Circuits Syst Signal Process (2014) 33:1451–1471
DOI 10.1007/s00034-013-9707-z

Robust H_∞ Output Tracking Control for a Class of Nonlinear Systems with Time-Varying Delays

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Received: 19 March 2013 / Revised: 5 November 2013 / Published online: 18 December 2013

Abstract This paper addresses the H_∞ output tracking problem for a class of nonlinear systems subjected to model uncertainties and with interval time-varying delay. The stability of the nonlinear time-delay system is analyzed with a novel delay-interval-dependent Lyapunov–Krasovskii functional. Compared to state-of-the-art criteria for linear and nonlinear time-delay systems, less conservative stability conditions are derived with the introduction of new delay-interval-dependent terms and the exploitation of the delay subintervals size. The proposed analysis considers that the delay derivative is either upper and lower bounded, bounded above only, or unbounded, i.e., no restrictions are cast upon the derivative. Numerical examples are provided to enlighten the importance and advantages of the present criterion which outperforms previous criteria in time-delay systems literature. Also, an additional example is provided to highlight the effectiveness of the proposed H_∞ output tracking control design technique for complex nonlinear systems with time-varying delay.

Keywords Nonlinear time-delay systems · Delay-dependent stability · H_∞ control · Output tracking

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1 Introduction

Time-delay systems belong to a class of infinite-dimensional systems often described by functional differential equations. The phenomena are encountered in various practical systems, e.g., biological, chemical systems, networked control systems, etc. [1, 10, 29]. They are usually employed in the description of propagation and transport phenomena, arising as feedback delays in control loops [19]. Since their existence can degrade systems performance and even cause instability, time-delay systems modeling, stability, and stabilization problems have emerged as a topic of significant interest to the control community, which is highlighted by several surveys and studies on the subject, see, e.g., [1, 10, 19, 29]. Among recent results, the following should be mentioned due to their contribution to time-delay systems analysis [5, 7, 8, 24, 37, 39]. Nonetheless, although being a fundamental issue in control theory, tracking performance and control have received little attention in time-delay systems literature, especially if we regard nonlinear time-delay systems. In this context, we investigate the H_∞ output tracking problem for a class of nonlinear systems subjected to model uncertainties and with input time-varying delay.

It is well recognized that the tracking problem is more general and challenging than stability and stabilization problems [8]. The main objective of the tracking control is to synthesize feedback controllers to make the output of a given plant asymptotically tracks a desired reference whereas ensuring disturbances attenuation properties. The importance of tracking is reflected by the extensive coverage with numerous applications in the areas of robot control, flight control, dynamic processes in industry, economics, etc.; see, e.g., [2, 16] and the references therein.

Nonetheless, existing results on time-delay systems rarely focus on tracking control problems. Indeed, time-delay systems literature contain several works on control design, e.g., [9, 22, 27, 35]; however, very few regard the tracking problem. Among these works, the following should be mentioned for their important contributions. In [15], the tracking for switched linear systems with delayed states has been investigated, but with no regard to the time-delay effects on the feedback-loop. The authors in [34] were the first to investigate the tracking problem with constant feedback delays, and their work has been extended to the H_∞ tracking control with time-varying delays in [5, 8]. The tracking control problem for nonlinear time-delay systems has been addressed in [13, 38]; however, the results are only valid for state-delayed systems, i.e., the time-delay effects on the feedback-loop were not considered. Since time-delay phenomena often arise as feedback delays in the control loop [19], the much more general and realistic scenario regarding the tracking of nonlinear systems with feedback delays still needs to be considered. To the best of the authors' knowledge, this scenario has never been considered and remains challenging. In this context, the introduction of less conservative stability techniques with the solution of this open problem are the major motivation of the present study.

During the last decade, various methods have been taken for deriving stability conditions for linear time-delay systems using different Lyapunov–Krasovskii functionals (LKFs). Particularly, a recent Lyapunov-based technique must be stressed for its significant contributions to delay-dependent stability analysis: the piecewise analysis method (PAM). The method has similar concepts to the discretized Lyapunov functionals technique (DLF) [10], although applied to time-varying delays, and has been successfully employed in recent literature, see, e.g., [6, 7, 24]. Still, we believe its potential has not been fully exploited, yet. Therefore, with novel less-restricted delay-interval-dependent LKFs, and by introducing the interval size information to the analysis, we improve the piecewise analysis method and considerably amend the stability results for time-delay systems.

In this context, the present paper brings an important contribution to the H_∞ output tracking analysis and control design for time-delay systems. The development of a novel delay-interval-dependent Lyapunov–Krasovskii functional, with the improved PAM, provides the conditions under which the prescribed H_∞ output tracking performance for a class of nonlinear uncertain time-delay system is achieved. The time-varying nonlinearities are assumed to be norm-bounded, satisfying a quadratic constraint. Moreover, it should be mentioned that the proposed tracking criterion, if particularized to stability analysis, also yields considerably superior results compared with state-of-the-art criteria for linear or nonlinear time-delay systems. The analysis is enriched with numerical examples that illustrates the advantages of our criteria, which outperform previous criteria in the literature, and with an additional example that shows the effectiveness of the proposed H_∞ output tracking control for nonlinear time-delay systems.

Notations Throughout the paper, the superscript 'T' stands for matrix transposition, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbb{R}^{n\times p}$ defines the set of all $n \times p$ real matrices. The notation diag$\{\cdots\}$ stands for a diagonal matrix, $P > 0$ means that P is symmetric and positive definite, and the symmetric term in a matrix is denoted by $*$. The notation $A|_{s\to b}$ stands for the limit of an s-dependent matrix A as $s \to b$. Matrices, if not explicitly stated, are assumed to have compatible dimensions.

2 Problem Formulation and Preliminaries

Consider a class of continuous-time nonlinear uncertain systems with time-varying delay:

$$\begin{cases} \dot{x}_p(t) = (A_p + \Delta A_p)x_p(t) + (B_p + \Delta B_p)u_p(t - d(t)) \\ \qquad\qquad + g(t, x_p(t), x_p(t - d(t))) + B_{p\omega}\omega(t), \\ y_p(t) = (C_p + \Delta C_p)x_p(t) + (D_p + \Delta D_p)u_p(t - d(t)), \quad t > 0, \\ x_p(t) = \rho(t), \qquad\qquad t \in [-\tau_{\max},\ 0], \end{cases} \tag{1}$$

where $x_p(t) \in \mathbb{R}^{r_{px}}$, $u_p(t) \in \mathbb{R}^{r_{pu}}$, $y_p(t) \in \mathbb{R}^{r_{py}}$ denote the plant's state, control input and output vectors, respectively, $\omega(t) \in \mathbb{R}^{r_{p\omega}}$ denotes the exogenous disturbance signal which is assumed to belong to $L_2[0, \infty)$. A_p, B_p, $B_{p\omega}$, C_p, D_p are constant matrices with appropriate dimensions, $\rho(t)$ describes the state's initial condition, and $g(t, x_p(t), x_p(t - d(t))) : \mathbb{R}_+ \times \mathbb{R}^{r_{px}} \times \mathbb{R}^{r_{px}} \to \mathbb{R}^{r_{px}}$ denotes a class of piecewise-continuous nonlinear functions in $t, x_p(t), x_p(t - d(t))$, which are assumed to satisfy the quadratic condition:

$$g^T\big(t, x_p(t), x_p\big(t-d(t)\big)\big)g\big(t, x_p(t), x_p\big(t-d(t)\big)\big)$$
$$\leq \alpha_1 x_p^T(t)H_1^T H_1 x_p(t) + \alpha_2 x_p^T\big(t-d(t)\big)H_2^T H_2 x_p\big(t-d(t)\big), \quad (2)$$

where α_1, α_2 are positive known bounding parameters of $g(t, x_p(t), x_p(t-d(t)))$, and H_1 and H_2 are known constant matrices. The systems uncertainties are assumed to be time-varying matrices

$$\begin{bmatrix}\Delta A_p & \Delta B_p\end{bmatrix} := \Xi_x \Delta(t)\begin{bmatrix}\Xi_A & \Xi_B\end{bmatrix}, \qquad \begin{bmatrix}\Delta C_p & \Delta D_p\end{bmatrix} := \Xi_y \Delta(t)\begin{bmatrix}\Xi_C & \Xi_D\end{bmatrix}, \quad (3)$$

where $\Xi_x, \Xi_A, \Xi_B, \Xi_y, \Xi_C, \Xi_D$ are known matrices with appropriate dimensions, and $\Delta(t)$ is an unknown time-varying matrix, which is Lebesgue measurable in t and satisfies $\Delta^T(t)\Delta(t) \leq I$.

We consider the reference signal, $y_r(t) \in \mathbb{R}^{r_{yr}}$, to be the output of the given linear system:

$$\dot{x}_r(t) = A_r x_r(t) + r(t), \quad (4a)$$

$$y_r(t) = C_r x_r(t), \quad (4b)$$

where $x_r(t), r(t) \in \mathbb{R}^{r_r}$ are the reference state vector and the energy bounded reference input, respectively, A_r is a Hurwitz matrix, and C_r is a constant matrix with appropriate dimensions.

Finally, the continuous function $d(t)$ denotes the time-varying delay which satisfies

$$\tau_{\min} \leq d(t) \leq \tau_{\max}, \quad (5a)$$

$$d_{\min} \leq \dot{d}(t) \leq d_{\max}, \quad (5b)$$

where the constants $0 \leq \tau_{\min} \leq \tau_{\max}$ and $d_{\min} \leq d_{\max}$ denote the bounding parameters of $d(t)$ and $\dot{d}(t)$, respectively. In this paper, we also consider the case when $d_{\min}$ is unknown, and when no restrictions are cast upon the delay derivative, i.e., when it is assumed to be fast-varying.

