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Fault Detection for Uncertain Fuzzy Systems Based on the Delta Operator Approach

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Abstract This paper investigates the problem of designing a robust fault-detection for uncertain T-S fuzzy models based on the delta operator approach. By means of the T-S fuzzy delta operator systems, a fuzzy fault detection filter system is constructed via the delta operator approach. The worst case fault sensitivity has been formulated in terms of linear matrix inequalities. The proposed fault-detection filter not only ensures the H_--gain from a fault signal to a residual signal greater than a prescribed value, but also guarantees the H_∞-gain from an exogenous input to a residual signal less than a prescribed value in terms of the solvability of linear matrix inequalities. The linear matrix inequalities can be solved by an effective algorithm. A numerical example is provided to illustrate the effectiveness of the proposed design techniques.

Keywords Fault detection · T-S fuzzy system · Delta operator system · Linear matrix inequality (LMI)

1 Introduction

In control systems, due to the unexpected variations in external surroundings, normal wear in components, or sudden changes in signals, there may appear different kinds of malfunction or imperfect behavior in normal operations, and people call them faults

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[35]. The objective of fault detection is to detect the fault signal accurately whenever it appears. In recent years, fault detection in dynamical systems has attracted considerable attention from many researchers due to the increasing demand for reliability and safety in industrial processes [12, 32], and [23]. There are also some recently published papers on robust H_∞-filtering, for example, [1, 31], and [29], and so on. In [3], the smallest nonzero singular value of the transfer function from fault to residual was used to evaluate the worst case fault sensitivity. For the purpose of fault detection, the H_- index defined as the smallest singular value of a transfer function matrix was proposed in [16]. A linear matrix inequality (LMI) approach to H_-/H_∞ fault-detection observers has been proposed both in [8] and [21]. Although many researchers have studied the problem of fault detection in linear systems with or without uncertainties for many years, the problem of fault detection in nonlinear systems remains an open research area.

One of the main difficulties in designing a fault-detection system for nonlinear dynamical systems is that a rigorous mathematical model may be very difficult to obtain, if not impossible. The T-S fuzzy model described by a family of fuzzy IF–THEN rules was first introduced in [24]. The T-S fuzzy model puts the complex nonlinear systems into a framework that interpolates some affine local models by a set of fuzzy membership functions. Based on this framework, a systematic analysis and design procedure for complex nonlinear systems can be possibly developed in view of the powerful control theories and techniques in linear systems. Therefore, many important results on T-S fuzzy systems have been reported, such as in [6, 30, 34], and [33], and the references therein. The T-S fuzzy model has attracted great interest from researchers, and a number of results have been reported in the literature, including stability analysis [15], H_∞-control [9], and state estimation [25]. An adaptive fuzzy sliding control method was used for a double-pendulum-and-cart system in [26]. Adaptive sliding mode control for nonlinear active suspension vehicle systems using T-S fuzzy approach has been investigated in [14]. Since T-S fuzzy models have provided a convenient way to study nonlinear systems, a feasible solution of the fault detection problem for nonlinear systems can be converted to that of fault detection for T-S fuzzy systems [20]. Two finite-frequency performance indices have been introduced to measure fault sensitivity and disturbance robustness in finite-frequency ranges in [27]. Reliable fuzzy control problem has been considered for active suspension systems with actuator delay and fault [13]. However, all the results above are not related to the case of fast sampling, which means that sampling periods are small in taking sample for continuous-time systems.

It is well known that discrete systems are suitable for computer realization and continuous systems are convenient for theoretical analysis. The shorter the sampling period, the better the system performances for discrete time control systems. Goodwin and Middleton constructed a delta operator instead of the traditional shift operator for sampling continuous systems at high sampling rate in [7] and [17]. Science then, the transformations between the delta operator and shift operator transfer function models have been highlighted [18]. Furthermore, the computational formulation, properties and applications of the delta operator systems have been illustrated [19]. The relationships between optimal realization sets for the shift operator and delta operator have been established in [11]. A structure in the shift operator and delta operator has been derived based on a polynomial-operator approach [10]. Especially, the book [28] has introduced some new achievements on the delta operator systems. However, to the best of our knowledge, there have been few papers on fault detection for T-S fuzzy systems via the delta operator approach, which motivates us to make an effort in this paper.

The aim of this paper is to design a robust fault-detection for uncertain T-S fuzzy models based on the delta operator approach. The worst case fault sensitivity has been formulated in terms of LMIs. The proposed fault-detection filter can ensure the $\mathcal{L}_2$-gain from a fault signal to a residual signal greater than a prescribed value. It can also guarantees the $\mathcal{L}_2$-gain from an exogenous input to a residual signal less than a prescribed value in terms of the solvability of LMIs. Some simulation results are provided to demonstrate the effectiveness of the obtained results.

This paper is organized as follows. In Sect. 2, system descriptions and definitions are presented. Section 3 presents the threshold design. Section 4 gives the main results for designing a robust fault detection in delta domain for the fuzzy system. Section 5 gives the filter algorithm in detail. In Sect. 6, we present numerical simulation results. Conclusions are given in Sect. 7.

Notation Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The notation $X > Y$ $(X \geq Y)$ means that the matrix $X - Y$ is positive definite ($X - Y$ is semi-positive definite, respectively). And $P > 0$ means that P is symmetric and positive-define; I is the identity matrix of appropriate dimension. For any matrix A, A^T denotes the transpose of matrix A, A^{-1} denotes the inverse of matrix A. The shorthand $\mathrm{diag}\{M_1, M_2, \dots, M_r\}$ denotes a block diagonal matrix with diagonal blocks being the matrices $M_1, M_2, \dots, M_r$.

2 System Description and Definitions

In the section, we consider the following uncertain fuzzy delta operator systems which are represented by the T-S fuzzy model composed of a set of fuzzy implications, and each implication is expressed by a linear system model. The ith rule of this T-S model is of the following form:

Plant Rule i

IF $v_1(t_k)$ is $M_{i1}$and … and $v_\vartheta(t_k)$ is $M_{i\vartheta}$, THEN

$$\delta x(t_k) = [A_i + \Delta A_i]x(t_k) + [B_i + \Delta B_i]w(t_k) + [G_i + \Delta G_i]f(t_k), \tag{1}$$

$$y(t_k) = [C_i + \Delta C_i]x(t_k) + [D_i + \Delta D_i]w(t_k) + [J_i + \Delta J_i]f(t_k) \tag{2}$$

where $i = 1, 2, \dots, r$, r is the number of IF–THEN rules, $v_i(t_k)$ are premise variables, M_{ij} $(j = 1, 2, \dots, \vartheta)$ are fuzzy sets, ϑ is the number of premise variables, $x(t_k) \in \mathbb{R}^n$ is the state vector with $x(0) = 0$, $w(t_k) \in \mathbb{R}^p$ and $f(t_k) \in \mathbb{R}^q$ are disturbances and faults, respectively, that belong to $\mathcal{L}_2[0, \infty]$. Matrices A_i, B_i, C_i, D_i, G_i, and J_i are of appropriate dimensions. Matrix functions ΔA_i, ΔB_i, ΔC_i, ΔD_i, ΔG_i, and ΔJ_i represent the time-varying uncertainties in the system and satisfy the following assumptions:

$$\Delta A_i = E_{1i} F\big(x(t_k), t_k\big) H_{1i}, \qquad \Delta B_i = E_{2i} F\big(x(t_k), t_k\big) H_{2i}, \tag{3}$$

$$\Delta C_i = E_{3i} F\big(x(t_k), t_k\big) H_{3i}, \qquad \Delta D_i = E_{4i} F\big(x(t_k), t_k\big) H_{4i}, \tag{4}$$

$$\Delta G_i = E_{5i} F\big(x(t_k), t_k\big) H_{5i}, \qquad \Delta J_i = E_{6i} F\big(x(t_k), t_k\big) H_{6i} \tag{5}$$

where H_{ji} and E_{ji} $(j = 1, \ldots, 6)$ are known matrices that characterize the structure of the uncertainties. Furthermore, the uncertainty satisfies

$$\big\| F\big(x(t_k), t_k\big) \big\| \le \rho \tag{6}$$

where ρ is a known positive constant. Let $\varpi_i(v(t_k)) = \prod_{k=1}^{\vartheta} M_{ik}(v_k(t_k))$ and

$$u_i\big(x(t_k)\big) = \left(\frac{\varpi_i(v(t_k))}{\sum_{i=1}^{r} \varpi_i(v(t_k))} \right) \tag{7}$$

where $M_{ik}(v_k(t_k))$ is the grade of membership of $v(t_k)$ in M_{ik}. In this paper, it is assumed that $\varpi_i(v(t_k)) \ge 0$ for $i = 1, 2, \ldots, r$ and $\sum_{i=1}^{r} \varpi_i(v(t_k)) > 0$ for all t_k. Therefore, $u_i(v(t_k)) \ge 0$ for $i = 1, 2, \ldots, r$ and $\sum_{i=1}^{r} u_i(v(t_k)) = 1$ for all t_k. Through, the use of fuzzy blending, the final output of the fuzzy delta operator system (1)–(2) is inferred as follows:

