k \in N\}$ are the impulse input delays satisfying $d = \max_k d_k < \infty$. The function $\phi \in PC([-r, 0], R^n)$ is the initial state of system and $\tau = \max\{r, d\}$. Let $D \subset R^n$ be an open set and $B(\rho) \subset D$ for some $\rho > 0$. We assume that, for each $i \in N_C$, given functionals $f_i : R^+ \times PC([-r, 0], D) \to R^n$ and $g_k : D \times D \to R^n$ satisfying $f_i(t, 0) \equiv g_k(t, 0) \equiv 0$. Moreover, we make the following assumptions on systems (2.1).

- (A1) $f_i(t, \psi)$ is composite-PC, i.e., for each $t_0 \in R^+$ and $\sigma > 0$, if $x \in PC([t_0 - r, t_0 + \sigma], R^n)$ and x is continuous at each $t \neq t_k$ in $(t_0, t_0 + \sigma]$, then the composite function h_i defined by $h_i(t) = f_i(t, x_t)$ is an element of the function class $PC([t_0, t_0 + \sigma], R^n)$.
- (A2) $f_i(t, \psi)$ is quasi-bounded, i.e., for each $t_0 \geq 0$ and $\sigma > 0$, and for each compact set $F \subset R^n$, there exist some $M_i > 0$ such that $|f_i(t, \psi)| \leq M_i$ for all $i \in N_c$ and $(t, \psi) \in [t_0, t_0 + \sigma] \times PC([-r, 0], F)$.
- (A3) For each fixed $t \in R^+$, $f_i(t, \psi)$ is a continuous function of ψ on $PC([-r, 0], R^n)$.
- (A4) There exist scalars $K_1^i > 0$ such that $|f_i(t, \psi)| \leq K_1^i \|\psi\|_r$ for any $\psi \in PC([-r, 0], B(\rho))$. Set $K_1 = \sup_{i\in N_c} K_1^i$.

(A5) There exist nonnegative bounded scalar sequences $\{h_{0k}\}, \{h_{1k}\}$ such that $|g_k(x, y) - x| \leq h_{0k}|x| + h_{1k}|y|$ for all $k \in N$ and all $x, y \in B(\rho)$. Set $\overline{h} = \sup_k(h_{0k} + h_{1k})$.

(A6) There exist scalars $K_2 > 0$ such that $|g_k(x, y_1) - g_k(x, y_2)| \leq K_2|y_1 - y_2|$ for all $k \in N$ and $y_1, y_2 \in B(\rho)$.

According to (A1)–(A3), it was shown that in [2], system (2.1) admits a solution $x(t, t_0, \phi)$ that exists in a maximal interval $[t_0 - \tau, t_0 + b)$ where $0 < b \leq +\infty$.

Definition 2.1 $V : R^+ \times B(\rho) \to R^+$ is said to belong to the class v_1 if

(1) V is continuous in each of the sets $\left[t_{k-1}, t_k\right) \times B(\rho)$ and for each $x, y \in R^n$, $k \in N$, $\lim_{(t,y)\to(t_k^-,x)} V(t, y) = V(t_k^-, x)$ exists;

(2) $V(t, x)$ is locally Lipschitzian in $x \in B(\rho)$, and for all $t \geq t_0$, $V(t, 0) \equiv 0$.

Definition 2.2 $V : R^+ \times PC([-\tau, 0], B(\rho)) \to R^+$ is said to belong to the class v_0 if

(1) V is continuous on $\left[t_{k-1}, t_k\right) \times PC([-\tau, 0], B(\rho))$ and for all $\psi, \phi \in PC([-\tau, 0], B(\rho))$, $\lim_{(t,\psi)\to(t_k^-,\phi)} V(t, \psi) = V(t_k^-, \phi)$ exists;

(2) $V(t, \psi)$ is locally Lipschitzian in ψ in each compact set in $PC([-\tau, 0], B(\rho))$, and for all $t \geq t_0$, $V(t, 0) \equiv 0$.

Definition 2.3 A functional $V(t, \psi) : R^+ \times PC([-\tau, 0], B(\rho)) \to R^+$ is said to belong to the class v_2 if $V(t, \psi) \in v_0$ and for any $x \in PC([t_0 - \tau, \infty], B(\rho)) = \{x : [t_0 - \tau, \infty] \to B(\rho)$ is piecewise continuous$\}$, $V(t, x_t)$ is continuous for $t \geq t_0$.

Definition 2.4 For a given impulsive time sequence $\{t_k\}$, the trivial solution of system (2.1) is said to be exponential stable if for any initial data $x_{t_0} = \phi$, there exist positive scalars ρ_0, M and λ such that $|\varphi||_\tau < \rho_0$ implies

$$|x(t, t_0, \phi)| \leq M\|\phi\|_\tau e^{-\lambda(t-t_0)}, \quad \forall t \geq t_0. \tag{2.2}$$

where $\rho_0 > 0$ is the upper bound of $\|\phi(\theta)\|$ over interval $[-\tau, 0]$.

To investigate the exponential stability of system (2.1), which has different modes of the continuous dynamics given by $\{f_i : i \in N_c\}$ and impulses given by $\{g_k : k \in N\}$, a family of multiple Lyapunov–Krasovskii functionals $\{V_i : i \in N_c\}$ are proposed, where each V_i is given by $V_i(t, \phi) = V_1^i(t, \phi(0)) + V_2^i(t, \phi)$. We shall assume that the family $\left\{V_1^i : i \in N_c\right\}$ are of class v_1 and the family $\left\{V_2^i : i \in N_c\right\}$ are of class v_2. Similar to the technique proposed in [22], the idea is still to break the Lyapunov–Krasovskii functionals V_i into two parts V_1^i and V_2^i, where V_1^i reflects the impulse effects and V_2^i is indifferent to impulses. So, the difficulties in analyzing the impulse effects using Lyapunov–Krasovskii functionals can be effectively overcome.

Definition 2.5 For the ith mode of system (2.1), for each $i \in N_C$ and $(t, \psi) \in R^+ \times PC([-\tau, 0], R^n)$, the upper right-hand derivative of $V_i(t, \phi)$ is defined by

$$D^+V_i(t, \phi) = \lim_{h\to 0^+} \sup \frac{1}{h}[V_i(t + h, x_{t+h}(t, \phi)) - V_i(t, \phi), \tag{2.3}$$

where $x(t, \phi)$ is a solution to the ith mode of system (2.1) satisfying $x_{t_0} = \phi$ and $x^{'}(t) = f_i(t, x_t)$ for $t \in (t_0, t_0 + h)$, where $h > 0$ is some positive number. To effectively analyze a family of multiple Lyapunov–Krasovskii functionals $\{V_i : i \in N_c\}$ for system (2.1), we define $v(t) = V_{i_k}(t, x_t) = V_1^i(t, x(t)) + V_2^i(t, x_t)$ for $t \in (t_k, t_{k+1})$.

3 Main Results

In this section, Lyapunov-based sufficient conditions for exponential stability of system (2.1) are developed. Our first result is concerned with exponential stability of system (2.1), in the case when all the subsystems governing the continuous dynamics of (2.1) are stable and the impulses, on the other hand, are destabilizing. Intuitively, the conditions in the following theorem consist of four aspects: (i) the Lyapunov–Krasovskii functionals satisfy certain positive definite and decrescent conditions; (ii) there exist some negative estimates of the upper right-hand derivatives of the functionals with respect to each stable mode of system (2.1); (iii) the dwell time of each mode of system (2.1) satisfy some lower bounds; (iv) the jumps induced by the destabilizing impulses satisfy certain growth conditions and the estimates on the decay rate of continuous dynamics and the delayed impulse; moreover, the estimates on the magnitude of the delayed impulses satisfy certain balancing conditions in terms of the decay.