Considering (1)–(5b) with a state feedback control law $u_p(t) = \bar{K}[x_p^T(t)\ x_r^T(t)]^T$, we obtain the augmented closed-loop nonlinear system with time-varying delay

$$\begin{aligned}\dot{x}(t) &= (\bar{A} + \Delta\bar{A})x(t) + (\bar{B} + \Delta\bar{B})\bar{K}x\big(t-d(t)\big) \\ &\quad + \bar{g}\big(t, x(t), x\big(t-d(t)\big)\big) + \bar{B}_\omega\bar{\omega}(t), \\ e(t) &= (\bar{C} + \Delta\bar{C})x(t) + (\bar{D} + \Delta\bar{D})\bar{K}x\big(t-d(t)\big),\end{aligned} \quad (6)$$

where $x^T(t) := [x_p^T(t)\ x_r^T(t)] \in \mathbb{R}^{r_x}$, $\bar{\omega}^T(t) := [\omega^T(t)\ r^T(t)] \in \mathbb{R}^{r_\omega}$,

$$\begin{aligned}&\bar{A} := \begin{bmatrix}A_p & 0\\ 0 & A_r\end{bmatrix}, \qquad \bar{B} := \begin{bmatrix}B_p\\ 0\end{bmatrix}, \qquad \bar{B}_\omega := \begin{bmatrix}B_\omega & 0\\ 0 & I\end{bmatrix}, \\ &\bar{C} := \begin{bmatrix}C_p & -C_r\end{bmatrix}, \qquad \bar{D} := D_p,\end{aligned} \quad (7a)$$

$$\begin{bmatrix}\Delta\bar{A} & \Delta\bar{B}\end{bmatrix} := \bar{\Xi}_x \Delta(t)\begin{bmatrix}\bar{\Xi}_A & \Xi_B\end{bmatrix}, \qquad \bar{\Xi}_x := \begin{bmatrix}\Xi_x^T & 0\end{bmatrix}^T, \qquad \bar{\Xi}_A := \begin{bmatrix}\Xi_A & 0\end{bmatrix}, \tag{7b}$$

$$\begin{bmatrix}\Delta\bar{C} & \Delta\bar{D}\end{bmatrix} := \bar{\Xi}_e \Delta(t)\begin{bmatrix}\bar{\Xi}_C & \Xi_D\end{bmatrix}, \qquad \bar{\Xi}_e := \Xi_y, \qquad \bar{\Xi}_C := \begin{bmatrix}\Xi_C & 0\end{bmatrix}, \tag{7c}$$

$$\bar{g}\big(t, x(t), x\big(t - d(t)\big)\big) := \begin{bmatrix}I & 0\end{bmatrix}^T g\big(t, x_p(t), x_p\big(t - d(t)\big)\big). \tag{7d}$$

The matrix $\bar{K}$ is the state-feedback controller, and $e(t) := y_p(t) - y_r(t)$ denotes the output tracking error.

Tracking Problem We desire the plant's output $y_p(t)$ to asymptotically track a given reference signal $y_r(t)$. Our purpose is therefore to design a robust state-feedback controller $\bar{K}$ such that the output tracking performance γ is ensured in the H_∞ sense.

Definition 1 For a prescribed scalar $\gamma > 0$, the nonlinear time-delay system (6) achieves H_∞ output tracking performance, if for any realization of the uncertainties $\Delta\bar{A}$, $\Delta\bar{B}$, $\Delta\bar{C}$, $\Delta\bar{D}$, the following hold:

1. The augmented closed-loop nonlinear system (6) with $\bar{\omega}(t) \equiv 0$ is asymptotically stable;
2. Under the assumption of zero initial condition, the disturbance effect on the tracking error is attenuated below a prescribed level γ, $\|e(t)\|_2 < \gamma\|\overline{w}(t)\|_2$, for all nonzero $\overline{w} \in L_2[0, \infty)$.

3 $\mathbf{H_\infty}$ Output Tracking Control Design

This section presents the main results of this paper. First, we divide the delay range $[\tau_{\min}, \tau_{\max}]$ into two equally spaced subintervals: $[\tau_1, \tau_2]$ and $[\tau_2, \tau_3]$, where $\tau_1 = \tau_{\min}$, $\tau_3 = \tau_{\max}$, and $\tau_2 = \frac{1}{2}(\tau_{\max} + \tau_{\min})$. Note that one can consider different partitioning strategies (e.g., in [23], τ_2 is defined to be anywhere between $\tau_{\min}$ and $\tau_{\max}$). Still, choosing equal subintervals, $\tau_3 - \tau_2 = \tau_2 - \tau_1$, adds more information to the analysis which is used to obtain less conservative criteria. In this context, we also define the auxiliary variable

$$\tau_\sigma := \tau_2 - \tau_1, \tag{8}$$

and the delay-interval-dependent indicator function $\chi_{[\tau_1,\tau_2]} : \mathbb{R}_+ \to \{0, 1\}$, which is assumed to be 1, if $d(t) \in [\tau_1, \tau_2]$, and $\chi_{[\tau_1,\tau_2]} = 0$, otherwise.

The indicator function enlightens the piecewise analysis method main contribution: the establishment of different linear matrix inequalities (LMIs) for each subinterval, reducing the conservatism which arises from the analysis of the delay range $[\tau_{\min}, \tau_{\max}]$. In this context, it is proposed the following delay-interval-dependent

LKF candidate

$$
\begin{aligned}
V(t) &= \sum_{i=1}^{3} V_i(t),\\
V_1(t) &= \chi_{[\tau_1,\tau_2]} x^T(t)\widehat{P}_1\big(d(t)\big)x(t) + (1-\chi_{[\tau_1,\tau_2]})x^T(t)\widehat{P}_2\big(d(t)\big)x(t),\\
V_2(t) &= \int_{t-d(t)}^{t-\tau_1} x^T(s)Qx(s)\,ds\\
&\quad + \int_{t-\frac{\tau_1}{2}}^{t} \big[x^T(s) \quad x^T(s-\tfrac{\tau_1}{2})\big] N_1 \big[x^T(s) \quad x^T(s-\tfrac{\tau_1}{2})\big]^T\,ds\\
&\quad + \left(\tau_\sigma - \frac{\tau_1}{2}\right)\int_{t-\tau_\sigma}^{t-\frac{\tau_1}{2}} x^T(s)N_2x(s)\,ds\\
&\quad + (\tau_\sigma-\tau_1)\int_{t-\tau_\sigma}^{t-\tau_1} x^T(s)N_3x(s)\,ds + \int_{t-\tau_\sigma}^{t}\varphi^T(s)N_4\varphi(s)\,ds,\\
V_3(t) &= \sum_{k=1}^{2}\Bigg(\frac{\tau_1}{2}\int_{-\frac{\tau_1}{2}k}^{-\frac{\tau_1}{2}(k-1)}\int_{t+\beta}^{t}\dot{x}^T(s)S_k\dot{x}(s)\,ds\,d\beta \qquad (9)\\
&\quad + \left(\tau_\sigma-\frac{\tau_1}{2}k\right)\int_{-\tau_\sigma}^{-\frac{\tau_1}{2}k}\int_{t+\beta}^{t}\dot{x}^T(s)S_{k+2}\dot{x}(s)\,ds\,d\beta\Bigg)\\
&\quad + \sum_{k=0}^{2}\left(\tau_\sigma\int_{-\tau_k-\tau_\sigma}^{-\tau_k}\int_{t+\beta}^{t}\dot{x}^T(s)Z_k\dot{x}(s)\,ds\,d\beta\right)\\
&\quad + \int_{-d(t)}^{0}\int_{t+\beta}^{t}\dot{x}^T(s)(R_1+R_2)\dot{x}(s)\,ds\,d\beta\\
&\quad + \int_{-\tau_3}^{-d(t)}\int_{t+\beta}^{t}\dot{x}^T(s)(R_3+R_4)\dot{x}(s)\,ds\,d\beta\\
&\quad + \chi_{[\tau_1,\tau_2]}\int_{-\tau_2}^{-d(t)}\int_{t+\beta}^{t}\dot{x}^T(s)(R_1-R_3)\dot{x}(s)\,ds\,d\beta\\
&\quad + (1-\chi_{[\tau_1,\tau_2]})\int_{-d(t)}^{-\tau_2}\int_{t+\beta}^{t}\dot{x}^T(s)(R_3-R_1)\dot{x}(s)\,ds\,d\beta,
\end{aligned}
$$

where $\varphi^T(s) = [x^T(s)\ x^T(s-\tau_1)\ x^T(s-\tau_2)]$, and the function matrices in $V_1(t)$ are defined as follows

$$
\begin{aligned}
\widehat{P}_1\big(d(t)\big) &:= \frac{d(t)-\tau_1}{\tau_2-\tau_1}\frac{P_1+P_2}{2} + \frac{\tau_2-d(t)}{\tau_2-\tau_1}P_1,\\
\widehat{P}_2\big(d(t)\big) &:= \frac{d(t)-\tau_2}{\tau_3-\tau_2}P_2 + \frac{\tau_3-d(t)}{\tau_3-\tau_2}\frac{P_1+P_2}{2}.
\end{aligned} \qquad (10)
$$

Note that the aforementioned delay-interval-dependent terms, firstly exploited in [7], together with the novel terms in $V_3(t)$, are continuously differentiable in t, and that the Lyapunov candidate (9) is positive definite if the following hold:

$$\begin{aligned}&P_1 > 0, \quad P_2 > 0, \quad Q \geq 0, \quad Z_0 > 0, \quad Z_1 > 0, \quad Z_2 > 0,\\ &N_i \geq 0, \quad S_i > 0, \quad i = \{1,2,3,4\}, \quad R_1 + R_2 > 0, \quad R_3 + R_4 > 0,\\ &Z_2 > \frac{1}{\tau_\sigma}(R_1 - R_3) > -Z_1.\end{aligned} \tag{11}$$

It is interesting to highlight that the novel delay-dependent, delay-derivative-dependent, and delay-interval-dependent LKF terms that are introduced in the Lyapunov candidate (9) lead to less-restricted stability conditions. The novel and enhanced stability analysis presented in the next subsection and the improvements illustrated in Sect. 4 stem mostly from this new set of Lyapunov terms.