$$\begin{aligned} \delta x(t_k) &= \big[A(u) + \Delta A(u)\big] x(t_k) + \big[B(u) + \Delta B(u)\big] w(t_k) \\ &\quad + \big[G(u) + \Delta G(u)\big] f(t_k), \end{aligned} \tag{8}$$

$$\begin{aligned} y(t_k) &= \big[C(u) + \Delta C(u)\big] x(t_k) + \big[D(u) + \Delta D(u)\big] w(t_k) \\ &\quad + \big[J(u) + \Delta J(u)\big] f(t_k) \end{aligned} \tag{9}$$

where

$$A(u) = \sum_{i=1}^{r} u_i A_i, \qquad B(u) = \sum_{i=1}^{r} u_i B_i, \qquad C(u) = \sum_{i=1}^{r} u_i C_i,$$

$$G(u) = \sum_{i=1}^{r} u_i G_i, \qquad D(u) = \sum_{i=1}^{r} u_i D_i, \qquad J(u) = \sum_{i=1}^{r} u_i J_i,$$

and

$$\Delta A(u) = E_1(u) F\big(x(t_k), t_k\big) H_1(u), \qquad \Delta B(u) = E_2(u) F\big(x(t_k), t_k\big) H_2(u),$$

$$\Delta C(u) = E_3(u) F\big(x(t_k), t_k\big) H_3(u), \qquad \Delta D(u) = E_4(u) F\big(x(t_k), t_k\big) H_4(u),$$

$$\Delta G(u) = E_5(u) F\big(x(t_k), t_k\big) H_5(u), \qquad \Delta J(u) = E_6(u) F\big(x(t_k), t_k\big) H_6(u)$$

with $E_\ell(u) F(x(t_k), t_k) H_\ell(u) = \sum_{i=1}^{r} u_i E_{\ell i} F(x(t_k), t_k) H_{\ell i}$, for $\ell = 1, 2, \ldots, 6$.

In this paper, we seek an nth-order fuzzy fault-detection filter as a residual generator that is inferred as the weighted average of the local models of the form

$$\delta \hat{x}(t_k) = \hat{A}(u) \hat{x}(t_k) + \hat{B}(u) y(t_k), \tag{10}$$

$$\hat{y}(t_k) = \hat{C} \hat{x}(t_k), \tag{11}$$

$$e(t_k) = y(t_k) - \hat{y}(t_k) \tag{12}$$

where $\hat{x}(t_k)$ is the filter's state vector, $e(t_k)$ is the residual signal, $\hat{A}(u)$, $\hat{B}(u)$, and $\hat{C}(u)$ are the matrix functions of appropriate dimensions, $\hat{y}(t_k)$ is the estimate of $y(t_k)$.

The state-space form of the fuzzy system model (8)–(9) with filter (10)–(12) is given by

$$\delta\check{x}(t_k) = \mathbb{A}_{cl}(u)\check{x}(t_k) + \mathbb{B}_w(u)w(t_k) + \mathbb{B}_f(u)f(t_k), \tag{13}$$

$$e(t_k) = \mathbb{C}_{cl}(u)\check{x}(t_k) + \mathbb{D}_{cl}(u)w(t_k) + \mathbb{J}_{cl}(u)f(t_k) \tag{14}$$

where

$$\check{x}(t_k) = \begin{bmatrix} x^T(t_k) & \hat{x}^T(t_k) \end{bmatrix}^T, \qquad \mathbb{A}_{cl}(u) = \begin{bmatrix} A(u) + \Delta A(u) & 0 \\ \hat{B}(u)[C(u) + \Delta C(u)] & \hat{A}(u) \end{bmatrix},$$

$$\mathbb{B}_w(u) = \begin{bmatrix} B(u) + \Delta B(u) \\ \hat{B}(u)[D(u) + \Delta D(u)] \end{bmatrix}, \qquad \mathbb{B}_{cl}(u) = \begin{bmatrix} G(u) + \Delta G(u) \\ \hat{B}(u)[J(u) + \Delta J(u)] \end{bmatrix},$$

$$\mathbb{C}_{cl}(u) = \begin{bmatrix} C(u) + \Delta C(u) & -\hat{C}(u) \end{bmatrix}, \qquad \mathbb{D}_{cl}(u) = \begin{bmatrix} D(u) + \Delta D(u) \end{bmatrix},$$

$$\mathbb{J}_{cl}(u) = \begin{bmatrix} J(u) + \Delta J(u) \end{bmatrix}.$$

Before ending this section, the following lemma will be used to prove our main results.

Lemma 1 [22] (The property of the delta operator) *For any time functions $x(t_k)$ and $y(t_k)$, the following holds*:

$$\delta\big(x(t_k)y(t_k)\big) = \delta\big(x(t_k)\big)y(t_k) + x(t_k)\delta\big(y(t_k)\big) + \mathrm{T}\delta\big(x(t_k)\big)\delta\big(y(t_k)\big)$$

where T is a sampling period.

3 Threshold Design

The fault detection problem for the delta operator systems can be viewed as finding the appropriate fault detection filter to make the system asymptotically stable, minimize the effects of disturbances, and enhance the effects of faults.

In order to detect the faults as in [5], the widely adopted approach is to choose an appropriate threshold J_{th} and determine the evaluation function $J_r(n)$, which is selected as

$$J_r(n) = \sqrt{\frac{\mathrm{T}}{n}\sum_{k=k_0}^{k_0+n} e^T(t_k)e(t_k)} \tag{15}$$

where k_0 denotes the initial evaluation time instant, n denotes the evaluation time steps. Based on this, the occurrence of faults can be detected by the following logic rule:

$$J_r(n) > J_{th} \quad \Rightarrow \quad \text{Fault} \quad \Rightarrow \quad \text{Alarm;}$$
$$J_r(n) \le J_{th} \quad \Rightarrow \quad \text{No Fault.}$$

Usually, a threshold function is chosen according to the test. It has been pointed out in [2] that there are many ways of defining evaluation functions and determining thresholds. We choose the threshold as discussed in the next section.

4 Robust Fuzzy Fault-Detection Filter Design

A good fault-detection filter should generate a residual signal that is sensitive to faults and simultaneously insensitive to disturbances and model uncertainties. Under the assumption that no false alarm is allowed, the threshold should be the maximal value of the evaluated output in the fault-free operating state.

4.1 Fault-Free Case

When $f(t_k) = 0$ (i.e., there are no faults), the fault-detection filter problem becomes a standard H_∞-filter design problem (fault-free case $f(t_k) = 0$), i.e., designing an H_∞-filter of the form (10)–(12) such that

$$\sum_{k=k_0}^{k_0+n} e^T(t_k)e(t_k) < \gamma \sum_{k=k_0}^{k_0+n} w^T(t_k)w(t_k). \tag{16}$$

It is evident that $\gamma > 0$ measures the influence of a fault-detection filter to disturbances under the fault-free case. The smaller the γ, the less sensitive the fault-detection filter to the disturbance. Note that under the assumptions (3)–(5) that $w(t_k)$ is bounded, i.e., $\sum_{t=0}^{T_d} w^T(t_k)w(t_k) \le M$, where M is a known scalar, the threshold can be chosen as $J_{th} = \sqrt{\frac{\mathrm{T}}{n}\gamma M}$.

With $f(t_k) = 0$, the state-space form of the fuzzy system model (8)–(9) with the filter (10)–(12) is given by

$$\begin{aligned} \delta\check{x}(t_k) &= \begin{bmatrix} A(u) & 0 \\ \hat{B}(u)C(u) & \hat{A}(u) \end{bmatrix} \check{x}(t_k) + \begin{bmatrix} \Delta A(u) & 0 \\ \hat{B}(u)\Delta C(u) & 0 \end{bmatrix} \check{x}(t_k) \\ &\quad + \begin{bmatrix} B(u) + \Delta B(u) \\ \hat{B}(u)\{D(u) + \Delta D(u)\} \end{bmatrix} w(t_k), \end{aligned} \tag{17}$$

$$e(t_k) = y(t_k) - \hat{y}(t_k) \tag{18}$$

where $\check{x}(t_k) = [x^T(t_k)\ \hat{x}^T(t_k)]^T$. Let us reexpress (17)–(18) in a more compact way as follows:

$$\delta\check{x}(t_k) = A_{cl}(u)\check{x}(t_k) + B_{cl}(u)\mathcal{R}^{-1}v(t_k) \tag{19}$$

where

$$v(t_k) = \mathcal{R}\begin{bmatrix} F(x(t_k, t_k)H_1(u)x(t_k)) \\ F(x(t_k, t_k)H_3(u)x(t_k)) \\ F(x(t_k, t_k)H_2(u)w(t_k)) \\ F(x(t_k), t_k)H_4(u)w(t_k)) \\ w(t_k) \end{bmatrix}, \qquad A_{cl}(u) = \begin{bmatrix} A(u) & 0 \\ \hat{B}(u)C(u) & \hat{A}(u) \end{bmatrix},$$

$$B_{cl}(u) = \begin{bmatrix} E_1(u) & 0 & E_2(u) & 0 & B(u) \\ 0 & \hat{B}(u)E_3(u) & 0 & \hat{B}(u)E_4(u) & \hat{B}(u)D(u) \end{bmatrix},$$

and $\mathcal{R} = \mathrm{diag}\{\alpha I, \alpha I, \gamma I, \gamma I, \gamma I\}$, where α and γ are positive constants, yet to be determined according to the following theorem.