Theorem 1 *Consider system (2.1) satisfying assumptions (A1)–(A6). Suppose that there exist a family of functions* $\left\{V_1^i : i \in N_c\right\}$ *of class* υ_1 *and a family of functionals* $\left\{V_2^i : i \in N_c\right\}$ *of class* υ_2*, positive scalars* $a, b, c, p_1, p_2 (p_1 \geq p_2 \geq 1), \mu_i, \beta_i$*, such that*

(i) $a|x|^{p_1} \leq V_1^i(t, x) \leq b|x|^{p_1}, \quad 0 \leq V_2^i(t, \phi) \leq c \, \|\phi\|_{\tau}^{p_2}$;
(ii) $D_i^+ V_i(t, \phi) \leq -\mu_i V_i(t, \phi)$;
(iii) for each $i \in N_c$ *and any* $k \in N$, $t_k - t_{k-1} \geq \beta_i$;
(iv) Define $\beta = \inf_{i \in N_c} \{\beta_i\}$. *If there exist positive scalars* $\varphi_i \geq 1$, L_1^i, L_2^i *such that*
  (1) $V_1^{\tilde{i}}(t, g_j(x, x)) \leq \varphi_i V_1^i(t^-, x), \quad V_2^{\tilde{i}}(t, \phi) \leq \varphi_i V_2^i(t, \phi)$*, for all* $t = t_k$;
  (2) $V_1^i(t, x+y) \leq L_1^i V_1^i(t, x) + L_2^i V_1^i(t, y)$ *for all* $t = t_k$ *and all* $x, y \in B(\rho)$, $x + y \in B(\rho)$*, and there exists a scalar* $d \geq 0$ *such that*

$$(L_1^i + 1)\varphi_i e^{-\mu_i \beta_i} + L_2^i (b/a)(K_2)^{p_1} (d l K_1 + l \bar{h})^{p_1} < 1, \tag{3.1}$$

where l *is a nonnegative integer satisfying* $l\beta \leq d \leq (l+1)\beta$*, then system (2.1) is exponential stable for any impulse input delays* $d_k \leq d$, $k \in N$;
(v) If there exist nonnegative bounded scalar sequences $\{\upsilon_{0k}\}$ *and* $\{\upsilon_{1k}\}$ *such that* $V_1^{\tilde{i}}(t, g_j(x, y)) \leq \upsilon_{0k} V_1^i(t^-, x) + \upsilon_{1k} V_1^i((t - d_k)^-, y)$, $V_2^{\tilde{i}}(t, \phi) \leq \upsilon_{0k} V_2^i(t^-, \phi)$ *for all* $t = t_k$ *and all* $x, y \in B(\rho)$*, and*

$$\sup_k \upsilon_{1k} < 1, \quad e^{-\mu_i \beta_i} < \inf_k ((1 - \upsilon_{1k})/\upsilon_{0k}), \tag{3.2}$$

then system (2.1) is exponential stable for any impulse input delays d_k, $k \in N$.

Proof In view of condition (3.1), we can choose positive constants $0 < \lambda < \mu_i$ and $\sigma > 0$ such that

$$e^{-(\mu_i-\lambda)\beta_i} < \sigma < \sigma_1, \tag{3.3}$$

where $\sigma_1 \triangleq \left[1-L_2^i(b/a)(K_2)^{p_1}(dlK_1 e^{\lambda(r+d)/p_1} + l\overline{h}e^{2\lambda d/p_1})^{p_1}\right]/(L_1^i+1)\varphi_i$. If conditions (3.2) holds, set $\overline{\upsilon}_1 = \sup_k \upsilon_{1k}$, $\sigma_2 = \inf_k((1-\upsilon_{1k})/\upsilon_{0k})$, then $\overline{\upsilon}_1 < 1$, and there exist scalars $\lambda \in \{0, \min\{\mu_i, \ln(1/\overline{\upsilon}_1)/d\}\}$ and $\sigma > 0$ such that

$$e^{-(\mu_i-\lambda)\beta_i} < \sigma < \sigma_2(1-\overline{\upsilon}_1 e^{\lambda d})/1-\overline{\upsilon}_1. \tag{3.4}$$

Let $x(t) = x(t, t_0, \phi)$ is a solution of (2.1) satisfying $x_{t_0} = \phi$, set $v_1(t) = V_1^{i_k}(t, x(t))$, $v_2(t) = V_2^{i_k}(t, x_t)$, and $v(t) = v_1(t) + v_2(t)$ for $t \in [t_k, t_{k+1})$, $k \in N$. It is clear that $v(t)$ defines a right-continuous function on $[t_0, \infty)$. We will prove that

$$v(t)e^{\lambda(t-t_0)} \le (b+c)\,\|\phi\|_\tau^{p_1}, \quad t \ge t_0. \tag{3.5}$$

Without loss of generality, we assume $\|\phi\|_\tau > 1$, $0 \le p_2 \le p_1$. We first prove that (3.5) holds for $t \in [t_0, t_1)$. When $t = t_0$, $v(t_0) = v_1(t_0, x(t_0)) + v_2(t_0, x_{t_0}) \le b\,\|\phi\|_\tau^{p_1} + c\,\|\phi\|_\tau^{p_2} \le (b+c)\,\|\phi\|_\tau^{p_1}$. It is clear that (3.5) holds for $t = t_0$. Define $t^* = \inf\left\{t \in [t_0, t_1), v(t)e^{\lambda(t-t_0)} > (b+c)\,\|\phi\|_\tau^{p_1} + \delta\right\}$, where $\delta > 0$ is an arbitrarily fixed number. It is obvious that $t^*(\delta) = t_1$ implies that $v(t)e^{\lambda(t-t_0)} \le (b+c)\,\|\phi\|_\tau^{p_1} + \delta$. So, if $t^*(\delta) = t_1$ for all $\delta > 0$, we must have (3.5) holds on $[t_0, t_1)$. If this is not true, i.e., $t^*(\delta^*) < t_1$ for some $\delta^* > 0$, one observes that $v(t^*)e^{\lambda(t^*-t_0)} = (b+c)\,\|\phi\|_\tau^{p_1} + \delta^* > 0$. Hence, from condition (ii), it follows that

$$\begin{aligned} D^+\left[v(t^*)e^{\lambda(t^*-t_0)}\right] &= e^{\lambda(t^*-t_0)}D^+v(t^*) + \lambda v(t^*)e^{\lambda(t^*-t_0)} \\ &\le -\mu_i v(t^*)e^{\lambda(t^*-t_0)} + \lambda v(t^*)e^{\lambda(t^*-t_0)} \\ &= -(\mu_i-\lambda)v(t^*)e^{\lambda(t^*-t_0)} < 0, \end{aligned}$$

which clearly contradicts with the choice of t^*. Therefore, Eq. (3.5) holds on $[t_0, t_1)$. Next under the assumption that (3.5) is satisfied on $[t_0, t_m)$ where $m \ge 1$, we will show that (3.5) is true on $[t_m, t_{m+1})$ as well. By condition (ii), we have

$$D^+v(t) \le -\mu_i v(t),$$

on $[t_{m-1}, t_m)$. On the other hand, from (3.5) on $[t_0, t_m)$, we can obtain

$$v(t_{m-1})e^{\lambda(t_{m-1}-t_0)} \le (b+c)\,\|\phi\|_\tau^{p_1}. \tag{3.6}$$

Integrating this differential inequality on $[t_{m-1}, t_m)$ and (3.6) gives

$$v(t_m^-) \le e^{-\mu_{i_{m-1}}(t_m-t_{m-1})}v(t_{m-1})$$

$$\leq e^{-\mu_{i_{m-1}}(t_m-t_{m-1})}e^{-\lambda(t_{m-1}-t_0)}(b+c)\,\|\phi\|_\tau^{p_1}$$
$$\leq e^{-(\mu_{i_{m-1}}-\lambda)(t_m-t_{m-1})}e^{-\lambda(t_m-t_0)}(b+c)\,\|\phi\|_\tau^{p_1}$$
$$\leq e^{-(\mu_{i_{m-1}}-\lambda)\beta_{i_{m-1}}}e^{-\lambda(t_m-t_0)}(b+c)\,\|\phi\|_\tau^{p_1}$$
$$< \sigma e^{-\lambda(t_m-t_0)}(b+c)\,\|\phi\|_\tau^{p_1}. \quad (3.7)$$

In the following, we distinguish two cases to prove $v(t_m)e^{\lambda(t_m-t_0)} \leq (b+c)\,\|\phi\|_\tau^{p_1}$.

Case (A) Assumption (iv) and (3.1) holds. From condition (iii) and $l\beta \leq d \leq (l+1)\beta$, it follows that there are at most l impulse times on $[t_k-d_k, t_k]$. We assume that these impulsive instants in $[t_m-d_m, t_m]$ are t_{m_i}, $i=1,2,\ldots,i_0, i_0 \leq l$. Combining condition (i) and $v(t)e^{\lambda(t-t_0)} \leq (b+c)\,\|\phi\|_\tau^{p_1}$ for $t \in [t_0, t_m]$ gives

$$|x(t)| \leq ((b+c)/a)^{1/p_1}\|\phi\|_\tau e^{-(\lambda/p_1)(t-t_0)}, \quad t \in [t_0, t_m). \quad (3.8)$$

According to (A4)–(A5) and (3.8), we can get

$$\left|x(t_m^-)-x((t_m-d_m)^-)\right| = \left|\int_{t_m-d_m}^{t_m^-}\dot{x}(s)\mathrm{d}s - \sum_{i=1}^{i_0}\Delta x(t_{m_i})\right|$$
$$\leq \sum_{i=1}^{i_0}\int_{t_m-d_m}^{t_m^-}|f_i(s,x_s)|\,\mathrm{d}s + \sum_{i=1}^{i_0}\left|g_{m_i}(x(t_{m_i}^-), x((t_{m_i}-d_{m_i})^-)) - x(t_{m_i}^-)\right|$$
$$\leq \sum_{i=1}^{i_0}K_1\int_{t_m-d_m}^{t_m^-}\|x_s\|_r\,\mathrm{d}s + \sum_{i=1}^{i_0}\left[h_{0k}\left|x(t_{m_i}^-)\right| + h_{1k}\left|x((t_{m_i}-d_{m_i})^-)\right|\right]$$
$$\leq (dlK_1e^{\lambda(r+d)/p_1} + l\overline{h}e^{2\lambda d/p_1})((b+c)/a)^{1/p_1}\|\phi\|_\tau e^{-(\lambda/p_1)(t_m-t_0)}. \quad (3.9)$$