Remark 1 Different strategies for the delay range partitioning have been previously considered, leading to different results. In [23], the dividing parameter τ_2 is manually defined at any point within the delay range, whereas in [36], an iterative optimization procedure based on the Nelder–Mead simplex algorithm [21] is proposed to define the partitioning points. Still, by simply choosing equally spaced subintervals, we add more information to the analysis. Indeed, the exact new partitioning subinterval size, together with the knowledge of the relationship between the system's states $x(t), x(t-\tau_\sigma), x(t-\tau_i)$, $i = \{1,2,3\}$, deem the new auxiliary variable τ_σ essential for the construction of Lyapunov candidate, as we can switch among these states solely by adding/subtracting a delay equal to τ_σ, e.g., $V_2(t)$. The amount of information and the relationship described solely by τ_σ can only stem from an equal partitioning technique, and that leads to an improved exploitation of the delayed states during the design of the Lyapunov candidate (10), e.g., by adding a delay τ_σ to $\varphi^T(t)$, we obtain $\varphi^T(t-\tau_\sigma) = [x^T(t-\tau_\sigma)\ x^T(s-\tau_2)\ x^T(s-\tau_3)]$. Furthermore, since we are adding non-diagonal terms to relate the states $x(t), x(t-\tau_\sigma), x(t-\tau_i)$, $i = \{1,2,3\}$, we indeed obtain less restrictive LMIs constraints, which in turn leads to less conservative results.

3.1 Robust H_∞ Output Tracking Performance analysis

In this subsection, we derive conditions under which the closed-loop uncertain nonlinear system (6) achieves H_∞ output tracking performance γ, namely, the augmented closed-loop system is asymptotically stable and satisfies the performance conditions described in Definition 1. The following result stems from the proposed Lyapunov candidate (9), and describes a novel robust criterion for the output tracking in the H_∞ sense.

Theorem 1 *For a prescribed* $\gamma > 0$, *given scalars* $\tau_{\min}, \tau_{\max}, d_{\min}, d_{\max}$ *such that* $0 \leq \tau_{\min} \leq \tau_{\max}$ *and* $d_{\min} < d_{\max}$, *and given controller gain* $\bar{K}$, *the augmented closed-loop nonlinear system* (6) *with time-varying delay satisfying* (5a)–(5b), *pa-*

rameter uncertainties and nonlinearities described in (7a)–(7d) and (2), respectively, achieves H_∞ output tracking performance γ if there exist positive scalars $\epsilon_{11}, \epsilon_{12}, \epsilon_{21}, \epsilon_{22}, \eta_1, \eta_2$, matrices $P_1, P_2, Q, Z_0, Z_1, Z_2 N_i, S_i, R_i$, $i = \{1, 2, 3, 4\}$, with appropriate dimensions, satisfying (11) and free-weighting matrices $\mathcal{F}_1, \mathcal{F}_2 \in \mathbb{R}^{8r_x \times r_x}$, $\mathcal{V}_1, \mathcal{V}_2 \in \mathbb{R}^{8r_x \times 2r_x}$ such that the following inequalities

$$\Omega_{1k}|_{\dot{d}(t)\to d_{\max}} < 0, \qquad \Omega_{2k}|_{\dot{d}(t)\to d_{\max}} < 0, \qquad S_k + U_1|_{\dot{d}(t)\to d_{\max}} > 0, \tag{12a}$$

$$\Omega_{1k}|_{\dot{d}(t)\to d_{\min}} < 0, \qquad \Omega_{2k}|_{\dot{d}(t)\to d_{\min}} < 0, \qquad S_k + U_1|_{\dot{d}(t)\to d_{\min}} > 0, \tag{12b}$$

hold for $k = \{1, 2\}$, where

$$\begin{aligned} \Omega_{1k} &= \begin{bmatrix} \Pi_1 + \Psi^{(1)}|_{d(t)\to\tau_k} & \tau_\sigma \mathcal{V}_1 J_k & \Upsilon_1 \\ * & -\tau_\sigma \Lambda_{1k} & 0 \\ * & * & \Sigma_1 \end{bmatrix}, \\ \Omega_{2k} &= \begin{bmatrix} \Pi_2 + \Psi^{(2)}|_{d(t)\to\tau_{(k+1)}} & \tau_\sigma \mathcal{V}_2 J_k & \Upsilon_2 \\ * & -\tau_\sigma \Lambda_{2k} & 0 \\ * & * & \Sigma_2 \end{bmatrix}, \end{aligned} \tag{13}$$

with $J_1 = [0\ I]^T$, $J_2 = [I\ 0]^T$ and

$$\begin{aligned}
&\Lambda_{11} = \tau_\sigma Z_1 + R_1 + R_4, \qquad \Gamma_1 = J_2(\mathbb{I}_2 - \mathbb{I}_5) + J_1(\mathbb{I}_6 - \mathbb{I}_2), \\
&\Sigma_\ell = \mathbb{I}_2 \bar{\Xi}_e \mathbb{I}_5^T - \operatorname{diag}\{\gamma^2 I; I; \epsilon_{\ell 1} I; \epsilon_{\ell 1} I; \epsilon_{\ell 2} I; \epsilon_{\ell 2} I; \eta_\ell\}, \\
&\Lambda_{12} = \tau_\sigma Z_1 + U_R, \qquad \Gamma_2 = J_2(\mathbb{I}_2 - \mathbb{I}_6) + J_1(\mathbb{I}_7 - \mathbb{I}_2), \\
&\Pi_\ell = \mathcal{F}_\ell \Gamma_x^T + \Gamma_x \mathcal{F}_\ell^T + \mathcal{V}_\ell \Gamma_\ell^T + \Gamma_\ell \mathcal{V}_\ell^T + \widetilde{H}_\ell, \\
&\Lambda_{21} = \tau_\sigma Z_2 + R_3 + R_4, \qquad \Gamma_x = \bar{A}^T \mathbb{I}_1 + (\bar{B}\bar{K})^T \mathbb{I}_2 - \mathbb{I}_3, \\
&\widetilde{H}_\ell = \eta_\ell \mathbb{I}_1 (\alpha_1 H_1^T H_1) \mathbb{I}_1^T + \eta_\ell \mathbb{I}_2 (\alpha_2 H_2^T H_2) \mathbb{I}_2^T, \\
&\Lambda_{22} = \tau_\sigma Z_2 + U_R + R_3 - R1, \qquad U_R = R_1 + \dot{d}(t) R_4 + (1 - \dot{d}(t)) R_2, \\
&U_1 = \frac{2}{\tau_1} U_R, \quad \textit{if } \tau_1 \neq 0, \textit{ or } U_1 = 0, \textit{ otherwise}, \\
&\Upsilon_\ell = \mathcal{F}_\ell (\bar{B}_\omega \mathbb{I}_1^T + \bar{\Xi}_x \mathbb{I}_3^T + \mathbb{I}_7^T) + (\bar{C}^T \mathbb{I}_1 + (\bar{D}\bar{K})^T \mathbb{I}_2) \mathbb{I}_2^T \\
&\qquad + \epsilon_{\ell 1} (\mathbb{I}_1 \bar{\Xi}_A^T + \mathbb{I}_2 (\bar{\Xi}_B \bar{K})^T) \mathbb{I}_4^T + \epsilon_{\ell 2} (\mathbb{I}_1 \bar{\Xi}_C^T + \mathbb{I}_2 (\bar{\Xi}_D \bar{K})^T) \mathbb{I}_6^T, \\
&\Psi^{(1)} = \widetilde{\Psi}(\dot{d}(t)) - (\mathbb{I}_6 - \mathbb{I}_7) \frac{1}{\tau_\sigma} \Lambda_{21} (\mathbb{I}_6 - \mathbb{I}_7)^T \\
&\qquad + \mathbb{I}_3 ((\tau_3 - d(t)) R_4 + d(t) R_2) \mathbb{I}_3^T + \mathbb{I}_1 \widehat{P}_1(d(t)) \mathbb{I}_3^T + \mathbb{I}_3 \widehat{P}_1(d(t)) \mathbb{I}_1^T, \\
&\Psi^{(2)} = \widetilde{\Psi}(\dot{d}(t)) - (\mathbb{I}_5 - \mathbb{I}_6) \frac{1}{\tau_\sigma} \Lambda_{12} (\mathbb{I}_5 - \mathbb{I}_6)^T + \mathbb{I}_3 ((\tau_3 - d(t)) R_4 + d(t) R_2) \mathbb{I}_3^T \\
&\qquad + \mathbb{I}_1 \widehat{P}_2(d(t)) \mathbb{I}_3^T + \mathbb{I}_3 \widehat{P}_2(d(t)) \mathbb{I}_1^T,
\end{aligned} \tag{14}$$