Theorem 1 *Consider the uncertain fuzzy delta operator system* (19). *Suppose there exist scalars* $\alpha > 0$ *and* $\gamma > 0$, *matrices* $X > 0$, $Y > 0$, $\mathcal{A}_{ij}$, *and* $\mathcal{B}_{ij}$ *satisfying*

$$X - Y > 0, \tag{20}$$

$$\begin{bmatrix} \Psi_{1ii} & \Psi_{2ii} & \Psi_{3ii} \\ * & \Psi_{4ii} & \Psi_{5ii} \\ * & * & \Psi_{6ii} \end{bmatrix} < 0, \tag{21}$$

$$\begin{bmatrix} \Psi_{1ij} & \Psi_{2ij} & \Psi_{3ij} \\ * & \Psi_{4ij} & \Psi_{5ij} \\ * & * & \Psi_{6ij} \end{bmatrix} + \begin{bmatrix} \Psi_{1ji} & \Psi_{2ji} & \Psi_{3ji} \\ * & \Psi_{4ji} & \Psi_{5ji} \\ * & * & \Psi_{6ji} \end{bmatrix} < 0 \tag{22}$$

where

$$\Psi_{1ij} = (\daleth - 2)\begin{bmatrix} Y & Y \\ Y & X \end{bmatrix},$$

$$\Psi_{2ij} = \begin{bmatrix} YA_i & YA_i \\ XA_i + \mathcal{B}_iC_j + (Y - X)\hat{A}_i & XA_i + \mathcal{B}_iC_j \end{bmatrix},$$

$$\Psi_{3ij} = \begin{bmatrix} YE_{1i} & 0 & YE_{2i} & 0 & YB_i \\ XE_{1i} & \mathcal{B}_iE_{3j} & XE_{2i} & \mathcal{B}_iE_{4j} & XB_i + \mathcal{B}_iD_j \end{bmatrix},$$

$$\Psi_{4ij} = \begin{bmatrix} YA_i + A_i^TY + \alpha\rho^2[H_{1i}^TH_{1j} + H_{3i}^TH_{3j}] & \mathcal{A}_{ij}^T \\ * & \Psi_{4ij}(2, 2) \end{bmatrix},$$

$$\Psi_{5ij} = \begin{bmatrix} YE_{1i} & 0 & YE_{2i} & 0 & YB_i \\ XE_{1i} & \mathcal{B}_iE_{3j} + \aleph C_i^TE_{3j} & XE_{2i} & \mathcal{B}_iE_{4j} + \aleph C_i^TE_{4j} & \Psi_{5ij}(2, 5) \end{bmatrix},$$

$$\Psi_{6ij} = -\begin{bmatrix} \alpha I & 0 & 0 & 0 & 0 \\ * & \alpha I - \aleph E_{3i}^TE_{3j} & 0 & -\aleph E_{3i}^TE_{4j} & -\aleph E_{3i}^TD_j \\ * & * & \gamma I & 0 & 0 \\ * & * & * & \gamma I - \aleph E_{4i}^TE_{4j} & \aleph E_{4i}^TD_j \\ * & * & * & * & \gamma I - \aleph D_{4i}^TD_{4j} \end{bmatrix}$$

with

$$\Psi_{4ij}(2,2) = A_i^T X + X A_i + \mathcal{B}_i C_j + C_i^T \mathcal{B}_j^T + \alpha\rho^2\big[H_{1i}^T H_{1j} + H_{3i}^T H_{3j}\big] + \aleph C_i^T C_j,$$

$$\Psi_{5ij}(2,5) = X B_i + \mathcal{B}_i D_j + \aleph C_i^T D_j$$

for $i = 1, 2, \ldots, r$, $\forall i < j \le r$, *where* $\aleph = 1 + \rho^2 \sum_{i=1}^r \sum_{j=1}^r \|H_{2i}^T H_{2j} + H_{4i}^T H_{4j}\|$. *Then*, (16) *is guaranteed. Moreover, the suitable robust filter parameters are given as follows*:

$$\hat{A}_{ij} = (Y - X)^{-1}\big\{-X A_i - \mathcal{B}_i C_j + \mathcal{A}_{ij} - A_i^T Y - \alpha\rho^2\big[H_{1i}^T H_{1j} + H_{3i}^T H_{3j}\big]\big\},$$

$$\hat{B}_i = (Y - X)^{-1}\mathcal{B}_i, \qquad \hat{C}_i = \mathcal{C}_i.$$

Proof Let us choose a Lyapunov function as

$$V\big(\check{x}(t_k)\big) = \check{x}^T(t_k) P \check{x}(t_k) \tag{23}$$

where P is a constant positive definite matrix. By using Lemma 1, we have

$$\delta V\big(\check{x}(t_k)\big) = \delta^T\big(\check{x}(t_k)\big) P \check{x}(t_k) + \check{x}^T(t_k) P \delta\big(\check{x}(t_k)\big) + \mathrm{T}\delta^T\big(\check{x}(t_k)\big) P \delta\big(\check{x}(t_k)\big). \tag{24}$$

For the positive definite real matrix P, one has

$$0 = -2\delta^T\big(\check{x}(t_k)\big) P\big[\delta\big(\check{x}(t_k)\big) - A_{cl}(u)\check{x}(t_k) - B_{cl}(u)\mathcal{R}^{-1}v(t_k)\big]. \tag{25}$$

Taking the delta operator manipulations on $V(\check{x}(t_k))$ along the closed-loop fuzzy system (19), we get

$$\begin{aligned}\delta V\big(\check{x}(t_k)\big) = {} & (\mathrm{T} - 2)\delta^T\big(\check{x}(t_k)\big) P \delta\big(\check{x}(t_k)\big) + \delta^T\big(\check{x}(t_k)\big) P A_{cl}(u)\check{x}(t_k) \\ & + \delta^T\big(\check{x}(t_k)\big) P B_{cl}(u)\mathcal{R}^{-1}v(t_k) + \check{x}^T(t_k)\big(A_{cl}(u)^T P \\ & + P A_{cl}(u)\big)\check{x}(t_k) + \check{x}^T(t_k) A_{cl}(u)^T P \delta\big(\check{x}(t_k)\big) \\ & + \check{x}^T(t_k) P B_{cl}(u)\mathcal{R}^{-1}v(t_k) + v^T(t_k)\mathcal{R}^{-1} B_{cl}^T(u) P \check{x}(t_k) \\ & + v^T(t_k)\mathcal{R}^{-1} B_{cl}^T(u) P \delta\big(\check{x}(t_k)\big).\end{aligned} \tag{26}$$

Let us examine the residual term

$$\begin{aligned}e(t_k)^T e(t_k) = {} & \big(y(t_k) - \hat{y}(t_k)\big)^T\big(y(t_k) - \hat{y}(t_k)\big) \\ = {} & \check{x}^T(t_k)\big[C(u) \quad -\hat{C}(u)\big]^T\big[C(u) \quad -\hat{C}(u)\big]\check{x}(t_k) \\ & + \check{x}^T(t_k)\big[C(u) \quad -\hat{C}(u)\big]^T \mathcal{D}(u)\mathcal{R}^{-1}v(t_k) \\ & + v^T(t_k)\mathcal{R}^{-1}\mathcal{D}^T(u)\big[C(u) \quad -\hat{C}(u)\big]\check{x}(t_k) \\ & + v^T(t_k)\mathcal{R}^{-1}\mathcal{D}^T(u)\mathcal{D}(u)\mathcal{R}^{-1}\mathcal{D}v(t_k)\end{aligned} \tag{27}$$

where $\mathcal{D}(u) = [\,0 \;\; E_3(u) \;\; 0 \;\; E_4(u) \;\; D(u)\,]$. Adding and subtracting $\aleph(y(t_k) - \hat{y}(t_k))^T(y(t_k) - \hat{y}(t_k))$ to (27) and from (26), it is obtained that

$$\begin{aligned}\delta V\big(\check{x}(t_k)\big) &= (\mathrm{T}-2)\delta^T\big(\check{x}(t_k)\big)P\delta\big(\check{x}(t_k)\big)+2\delta^T\big(\check{x}(t_k)\big)PA_{cl}(u)\check{x}(t_k)\\ &\quad+2\delta^T\big(\check{x}(t_k)\big)PB_{cl}(u)\mathcal{R}^{-1}v(t_k)+\check{x}^T(t_k)\big\{2A_{cl}(u)^TP\\ &\quad+\aleph\big[C(u)\quad -\hat{C}(u)\big]^T\big[C(u)\quad -\hat{C}(u)\big]\big\}\check{x}(t_k)+\aleph v^T(t_k)\\ &\quad\times\mathcal{R}^{-1}B_{cl}^T(u)P\delta\check{x}(t_k)+2\check{x}^T(t_k)\big\{PB_{cl}(u)+\aleph\big[C(u)\quad -\hat{C}(u)\big]^T\\ &\quad\times\mathcal{D}(u)\big\}\mathcal{R}^{-1}v(t_k)+\aleph v^T(t_k)\mathcal{R}^{-1}\mathcal{D}^T(u)\mathcal{D}(u)\mathcal{R}^{-1}v(t_k)\\ &\quad-\aleph\big(y(t_k)-\hat{y}(t_k)\big)^T\big(y(t_k)-\hat{y}(t_k)\big). \end{aligned}\tag{28}$$