Based (A6) and (3.9), it follows that

$$|\Delta g_m|^{p_1} \triangleq \left|g_m(x(t_m^-), x((t_m-d_m)^-)) - g_m(x(t_m^-), x(t_m^-))\right|^{p_1}$$
$$\leq (K_2)^{p_1}(dlK_1e^{\lambda(r+d)/p_1} + l\overline{h}e^{2\lambda d/p_1})^{p_1}((b+c)/a)\|\phi\|_\tau e^{-\lambda(t_m-t_0)}. (3.10)$$

In light of condition (i), (iv), and (3.10), one gets

$$v_1(t_m, g_m(x(t_m^-), x((t_m-d_m)^-))) = v_1(t_m, g_m(x(t_m^-), x(t_m^-)) + \Delta g_m)$$
$$\leq L_1^{i_m}\varphi_i v_1\left(t_m, g_m\left(x(t_m^-), x(t_m^-)\right)\right) + L_2^{i_m}v_1(t_m, \Delta g_m)$$
$$\leq L_1^{i_m}\varphi_i v_1\left(t_m, g_m\left(x(t_m^-), x(t_m^-)\right)\right)$$
$$+L_2^{i_m}(K_2)^{p_1}b(dlK_1e^{\lambda(r+d)/p_1} + l\overline{h}e^{2\lambda d/p_1})^{p_1}((b+c)/a)\,\|\phi\|_\tau^{p_1}e^{-\lambda(t_m-t_0)}. \quad (3.11)$$

According to the definition of $v_2(t)$, we have

$$v_2(t_m) \leq \varphi_{i_m} v_2(t_m^-). \tag{3.12}$$

Combining (3.11) and (3.12), and taking into account condition (iii) and (3.3), (3.7), we obtain that

$$\begin{aligned} v(t_m) &= v_1(t_m) + v_2(t_m) = v_1(t_m, g_m(x(t_m^-), x((t_m - d_m)^-))) + v_2(t_m) \\ &\leq L_1^{i_m} \varphi_i v_1(t_m^-, x(t_m^-)) + \varphi_i v_2(t_m^-) + L_2^{i_m} v_1(t_m, \Delta g_m) \\ &\leq (L_1^{i_m} + 1)\varphi_{i_m} v(t_m^-) + L_2^{i_m} b|\Delta g_m|^{p_1} \\ &\leq \sigma(L_1^{i_m} + 1)\varphi_{i_m}(b + c) \|\phi\|_\tau^{p_1} e^{-\lambda(t_m - t_0)} \\ &\quad + L_2^{i_m}(b/a)(K_2)^{p_1}(dlK_1 e^{\lambda(r+d)/p_1} + l\overline{h} e^{2\lambda d/p_1})^{p_1}(b+c) \|\phi\|_\tau^{p_1} e^{-\lambda(t_m - t_0)} \\ &\leq \left[\sigma(L_1^{i_m} + 1)\varphi_{i_m} + L_2^{i_m}(b/a)(K_2)^{p_1}(dlK_1 e^{\lambda(r+d)/p_1} + l\overline{h} e^{2\lambda d/p_1})^{p_1}\right] \\ &\quad (b + c) \|\phi\|_\tau^{p_1} e^{-\lambda(t_m - t_0)} \\ &\leq (b + c) \|\phi\|_\tau^{p_1} e^{-\lambda(t_m - t_0)}, \end{aligned} \tag{3.13}$$

i.e., (3.5) holds for $t = t_m$.

Case (B) Condition (v) and (3.2) hold. By (v), it is obtained from (3.4) and (3.7) that

$$\begin{aligned} v(t_m)e^{\lambda(t_m - t_0)} &= (v_1(t_m) + v_2(t_m))e^{\lambda(t_m - t_0)} \\ &\leq \left[\upsilon_{0m} v_1(t_m^-) + \upsilon_{1m} v_1((t_m - d_m)^-) + \upsilon_{0m} v_2(t_m^-)\right] e^{\lambda(t_m - t_0)} \\ &\leq \left[\upsilon_{0m} v(t_m^-) + \upsilon_{1m} v((t_m - d_m)^-)\right] e^{\lambda(t_m - t_0)} \\ &\leq (\sigma \upsilon_{0m} + \upsilon_{1m} e^{\lambda d})(b + c) \|\phi\|_\tau^{p_1} \\ &= \left[(1 - \upsilon_{1m})(\sigma/\sigma_2) + \upsilon_{1m} e^{\lambda d}\right](b + c) \|\phi\|_\tau^{p_1} \\ &\leq \left[(\sigma/\sigma_2) + \overline{\upsilon_1}(e^{\lambda d} - \sigma/\sigma_2)\right](b + c) \|\phi\|_\tau^{p_1} \\ &\leq (b + c) \|\phi\|_\tau^{p_1}, \end{aligned} \tag{3.14}$$

i.e., (3.5) holds for $t = t_m$.

Applying the argument used to show (3.5) over $[t_0, t_1)$, we can prove that (3.5) is true on $[t_m, t_{m+1})$. By induction, (3.5) is true for all $t \geq t_0$. Rewrite (3.1) as

$$v(t) \leq (b + c) \|\phi\|_\tau^{p_1} e^{-\lambda(t - t_0)}.$$

By condition (i), one gets

$$|x(t)| \leq ((b + c)/a)^{1/p_1} \|\phi\|_\tau e^{-(\lambda/p_1)(t - t_0)}, \quad t \geq t_0.$$

From Definition 2.4, it is concluded that system (2.1) is exponential stable. The above estimate also establishes boundedness of the state, which further implies global existence of solutions [2]. The proof is complete.

Remark 1 Observe that condition (iv)(1) in Theorem 1, $\widetilde{V_1^i}(t_k)$ is used to define the Lyapunov function which reflects the impulse effects and $\widetilde{V_2^i}(t_k)$ is used to define the Lyapunov function which is indifferent to impulses at constant t_k, respectively. Correspondingly, the items $V_1^i(t_k)$ and $V_2^i(t_k)$ stands for the value at constant t_k^-, respectively.

Remark 2 From condition (ii) in Theorem 1, it is seen that each of the continuous dynamics is exponential sable. On the other hand, from condition (iv), we see that each of the discrete dynamics can be destabilizing ($\varphi_i \geq 1$). It is well known that a well-behaved system may lose its stability due to uncontrolled impulsive input. Theorem 1 shows if system (2.1) satisfies a dwell-time lower-bound condition given by condition (iii), then it is exponential stable. It implies that, if the impulses and switching occur not too frequently, the exponential stability of a time-delay impulsive switched system with stable continuous dynamics can be preserved under destabilizing impulsive perturbations. So, the upper bound of dwell time is not necessary.

Remark 3 In the proof of Theorem 1, the effect of delayed impulses satisfying condition [iv,(1)] can be used to achieve (3.13). If no impulse occurs, i.e., $d = 0$, then assumption [iv,(2)] will be removed and condition (3.1) can be rewritten as $\varphi_i e^{-\mu_i \beta_i} < 1$. The condition is reduced to that of hybrid systems without delayed impulse. Therefore, Theorem 1 is the generalization of exponential stability of nonlinear hybrid systems with/without delayed impulse. Moreover, condition (3.1) is divided into two parts in which the first part reflects the effect of the delay-free impulse and the decay rate of the continuous dynamics, while the second part reveals the effects of delayed impulses. Condition (v) investigates the influence of the magnitude of the delayed impulses. If the magnitude is sufficiently small satisfying (3.2), then the delayed impulses cannot destroy the stability.

Remark 4 Besides, condition (iv) characterizes possible jumps in terms of the multiple Lyapunov–Krasovskii functionals V_i at the impulsive and switching times. Actually, when there is only impulse or there is only switching, the comparison condition among the multiple Lyapunov–Krasovskii functionals is necessary to analyze the stability of systems based the dwell-time approach.