$$\begin{aligned}\widetilde{\Psi}\big(\dot{d}(t)\big) = \mathbb{I}_3\Bigg(\left(\frac{\tau_1}{2}\right)^2(S_1+S_2) + \left(\tau_\sigma - \frac{\tau_1}{2}\right)^2 S_3 + (\tau_\sigma-\tau_1)^2 S_4 \\ + \tau_\sigma^2(Z_0+Z_1+Z_2) + \tau_\sigma R_3 + \tau_2 R_1\Bigg)\mathbb{I}_3^T + \begin{bmatrix}\mathbb{I}_1 & \mathbb{I}_4\end{bmatrix} N_1 \begin{bmatrix}\mathbb{I}_1 & \mathbb{I}_4\end{bmatrix}^T \\ - \begin{bmatrix}\mathbb{I}_4 & \mathbb{I}_5\end{bmatrix} N_1 \begin{bmatrix}\mathbb{I}_4 & \mathbb{I}_5\end{bmatrix}^T + \begin{bmatrix}\mathbb{I}_1 & \mathbb{I}_5 & \mathbb{I}_6\end{bmatrix} N_4 \begin{bmatrix}\mathbb{I}_1 & \mathbb{I}_5 & \mathbb{I}_6\end{bmatrix}^T \\ - \begin{bmatrix}\mathbb{I}_8 & \mathbb{I}_6 & \mathbb{I}_7\end{bmatrix} N_4 \begin{bmatrix}\mathbb{I}_8 & \mathbb{I}_6 & \mathbb{I}_7\end{bmatrix}^T - (\mathbb{I}_1-\mathbb{I}_8)Z_0(\mathbb{I}_1-\mathbb{I}_8)^T \\ - (\mathbb{I}_1-\mathbb{I}_4)(S_1+U_1)(\mathbb{I}_1-\mathbb{I}_4)^T - (\mathbb{I}_4-\mathbb{I}_5)(S_2+U_1)(\mathbb{I}_4-\mathbb{I}_5)^T \\ - (\mathbb{I}_8-\mathbb{I}_4)S_3(\mathbb{I}_8-\mathbb{I}_4)^T - (\mathbb{I}_8-\mathbb{I}_5)S_4(\mathbb{I}_8-\mathbb{I}_5)^T \\ + \operatorname{diag}\Bigg\{\frac{\dot{d}(t)}{\tau_\sigma}\frac{P_2-P_1}{2}; -\big(1-\dot{d}(t)\big)Q; 0; \left(\tau_\sigma - \frac{\tau_1}{2}\right)N_2; \\ \big(Q + (\tau_\sigma-\tau_1)N_3\big); 0; 0; -\left(\tau_\sigma - \frac{\tau_1}{2}\right)N_2 - (\tau_\sigma-\tau_1)N_3\Bigg\}.\end{aligned}$$

The matrices $\mathbb{I}_i$, $i = \{1, 2, \dots, 8\}$, *are block entry matrices with eight elements*, *e.g.*, $\mathbb{I}_4^T = [0\;0\;0\;I\;0\;0\;0\;0]$.

Remark 2 It is also interesting to consider two particular cases regarding the delay and its derivative information: the case when the time-delay derivative lower bound is unknown, and the case when there exist no information concerning the delay derivative, i.e., fast-varying delays. Theorem 1 can be easily adapted to deal with both cases. For the first case, if we take the conditions $P_2 > P_1$, $R_2 > R_4$ instead of (12b), then Theorem 1 becomes suitable for the analysis when the lower bound, $d_{\min}$, is unknown. Note that, if the above conditions and (12a) hold, then (12b) will be satisfied regardless of $d_{\min}$. An evident consequence is the needlessness of the derivative lower bound information for the resulting performance conditions. For the latter case, by assuming $P_1 = P_2$, and null Q, R_2, R_4 matrices, all the time-delay derivative information is removed from Theorem 1, and the criterion will thus be suitable for the analysis with fast-varying delays. Moreover, it should be mentioned that Theorem 1 can also be applied for nonlinear/linear time-delay systems if one simply takes $\bar{B}_\omega$, $\bar{C}$, $\bar{D}$ to be null matrices.

Theorem 1 presents conditions which guarantee the H_∞ output tracking performance for nonlinear time-delay systems. The results stem from a novel delay-dependent Lyapunov–Krasovskii functional that enhances the delay fractioning and the piecewise analysis. With interval-dependent terms and by further exploiting the delay partitioning information, we have weakened the positiveness constraints upon new functional terms and matrices, whereas maintaining (9) positive definite and continuously differentiable. The proposed method therefore increases the flexibility analysis upon some matrices and relaxes resulting LMIs conditions, yielding in a considerably reduction of conservatism, even if compared with state-of-the-art results for linear time-delay systems stability analysis.

3.2 Robust H_∞ Output Tracking Controller Design

For the H_∞ output tracking control, we seek conditions for the design of a state-feedback gain $\bar{K}$ which leads the nonlinear time-delay systems output to asymptotically track a desired reference while ensuring disturbance attenuation properties in the H_∞ sense. The next theorem provides a solution for the above-mentioned problem, which is non-convex due to the existence of the variable $\bar{K}$. The main idea is to transform the non-convex problem into a rank minimization problem which may be approximated by a sequence of semi-definite problems involving the trace minimization of certain variables. The H_∞ output tracking control problem is then solved through the use of the cone complementarity linearization algorithm (CCLA) from [4].

Theorem 2 *For a prescribed* $\gamma > 0$, *and given* $\tau_{\min}, \tau_{\max}, d_{\min}, d_{\max}$, *there exist a feedback gain* $\bar{K}$ *such that the resulting closed-loop nonlinear system* (6) *with input time-varying delay satisfying* (5a)–(5b), *uncertainties and nonlinearities described in* (7a)–(7d) *and* (2), *respectively, achieves* H_∞ *output tracking performance* γ *if there exist positive scalars* $\hat{\epsilon}_{11}, \hat{\epsilon}_{12}, \hat{\epsilon}_{21}, \hat{\epsilon}_{22}, \hat{\eta}_1, \hat{\eta}_2$, *matrices* $P_1, P_2, Q, Z_0, \mathrm{Z}_1, \mathrm{Z}_2 N_i, S_i, R_i$, $i = \{1, 2, 3, 4\}$, *satisfying* (11); *free-weighting matrices* $\mathcal{V}_1, \mathcal{V}_2 \in \mathbb{R}^{8r_x \times 2r_x}$, $\mathsf{Y} \in \mathbb{R}^{r_u \times r_x}$; *and definite positive matrices* $\mathsf{X}, \mathrm{F}_j, \mathfrak{M}_j, \mathfrak{N}_j \in \mathbb{R}^{r_x \times r_x}$, $j = \{1, 2\}$, *such that the global minimum of the optimization problem*

$$\min \quad \mathrm{tr}\{\mathsf{X}\widetilde{\mathsf{X}} + \mathfrak{N}_1\widetilde{\mathfrak{N}}_1 + \mathfrak{N}_2\widetilde{\mathfrak{N}}_2 + \mathfrak{M}_1\widetilde{\mathfrak{M}}_1 + \mathfrak{M}_2\widetilde{\mathfrak{M}}_2\}, \tag{15}$$

subject to

$$\begin{bmatrix} \mathsf{X} & I \\ I & \widetilde{\mathsf{X}} \end{bmatrix} \geq 0, \quad \begin{bmatrix} \mathfrak{M}_k & I \\ I & \widetilde{\mathfrak{M}}_k \end{bmatrix} \geq 0, \quad \begin{bmatrix} \mathfrak{N}_k & I \\ I & \widetilde{\mathfrak{N}}_k \end{bmatrix} \geq 0, \tag{16}$$
$$\begin{bmatrix} \widetilde{\mathsf{X}} & F_k \\ * & \widetilde{\mathfrak{M}}_k \end{bmatrix} \geq 0, \quad \begin{bmatrix} F_k & \widetilde{\mathsf{X}} \\ * & \widetilde{\mathfrak{N}}_k \end{bmatrix} \geq 0,$$

and (12a)–(12b), *for* $k = \{1, 2\}$, *is equal to* $5r_x$, *where*

$$\Omega_{\ell k} = \begin{bmatrix} \widehat{\Pi}_\ell + \Psi^{(\ell)}|_{d(t)\to\tau_k} & \tau_\sigma \mathcal{V}_\ell J_k & \widehat{\Upsilon}_\ell \ [\ \widehat{\Gamma}_x & \mathbb{I}_1 \mathsf{X} H_1^T + \mathbb{I}_2 \mathsf{X} H_2^T\,] \\ * & -\tau_\sigma \Lambda_{\ell k} & 0 \ \ [0 & 0] \\ * & * & \widehat{\Sigma}_\ell \ [\widehat{\Gamma}_\Delta & 0] \\ * & * & * \ \ \mathrm{diag}\{-\mathfrak{M}_\ell; -(\hat{\eta}_\ell \alpha_\ell^{-1}) I\} \end{bmatrix},$$