Now let us determine an upper bound for the term $v^T(t_k)\mathcal{R}^{-1}v(t_k)$ by using the triangular inequality as follows:

$$\begin{aligned}v^T(t_k)\mathcal{R}^{-1}v(t_k) &= \begin{bmatrix}F(x(t_k),t_k)H_1(u)x(t_k)\\ F(x(t_k),t_k)H_3(u)x(t_k)\\ F(x(t_k),t_k)H_2(u)w(t_k)\\ F(x(t_k),t_k)H_4(u)w(t_k)\\ w(t_k)\end{bmatrix}^T \mathcal{R}\begin{bmatrix}F(x(t_k),t_k)H_1(u)x(t_k)\\ F(x(t_k),t_k)H_3(u)x(t_k)\\ F(x(t_k),t_k)H_2(u)w(t_k)\\ F(x(t_k),t_k)H_4(u)w(t_k)\\ w(t_k)\end{bmatrix}\\ &\le \alpha\rho^2\check{x}^T(t_k)\big\{H_1^T(u)H_1(u)+H_3^T(u)H_3(u)\big\}\check{x}(t_k)\\ &\quad+\gamma w^T(t_k)\big\{I+\rho^2\big(H_2^T(u)H_2(u)+H_4^T(u)H_4(u)\big)\big\}w(t_k).\end{aligned}$$

Knowing that $\|I+\rho^2(H_2^T(u)H_2(u)+H_4^T(u)H_4(u))\|\le\aleph$, we have

$$v^T(t_k)\mathcal{R}^{-1}v(t_k)\le\alpha\check{x}^T(t_k)C_{cl}^T(u)C_{cl}(u)\check{x}(t_k)+\aleph\gamma w^T(t_k)w(t_k)\tag{29}$$

where

$$C_{cl}(u)=\rho^2\begin{bmatrix}H_1(u) & 0\\ H_3(u) & 0\end{bmatrix}.$$

Adding and subtracting $v^T(t_k)\mathcal{R}^{-1}v(t_k)$ to (28), we obtain

$$\begin{aligned}\delta V\big(\check{x}(t_k)\big) &\le \bar{x}^T(t_k)\begin{bmatrix}(\mathrm{T}-2)P & PA_{cl}(u) & PB_{cl}(u)\\ * & \Pi_1 & \Pi_2\\ * & * & -(\mathcal{R}-\aleph\mathcal{D}^T(u)\mathcal{D}(u))\end{bmatrix}\bar{x}(t_k)\\ &\quad-\aleph\big(y(t_k)-\hat{y}(t_k)\big)^T\big(y(t_k)-\hat{y}(t_k)\big)+\aleph\gamma w^T(t_k)w(t_k)\end{aligned}\tag{30}$$

where

$$\begin{aligned}\bar{x}(t_k) &= \begin{bmatrix}\delta(\check{x}(t_k))\\ \check{x}(t_k)\\ \mathcal{R}^{-1}v(t_k)\end{bmatrix},\\ \Pi_1 &= A_{cl}^T(u)P+PA_{cl}(u)+\aleph\big[C(u)\quad -\hat{C}(u)\big]^T\big[C(u)\quad -\hat{C}(u)\big]\\ &\quad+\alpha C_{cl}^T(u)C_{cl}(u),\\ \Pi_2 &= PB_{cl}(u)+\aleph\big[C(u)\quad -\hat{C}(u)\big]^T\mathcal{D}(u).\end{aligned}$$

Following [4], without loss of generality, we partition P as

$$P = \begin{bmatrix} X & Y - X \\ Y - X & X - Y \end{bmatrix}. \tag{31}$$

Utilizing (31) and letting

$$\breve{x}(t_k) = \begin{bmatrix} \begin{bmatrix} 0 & I \\ I & -I \end{bmatrix} \delta\breve{x}(t_k) \\ \begin{bmatrix} 0 & I \\ I & -I \end{bmatrix} \breve{x}(t_k) \\ \mathcal{R}^{-1} v(t_k) \end{bmatrix},$$

we have the following inequality

$$\delta V\big(\breve{x}(t_k)\big) \le \breve{x}^T(t_k) \begin{bmatrix} \Phi_1(u) & \Phi_2(u) & \Phi_3(u) \\ * & \Phi_4(u) & \Phi_5(u) \\ * & * & \Phi_6(u) \end{bmatrix} \breve{x}(t_k) - \aleph\big(y(t_k) - \hat{y}(t_k)\big)^T \big(y(t_k) - \hat{y}(t_k)\big) + \aleph\gamma w^T(t_k) w(t_k) \tag{32}$$

where

$$\Phi_1(u) = (\daleth - 2)\begin{bmatrix} Y & Y \\ Y & X \end{bmatrix},$$

$$\Phi_2(u) = \begin{bmatrix} YA(u) & YA(u) \\ XA(u) + \mathcal{B}(u)C(u) + (Y - X)\hat{A}(u) & XA(u) + \mathcal{B}(u)C(u) \end{bmatrix},$$

$$\Phi_3(u) = \begin{bmatrix} YE_1(u) & 0 & YE_2(u) & 0 & YB(u) \\ XE_1(u) & \mathcal{B}(u)E_3(u) & XE_2(u) & \mathcal{B}(u)E_4(u) & XB(u) + \mathcal{B}(u)D(u) \end{bmatrix},$$

$$\Phi_4(u) = \begin{bmatrix} YA(u) + A^T(u)Y + \alpha\rho^2[H_1^T(u)H_1(u) + H_3(u)^T H_3(u)] & (\mathcal{A}(u))^T \\ * & \Phi_4(2,2) \end{bmatrix},$$

$$\Phi_5(u) = \begin{bmatrix} YE_1(u) & 0 & YE_2(u) & 0 & YB(u) \\ XE_1(u) & \Phi_5(2,2) & XE_2(u) & \Phi_5(2,4) & \Phi_5(2,5) \end{bmatrix},$$

$$\Phi_6(u) = -\big(\mathcal{R} - \aleph\mathcal{D}^T(u)\mathcal{D}(u)\big)$$

with

$$\Phi_4(2,2) = A^T(u)X + XA(u) + \mathcal{B}(u)C(u) + C^T(u)\mathcal{B}^T(u) + \aleph C^T(u)C(u) + \alpha\rho^2\big[H_1^T(u)H_1(u) + H_3^T(u)H_3(u)\big],$$

$$\Phi_5(2,2) = \mathcal{B}(u)E_3(u) + \aleph C^T(u)E_3,$$

$$\Phi_5(2,4) = \mathcal{B}(u)E_4(u) + \aleph C^T(u)E_4(u),$$

$$\Phi_5(2,5) = XB(u) + \mathcal{B}(u)D(u) + \aleph C^T(u)D(u),$$

and

$$\begin{aligned}\mathcal{A}(u) &= XA(u) + \mathcal{B}(u)C(u) + (Y - X)\hat{A}(u) + A(u)^T Y \\ &\quad + \alpha\rho^2\big[H_1^T(u)H_1(u) + H_3^T(u)H_3(u)\big], \\ \mathcal{B}(u) &= (Y - X)\hat{B}(u), \qquad \mathcal{C}(u) = \hat{C}(u).\end{aligned}$$

Note that

$$\begin{bmatrix} \Phi_1(u) & \Phi_2(u) & \Phi_3(u) \\ * & \Phi_4(u) & \Phi_5(u) \\ * & * & \Phi_6(u) \end{bmatrix} = \sum_{i=1}^{r}\sum_{j=1}^{r} u_i u_j \begin{bmatrix} \Phi_{1ij} & \Phi_{2ij} & \Phi_{3ij} \\ * & \Phi_{4ij} & \Phi_{5ij} \\ * & * & \Phi_{6ij} \end{bmatrix}.$$

Considering (20)–(22), we have that

$$\delta V\big(\check{x}(t_k)\big) \le -\aleph\big(y(t_k) - \hat{y}(t_k)\big)^T\big(y(t_k) - \hat{y}(t_k)\big) + \aleph\gamma w^T(t_k)w(t_k). \tag{33}$$

Integrating both sides of (33) yields

$$\sum_{k=k_0}^{k_0+n} \delta V\big(\check{x}(t_k)\big) \le \sum_{k=k_0}^{k_0+n}\big\{-\aleph\big(y(t_k) - \hat{y}(t_k)\big)^T\big(y(t_k) - \hat{y}(t_k)\big) + \aleph\gamma w^T(t_k)w(t_k)\big\},$$

or

$$\begin{aligned}\delta V\big(\check{x}(\mathrm{T}_d)\big) - \delta V\big(\check{x}(0)\big) \le \sum_{k=k_0}^{k_0+n}\big\{&-\aleph\big(y(t_k) - \hat{y}(t_k)\big)^T\big(y(t_k) - \hat{y}(t_k)\big) \\ &+ \aleph\gamma w^T(t_k)w(t_k)\big\}.\end{aligned}$$