Corollary 1 *Assume that hypotheses (A1)–(A6) are satisfied and there a family of functions $\left\{V_1^i : i \in N_c\right\}$ of class υ_1 and a family of functionals $\left\{V_2^i : i \in N_c\right\}$ of class υ_2, positive scalars $a, b, c, p_1, p_2 (p_1 \geq p_2 \geq 1), \lambda$, such that conditions (ii) and (3.1), (3.2) in Theorem 1 are replaced by the following (ii)* $D_i^+ V_i(t, \phi) \leq 0$;*

$$(L_1^i + 1)\varphi_i e^{\lambda(t_k - t_{k-1})} + L_2^i (b/a)(K_2)^{p_1} (dl K_1 + l\bar{h})^{p_1} < 1, \tag{3.15}$$

$$\sup_k \upsilon_{1k} < 1, \quad e^{\lambda(t_k - t_{k-1})} < \inf_k \left((1 - \upsilon_{1k})/\upsilon_{0k}\right), \tag{3.16}$$

and all other assumptions remain the same. Then system (2.1) is exponentially stable for any impulse input delays d_k, $k \in N$.

Proof In light of the procedure of Theorem 1, the result can be derived.

Remark 5 From Theorem 1 and Corollary 1, we see that when the underlying continuous system is stable, impulses are not required to be very frequent. Thus, the upper bound on the time interval between consecutive impulses and switching is not needed.

In the second part, we proceed to consider the exponential stability of systems (2.1). It is supposed that all the subsystems governing the continuous dynamics of (2.1) can be unstable while the impulse are stabilizing. Intuitively, the conditions in the following theorem consist of aspects: (i) the Lyapunov–Krasovskii functionals satisfy certain positive definite and decrescent conditions; (ii) there exist some positive estimates of the upper right-hand derivatives of the functionals with respect to each unstable mode of system (2.1); (iii) the dwell time of each mode of system (2.1) satisfy some supper bounds; (iv) the jumps induced by the stabilizing impulses satisfy certain diminishing conditions; moreover, the estimates on the magnitude of the delayed impulse satisfy certain balancing conditions in terms of the growth rate of the continuous dynamics and the dwell-time upper bounds.

Theorem 2 *Consider system (2.1) satisfying assumptions (A1)–(A6). Suppose that there exist a family of functions* $\left\{V_1^i : i \in N_c\right\}$ *of class* υ_1 *and a family of functionals* $\left\{V_2^i : i \in N_c\right\}$ *of class* υ_2*, positive scalars* $a, b, c, p_1, p_2 (p_1 \geq p_2 \geq 1), \mu_i (\leq 0), \kappa_i, \beta_0 = \inf_{i \in N_c}\{\beta_i\}, \beta_1 = \sup_{i \in N_c}\{\beta_i\}$*, such that*

(i) $a|x|^{p_1} \leq V_1^i(t, x) \leq b|x|^{p_1}, \quad 0 \leq V_2^i(t, \phi) \leq c\,\|\phi\|_\tau^{p_2}$*;*

(ii) $D_i^+ V_i(t, \phi) \leq -\mu_i V_i(t, \phi)$*;*

(iii) for each $i \in N_c$ *and any* $k \in N$, $\beta_0 \leq t_k - t_{k-1} \leq \beta_1$*;*

(iv) If there exist positive scalars $\varphi_i < 1, L_1^i, L_2^i, \kappa_i$ *such that*

(1) $V_1^{\tilde{i}}(t, g_k(x, x)) \leq \varphi_i V_1^i(t^-, x), \quad V_2^{\tilde{i}}(t, \phi) \leq \kappa_i \sup_{-r \leq s \leq 0} V_1^i(t + s, \phi(s))$ *, for all* $t = t_k$*;*

(2) $V_1^i(t, x + y) \leq L_1^i V_1^i(t, x) + L_2^i V_1^i(t, y)$ *for all* $t = t_k$ *and all* $x, y \in B(\rho), x + y \in B(\rho)$ *, and there exists a scalar* $d \geq 0$ *such that*

$$e^{\mu_i \beta_1} > (L_1^i \varphi_i + \kappa_i e^{\lambda r}) + L_2^i (b/a)(K_2)^{p_1} (dlK_1 + l\overline{h})^{p_1}, \tag{3.17}$$

where l *is a nonnegative integer satisfying* $l\beta_0 \leq d \leq (l+1)\beta_0$*, then system (2.1) is exponential stable for any impulse input delays* $d_k \leq d, k \in N$*;*

(v) If there exist nonnegative bounded scalar sequences $\{\upsilon_{0k}\}$ *and* $\{\upsilon_{1k}\}$ *such that* $V_1^{\tilde{i}}(t, g_k(x, y)) \leq \upsilon_{0k} V_1^i(t^-, x) + \upsilon_{1k} V_1^i((t - d_k)^-, y)$*;* $V_2^{\tilde{i}}(t, \phi) \leq \kappa_i \sup_{-r \leq s \leq 0} V_1^i(t + s, \phi(s))$ *for all* $t = t_k$ *and all* $x, y \in B(\rho)$*, and*

$$e^{\mu_i \beta_1} > \sup_k (\overline{\upsilon}_{0k} + \upsilon_{1k}), \tag{3.18}$$

where $\overline{\upsilon}_{0k} = \upsilon_{0k} + \rho_i$ *and* $\rho_i = \kappa_i e^{\lambda r}$*, then system (2.1) is exponential stable for any impulse input delays* $d_k, k \in N$*. Moreover, suppose that* $\sup_{i \in N_c} \varphi_i < \infty$*,* $\inf_{i \in N_c} \delta_i > 0$*, then system (2.1) is uniformly exponential stable over* $\bigcap_{i \in N_c} \{t_k - t_{k-1} \geq \beta_i, i_k = i\}$*.*

Proof If condition (3.17) holds, then there exist positive scalars $\lambda, \sigma > 0$ such that

$$e^{(-\mu_i+\lambda)\beta_1} < \sigma < \eta_i, \tag{3.19}$$

where $\eta_i \triangleq \left[(L_1^i\varphi_i + \rho_i) + L_2^i(b/a)(K_2)^{p_1}(dlK_1e^{\lambda(r+d)/p_1} + l\overline{h}e^{2\lambda d/p_1})^{p_1}\right]^{-1}$ and $\rho_i = \kappa_i e^{\lambda r}$. If conditions (3.18) holds, set $\overline{\upsilon}_1 = \sup_k \upsilon_{1k}, \sigma_2 = \sup_k(\overline{\upsilon}_{0k} + \upsilon_{1k})$, then $\sigma_2 < 1$, and there exist scalars $\lambda, \sigma > 0$ such that

$$e^{(-\mu_i+\lambda)\beta_1} < \sigma < \left[\sigma_2 + \overline{\upsilon}_1(e^{\lambda d} - 1)\right]^{-1}. \tag{3.20}$$

Let $x(t) = x(t, t_0, \phi)$ is a solution of (2.1) satisfying $x_{t_0} = \phi$, set $v_1(t) = V_1^{i_k}(t, x(t))$, $v_2(t) = V_2^{i_k}(t, x_t)$, and $v(t) = v_1(t) + v_2(t)\ t \in \left[t_k, t_{k+1}\right), for\ k \in N$. We will prove that

$$v(t)e^{\lambda(t-t_0)} \le \sigma(b+c)\,\|\phi\|_\tau^{p_1}, \quad t \ge t_0. \tag{3.21}$$

We first prove that (3.21) holds for $t \in [t_0, t_1)$. The first thing to do is to show (3.21) holds when $t = t_0$. One has

$$\begin{aligned} v(t_0) &= v_1(t_0, x(t_0)) + v_2(t_0, x_{t_0}) \\ &\le b\,\|\phi\|_\tau^{p_1} + c\,\|\phi\|_\tau^{p_2} \le (b+c)\,\|\phi\|_\tau^{p_1} < \sigma(b+c)\,\|\phi\|_\tau^{p_1}. \end{aligned}$$

It is clear that (3.21) holds for $t = t_0$. Suppose for the sake of contradiction that $v(t)e^{\lambda(t-t_0)} > \sigma(b+c)\,\|\phi\|_\tau^{p_1}$ for some $t \in [t_0, t_1)$. Let $t^* = \inf\left\{t \in [t_0, t_1), v(t)e^{\lambda(t-t_0)} \ge \sigma(b+c)\,\|\phi\|_\tau^{p_1}\right\}$. It is clear that $t^* \in (t_0, t_1)$ and $v(t^*)e^{\lambda(t^*-t_0)} = \sigma(b+c)\,\|\phi\|_\tau^{p_1}$. Now suppose that $\overline{t} = \sup\{t \in [t_0, t^*), v(\overline{t})e^{\lambda(\overline{t}-t_0)} \le \sigma(b+c)\|\phi\|_\tau^{p_1}\}$, we claim that $\overline{t} \in (t_0, t^*)$ and $v(\overline{t})e^{\lambda(\overline{t}-t_0)} = \sigma(b+c)\,\|\phi\|_\tau^{p_1}$. From condition (ii), we have $D^+v(s) \le -\mu_i v(s)$ for $s \in \left[\overline{t}, t^*\right]$. Integrating this differential inequality on $\left[\overline{t}, t^*\right]$ gives

$$\begin{aligned} v(t^*) &\le e^{-\mu_0(t^*-\overline{t})}v(\overline{t}) = e^{-\mu_0(t^*-\overline{t})}e^{-\lambda(\overline{t}-t_0)}(b+c)\,\|\phi\|_\tau^{p_1} \\ &\le e^{-\mu_0(t^*-\overline{t})}e^{-\lambda(t^*-t_0)}e^{\lambda(t^*-\overline{t})}(b+c)\,\|\phi\|_\tau^{p_1} \\ &\le e^{(-\mu_0+\lambda)\beta_1}e^{-\lambda(t^*-t_0)}(b+c)\,\|\phi\|_\tau^{p_1} \\ &< \sigma e^{-\lambda(t^*-t_0)}(b+c)\,\|\phi\|_\tau^{p_1}, \end{aligned}$$

which clearly contradicts with $v(t^*) = \sigma(b+c)\,\|\phi\|_\tau^{p_1}\,e^{\lambda(t^*-t_0)}$. So (3.21) holds for $t \in [t_0, t_1)$.