for $\ell, k = \{1, 2\}$,

$$\begin{aligned} \widehat{\Pi}_\ell &= \mathbb{I}_1 \widehat{\Gamma}_x^T + \widehat{\Gamma}_x \mathbb{I}_1^T + \mathcal{V}_\ell \Gamma_\ell^T + \Gamma_\ell \mathcal{V}_\ell^T + \mathbb{I}_3(\mathsf{X} - 4\mathfrak{N}_\ell)\mathbb{I}_3^T, \\ \widehat{\Gamma}_\Delta^T &= \bar{B}_\omega \mathbb{I}_1^T + \hat{\epsilon}_{\ell 1} \bar{\Xi}_x \mathbb{I}_3^T + \hat{\eta}_\ell \mathbb{I}_7^T, \\ \widehat{\Sigma}_\ell &= \mathbb{I}_2(\hat{\epsilon}_{\ell 2} \bar{\Xi}_e)\mathbb{I}_5^T - \mathrm{diag}\left\{\gamma^2 I; I; \hat{\epsilon}_{\ell 1} I; \hat{\epsilon}_{\ell 1} I; \hat{\epsilon}_{\ell 2} I; \hat{\epsilon}_{\ell 2} I; \hat{\eta}_\ell\right\}, \\ \widehat{\Gamma}_x^T &= \bar{A}\mathsf{X}\mathbb{I}_1^T + \bar{B}\mathsf{Y}\mathbb{I}_2^T - \mathsf{X}\mathbb{I}_3^T, \end{aligned} \tag{17}$$

$$\widehat{\Upsilon}_\ell = \mathbb{I}_1 \widehat{\Gamma}_\Delta^T + \widehat{\Gamma}_e \mathbb{I}_2^T + \left(\mathbb{I}_1 \mathsf{X} \bar{\Xi}_A^T + \mathbb{I}_2 \mathsf{Y}^T \bar{\Xi}_B^T\right)\mathbb{I}_4^T + \left(\mathbb{I}_1 \mathsf{X} \bar{\Xi}_C^T + \mathbb{I}_2 \mathsf{Y}^T \bar{\Xi}_D^T\right)\mathbb{I}_6^T,$$
$$\widehat{\Gamma}_e^T = \bar{C}\mathsf{X}\mathbb{I}_1^T + \bar{D}\mathsf{Y}\mathbb{I}_2^T,$$

with $\Psi^{(1)}, \Psi^{(2)}, J_1, J_2, \Lambda_{11}, \Lambda_{12}, \Lambda_{21}, \Lambda_{22}, \Gamma_1, \Gamma_2$, *and* $\mathbb{I}_i, i = \{1, 2, \dots, 8\}$, *defined in* (14). *Moreover, if the above conditions are satisfied, the stabilizing controller gain is given by* $\bar{K} = \mathsf{Y}\mathsf{X}^{-1}$.

Proof The proposed stabilization technique is based on the results from Theorem 1 and [4, 20]. The basic idea is to redefine the non-convex problem in a particular manner to obtain nonlinear equalities constraints, e.g., $\mathsf{X}\widetilde{\mathsf{X}} = I$, which are proved to be satisfied if a rank minimization problem involving the matrices trace is solved. To obtain such results, we first set $\mathcal{F}_\ell := [\mathsf{X}^{-1}\ 0\ F_\ell\ 0\ \dots\ 0]$, and pre- and post-multiply (12a)–(12b) by $\mathcal{D}_\ell := \mathrm{diag}\{\mathsf{X}; \dots; \mathsf{X}; 0; 0; \hat{\epsilon}_{\ell 1}; \hat{\epsilon}_{\ell 1}; \hat{\epsilon}_{\ell 2}; \hat{\epsilon}_{\ell 2}; \hat{\eta}_\ell\}$, where $\hat{\epsilon}_{\ell k} := \epsilon_{\ell k}^{-1}$ and $\hat{\eta}_\ell := \eta_\ell^{-1}$, for $\ell, k = \{1, 2\}$. Note that all the variables in (11), which exclusively appear in $\Psi^{(\ell)}$ and $\Lambda_{\ell k}$, are pre- and post-multiplied by X. Therefore, they can be easily redefined in such a manner that $\Psi^{(\ell)} \leftarrow \mathsf{X}\Psi^{(\ell)}\mathsf{X}$ and $\Lambda_{\ell k} \leftarrow \mathsf{X}\Lambda_{\ell k}\mathsf{X}$. Similar argument is valid for the slack-matrices $\mathcal{V}_\ell$, $\ell = \{1, 2\}$. Thus, we have

$$\mathcal{D}_\ell \Omega_{\ell k} \mathcal{D}_\ell = \begin{bmatrix} \mho_\ell + \Psi^{(\ell)} & \tau_\sigma \mathcal{V}_\ell J_k & \widehat{\Upsilon}_\ell \\ * & -\tau_\sigma \Lambda_{\ell k} & 0 \\ * & * & \widehat{\Sigma}_\ell \end{bmatrix} + \vartheta_\ell^T \beta_\ell + \beta_\ell^T \vartheta_\ell, \quad \text{for } \ell, k = \{1, 2\}, \tag{18}$$

with $\vartheta_\ell := [(\bar{A}\mathsf{X}\mathbb{I}_1^T + \bar{B}\mathsf{Y}\mathbb{I}_2^T)\ 0\ \widehat{\Gamma}_\Delta^T]$, $\beta_\ell := [F_\ell \mathsf{X}\mathbb{I}_3^T\ 0\ 0]$, and

$$\mho_\ell := \mathbb{I}_1 \widehat{\Gamma}_x^T + \widehat{\Gamma}_x \mathbb{I}_1^T + \mathcal{V}_\ell \Gamma_\ell^T + \Gamma_\ell \mathcal{V}_\ell^T - 2\mathbb{I}_3 (\mathsf{X} F_\ell \mathsf{X})\mathbb{I}_3^T + \hat{\eta}_\ell^{-1}\left(\mathbb{I}_1\left(\alpha_1 \mathsf{X} H_1^T H_1 \mathsf{X}\right)\mathbb{I}_1^T + \mathbb{I}_2\left(\alpha_2 \mathsf{X} H_2^T H_2 \mathsf{X}\right)\mathbb{I}_2^T\right).$$

Now, using Park–Moon's inequality [20], we have

$$\vartheta_\ell^T \beta_\ell + \beta_\ell^T \vartheta_\ell \le \left(\vartheta_\ell^T - \beta_\ell^T F_\ell^{-1}\right) F_\ell \mathsf{X} F_\ell \left(\vartheta_\ell - F_\ell^{-1}\beta_\ell\right) + \beta_\ell^T (F_\ell \mathsf{X} F_\ell)^{-1} \beta_\ell - 2\beta_\ell^T F_\ell^{-1} \beta_\ell. \tag{19}$$

Now, we introduce additional variables $\mathfrak{M}_\ell$ and $\mathfrak{N}_\ell$, such that $\mathfrak{M}_\ell - (F_\ell \mathsf{X} F_\ell)^{-1} \le 0$ and $\mathfrak{N}_\ell - \mathsf{X} F_\ell \mathsf{X} \le 0$. Using Schur Lemma in (18)–(19), we have the conditions in (16)–(17) and, additionally, $\mathsf{X}\widetilde{\mathsf{X}} = I$, $\mathfrak{M}_\ell \widetilde{\mathfrak{M}}_\ell = I$, $\mathfrak{N}_\ell \widetilde{\mathfrak{N}}_\ell = I$, which are proved to be satisfied if the minimization problem (15) is solved. □

To solve the nonlinear optimization problem (15), we use a modified CCL algorithm.

The first and every Step 2 of Algorithm 1 are simple LMIs problems. Hence, interior point based algorithms can solve the set of convex problems in polynomial time. The predefined constants $k_{\lim}$ and $e_{\lim}$ denote the maximum number of iterations and the threshold for the convergence rate, respectively.

The sequence O_k is monotonically decreasing and, according to [3, 4, 12], the algorithm shows excellent search performance and converges for a wide set of problems when properly set. Therefore, we expect (15) to converge to $5r_x$, when feasible.

Algorithm 1 H_∞ output tracking controller design procedure

1. Find a feasible solution for the convex LMIs conditions in Theorem 2 (without the optimization problem). If none are found, exit. Else, set $\mathsf{X}^0 = \mathsf{X}$, $\widetilde{\mathsf{X}}^0 = \widetilde{\mathsf{X}}$, $\mathfrak{N}_i^0 = \mathfrak{N}$, $\widetilde{\mathfrak{N}}_i^0 = \widetilde{\mathfrak{N}}$, $\mathfrak{M}_i^0 = \mathfrak{M}$, $\widetilde{\mathfrak{M}}_i^0 = \widetilde{\mathfrak{M}}$, $i = \{1,2\}$; and $k = 1$.
2. For $k < k_{\lim}$, find $\mathsf{X}^{k+1} = \mathsf{X}$, $\widetilde{\mathsf{X}}^{k+1} = \widetilde{\mathsf{X}}$, $\mathfrak{N}_i^{k+1} = \mathfrak{N}$, $\widetilde{\mathfrak{N}}_i^{k+1} = \widetilde{\mathfrak{N}}$, $\mathfrak{M}_i^{k+1} = \mathfrak{M}$, $\widetilde{\mathfrak{M}}_i^{k+1} = \widetilde{\mathfrak{M}}$, $i = \{1,2\}$, that solve the LMIs conditions in Theorem 2 with the following linear minimization

$$O_k := \mathrm{tr}\big\{\epsilon^{-1}\big(\mathsf{X}^k\mathsf{X} + \widetilde{\mathsf{X}}^k\widetilde{\mathsf{X}} + \mathfrak{N}_i^k\mathfrak{N}_i + \widetilde{\mathfrak{N}}_i^k\widetilde{\mathfrak{N}}_i + \mathfrak{M}_i^k\mathfrak{M}_i + \widetilde{\mathfrak{M}}_i^k\widetilde{\mathfrak{M}}_i\big)\big\},\ i = \{1,2\}.$$

3. If $\|O_k - O_{k-1}\| < e_{\lim}$, where $e_{\lim} > 0$ is a predefined parameter, move to Step 4, else, set $k = k + 1$ and go to Step 2.
4. Stopping criterion: reconstruct the stabilizing controller $\bar{K} = \mathsf{Y}\mathsf{X}^{-1}$. If (12a)–(12b) is feasible, exit. Otherwise, set $k = k + 1$, reduce $e_{\lim}$, and go back to Step 2.