Using the fact that $\check{x} = 0$ and $\delta V(\check{x}(t_k)) > 0$ for all $t_k \neq 0$, we have

$$\sum_{k=k_0}^{k_0+n}\big(y(t_k) - \hat{y}(t_k)\big)^T\big(y(t_k) - \hat{y}(t_k)\big) \le \gamma \sum_{k=k_0}^{k_0+n} w^T(t_k)w(t_k).$$

Hence, (16) is guaranteed, which is equal to $J_r(n) < J_{th}$. □

4.2 Disturbance-Free Case

Before presenting our main result, we consider the disturbance-free case ($w(t_k) = 0$) and design a fault-detection H_- filter such that

$$\sum_{k=k_0}^{k_0+n} e^T(t_k)e(t_k) > \beta \sum_{k=k_0}^{k_0+n} f^T(t_k)f(t_k) \tag{34}$$

where β measures the sensitivity of a fault-detection filter to faults under the disturbance-free case. The larger the β, the more sensitive the fault-detection filter to the faults. The threshold can be chosen as

$$J_{th} = \sqrt{\frac{\mathrm{T}}{n}\beta\sum_{k=k_0}^{k_0+n} f^T(t_k)f(t_k)}.$$

When the disturbance input is zero, the state-space form of the fuzzy system model (8)–(9) with the filter (10)–(12) is given by

$$\delta\check{x}(t_k) = \begin{bmatrix} A(u) & 0 \\ \hat{B}(u)C(u) & \hat{A}(u) \end{bmatrix}\check{x}(t_k) + \begin{bmatrix} \Delta A(u) & 0 \\ \hat{B}(u)\Delta C(u) & 0 \end{bmatrix}\check{x}(t) + \begin{bmatrix} B(u) + \Delta B(u) \\ \hat{B}(u)\{J(u) + \Delta J(u)\} \end{bmatrix} f(t_k), \tag{35}$$

$$e(t_k) = y(t_k) - \hat{y}(t_k) \tag{36}$$

where $\check{x}(t_k) = [x^T(t_k)\ \hat{x}^T(t_k)]^T$. The closed-loop fuzzy delta operator system (35)–(36) is reexpressed as follows:

$$\delta\check{x}(t_k) = \tilde{A}_{cl}(u)\check{x}(t_k) + \tilde{B}_{cl}\tilde{\mathcal{R}}^{-1}v(t_k) \tag{37}$$

with

$$\tilde{v}(t_k) = \tilde{\mathcal{R}}\begin{bmatrix} F(x(t_k), t_k)H_1(u)x(t_k)) \\ F(x(t_k), t_k)H_3(u)x(t_k)) \\ F(x(t_k), t_k)H_5(u)f(t_k)) \\ F(x(t_k), t_k)H_6(u)f(t_k)) \\ f(t_k) \end{bmatrix}, \qquad \tilde{A}_{cl}(u) = \begin{bmatrix} A(u) & 0 \\ \hat{B}(u)C(u) & \hat{A}(u) \end{bmatrix},$$

$$\tilde{B}_{cl}(u) = \begin{bmatrix} E_1(u) & 0 & E_5(u) & 0 & G(u) \\ 0 & \hat{B}(u)E_3(u) & 0 & \hat{B}(u)E_6(u) & \hat{B}(u)J(u) \end{bmatrix}$$

where $\tilde{\mathcal{R}} = \mathrm{diag}\{\alpha I, \alpha I, \beta I, \beta I, \beta I\}$, with α and β being positive constants, yet to be determined according to the following theorem.

Theorem 2 *Consider the uncertain fuzzy system* (37). *Suppose there exist scalars* $\alpha > 0$, $\beta > 0$, *matrices* $X > 0$, $Y > 0$, $\mathcal{A}_{ij}$, *and* $\mathcal{B}_{ij}$ *satisfying*

$$X - Y > 0, \tag{38}$$

$$\begin{bmatrix} \tilde{\Psi}_{1ii} & \tilde{\Psi}_{2ii} & \tilde{\Psi}_{3ii} \\ * & \tilde{\Psi}_{4ii} & \tilde{\Psi}_{5ii} \\ * & * & \tilde{\Psi}_{6ii} \end{bmatrix} < 0, \tag{39}$$

$$\begin{bmatrix} \tilde{\Psi}_{1ij} & \tilde{\Psi}_{2ij} & \tilde{\Psi}_{3ij} \\ * & \tilde{\Psi}_{4ij} & \tilde{\Psi}_{5ij} \\ * & * & \tilde{\Psi}_{6ij} \end{bmatrix} + \begin{bmatrix} \Psi_{1ji} & \tilde{\Psi}_{2ji} & \tilde{\Psi}_{3ji} \\ * & \tilde{\Psi}_{4ji} & \tilde{\Psi}_{5ji} \\ * & * & \tilde{\Psi}_{6ji} \end{bmatrix} < 0 \tag{40}$$

where

$$\tilde{\Psi}_{1ij} = (\mathrm{T} - 2)\begin{bmatrix} Y & Y \\ Y & X \end{bmatrix},$$

$$\tilde{\Psi}_{2ij} = \begin{bmatrix} YA_i & YA_i \\ XA_i + \mathcal{B}_i C_j + (Y - X)\hat{A}_i & XA + \mathcal{B}_i C_j \end{bmatrix},$$

$$\tilde{\Psi}_{3ij} = \begin{bmatrix} YE_{1i} & 0 & YE_{5i} & 0 & YG_i \\ XE_{1i} & \mathcal{B}_i E_{3j} & XE_{5i} & \mathcal{B}_i E_{6j} & XG_i + \mathcal{B}_i J_i \end{bmatrix},$$

$$\tilde{\Psi}_{4ij} = \begin{bmatrix} YA_i + A_i^T Y + \alpha\rho^2[H_{1i}^T H_{1j} + H_{3i}^T H_{3j}] & \mathcal{A}_{ij}^T \\ * & \tilde{\Psi}_{4ij}(2,2) \end{bmatrix},$$

$$\tilde{\Psi}_{5ij} = \begin{bmatrix} YE_{1i} & 0 & YE_{5i} & 0 & YG_i \\ XE_{1i} & \mathcal{B}_i E_{3j} - C_i^T E_{3j} & XE_{5i} & \mathcal{B}_i E_{6j} - C_i^T E_{6j} & \tilde{\Psi}_{5ij}(2,5) \end{bmatrix},$$

$$\tilde{\Psi}_{6ij} = -\begin{bmatrix} \alpha I & 0 & 0 & 0 & 0 \\ * & \alpha I + E_{3i}^T E_{3j} & 0 & E_{3i}^T E_{6j} & E_{3i}^T J_j \\ * & * & \beta I & 0 & 0 \\ * & * & * & \beta I + E_{6i}^T E_{6j} & \tilde{\aleph} E_{6i}^T J_j \\ * & * & * & * & J_i^T J_j - \beta(1 + \tilde{\aleph}) I \end{bmatrix}$$

with

$$\tilde{\Psi}_{4ij}(2,2) = A_i^T X + XA_i + \mathcal{B}_i C_j + C_i^T \mathcal{B}_j^T + \alpha\rho^2\big[H_{1i}^T H_{1j} + H_{3i}^T H_{3j}\big] - C_i^T C_j,$$

$$\tilde{\Psi}_{5ij}(2,5) = XG_i + \mathcal{B}_i J_j - C_i^T J_j$$

for $i = 1, 2, \ldots, r$, $\forall i < j \le r$, *where* $\tilde{\aleph} = \rho^2 \sum_{i=1}^r \sum_{j=1}^r \|H_{5i}^T H_{5j} + H_{6i}^T H_{6j}\|$. *Then* (34) *is guaranteed. Moreover, the suitable robust filter parameters are given as follows*:

$$\hat{A}_{ij} = (Y - X)^{-1}\big\{-XA_i - \mathcal{B}_i C_j + \mathcal{A}_{ij} - A_i^T Y - \alpha\rho^2\big[H_{1i}^T H_{1j} + H_{3i}^T H_{3j}\big]\big\},$$

$$\hat{B}_i = (Y - X)^{-1}\mathcal{B}_i, \qquad \hat{C}_i = \mathcal{C}_i.$$

Proof Let us choose a Lyapunov function

$$V\big(\check{x}(t_k)\big) = \check{x}^T(t_k) P \check{x}(t_k) \tag{41}$$

where P is a constant positive definite matrix. By using Lemma 1, we have

$$\delta V\big(\check{x}(t_k)\big) = \delta^T\big(\check{x}(t_k)\big) P \check{x}(t_k) + \check{x}(t_k) P \delta\big(\check{x}(t_k)\big) + \mathrm{T}\delta^T\big(\check{x}(t_k)\big) P \delta\big(\check{x}(t_k)\big). \tag{42}$$

For the positive definite real matrix P, one has

$$0 = -2\delta^T\big(\check{x}(t_k)\big) P\big[\delta\big(\check{x}(t_k)\big) - A_{cl}(u)\check{x}(t_k) - \tilde{B}_{cl}(u)\tilde{\mathcal{R}}^{-1}\tilde{v}(t_k)\big]. \tag{43}$$