Suppose that (3.21) holds for $t \in [t_0, t_m)$ where $m \in \{1, 2, \ldots k-1\}$. Then we will show that (3.21) holds on $\left[t_m, t_{m+1}\right)$ as well. Combining condition (i) and $v(t)e^{\lambda(t-t_0)} \le \sigma(b+c)\,\|\phi\|_\tau^{p_1}$ for $t \in [t_0, t_m]$ gives

$$|x(t)| \le \sigma^{1/p_1}((b+c)/a)^{1/p_1}\|\phi\|_\tau e^{-(\lambda/p_1)(t-t_0)}, \quad t \in (t_0, t_m). \tag{3.22}$$

From condition (iii) and $l\beta_0 \leq d \leq (l+1)\beta_0,\ i \in N_c$, it follows that there are at most l impulse times on $[t_m - d_m, t_m]$. Then applying the same argument as used in the proof of (3.13) and (3.14), from (3.17) and (3.22), eventually, this would lead to

$$\begin{aligned}
v(t_m) &= v_1(t_m) + v_2(t_m) = v_1(t_m, g_m(x(t_m^-), x((t_m - d_m)^-))) + v_2(t_m) \\
&\leq L_1^{i_m}\varphi_{i_m} v_1(t_m^-, x(t_m^-)) + \kappa_{i_m} \sup_{-r\leq s<0} |v_1(t_m+s)| + L_2^i v_1(t_m, \Delta g_m) \\
&\leq (L_1^{i_m}\varphi_{i_m} + \kappa_{i_m} e^{\lambda r}) v(t_m^-) + L_2^{i_m} b |\Delta g_m|^{p_1} \\
&\leq \sigma (L_1^{i_m}\varphi_{i_m} + \kappa_{i_m} e^{\lambda r})(b+c)\, \|\phi\|_\tau^{p_1}\, e^{-\lambda(t_m - t_0)} \\
&\quad + \sigma L_2^{i_m}(b/a)(K_2)^{p_1}(dlK_1 e^{\lambda(r+d)/p_1} + l\overline{h} e^{2\lambda d/p_1})^{p_1}(b+c)\, \|\phi\|_\tau^{p_1}\, e^{-\lambda(t_m - t_0)} \\
&\leq \Big[(L_1^{i_m}\varphi_{i_m} + \kappa_{i_m} e^{\lambda r}) + L_2^{i_m}(b/a)(K_2)^{p_1}(dlK_1 e^{\lambda(r+d)/p_1} + l\overline{h} e^{2\lambda d/p_1})^{p_1}\Big] \\
&\qquad \sigma(b+c)\, \|\phi\|_\tau^{p_1}\, e^{-\lambda(t_m - t_0)} \\
&\leq (b+c)\, \|\phi\|_\tau^{p_1}\, e^{-\lambda(t_m - t_0)} \leq \sigma(b+c)\, \|\phi\|_\tau^{p_1}\, e^{-\lambda(t_m - t_0)},
\end{aligned} \tag{3.23}$$

or

$$\begin{aligned}
v(t_m)e^{\lambda(t_m - t_0)} &= (v_1(t_m) + v_2(t_m))e^{\lambda(t_m - t_0)} \\
&\leq \Big[\upsilon_{0m} v_1(t_m^-) + \upsilon_{1m} v_1((t_m - d_m)^-) + \kappa_{i_m} \sup\nolimits_{-r\leq s<0} v_1(t_m + s)\Big] e^{\lambda(t_m - t_0)} \\
&\leq \Big[(\upsilon_{0m} + \kappa_{i_m} e^{\lambda r}) v(t_m^-) + \upsilon_{1m} v((t_m - d_m)^-)\Big] e^{\lambda(t_m - t_0)} \\
&\leq (\overline{\upsilon}_{0m} + \upsilon_{1m} e^{\lambda d}) \sigma (b+c)\, \|\phi\|_\tau^{p_1} \\
&\leq \Big[\sigma_2 + \overline{\upsilon}_1 (e^{\lambda d} - 1)\Big] \sigma (b+c)\, \|\phi\|_\tau^{p_1} \\
&\leq (b+c)\, \|\phi\|_\tau^{p_1} \leq \sigma(b+c)\, \|\phi\|_\tau^{p_1}.
\end{aligned} \tag{3.24}$$

Finally, it is concluded that

$$v(t_m)e^{\lambda(t_m - t_0)} \leq \sigma(b+c)\, \|\phi\|_\tau^{p_1}. \tag{3.25}$$

Hence, from induction, (3.21) is true for all $t \geq t_0$. From Definition 2.4, it is shown that system (2.1) is exponential stable. The proof is complete.

Corollary 2 *Suppose that there exist positive constants $a, b, c, p_1, p_2, \mu_i, \beta_0, \beta_1 (i \in N_c)$, such that conditions [iv (1)] and conditions (v) in Theorem 2 are replaced by the following (iv)* If there exist positive scalars $\varphi_i < 1, L_1^i, L_2^i, \kappa_i$ such that (1) $\widetilde{V_1^i}(t, g_j(x,x)) \leq \varphi_i V_1^i(t^-, x)$ for all $t = t_k$; (v)* If there exist nonnegative bounded scalar sequences $\{\upsilon_{0k}\}$ and $\{\upsilon_{1k}\}$ such that $\widetilde{V_1^i}(t, g_k(x,y)) \leq \upsilon_{0k} V_1^i(t^-, x) + \upsilon_{1k} V_1^i((t - d_k)^-, y)$ for all $t = t_k$ and all $x, y \in B(\rho)$, and*

$$e^{\mu_i \beta_1} > \sup_k (\overline{\upsilon}_{0k} + \upsilon_{1k}), \tag{3.26}$$

where $\overline{\upsilon}_{0k} = \upsilon_{0k} + \kappa_i e^{\lambda r}$ *in which* $\overline{\upsilon}_{0k} = \upsilon_{0k} + \rho_i$ *and* $\rho_i = \kappa_i e^{\lambda r}$*, and all other conditions remain the same. Then system* (2.1) *is exponential stable for any impulse input delays* $d_k, k \in N$.

Proof It suffices to verify that condition (iv) in Theorem 2 is satisfied with $\kappa_i = \frac{c}{a}$. Correspondingly, the item $\rho_i = \kappa_i e^{\lambda r}$ in the proof is replaced by $\rho_i = \frac{c}{a} e^{\lambda r}$.

Remark 6 From condition (ii) in Theorem 2, it is seen that each of the continuous dynamics can be unstable ($-\mu_i > 0$). Nevertheless, from condition (iv), we see that each of the discrete dynamics is stabilizing ($\varphi_i < 1$). Theorem 2 shows if system (2.1) satisfies a dwell-time upper bound condition given by condition (iii), then it is exponential stable. It implies that, the impulses and switching must be frequent and their amplitude must be suitably related to the growth rate of V_i. In other words, a time-delay impulsive switched system with unstable continuous dynamics can be impulsively stabilized in the exponential stability sense.

Remark 7 Condition (iv) in Theorem 2 characterizes the key distinction of the idea of impulsive stabilization of time-delay systems using the method of Lyapunov–Krasovskii functionals. By condition (iv), it is only needed that the function part of V_i(i.e., V_1^i) is stabilized by the impulses ($\varphi_i < 1$). Because at a discrete time, there exist impulse, so we cannot always expect the impulse bring the value of a purely functional part of V_2(i.e., V_2^i) down. The tuning parameter κ_i in V_2^i establishes the relationship between the functional part of V_i and $V_{\tilde{i}}$. Of course, if condition (iv) $V_2^{\tilde{i}}(t, \phi) \le \kappa_i \sup_{-r \le s \le 0} V_1^i(t+s, \phi(s))$ is changed into $V_2^{\tilde{i}}(t, \phi) \le \varphi_i V_2^i(t^-, \phi)$, the result is still hold. But it is obviously observed that the result is more conservative than Theorem 2.