Indeed, numerical experience reported shows that it is extremely efficient and fails to compute the global optimum in very few cases [3], usually due to a small number of iterations [12].

Remark 3 The results from Theorem 2 with proper modifications, as stressed in Remark 2, are also valid when the delay derivative lower bound is unknown and for fast-varying delays.

4 Numerical Examples

This section presents different benchmark examples[1] that illustrate the effectiveness of the proposed criteria. First, we investigate the advantages of applying Theorem 1 for the stability analysis of linear time-delay systems, i.e., when $\omega(t) \equiv 0$, and the reference output (4a)–(4b) is null. In the second, we show the improvements from the proposed criteria for H_∞ performance analysis. Finally, we present a simulation to illustrate the effectiveness of the proposed H_∞ output tracking control criterion for a class of nonlinear time-delay systems.

Example 1 Consider the following linear system with time-varying delay

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x\big(t - d(t)\big).$$

Assuming $\tau_{\min} = 0$, the maximum allowable upper bound for $\tau_{\max}$ from Theorem 1 and from the literature [7, 14, 25, 36] are listed in Table 1. Particularly, the results from [36] are only feasible for $\tau_{\min} = 0$ and full knowledge about the delay

[1] All numerical tests have been performed with an Intel Core i7, CPU 870@2.93 GHz, 8 GB RAM, using Matlab with SeDuMi [26] and YALMIP [18]. The configuration constants in Algorithm 1 have been set to $k_{\lim} = 300$ and $e_{\lim} = 10^{-3}$.

Table 1 Admissible τ_{max} value for $\tau_{min} = 0$ and given d_{min} and d_{max} (Example 1)

Method		Unknown d_{min}		$d_{min} = -d_{max}$	
		($d_{max} = 0.5$)	($d_{max} = 0.9$)	($d_{max} = 0.5$)	($d_{max} = 0.9$)
Park and Ko (2007) [25]		2.337	1.873	–	–
Kim (2011) [14]		2.33	1.88	–	–
Fridman et al. (2009) [7]	Theorem 1	2.410	2.118	2.451	2.175
	Theorem 2	2.337	1.872	2.337	1.872
Zhang and Liu (2011) [36][a]	$m = 1$	2.29	1.48	2.408	1.523
	$m = 2$	2.37	1.50	2.590	1.559
Theorem 1		2.410	2.120	2.501	2.188

[a]The notation 'm' stands for the number of delay range ($[\tau_{min}, \tau_{max}]$) partitions

Table 2 Allowable upper bound value of τ_{max} for fast-varying delays and various τ_{min} (Example 1)

Method		τ_{min}				
		1	2	3	4	5
Shao (2009) [30]		1.874	2.505	3.259	4.074	–
Tang et al. (2012) [33]		2.045	2.605	3.310	4.088	–
Sun et al. (2010) [32]		–	2.567	3.341	4.169	5.028
Qian and Liu (2013) [28]		–	2.690	3.410	4.200	5.030
Souza (2013) [31]		–	–	3.418	4.210	5.044
Liu et al. (2012) [17]		2.092	2.699	3.419	4.210	5.044
Fridman et al. (2009) [7]	Theorem 1	2.169	2.646	3.321	4.090	–
	Theorem 2	2.120	2.724	3.458	4.257	5.097
Guo et al. (2012) [11]		2.120	2.712	3.457	4.257	5.097
Figueredo et al. (2011) [6]		2.216	2.750	3.462	4.257	5.097
Theorem 1		2.359	2.800	3.482	4.266	5.101

derivative. Hence, the derivative lower bound for [36] is set to $d_{min} = -3$ instead of regarded to be unknown. From Table 1, it is clear that Theorem 1 provides much less conservative results for larger (or unknown) delay derivative bounds. The results from the proposed criterion are only outperformed by [36], and only for very slow-varying delays and a larger number of delay partitions ($m = 2$ partitions compared to 1 from Theorem 1). Indeed, for all other conditions, Theorem 1 provides considerably superior results than [36], e.g., for $|\dot{d}(t)| \leq 0.9$, the improvement over [36] is higher than 40 %. This illustrates the importance of the proposed method for the analysis of linear time-delay systems.

Now, assuming fast-varying delays, the maximum values for τ_{max} which maintain the time-delay system stability are listed in Table 2. The results compared to state-of-the-art criteria in the literature enlighten the advantages of Theorem 1, when particularized to the stability analysis of linear time-delay systems. Moreover, compared to the results from different authors [7, 11, 17, 28, 30–33], the improvements

Table 3 Minimum value of γ for different values of α_1 (Example 2)

Methods	$\alpha_1^2 = 0.05$	$\alpha_1^2 = 0.10$	$\alpha_1^2 = 0.15$	$\alpha_1^2 = 0.2$	$\alpha_1^2 = 0.25$
Orihuela et al. (2011) [24]	0.92	1.20	1.60	2.66	7.70
Theorem 1	0.816	0.906	1.038	1.253	1.706
Improvements	(13 %)	(33 %)	(54 %)	(112 %)	(351 %)

from the proposed method become even more evident, e.g., for $\tau_{\min} = 1$, the delay interval size is 16 % larger than the results from [7] (Theorem 1), and 21 % larger than the results from [11] and [7], Theorem 2.

Example 2 Consider the following nonlinear time-delay systems

$$\dot{x}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 0.99 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -3.715 & -3.514 \end{bmatrix} x\big(t - d(t)\big) + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \omega(t) + \bar{g}\big(t, x(t), x\big(t - d(t)\big)\big),$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -0.03715 & -0.03514 \end{bmatrix} x\big(t - d(t)\big),$$

and with nonlinearity $\bar{g}(t, x(t), x(t - d(t)))$ satisfying (2) with $H_1 = [1\ 0]$ and $H_2 = 0$. To allow comparison with existing methods, the reference signal $y_r(t)$ is considered to be null.

Assuming fast-varying delay and $d(t) \in [0, 0.2509]$, Table 3 presents the values for the noise to error attenuation, γ, for different values of the bounding parameter α_1. From Table 3, it can be seen that the results from Theorem 1 are considerably less conservative than the ones from the state-of-the-art criterion given by [24]. Moreover, in order to compare with different criteria, we also consider the case when there are no external disturbances, $\omega(t) \equiv 0$. In this particular case, the maximal bounding parameters obtained with [27] and [24] are $\alpha_1^2 = 0.164$ and $\alpha_1^2 = 0.276$, respectively, whereas using Theorem 1, we find a maximum bound $\alpha_1^2 = 0.365$.

Example 3 Now, consider the example of a satellite system [8] modeled by rigid bodies joined through a link with torque 0.09 N m and yaw angles denoted by θ_1 and θ_2. Differently from [8], a more realistic scenario is obtained with a nonlinear viscous damping $f = 0.04 + g(t, \theta(t), \theta(t - d(t)))$ N s/m. Taking the angular position θ_2 as the system's output $y_p(t)$, the state-space representation is derived as

$$\begin{bmatrix} \dot{\theta}_1(t) \\ \dot{\theta}_2(t) \\ \ddot{\theta}_1(t) \\ \ddot{\theta}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.09 & 0.09 & -0.04 & 0.04 \\ 0.09 & -0.09 & 0.04 & -0.04 \end{bmatrix} \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \\ \dot{\theta}_1(t) \\ \dot{\theta}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u\big(t - d(t)\big) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ g(t, \theta, \theta_d) \end{bmatrix},$$

Table 4 Minimum achievable value of γ for different values of α_1 (Example 3)

	$\alpha_1=1$	$\alpha_1=5$	$\alpha_1=10$	$\alpha_1=20$	$\alpha_1=30$	$\alpha_1=40$	$\alpha_1=50$
$\min\gamma$	0.395	0.429	0.470	0.575	0.705	0.990	1.921

$$y_p(t)=\begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}\theta_1(t) & \theta_2(t) & \dot\theta_1(t) & \dot\theta_2(t)\end{bmatrix}^T,$$

with the nonlinearity $g(t,\theta,\theta_d)=\sin(\theta_1(t))\mathrm{sgn}(h(\theta))\sqrt{\alpha_1|h(\theta)|}$, $h(\theta)=\dot\theta_1^2(t)+2\dot\theta_1(t)\dot\theta_2(t)+\frac{1}{2}\dot\theta_2^2(t)$, satisfying the quadratic constraint (2) for $H_1=[0\ 0\ 0.1\ 0.1]$, $H_2=0$. The reference model is defined as