Taking the delta operator manipulations on $V(\check{x}(t_k))$ along the closed-loop system (37), we get

$$\delta V\big(\check{x}(t_k)\big) = (\mathrm{T} - 2)\delta^T\big(\check{x}(t_k)\big) P \delta\big(\check{x}(t_k)\big) + \delta^T\big(\check{x}(t_k)\big) P A_{cl}(u)\check{x}(t_k)$$

$$+\delta^T\big(\check{x}(t_k)\big)P\tilde{B}_{cl}(u)\tilde{\mathcal{R}}^{-1}\tilde{v}(t_k)+\check{x}^T(t_k)\big(A_{cl}(u)^TP$$
$$+PA_{cl}(u)\big)\check{x}(t_k)+\check{x}^T(t_k)A_{cl}(u)^TP\delta\big(x(t_k)\big)$$
$$+\check{x}^T(t_k)P\tilde{B}_{cl}(u)\tilde{\mathcal{R}}^{-1}\tilde{v}(t_k)+\tilde{v}^T(t_k)\tilde{\mathcal{R}}^{-1}\tilde{B}_{cl}^T(u)P\check{x}(t_k)$$
$$+\tilde{v}^T(t_k)\tilde{\mathcal{R}}^{-1}\tilde{B}_{cl}^T(u)P\delta\big(\check{x}(t_k)\big). \tag{44}$$

Let us examine the residual term

$$\begin{aligned}
e(t_k)^Te(t_k) &= \big(y(t_k)-\hat{y}(t_k)\big)^T\big(y(t_k)-\hat{y}(t_k)\big)\\
&= \check{x}^T(t_k)\big[C(u)\quad -\hat{C}(u)\big]^T\big[C(u)\quad -\hat{C}(u)\big]\check{x}(t_k)\\
&\quad+\check{x}^T(t_k)\big[C(u)\quad -\hat{C}(u)\big]^T\mathcal{J}(u)\tilde{\mathcal{R}}^{-1}\tilde{v}(t_k)\\
&\quad+\tilde{v}^T(t_k)\tilde{\mathcal{R}}^{-1}\mathcal{J}^T(u)\big[C(u)\quad -\hat{C}(u)\big]\check{x}(t_k)\\
&\quad+\tilde{v}^T(t_k)\tilde{\mathcal{R}}^{-1}\mathcal{J}^T(u)\mathcal{J}(u)\tilde{\mathcal{R}}^{-1}\tilde{v}(t_k)
\end{aligned}\tag{45}$$

where $\mathcal{J}(u)=[\,0\;E_3(u)\;0\;E_6(u)\;J(u)\,]$. Adding and subtracting $(y(t_k)-\hat{y}(t_k))^T(y(t_k)-\hat{y}(t_k))$ to and from (44), one obtains

$$\begin{aligned}
\delta V\big(\check{x}(t_k)\big) &= (\mathrm{T}-2)\delta^T\big(\check{x}(t_k)\big)P\delta\big(\check{x}(t_k)\big)+2\delta^T\big(\check{x}(t_k)\big)PA_{cl}(u)\check{x}(t_k)\\
&\quad+2\delta^T\big(\check{x}(t_k)\big)P\tilde{B}_{cl}(u)\tilde{\mathcal{R}}^{-1}\tilde{v}(t_k)+\tilde{v}^T(t_k)\tilde{\mathcal{R}}^{-1}\tilde{B}_{cl}^T(u)P\delta\check{x}(t_k)\\
&\quad+\check{x}^T(t_k)\big\{2A_{cl}(u)^TP-\big[C(u)\quad -\hat{C}(u)\big]^T\big[C(u)\quad -\hat{C}(u)\big]\big\}\check{x}(t_k)\\
&\quad+2\check{x}^T(t_k)\big\{P\tilde{B}_{cl}(u)-\big[C(u)\quad -\hat{C}(u)\big]^T\mathcal{J}(u)\big\}\tilde{\mathcal{R}}^{-1}\tilde{v}(t_k)\\
&\quad-\tilde{v}^T(t_k)\tilde{\mathcal{R}}^{-1}\mathcal{J}^T(u)\mathcal{J}(u)\tilde{\mathcal{R}}^{-1}\tilde{v}(t_k)\\
&\quad+\big(y(t_k)-\hat{y}(t_k)\big)^T\big(y(t_k)-\hat{y}(t_k)\big).
\end{aligned}\tag{46}$$

Now let us determine an upper bound for the term $\tilde{v}^T(t_k)\mathcal{Q}\tilde{\mathcal{R}}^{-1}\tilde{v}(t_k)$, where $\mathcal{Q}=\mathrm{diag}\{I,I,I,I,0\}$. Using the triangular inequality, we have

$$\begin{aligned}
\tilde{v}^T(t_k)\mathcal{Q}\tilde{\mathcal{R}}^{-1}\tilde{v}(t_k) &= \begin{bmatrix} F(x(t_k),t_k)H_1(u)x(t_k)\\ F(x(t_k),t_k)H_3(u)x(t_k)\\ F(x(t_k),t_k)H_5(u)f(t_k)\\ F(x(t_k),t_k)H_6(u)f(t_k)\\ f(t_k)\end{bmatrix}^T \tilde{\mathcal{R}}\mathcal{Q}\begin{bmatrix} F(x(t_k),t_k)H_1(u)x(t_k)\\ F(x(t_k),t_k)H_3(u)x(t_k)\\ F(x(t_k),t_k)H_5(u)f(t_k)\\ F(x(t_k),t_k)H_6(u)f(t_k)\\ f(t_k)\end{bmatrix}\\
&\le \alpha\rho^2\check{x}^T(t_k)\big\{H_1^T(u)H_1(u)+H_3^T(u)H_3(u)\big\}\check{x}(t_k)\\
&\quad+\beta\rho^2f^T(t_k)\big\{H_5^T(u)H_5(u)+H_6^T(u)H_6(u)\big\}f(t_k).
\end{aligned}$$

Knowing that $\|\rho^2(H_5^T(u)H_5(u)+H_6^T(u)H_6(u))\|\le\tilde{\aleph}$, we have

$$\tilde{v}^T(t_k)\mathcal{Q}\tilde{\mathcal{R}}^{-1}\tilde{v}(t_k)\le\alpha\check{x}^T(t_k)C_{cl}^T(u)C_{cl}(u)\check{x}(t_k)+\aleph\beta f^T(t_k)f(t_k)$$

where

$$C_{cl}(u) = \rho^2 \begin{bmatrix} H_1(u) & 0 \\ H_3(u) & 0 \end{bmatrix}.$$

Adding and subtracting $\tilde{v}^T(t_k)\mathcal{Q}\tilde{\mathcal{R}}^{-1}\tilde{v}(t_k)$ to and from (46), we obtain the following inequality:

$$\delta V\big(\check{x}(t_k)\big) \le \bar{x}^T(t_k) \begin{bmatrix} (\mathrm{T}-2)P & PA_{cl}(u) & P\tilde{B}_{cl}(u) \\ * & \Pi_1 & \Pi_2 \\ * & * & -(\mathcal{J}^T(u)\mathcal{J}(u)\tilde{\mathcal{R}} + \mathcal{Q}\tilde{\mathcal{R}}) \end{bmatrix} \bar{x}(t_k) + \big(y(t_k) - \hat{y}(t_k)\big)^T \big(y(t_k) - \hat{y}(t_k)\big) + \tilde{\aleph}\beta f^T(t_k) f(t_k) \tag{47}$$

where

$$\bar{x}(t_k) = \begin{bmatrix} \delta(\check{x}(t_k)) \\ \check{x}(t_k) \\ \tilde{\mathcal{R}}^{-1}\tilde{v}(t_k) \end{bmatrix},$$

$$\Pi_1 = A_{cl}^T(u)P + PA_{cl}(u) + \alpha C_{cl}^T(u)C_{cl}(u) - \begin{bmatrix} C(u) & -\hat{C}(u) \end{bmatrix}^T \begin{bmatrix} C(u) & -\hat{C}(u) \end{bmatrix},$$

$$\Pi_2 = P\tilde{B}_{cl}(u) - \begin{bmatrix} C(u) & -\hat{C}(u) \end{bmatrix}^T \mathcal{J}(u).$$

Adding and subtracting $\beta(1+\tilde{\aleph}) f^T(t_k) f(t_k)$ to and from (47), we obtain the inequality

$$\delta V\big(\check{x}(t_k)\big) \le \bar{x}^T(t_k) \begin{bmatrix} (\mathrm{T}-2)P & PA_{cl}(u) & P\tilde{B}_{cl}(u) \\ * & \Pi_1 & \Pi_2 \\ * & * & -(\mathcal{J}^T(u)\mathcal{J}(u)\tilde{\mathcal{R}} + \mathcal{Q}\tilde{\mathcal{R}} - \mathcal{U}) \end{bmatrix} \bar{x}(t_k) + \big(y(t_k) - \hat{y}(t_k)\big)^T \big(y(t_k) - \hat{y}(t_k)\big) - \beta f^T(t_k) f(t_k)$$

where $\mathcal{U} = \mathrm{diag}\{0, 0, 0, 0, \beta(1+\tilde{\aleph})I\}$ and

$$\Pi_1 = A_{cl}^T(u)P + PA_{cl}(u) + \alpha C_{cl}^T(u)C_{cl}(u) - \begin{bmatrix} C(u) & -\hat{C}(u) \end{bmatrix}^T \begin{bmatrix} C(u) & -\hat{C}(u) \end{bmatrix},$$

$$\Pi_2 = P\tilde{B}_{cl}(u) - \begin{bmatrix} C(u) & -\hat{C}(u) \end{bmatrix}^T \mathcal{J}(u).$$