Remark 8 Noting $\mu_i \le 0$ and from (3.17), it is easily obtained that $L_1^i \varphi_i + \kappa_i e^{\lambda r} < 1$. It appears that the state delay size r and the factor κ_i have to be sufficiently small. However, as we can from example 1, we may always depend on a tuning parameter as the coefficient of V_2^i and hence make κ_i sufficiently small. By using this technique, the restriction on the delay size r can be also be resolved. Moreover, when state delay r is very small, by using the design, information of state delay is not be ignored which may cause system instability.

Now consider the following time-varying nonlinear impulsive switched time-delay system

$$\begin{aligned} x'(t) &= A_i(t)x(t) + F(t, x(t), x(t - r_1(t)), x(t - r_2(t)), \ldots, x(t - r_m(t))), \\ &\qquad t > t_0, t \ne t_k, \\ x(t) &= C_{0k}x(t^-) + C_{1k}x((t - d_k)^-), \quad t = t_k, \end{aligned} \tag{3.27}$$

where $x(t) \in R^n$ is the state, $A_i(t) : R^+ \to R^{n\times n}$, $F_i : R^+ \times R^n \times \cdots \times R^n$ are continuous functions, and $C_{0k}, C_{1k} \in R^{n\times n}$. And

$$|F(t, x, y_1, \ldots y_m)| \le M_0(t)\,|x| + \sum_{j=1}^{m} M_j(t)\,|y_j|, \forall x, y_1, \ldots, y_m \in R^n, \tag{3.28}$$

where $M_j(t)$, $j = 0, 1, \ldots, m$ are nonnegative continuous functions. The delays $r_i(t)$ are continuous and satisfy $0 \le r_i(t) \le r, i = 1, 2, \ldots, m$.

Corollary 3 *Consider system (3.27) satisfying (3.28). Set* $c_{ik} = |C_{ik}|$, $i = 0, 1, \bar{c}_{0k}= |C_{0k} - I|$. *Suppose that there exist positive scalars* $c_0, c_1, \mu_0^i, \mu_1^i (i \in S), \overline{M}, \bar{c} \ge 1$ *such that* $\sup_k c_{ik} = c_i, i = 0, 1$,
$\sup_k |C_{0k} + C_{1k}| \le \bar{c}, \lambda_{\max}((A_i(t) + A_i^T(t))/2) + M_0(t) \le \mu_1^i, \sum_{j=1}^m M_j(t) \le \overline{M}$.

(i) If $c_1 < 1,\ 1 < c_0$, *and there exists scalars* $\beta_i > 0$ *such that*

$$2\mu_1^i + \max\left\{1, \sup_k \frac{c_{0k}}{1 - c_{1k}}\right\}\overline{M} + \frac{2}{\beta_i}\ln\max\left\{1, \sup_k \frac{c_{0k}}{1 - c_{1k}}\right\} < 0, \quad (3.29)$$

then system (3.27) is exponential stable for any bounded impulse input delays $d_k, k \in N$.

(ii) If

$$|A_i(t)| + M_0(t) \le \mu_0^i, \quad t \ge 0. \quad (3.30)$$

and there exist scalars $\beta_i > 0$ *such that*

$$w_0^i \overset{\Delta}{=} c_1\left[dl(\mu_0^i + \overline{M}) + l\sup_k(\bar{c}_{0k} + c_{1k})\right] < 1, \quad (3.31)$$

$$\mu_1^i + \max\left\{1, \frac{\sqrt{2}\bar{c}}{1 - w_0^i}\right\}\overline{M} + \frac{1}{\beta_i}\ln\max\left\{1, \frac{\sqrt{2}\bar{c}}{1 - w_0^i}\right\} < 0, \quad (3.32)$$

where l *is a nonnegative integer satisfying* $l\beta_0 \le d \le (l+1)\beta_0$, *then system (3.27) is exponential stable for any bounded impulse input delays* $d_k, k \in N$.

(iii) If there exist scalars $\alpha \in (0, 1)$ *and* $\beta_i > 0$ *such that*

$$\sup_k(c_{0k} + c_{1k}) \le \alpha, \quad (3.33)$$

$$2\mu_1^i + \frac{1}{\alpha}\overline{M} - \frac{2}{\beta_i}\ln\frac{1}{\alpha} < 0, \quad (3.34)$$

then system (3.27) is s exponential stable for any bounded impulse input delays $d_k, k \in N$.

(iv) Under the same condition as in (3.30), suppose that there exist scalars $\alpha \in (0, 1)$ *and* $0 \le \beta_0 \le \beta_1$ *such that (3.31) is satisfied, where* l *satisfying* $l\beta_0 \le d \le (l+1)\beta_0$, *moreover,*

$$\bar{c} \le \alpha \le 1 - w_0^i, \quad (3.35)$$

$$\mu_1^i + \frac{1}{\alpha + w_0^i}\overline{M} + \frac{1}{\beta_1}\ln\frac{1}{\alpha + w_0^i} < 0, \quad (3.36)$$

then system (3.27) is exponential stable for any bounded impulse input delays $d_k, k \in N$.

Proof We only prove part(ii) due to space limitations. The other parts can be obtained in light of part (ii). One can verify that for systems (3.27), conditions (A1)–(A6) are satisfied with $\rho = \infty$, $K_1^i = \mu_0^i + \overline{M}$, $h_{0k} = \overline{c}_{0k}$, $h_{1k} = c_{1k}$, $\overline{h} = \sup_k(\overline{c}_{0k} + c_{1k})$, $K_2 = c_1$. Condition (3.31)–(3.32) imply that there exist scalars $k > 0$ and $\varepsilon > 0$ such that

$$\max\left\{1, \frac{\sqrt{2\overline{c}}}{1-w_0^i}\right\} < k + \frac{1}{k} < \frac{1}{\overline{M}}\left[\mu_1^i + \min\left\{0, \frac{1}{\beta_i}\ln\frac{1-w_0^i}{\sqrt{2\overline{c}}}\right\} - \varepsilon\right]. \tag{3.37}$$

Set $w_1^i = -\min\left\{0, \frac{1}{\beta_i}\ln\frac{1-w_0^i}{\sqrt{2\overline{c}}}\right\} + \varepsilon$, then $w_1^i > 0$, and

$$\sqrt{2}\alpha_i e^{-w_1^i\beta_i} + w_0^i < 1, \tag{3.38}$$

where $\alpha_i = \max\left\{\sqrt{2\overline{c}}, 1-w_0^i\right\}$. Choose the Lyapunov–Krasovskii functionals $V_i = V_1^i + V_2^i$, where $V_1^i = |x|^2$, $V_2^i = \frac{\overline{M}}{k}\sup_{-r\le s\le 0}\int_{-r}^0 |\phi(s)|^2 ds$. For any $0 < t \neq t_k$, by (3.37), we have

$$\begin{aligned}
D^+V_i(x(t)) &= 2x^T(t)\left(A(t)x(t) + f\left(t, x(t), x(t-r_1(t)), \ldots x(t-r_m(t))\right)\right)\\
&\quad + \tfrac{\overline{M}}{k}|x(t)|^2 - \tfrac{\overline{M}}{k}\sup_{-r\le s\le 0}|x(t+s)|^2\\
&\le \lambda_{\max}(A(t)+A^T(t))|x(t)|^2 + 2M_0(t)|x(t)|^2\\
&\quad + \textstyle\sum_{j=1}^m M_j(t)2\,|x(t)|\,\left|x(t-r_j(t))\right| + \tfrac{\overline{M}}{k}|x(t)|^2\\
&\quad - \tfrac{\overline{M}}{k}\sup_{-r\le s\le 0}|x(t+s)|^2\\
&\le 2\mu_1^i\,|x(t)|^2 + k\overline{M}|x(t)|^2 + \tfrac{\overline{M}}{k}\sup_{-r\le s\le 0}|x(t+s)|^2 + \tfrac{\overline{M}}{k}|x(t)|^2\\
&\quad - \tfrac{\overline{M}}{k}\sup_{-r\le s\le 0}|x(t+s)|^2\\
&\le 2\mu_1^i\,|x(t)|^2 + (k+\tfrac{1}{k})\overline{M}|x(t)|^2\\
&\le -2w_1^i V_i(x(t)).
\end{aligned}$$

That is, condition (ii) in Theorem 1 holds with $\mu_i = 2w_1^i$.