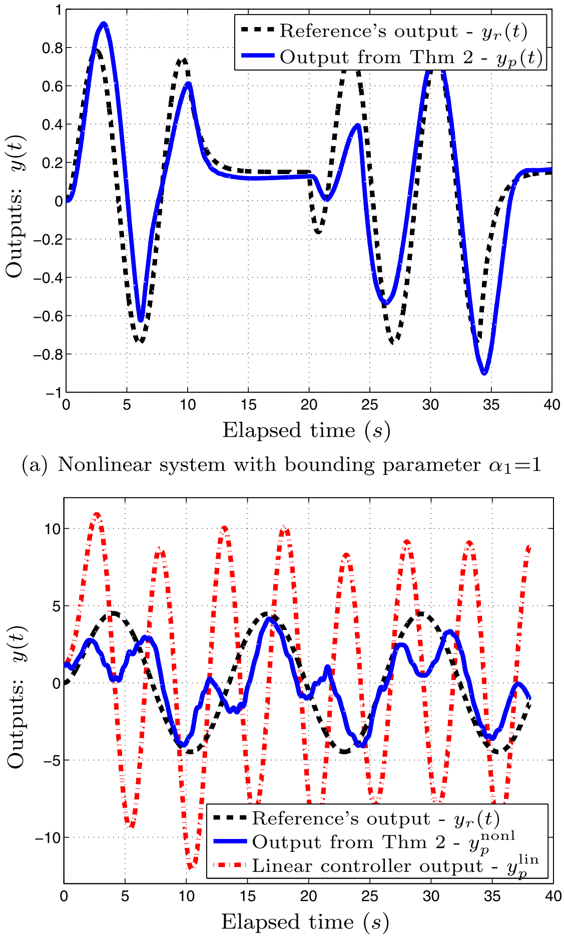
$$\dot x_r(t)=-x_r(t)+r(t), \quad \text{and} \quad y_r(t)=0.5x_r(t).$$

In this context, we assume a fast-varying delay with $\tau_{\min}=0.08$ s, $\tau_{\max}=0.1$ s. Our purpose is to find a stabilizing controller $\bar K$, which makes the system's output, $y_p(t)$, asymptotically track the signal $y_r(t)$, while minimizing the upper bound of the disturbance attenuation, γ. From Algorithm 1, the minimal values of γ for different values of the bounding parameter α_1 are listed in Table 4.

Simulation In the first scenario, regard a fast-varying delay with $d(t)\in[0.08,0.1]$, and $\alpha_1=1$. From Theorem 2 and Algorithm 1, we find the stabilizing controller $\bar K=-[18.79\ 939.80\ 10.60\ 688.47\ \ -213.13]$, which yields an H_∞ performance with $\gamma=0.395$. The disturbances are assumed $\omega(t)=0.5\sin(0.3t)$, for $6\le t\le 24$ or $t>37$, and $\omega(t)=2.75\sin(0.7t)$, otherwise; and $r(t)=0.3$, for $10\le t\le 20$ or $t>33$, and $r(t)=2\sin(0.9t)$, otherwise. The prescribed delay is achieved using uniform distribution random delay.

The system's and the reference output are presented in Fig. 1(a). It is clear that $y_p(t)$ successfully tracks the reference signal, despite the nonlinearity, $g(t,\theta(t),\theta(t-d(t)))$, and the time-varying delay. Moreover, considering zero initial conditions and numerically computing $\|\bar\omega(t)\|_2$ and $\|e(t)\|_2$, we find the simulated disturbance attenuation, $\frac{\|e(t)\|_2}{\|\bar\omega(t)\|_2}=0.314$, which is smaller than the minimum upper bound $\gamma=0.395$ obtained with Theorem 2, highlighting the effectiveness of the proposed method.

To further enlighten the importance of Theorem 2, we propose a second scenario with a higher nonlinear influence, i.e., we will increase the value of α_1 to $\alpha_1=50$. From Theorem 2 and Algorithm 1 for $\alpha_1=50$, we find a stabilizing controller that yields an output ($y_p^{\mathrm{nonl}}(t)$) with an H_∞ performance of $\gamma=1.921$. In this scenario, we also define a linear controller yielding an output ($y_p^{\mathrm{lin}}(t)$) that satisfies the H_∞ performance of $\gamma=1.921$; however, ignoring the existence of the nonlinear term $g(t,\theta,\theta_d)$. The disturbances are assumed $\omega(t)=9\sin(1.25t)$, $r(t)=10\sin(0.5t)$. The results, presented in Fig. 1(b), clearly demonstrate that ($y_p^{\mathrm{lin}}(t)$) from the linear controller (ignoring the existence of nonlinearities) is unable to track the desired reference, whereas ($y_p^{\mathrm{nonl}}(t)$) from Theorem 2 with $\alpha_1=50$ successfully tracks the reference with disturbance attenuation below the prescribed γ. The analysis illustrates the advantages and effectiveness of explicitly considering nonlinearities in the synthesis of controllers.



(a) Nonlinear system with bounding parameter α_1=1

(b) Nonlinear system with bounding parameter α_1=50

Fig. 1 The plot (**a**) shows the system's output tracking the reference output, whereas in (**b**), it is shown the desired reference $y_r(t)$, the plant's output with the linear controller ($y_p^{\text{lin}}(t)$), and the output from Theorem 2 with $\alpha_1 = 50$, ($y_p^{\text{nonl}}(t)$). Note the linear controller fails in tracking the reference, whereas $y_p^{\text{nonl}}(t)$ successfully tracks $y_r(t)$

5 Conclusion

In this paper, the H_∞ output tracking problem for nonlinear uncertain time-delay systems was investigated, and novel criteria were derived for the performance analysis and control design. With a novel Lyapunov–Krasovskii functional based on a delay partition, we improved the piecewise analysis method and introduced delay-interval-dependent terms exploiting the delay partitioning subintervals size. The resulting H_∞ output tracking performance and control criteria showed to be considerably less conservative than existing methods. The proposed technique, if particularized to the stability analysis of linear/nonlinear time-delay systems, also yielded superior results compared to state-of-the-art criteria in the literature. These advantages in terms of conservatism reduction were further illustrated with numerical examples. Two different simulation scenarios were also provided to demonstrate the effectiveness and the importance of the proposed H_∞ output tracking control criterion for nonlinear time-delay systems.

Acknowledgements The authors were supported by research grants funded by the National Council of Technological and Scientific Development—CNPq, and by the Coordination for the Improvement of Higher Education Personnel—CAPES, Brazil.

Appendix: Proof of Theorem 1

This appendix presents the proof of Theorem 1. First, we shall take the time derivative of (9) with respect to t along the trajectory of $x(t)$:

$$
\begin{aligned}
\dot{V}_1(t) &= \dot{d}(t)\frac{1}{2\tau_\sigma}x^T(t)(P_2-P_1)x(t) + 2\dot{x}^T(t)\big(\chi_{[\tau_1,\tau_2]}\widehat{P}_1\big(d(t)\big) \\
&\quad + (1-\chi_{[\tau_1,\tau_2]})\widetilde{P}_2\big(d(t)\big)\big)x(t), \\
\dot{V}_2(t) &= x^T(t-\tau_1)Qx(t-\tau_1) - \big(1-\dot{d}(t)\big)x^T\big(t-d(t)\big)Qx\big(t-d(t)\big) \\
&\quad + \begin{bmatrix} x(t) \\ x(t-\frac{\tau_1}{2}) \end{bmatrix}^T N_1 \begin{bmatrix} x(t) \\ x(t-\frac{\tau_1}{2}) \end{bmatrix} - \begin{bmatrix} x(t-\frac{\tau_1}{2}) \\ x(t-\tau_1) \end{bmatrix}^T N_1 \begin{bmatrix} x(t-\frac{\tau_1}{2}) \\ x(t-\tau_1) \end{bmatrix} \\
&\quad + \begin{bmatrix} x(t) \\ x^T(t-\tau_1) \\ x(t-\tau_2) \end{bmatrix}^T N_4 \begin{bmatrix} x(t) \\ x^T(t-\tau_1) \\ x(t-\tau_2) \end{bmatrix} - \begin{bmatrix} x(t-\tau_\sigma) \\ x^T(t-\tau_2) \\ x(t-\tau_3) \end{bmatrix}^T N_4 \begin{bmatrix} x(t-\tau_\sigma) \\ x^T(t-\tau_2) \\ x(t-\tau_3) \end{bmatrix} \\
&\quad + x^T\left(t-\frac{\tau_1}{2}\right)\left[\left(\tau_\sigma-\frac{\tau_1}{2}\right)N_2\right]x\left(t-\frac{\tau_1}{2}\right) \\
&\quad - x^T(t-\tau_\sigma)\left[\left(\tau_\sigma-\frac{\tau_1}{2}\right)N_2\right]x(t-\tau_\sigma) \\
&\quad + x^T(t-\tau_1)\big[(\tau_\sigma-\tau_1)N_3\big]x(t-\tau_1) \\
&\quad - x^T(t-\tau_\sigma)\big[(\tau_\sigma-\tau_1)N_3\big]x(t-\tau_\sigma), \\
\dot{V}_3(t) &= \dot{x}^T(t)\left(\left(\frac{\tau_1}{2}\right)^2(S_1+S_2) + \left(\tau_\sigma-\frac{\tau_1}{2}\right)^2 S_3 + (\tau_\sigma-\tau_1)^2 S_4\right. \\
&\quad \left. + \tau_\sigma{}^2(Z_0+Z_1+Z_2) + \tau_2R_1 + d(t)R_2 + \tau_\sigma R_3 + \big(\tau_3-d(t)\big)R_4\right)\dot{x}(t) \\
&\quad - \sum_{k=1}^{2}\frac{\tau_1}{2}\int_{t-\frac{\tau_1}{2}k}^{t-\frac{\tau_1}{2}(k-1)}\dot{x}^T(s)S_k\dot{x}(s)\,ds - \left(\tau_\sigma-\frac{\tau_1}{2}k\right) \\
&\quad \times \int_{t-\tau_\sigma}^{t-\frac{\tau_1}{2}k}\dot{x}^T(s)S_{k+2}\dot{x}(s)\,ds - \sum_{k=0}^{2}\tau_\sigma\int_{t-\tau_k-\tau_\sigma}^{t-\tau_k}\dot{x}^T(s)Z_k\dot{x}(s)\,ds \\
&\quad - \int_{t-d(t)}^{t}\dot{x}^T(s)U_R\dot{x}(s)\,ds - \int_{t-\tau_3}^{t-d(t)}\dot{x}^T(s)(R_3+R_4)\dot{x}(s)\,ds - \chi_{[\tau_1,\tau_2]} \\
&\quad \times \int_{t-\tau_2}^{t-d(t)}\dot{x}^T(s)(R_1-R_3)\dot{x}(s)\,ds \\
&\quad - (1-\chi_{[\tau_1,\tau_2]})\int_{t-d(t)}^{t-\tau_2}\dot{x}^T(s)(R_3-R_1)\dot{x}(s)\,ds,
\end{aligned}
\tag{20}
$$