Following [4], without loss of generality, we partition P as

$$P = \begin{bmatrix} X & Y - X \\ Y - X & X - Y \end{bmatrix}. \tag{48}$$

Utilizing (48) and letting

$$\breve{x}(t_k) = \begin{bmatrix} \begin{bmatrix} 0 & I \\ I & -I \end{bmatrix} \delta\breve{x}(t_k) \\ \begin{bmatrix} 0 & I \\ I & -I \end{bmatrix} \breve{x}(t_k) \\ \tilde{\mathcal{R}}^{-1}\tilde{v}(t_k) \end{bmatrix},$$

we have the inequality

$$\delta V\big(\breve{x}(t_k)\big) \le \breve{x}^T(t_k) \begin{bmatrix} \tilde{\Phi}_1(u) & \tilde{\Phi}_2(u) & \tilde{\Phi}_3(u) \\ * & \tilde{\Phi}_4(u) & \tilde{\Phi}_5(u) \\ * & * & \tilde{\Phi}_6(u) \end{bmatrix} \breve{x}(t_k) + \big(y(t_k) - \hat{y}(t_k)\big)^T \big(y(t_k) - \hat{y}(t_k)\big) - \beta f^T(t_k) f(t_k)$$

where

$$\tilde{\Phi}_1(u) = (\mathrm{T} - 2) \begin{bmatrix} Y & Y \\ Y & X \end{bmatrix},$$

$$\tilde{\Phi}_2(u) = \begin{bmatrix} YA(u) & YA(u) \\ XA(u) + \mathcal{B}(u)C(u) + (Y - X)\hat{A}(u) & XA(u) + \mathcal{B}(u)C(u) \end{bmatrix},$$

$$\tilde{\Phi}_3(u) = \begin{bmatrix} YE_1(u) & 0 & YE_5(u) & 0 & YG(u) \\ XE_1(u) & \mathcal{B}(u)E_3(u) & XE_5(u) & \mathcal{B}(u)E_6(u) & XG(u) + \mathcal{B}(u)J(u) \end{bmatrix},$$

$$\tilde{\Phi}_4(u) = \begin{bmatrix} YA(u) + A^T(u)Y + \alpha\rho^2[H_1^T(u)H_1(u) + H_3(u)^T H_3(u)] & (*)^T \\ \mathcal{A}(u) & \tilde{\Phi}_4(2,2) \end{bmatrix},$$

$$\tilde{\Phi}_5(u) = \begin{bmatrix} YE_1(u) & 0 & YE_5(u) & 0 & YG(u) \\ XE_1(u) & \tilde{\Phi}_5(2,2) & XE_5(u) & \tilde{\Phi}_5(2,4) & \tilde{\Phi}_5(2,5) \end{bmatrix},$$

$$\tilde{\Phi}_6(u) = -\big(\mathcal{J}^{\mathbf{T}}(u)\mathcal{J}(u) + \mathcal{Q}\tilde{\mathcal{R}} - \mathcal{U}\big)$$

with

$$\tilde{\Phi}_4(2,2) = A^T(u)X + XA(u) + \mathcal{B}(u)C(u) + C^T(u)\mathcal{B}^T(u) - C^T(u)C(u) + \alpha\rho^2\big[H_1^T(u)H_1(u) + H_3^T(u)H_3(u)\big],$$

$$\tilde{\Phi}_5(2,2) = \mathcal{B}(u)E_3(u) - C^T(u)E_3,$$

$$\tilde{\Phi}_5(2,4) = \mathcal{B}(u)E_6(u) - C^T(u)E_6(u),$$

$$\tilde{\Phi}_5(2,5) = XG(u) + \mathcal{B}(u)J(u) - C^T(u)J(u),$$

and

$$\mathcal{A}(u) = XA(u) + \mathcal{B}(u)C(u) + (Y - X)\hat{A}(u) + A(u)^T Y + \alpha\rho^2\big[H_1^T(u)H_1(u) + H_3^T(u)H_3(u)\big],$$

$$\mathcal{B}(u) = (Y - X)\hat{B}(u), \qquad \mathcal{C}(u) = \hat{C}(u).$$

Note that

$$\begin{bmatrix} \tilde{\Phi}_1(u) & \tilde{\Phi}_2(u) & \tilde{\Phi}_3(u) \\ * & \tilde{\Phi}_4(u) & \tilde{\Phi}_5(u) \\ * & * & \tilde{\Phi}_6(u) \end{bmatrix} = \sum_{i=1}^{r}\sum_{j=1}^{r} u_i u_j \begin{bmatrix} \tilde{\Phi}_{1ij} & \tilde{\Phi}_{2ij} & \tilde{\Phi}_{3ij} \\ * & \tilde{\Phi}_{4ij} & \tilde{\Phi}_{5ij} \\ * & * & \tilde{\Phi}_{6ij} \end{bmatrix}.$$

Considering (38)–(40), we have

$$\delta V\big(\breve{x}(t_k)\big) \le \big(y(t_k) - \hat{y}(t_k)\big)^T \big(y(t_k) - \hat{y}(t_k)\big) - \beta f^T(t_k) f(t_k). \tag{49}$$

Integrating both sides of (49) yields

$$\sum_{k=k_0}^{k_0+n} \delta V\big(\breve{x}(t_k)\big) \le \sum_{k=k_0}^{k_0+n} \big\{\big(y(t_k) - \hat{y}(t_k)\big)^T \big(y(t_k) - \hat{y}(t_k)\big) - \beta f^T(t_k) f(t_k)\big\},$$

or

$$\delta V\big(\breve{x}(T_d)\big) - \delta V\big(\breve{x}(0)\big) \le \sum_{k=k_0}^{k_0+n} \big\{\big(y(t_k) - \hat{y}(t_k)\big)^T \big(y(t_k) - \hat{y}(t_k)\big) - \beta f^T(t_k) f(t_k)\big\}.$$

Using the fact that $\breve{x} = 0$ and $\delta V(\breve{x}(t_k)) > 0$ for all $t_k \neq 0$, we have

$$\sum_{k=k_0}^{k_0+n} \big(y(t_k) - \hat{y}(t_k)\big)^T \big(y(t_k) - \hat{y}(t_k)\big) > \beta \sum_{k=k_0}^{k_0+n} f^T(t_k) f(t_k).$$

Hence, (34) is guaranteed, which is equal to $J_r(n) > J_{th}$. □

4.3 Stability Analysis

When the disturbance input and the fault are zero ($w(t_k) = 0$, $f(t_k) = 0$), the state-space form of the fuzzy system model (8)–(9) with the filter (10)–(12) is given by

$$\delta\breve{x}(t_k) = \begin{bmatrix} A(u) & 0 \\ \hat{B}(u)C(u) & \hat{A}(u) \end{bmatrix} \breve{x}(t_k) + \begin{bmatrix} \Delta A(u) & 0 \\ \hat{B}(u)\Delta C(u) & 0 \end{bmatrix} \breve{x}(t_k) \tag{50}$$

in which $\breve{x}(t_k) = [x^T(t_k)\ \hat{x}^T(t_k)]^T$. The closed-loop system (50) can be reexpressed as

$$\delta\breve{x}(t_k) = \breve{A}_{cl}(u)\breve{x}(t_k) + \breve{B}_{cl}(u)\breve{\mathcal{R}}^{-1}\breve{v}(t_k) \tag{51}$$

where

$$\breve{A}_{cl}(u) = \begin{bmatrix} A(u) & 0 \\ \hat{B}(u)C(u) & \hat{A}(u) \end{bmatrix}, \qquad \breve{v}(t_k) = \breve{\mathcal{R}} \begin{bmatrix} F(x(t_k), t_k)H_1(u)x(t_k)) \\ F(x(t_k), t_k)H_3(u)x(t_k)) \end{bmatrix},$$

$$\breve{B}_{cl} = \begin{bmatrix} E_1(u) & 0 \\ 0 & \hat{B}(u)E_3(u) \end{bmatrix},$$

and $\breve{\mathcal{R}} = \mathrm{diag}\{\alpha I, \alpha I\}$ are positive constants, yet to be determined according to the following theorem.