Next we consider condition (iv). From the definition of $V_i(x(t))$, we have $\widetilde{V_1^i}(t, C_{0k}x + C_{1k}x) \le |C_{0k} + C_{1k}|^2|x|^2 \le \overline{c}|x|^2$, which implies that condition [(iv (1)] holds with $\varphi_i = \overline{c}$, and $\widetilde{V_2^i}(t,\phi) = \frac{\overline{M}}{k}\sup_{-r\le s\le 0}\int_{-r}^0 |\phi(s)|^2 ds \le \varphi_i V_2^i(t,\phi)$. Since $V_1^i(t, x_1+x_2) = |x_1+x_2|^2 \le (1+\varepsilon)|x_1|^2 + (1+\frac{1}{\varepsilon})|x_2|^2$ for any $\varepsilon > 0$, condition [iv (2)] holds with $L_1^i = 1+\varepsilon$, $L_2^i = 1+\frac{1}{\varepsilon}$. By choosing $\varepsilon = w_0^i(\alpha_i e^{-w_1^i\beta_i})^{-1}$ and using (3.32), we have that $(L_1^i+1)\varphi_i e^{-\mu_i\beta_i} + L_2^i(b/a)(K_2)^{p_1}(dlK_1 + l\overline{h})^{p_1} = (2+\varepsilon)(\alpha_i e^{-w_1^i\beta_i})^2 + (1+\frac{1}{\varepsilon})(w_0^i)^2 \le (\sqrt{2}\alpha_i e^{-w_1^i\beta_i} + w_0^i)^2 < 1$. This implies (3.1) holds. Therefore, by Theorem 1, system (3.27) is exponential stable.

4 Numerical Examples

The applicability of the results derived in the preceding section is illustrated by the following three examples.

Example 1 Consider the nonlinear impulsive time-delay switched system as follows Mode 1, i.e., $i_k = 1$;

$$\begin{aligned}
&x'_1(t) = -x_2(t)\sin(x_1(t-r)) - 6x_1(t) + 0.5x_2^2(t-r), \quad t \neq t_k,\\
&x'_2(t) = 0.8x_1(t)\sin(x_1(t-r)) - 6x_2(t) + 0.4x_2^2(t-r), \quad t \neq t_k,\\
&x_1(t) = x_1(t^-) + c_1x_1((t-d_k)^-), \quad t = t_k,\\
&x_2(t) = x_2(t^-) + c_1x_2((t-d_k)^-), \quad t = t_k.
\end{aligned}$$

Mode 2, i.e., $i_k = 2$;

$$\begin{aligned}
&x'_1(t) = -x_2(t)\sin(x_1(t-r)) - 3x_1(t) + 0.5x_2^2(t-r), \quad t \neq t_k,\\
&x'_2(t) = x_1(t)\sin(x_1(t-r)) - 3x_2(t) + 0.5x_2^2(t-r), \quad t \neq t_k,\\
&x_1(t) = x_1(t^-) + c_2x_1((t-d_k)^-), \quad t = t_k,\\
&x_2(t) = x_2(t^-) + c_2x_2((t-d_k)^-), \quad t = t_k.
\end{aligned} \tag{4.1}$$

where $r \geq 0, d_k \in [0, d]$, and $\beta_1 = 0.6, \beta_2 = 1.2$. By choosing $\rho = 1$, condition (A1)–(A3) is clearly satisfied. It is easy to see that (A5) is satisfied with $h_{0k} = 0$ and $h_{1k} = c_1$, and (A5) is satisfied with $K_2 = c_1$. So $\overline{h} = c_1$. Moreover, for any $\phi \in PC([-r, 0], B(\rho))$, one has that $|f_1(t,\phi)|^2 \leq 47.3144\|\phi\|^2, |f_2(t,\phi)|^2 \leq 17.156\|\phi\|^2$ which implies $K_1^1 = \sqrt{47.3144}$, $K_1^2 = \sqrt{17.156}$.

According to Theorem 1, our purpose here is to find the upper bound of c_1 such that system (4.1) is exponential stable for any bounded impulse input delays d_k. Choose the Lyapunov–Krasovskii functionals $V_i(t,\phi) = V_1^i + V_2^i$. For $i = 1$, we have $V_1^1(t,\phi(0)) = \frac{1}{2}\phi_1^2(0) + \frac{1}{1.6}\phi_2^2(0)$ and $V_2^1(t,\phi) = 0.5\sqrt{2}\int_{-r}^0 \phi^2(s)\left[k_1 + 1 + \frac{k_1 s}{r}\right]ds$ for $0 < k_1 < 9.6082$ with $a = \frac{1}{2}, b = \frac{1}{1.6}$. For $i = 2$, we choose $V_1^2(t,\phi(0)) = \frac{1}{2}\phi_1^2(0) + \frac{1}{1.6}\phi_2^2(0)$ and $V_2^2(t,\phi) = 0.5\sqrt{2}\int_{-r}^0 \phi^2(s)\left[k_2 + 1 + \frac{k_2 s}{r}\right]\mathrm{d}s$ with $a = \frac{1}{2}, b = \frac{1}{2}$ for $0 < k_2 < 3.2433$. It follows that

$$\begin{aligned}
D^+V_1(t,\phi) &= -6\phi_1^2(0) - \tfrac{12}{1.6}\phi_2^2(0) + k\phi_2^2(0) + 0.5(\phi_1(0) + \phi_2(0))\phi_2^2(-r)\\
&\quad + 0.5\sqrt{2}(k+1)\phi_2^2(0) - 0.5\sqrt{2}\phi_2^2(-r) - \tfrac{k}{r}0.5\sqrt{2}\int_{-r}^0 \phi^2(s)\mathrm{d}s\\
&\leq -6\phi_1^2(0) - \left(\tfrac{12-0.8\sqrt{2}(k+1)}{1.6}\right)\phi_2^2(0) + (0.5\sqrt{2}\,|\phi(0)| - 0.5\sqrt{2})\phi_2^2(-r)\\
&\quad - \tfrac{k}{r}0.5\sqrt{2}\int_{-r}^0 \phi^2(s)\mathrm{d}s\\
&\leq -\left[12 - 0.8\sqrt{2}(k+1)\right] V_1^1(t,\phi(0)) - \tfrac{k}{(k+1)r}V_2^1(t,\phi)\\
&\leq -\min\left\{12 - 0.8\sqrt{2}(k+1), \tfrac{k}{(k+1)r}\right\} V_1(t,\phi),
\end{aligned}$$

which implies that condition (ii) holds with $\mu_1 = \min\{12 - 0.8\sqrt{2}(k+1), \frac{k}{(k+1)r}\}$. Similarly, we can obtain $\mu_2 = \min\{6 - \sqrt{2}(k_2+1), \frac{k_2}{(k_2+1)r}\}$.

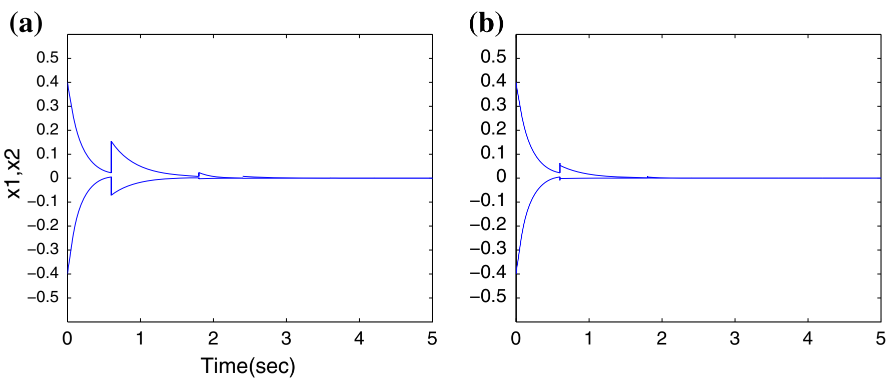


Fig. 1 Example 1: State response of system (4.1) under delayed impulsive perturbations

One can check that for any $\varepsilon > 0$, condition (v) is satisfied with $\upsilon_{0k} = 1 + \varepsilon, \upsilon_{1k} = 1 + \frac{1}{\varepsilon}$. By choosing $\varepsilon = c_1 e^{\frac{\mu_1}{2}\beta_1}$, it can be proved that condition (3.2) holds if and only if $c_1 < 1$ and $e^{-\mu_1\beta_1} + 2c_1 e^{-\frac{\mu_1}{2}\beta_1} + c_1^2 - 1 < 0$. This means that when $k_1 = 1.5, c_1 < 0.7631$, system (4.1) is exponential stable for any bounded impulse delays $\{d_k\}$. For $i = 2$, by using similar analysis, it is followed that when $k_2 = 1.5, c_2 < 0.6988$, system (4.1) is exponential stable for any bounded impulse delays $\{d_k\}$.