where $\widehat{P}_1(d(t))$, $\widehat{P}_2(d(t))$ are defined in (10), and U_R in (14). Considering the first subinterval, $\chi_{[\tau_1,\tau_2]} = 1$, suppose we expand the integral terms, taking the fact that $\tau_1 \le d(t) \le \tau_2$. Applying Jensen's inequality [10], and after some manipulation, we obtain

$$\dot{V}(t)|_{d(t)<\tau_2} \le \zeta_1^T(t)\,\mathrm{diag}\big\{\Psi^{(1)}; -\big(d(t)-\tau_1\big)\Lambda_{12}; -\big(\tau_2-d(t)\big)\Lambda_{11}; \big\}\zeta_1(t), \quad (21)$$

where $\zeta_x^T := [x^T(t)\ x^T(t-d(t))\ \dot{x}^T(t)\ x^T(t-\frac{\tau_1}{2})\ x^T(t-\tau_1)\ x^T(t-\tau_2)\ x^T(t-\tau_3)\ x^T(t-\tau_\sigma)]$, $\zeta_1^T := [\zeta_x^T\ \xi_{1d}^T\ \xi_{d2}^T]$, $\xi_{1d} := \frac{1}{d(t)-\tau_1}\int_{t-d(t)}^{t-\tau_1}\dot{x}(s)\,ds$, $\xi_{d2} := \frac{1}{\tau_2-d(t)} \times \int_{t-\tau_2}^{t-d(t)}\dot{x}(s)\,ds$ with $\lim_{d(t)\to\tau_1}\xi_{1d} = \dot{x}(t-\tau_1)$, and $\lim_{d(t)\to\tau_2}\xi_{d2} = \dot{x}(t-\tau_2)$.

Now, from Leibniz–Newton formula for definite integrals with (6), we introduce the following null expressions:

$$\begin{aligned}
&2\zeta_x^T\mathcal{F}_1\big((\bar{A}+\Delta\bar{A})x(t)+(\bar{B}+\Delta\bar{B})\bar{K}x\big(t-d(t)\big)\\
&\quad+\bar{B}_\omega\bar{\omega}(t)+\bar{g}\big(t,x(t),x\big(t-d(t)\big)\big)-\dot{x}(t)\big)=0,\\
&2\zeta_x^T\mathcal{V}_1\big(J_1\big(x(t-\tau_2)-x\big(t-d(t)\big)+\big(\tau_2-d(t)\big)\xi_{d2}\big)\\
&\quad+J_2\big(x\big(t-d(t)\big)-x(t-\tau_1)+\big(d(t)-\tau_1\big)\xi_{1d}\big)\big)=0,
\end{aligned}$$

where J_1, J_2 are defined in (14). Moreover, applying the inequality

$$\begin{aligned}
&2\zeta_x^T\mathcal{F}_1\bar{g}\big(t,x(t),x\big(t-d(t)\big)\big)\\
&\quad\le \eta_1^{-1}\zeta_x^T\mathcal{F}_1\mathcal{F}_1^T\zeta_x+\eta_1\bar{g}^T\big(t,x(t),x\big(t-d(t)\big)\big)\bar{g}\big(t,x(t),x\big(t-d(t)\big)\big),
\end{aligned}$$

which arises from [20], with (2), and adding the expression

$$-e^T(t)e(t)+\gamma^2\bar{\omega}^T(t)\bar{\omega}(t)-\gamma^2\bar{\omega}^T(t)\bar{\omega}(t)+\zeta_x^T\Gamma_{cd}^T\Gamma_{cd}\zeta_x=0,$$

with $\Gamma_{cd}^T := \mathbb{I}_1(\bar{C}+\Delta\bar{C})+\mathbb{I}_2(\bar{D}+\Delta\bar{D})\bar{K}$, we have

$$\dot{V}(t)|_{d(t)<\tau_2}+e^T(t)e(t)-\gamma^2\bar{\omega}^T(t)\bar{\omega}(t)=\big[\zeta_1^T\quad\bar{\omega}^T\big]\Omega_1\big[\zeta_1^T\quad\bar{\omega}^T\big]^T, \quad (22)$$

with

$$\Omega_1=\begin{bmatrix}\Psi^{(1)}+\Pi_1+2\mathcal{F}_1(\Delta\bar{A}\mathbb{I}_1^T+\Delta\bar{B}\bar{K}\mathbb{I}_2^T)+\Gamma_{cd}^T\Gamma_{cd} & [\mathcal{V}_1((d(t)-\tau_1)J_2+(\tau_2-d(t))J_1\ \mathcal{F}_1\bar{B}\omega]\\ * & -\mathrm{diag}\{(d(t)-\tau_1)\Lambda_{12};(\tau_2-d(t))\Lambda_{11};\gamma^2 I\}\end{bmatrix}.$$

Now, we shall consider the matrices that arise from the analysis of Ω_1 for $d(t)\to\tau_1$ and $d(t)\to\tau_2$. It is straightforward to conclude that $[\zeta_1^T\ \bar{\omega}^T]\Omega_1[\zeta_1^T\ \bar{\omega}^T]^T$ may be written as $\frac{\tau_2-d(t)}{\tau_2-\tau_1}\zeta_{11}^T(t)\Omega_1|_{d(t)\to\tau_1}\zeta_{11}(t)+\frac{d(t)-\tau_1}{\tau_2-\tau_1}\zeta_{12}^T(t)\Omega_1|_{d(t)\to\tau_2}\zeta_{12}(t)$, where $\zeta_{11}^T(t) := [\zeta_x^T\ \gamma_{d2}^T\ \bar{\omega}(t)]$ and $\zeta_{12}^T(t) := [\zeta_x^T\ \gamma_{1d}^T\ \bar{\omega}(t)]$. This analysis enlightens the convex properties of Ω_1 regarding $d(t)$, which, in turn, implies that the matrix is negative definite only if the vertices are.

Moreover, using the Schur Lemma with the term $\Gamma_{cd}^T \Gamma_{cd}$, and applying Park–Moon's inequality, yields

$$2\vartheta_k^T \Delta\beta_k \leq \epsilon_{1k}\vartheta_k^T \vartheta_k + \epsilon_{1k}^{-1}\beta_k^T \beta_k \quad \text{for } k = \{1, 2\}$$

with $\vartheta_1 = [(\mathcal{F}_1 \bar{\Xi}_x)^T \;\; 0]$, $\vartheta_2 = [(\mathbb{I}_1 \bar{\Xi}_C^T + \mathbb{I}_2(\bar{\Xi}_D \bar{K})^T)^T \;\; 0]$, $\beta_1 = [(\bar{\Xi}_A \mathbb{I}_1^T + \bar{\Xi}_B \bar{K}\mathbb{I}_2^T)^T \;\; 0]$, and $\beta_2 = [0 \;\; \bar{\Xi}_e]$. Then, from Schur's Lemma, we have the matrices Ω_{11} and Ω_{12}, described in (13). Therefore, it easy to see that Ω_1 is negative definite if $\Omega_{11} < 0$ and $\Omega_{12} < 0$ hold. Also, given (5b), we have that the matrices are convex in $\dot{d}(t) \in [d_{\min}, d_{\max}]$.

Therefore, if the conditions in Theorem 1 are satisfied, then

$$\dot{V}(t)|_{d(t)<\tau_2} + e^T(t)e(t) - \gamma^2 \bar{\omega}^T(t)\bar{\omega}(t) < 0$$

holds for $\chi_{[\tau_1,\tau_2]} = 1$. Furthermore, using exactly the same arguments of the former case, we may prove that analogous results can be derived for $\chi_{[\tau_1,\tau_2]} = 0$, i.e., $\tau_2 < d(t) < \tau_3$. In this context, it is easy to conclude that if the conditions in Theorem 1 are satisfied, then $\dot{V}(t)|_{d(t)<\tau_2} < 0$ and $\dot{V}(t)|_{d(t)>\tau_2} < 0$ must hold for $\bar{\omega}(t) \equiv 0$. Since (9) is continuously differentiable, the nonlinear system is robustly asymptotically stable for $\bar{\omega}(t) \equiv 0$. Moreover, we also have $\dot{V}(t) + e^T(t)e(t) - \gamma^2\bar{\omega}^T(t)\bar{\omega}(t) < 0$. Thus, integrating the inequality, from 0 to ∞, yields

$$V(\infty) - V(0) + \int_0^{\infty} \left[e^T(t)e(t) - \gamma^2\bar{\omega}^T(t)\bar{\omega}(t)\right] dt < 0.$$

Assuming zero initial conditions, and the positiveness of $V(t)$, $t \in (0, \infty]$, it is easy to see that $\int_0^{\infty}[e^T(t)e(t) - \gamma^2\bar{\omega}^T(t)\bar{\omega}(t)]\, dt < 0$, and thus $\|e(t)\|_2 < \gamma\|\bar{w}(t)\|_2$. Therefore, the conditions in Definition 1 are satisfied. □

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