Next, turning to using Theorem 1, for the case of $c_1 = c_2 = 0.8$, we proceed to determine the upper bound d of impulse input delays d_k such that system (4.1) is exponential stable for any $d_k \le d$. It is noted that condition (iv) is satisfied with $\varphi_1 = (1.6)^2$ and $L_1^1 = 1 + \varepsilon, L_2^1 = 1 + \frac{1}{\varepsilon}$. Then, define $\upsilon_0 = \sqrt{b/a} K_2(dlK_1 + l\bar{h})$ and choose $\varepsilon = \sqrt{2}\upsilon_0/\sqrt{\upsilon_1}\upsilon_1^1$ where $\upsilon_1^1 = e^{-\frac{\mu_1}{2}\beta_1}$. In this case, condition (3.1) is satisfied if and only if $\upsilon_0 < 1$ and $\sqrt{2\varphi_1 \upsilon_1^1} + \upsilon_0 < 1$. Then we can obtain $\upsilon_0^1 < 0.4639$ which implies $d < 0.0543$. By using similar analysis, it is followed that $d < 0.0373$. Thus, according to above result, it follows that system (4.1) is exponential stable for any $d_k \le d = 0.0309$. Finally, Fig. 1 displays the simulation results of system (4.1) is given with $r = 1, c_1 = c_2 = 0.8$ and $d = 0.03$.

Example 2 Consider the nonlinear impulsive delay system with time delay at impulsive moment

Mode 1, i.e., $i_k = 1$;

$$\begin{aligned} x'(t) &= \begin{bmatrix} \frac{1-|\sin t|}{2} & -0.1t, \\ -0.1t & \frac{1-|\sin t|}{2} \end{bmatrix} + f(t, x(t), x(t-r)), & t \ne t_k, \\ x(t) &= 0.1x(t^-) + g_k(x(t^-), x(t-d)^-). & t = t_k. \end{aligned} \tag{4.2}$$

Mode 2, i.e., $i_k = 2$;

$$\begin{aligned} x'(t) &= \begin{bmatrix} \frac{1-|\sin t|}{4} & -0.2t, \\ -0.2t & \frac{1-|\sin t|}{4} \end{bmatrix} + f(t, x(t), x(t-r)), & t \ne t_k, \\ x(t) &= 0.1x(t^-) + g_k(x(t^-), x(t-d)^-). & t = t_k. \end{aligned} \tag{4.3}$$

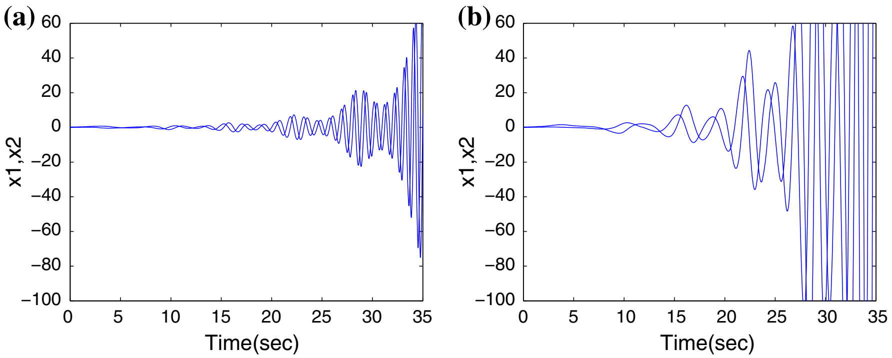
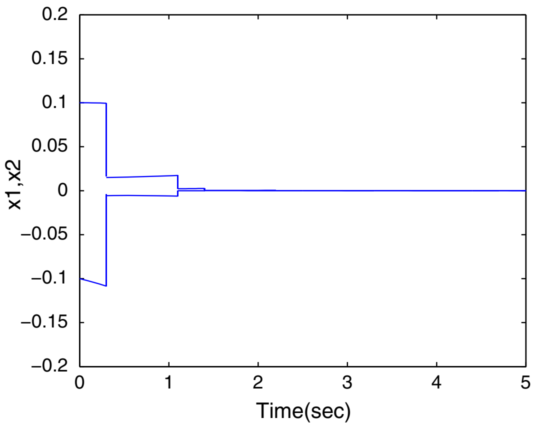


Fig. 2 Example 2: State response of system (4.2) and (4.3) without impulses

Fig. 3 Example 2: State response of system (4.2) and (4.3) with impulses



where $\beta_1 = 0.3$, $\beta_2 = 0.8$. $f : R^+ \times R^2 \times R^2 \to R^2$ and $g_k : R^2 \times R^2 \to R^2$ are continuous and satisfying the condition: $|f(t, x, y)| \leq \frac{1}{2}|\sin t||x| + \frac{1}{2}|y|$, $|g_k(x, y)| \leq c_{0k}|x| + c_{1k}|y|$. When $r = d = 0.2$, by the result of [5], the upper bound α of $c_{0k} + c_{1k}$ for exponential stability of system (4.2) is $\alpha < 0.2326$. And, using part (iii) of our result (Corollary 3) and noticing that $\mu_1^1 = \mu_1^2 = 0.5$, $\overline{M} = 0.5$, it is obtain that when $\alpha < 0.4130$, system (4.2) and (4.3) is exponentially stable. This shows that our results are less conservative than those in [5] and [21]. To illustrate our conditions numerically, we choose the functions as

$$f(t, x(t), x(t-r)) = \begin{bmatrix} 0.5(\sin(t)x_1(t) + x_2(t-0.2)) \\ 0.5(\sin(t)x_2(t) + x_1(t-0.2)) \end{bmatrix},$$

$$g_k(x(t^-), x(t-d)^-) = \begin{bmatrix} 0.05\sqrt{\left|x_1(t_k^-)x_2(t_k^- - 0.2)\right|} \\ 0.05\sqrt{\left|x_2(t_k^-)x_1(t_k^- - 0.2)\right|} \end{bmatrix},$$

so $c_{0k} = c_{1k} = 0.05$. Figure 2 shows that the corresponding subsystem without impulses is unstable, but it can be exponentially stabilized by impulses, as shown in Fig. 3.

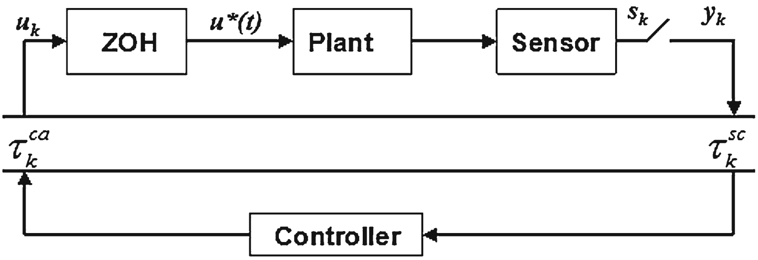


Fig. 4 Schematic overview of the networked control system

Example 3 An example is given to illustrate the studied systems is of actual background in the following (Fig. 4).

The two-channel network control systems (NCSs) are schematically depicted in Fig. 4. It consists of a continuous-time plant and a discrete-time controller, which receives information from the plant only at the k-th sampling instant s_k. The sensor acts in a time-driven (though variable) fashion and the controller and actuator [including the zero-order-hold (ZOH) in Fig. 4] act in an event-driven fashion in the sense that the controller and the actuator update their outputs as soon as they receive a new sample. In this example, plant is usually assumed to take the form:

$$x^{'}(t) = A_i t + B_i u^*(t), \quad u^*(t) = u_k.$$

and the output y_k is transmitted through a digital communication network, i.e., $y_k = C_k x(t_k^-)$. Owing to the fact that computation time and network-induced delays result in sensor-to-controller delay (τ_k^{sc}) and controller-to-actuator delay (τ_k^{ca}), the k-th control input update time t_k at which the k-th sample arrives to the destination may be greater than s_k. Denote d_k the k-th total delay with $d_k = \tau_k^{sc} + \tau_k^{ca}$. Then, we have $t_k = s_k + d_k$. Assume that we adopt the impulsive control law of $\Delta x(t_k) = B_k u_k$ to stabilize the plant. If the control input u_k is taken as the form of $u_k = K y_k$, then the above form of impulses should be modified as follows

$$\Delta x(t) = B_k K x((t - d_k)^-), \quad t = t_k.$$

Then, in this paper, the more general impulses are given:

$$x(t^+) = B_{1k} x(t^-) + B_{2k} x((t - d_k)^-), \quad t = t_k.$$

5 Conclusions

In this paper, a method of multiple Lyapunov–Krasovskii functionals has been applied to deal with the effects of delayed impulses on exponential stability of impulsive and switching time-delay systems. We have established sufficient conditions for exponential stability of time-delay systems with destabilizing impulses and stabilizing impulses. When the delayed impulses are sufficiently small, the exponential stability of impulsive and switching time-delay systems is robust. If the magnitude of the delayed impulses is sufficiently small, then under some conditions, the exponential stability properties can be derived irrespective of the size of the impulse input delays. It is shown that if the magnitude of the delayed impulses is sufficiently small, destabilizing delayed impulses cannot destroy the stability under some conditions. Applying our results to a class of nonlinear impulsive and switching time-delay systems, the deduced new stability criteria can relax some restrictions on delays and impulses imposed by the existing results. Two illustrative examples have been provided to demonstrate the main theoretical results.

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