Theorem 3 *Consider the uncertain fuzzy delta operator system* (51). *Suppose there exist scalars* $\alpha > 0$ *and* $\beta > 0$, *matrices* $X > 0$, $Y > 0$, $\mathcal{A}_{ij}$, *and* $\mathcal{B}_{ij}$ *satisfying*

$$X - Y > 0, \tag{52}$$

$$\begin{bmatrix} \breve{\Psi}_{1ii} & \breve{\Psi}_{2ii} & \breve{\Psi}_{3ii} \\ * & \breve{\Psi}_{4ii} & \breve{\Psi}_{5ii} \\ * & * & \breve{\Psi}_{6ii} \end{bmatrix} < 0, \tag{53}$$

$$\begin{bmatrix} \breve{\Psi}_{1ij} & \breve{\Psi}_{2ij} & \breve{\Psi}_{3ij} \\ * & \breve{\Psi}_{4ij} & \breve{\Psi}_{5ij} \\ * & * & \breve{\Psi}_{6ij} \end{bmatrix} + \begin{bmatrix} \breve{\Psi}_{1ji} & \breve{\Psi}_{2ji} & \breve{\Psi}_{3ji} \\ * & \breve{\Psi}_{4ji} & \breve{\Psi}_{5ji} \\ * & * & \breve{\Psi}_{6ji} \end{bmatrix} < 0 \tag{54}$$

where

$$\breve{\Psi}_{1ij} = (\mathrm{T} - 2)\begin{bmatrix} Y & Y \\ Y & X \end{bmatrix},$$

$$\breve{\Psi}_{2ij} = \begin{bmatrix} YA_i & YA_i \\ XA_i + \mathcal{B}_iC_j + (Y - X)\hat{A}_i & XA_i + \mathcal{B}_iC_j \end{bmatrix},$$

$$\breve{\Psi}_{3ij} = \begin{bmatrix} YE_{1i} & 0 \\ XE_{1i} & \mathcal{B}_iE_{3j} \end{bmatrix}, \qquad \breve{\Psi}_{4ij} = \begin{bmatrix} \breve{\Psi}_{4ij}(1,1) & \mathcal{A}_{ij}^T \\ * & \breve{\Psi}_{4ij}(2,2) \end{bmatrix},$$

$$\breve{\Psi}_{5ij} = \begin{bmatrix} YE_{1i} & 0 \\ XE_{1i} & \mathcal{B}_iE_{3j} + C_i^TE_{3j} \end{bmatrix}, \qquad \breve{\Psi}_{6ij} = -\begin{bmatrix} \alpha I & 0 \\ * & \alpha I - E_{3i}^TE_{3j} \end{bmatrix}$$

with

$$\breve{\Psi}_{4ij}(1,1) = YA_i + A_i^TY + \alpha\rho^2\left[H_{1i}^TH_{1j} + H_{3i}^TH_{3j}\right],$$
$$\breve{\Psi}_{4ij}(2,2) = A_i^TX + XA_i + \mathcal{B}_iC_j + C_i^T\mathcal{B}_j^T + \alpha\rho^2\left[H_{1i}^TH_{1j} + H_{3i}^TH_{3j}\right] + C_i^TC_j$$

for $i = 1, 2, \ldots, \gamma$, $\forall i < j \le r$. *Then system* (51) *is asymptotically stable. Moreover, the suitable robust filter parameters are given as follows*:

$$\hat{A}_{ij} = (Y - X)^{-1}\left\{-XA_i - \mathcal{B}_iC_j + \mathcal{A}_{ij} - A_i^TY - \alpha\rho^2\left[H_{1i}^TH_{1j} + H_{3i}^TH_{3j}\right]\right\},$$
$$\hat{B}_i = (Y - X)^{-1}\mathcal{B}_i, \qquad \hat{C}_i = \mathcal{C}_i.$$

Proof Let us choose a Lyapunov function

$$V\left(\breve{x}(t_k)\right) = \breve{x}^T(t_k)P\breve{x}(t_k)$$

where P is a constant positive definite matrix. By using Lemma 1, we have

$$\delta V\left(\breve{x}(t_k)\right) = \delta^T\left(\breve{x}(t_k)\right)P\breve{x}(t_k) + \breve{x}(t_k)P\delta\left(\breve{x}(t_k)\right) + \mathrm{T}\delta^T\left(\breve{x}(t_k)\right)P\delta\left(\breve{x}(t_k)\right). \tag{55}$$

For the positive definite real matrix P, one has

$$0 = -2\delta^T\left(\breve{x}(t_k)\right)P\left[\delta\left(\breve{x}(t_k)\right) - A_{cl}(u)\breve{x}(t_k) - \breve{B}_{cl}(u)\breve{\mathcal{R}}^{-1}\breve{v}(t_k)\right]. \tag{56}$$

Taking the time derivative on $V(\breve{x}(t_k))$ along the closed-loop system (51), we get

$$\begin{aligned}\delta V\big(\breve{x}(t_k)\big) &= (\mathrm{T}-2)\delta^T\big(\breve{x}(t_k)\big)P\delta\big(\breve{x}(t_k)\big) + \delta^T\big(\breve{x}(t_k)\big)PA_{cl}(u)\breve{x}(t_k)\\ &+ \delta^T\big(\breve{x}(t_k)\big)\breve{B}_{cl}(u)\breve{\mathcal{R}}^{-1}\breve{v}(t_k) + \breve{x}^T(t_k)P\breve{B}_{cl}(u)\breve{\mathcal{R}}^{-1}\breve{v}(t_k)\\ &+ \breve{x}^T(t_k)A_{cl}^T(u)P\delta\big(\breve{x}(t_k)\big) + \breve{x}^T(t_k)\big(A_{cl}^T(u)P + PA_{cl}(u)\big)\breve{x}(t_k)\\ &+ \breve{v}^T(t_k)\breve{\mathcal{R}}^{-1}\breve{B}_{cl}^T(u)P\breve{x}(t_k) + \breve{v}^T(t_k)\breve{\mathcal{R}}^{-1}\breve{B}_{cl}^T(u)P\delta\big(\breve{x}(t_k)\big). \end{aligned} \tag{57}$$

Let us examine the residual term

$$\begin{aligned} e(t_k)^T e(t_k) &= \big(y(t_k) - \hat{y}(t_k)\big)^T\big(y(t_k) - \hat{y}(t_k)\big)\\ &= \breve{x}^T(t_k)\big[C(u) \quad -\hat{C}(u)\big]^T\big[C(u) \quad -\hat{C}(u)\big]\breve{x}(t_k)\\ &\quad + \breve{x}^T(t_k)\big[C(u) \quad -\hat{C}(u)\big]^T\mathcal{O}(u)\breve{\mathcal{R}}^{-1}\breve{v}(t_k)\\ &\quad + \breve{v}^T(t_k)\breve{\mathcal{R}}^{-1}\mathcal{O}^T(u)\big[C(u) \quad -\hat{C}(u)\big]\breve{x}(t_k)\\ &\quad + \breve{v}^T(t_k)\breve{\mathcal{R}}^{-1}\mathcal{O}^T(u)\mathcal{O}(u)\breve{\mathcal{R}}^{-1}\mathcal{O}(u)\breve{v}(t_k) \end{aligned} \tag{58}$$

where $\mathcal{O}(u) = [0 \ E_3(u)]$. Adding and subtracting $(y(t_k) - \hat{y}(t_k))^T(y(t_k) - \hat{y}(t_k))$ to and from (57), one obtains

$$\begin{aligned}\delta V\big(\breve{x}(t_k)\big) &= (\mathrm{T}-2)\delta^T\big(\breve{x}(t_k)\big)P\delta\big(\breve{x}(t_k)\big) + 2\delta^T\big(\breve{x}(t_k)\big)PA_{cl}(u)\breve{x}(t_k)\\ &+ 2\delta^T\big(\breve{x}(t_k)\big)P\breve{B}_{cl}(u)\breve{\mathcal{R}}^{-1}\breve{v}(t_k) + \breve{x}^T(t_k)\big\{2A_{cl}(u)^T P\\ &+ \big[C(u) \quad -\hat{C}(u)\big]^T\big[C(u) \quad -\hat{C}(u)\big]\big\}\breve{x}(t_k) + \breve{v}^T(t_k)\breve{\mathcal{R}}^{-1}\breve{B}_{cl}^T(u)\\ &P\delta\breve{x}(t_k) + 2\breve{x}^T(t_k)\big\{P\breve{B}_{cl}(u) + \big[C(u) \quad -\hat{C}(u)\big]^T\mathcal{O}(u)\big\}\breve{\mathcal{R}}^{-1}\breve{v}(t_k)\\ &+ \breve{v}^T(t_k)\breve{\mathcal{R}}^{-1}\mathcal{O}^T(u)\mathcal{O}(u)\breve{\mathcal{R}}^{-1}\breve{v}(t_k)\\ &- \big(y(t_k) - \hat{y}(t_k)\big)^T\big(y(t_k) - \hat{y}(t_k)\big). \end{aligned} \tag{59}$$

By using the triangular inequality, determine an upper bound for the term $\breve{v}^T(t_k)$ $\breve{\mathcal{R}}^{-1}\breve{v}(t_k)$ as

$$\begin{aligned}\breve{v}^T(t_k)\breve{\mathcal{R}}^{-1}\breve{v}(t_k) &= \begin{bmatrix} F(x(t_k), t_k)H_1(u)x(t_k)\\ F(x(t_k), t_k)H_3(u)x(t_k)\end{bmatrix}^T \breve{\mathcal{R}} \begin{bmatrix} F(x(t_k), t_k)H_1(u)x(t_k)\\ F(x(t_k), t_k)H_3(u)x(t_k)\end{bmatrix}\\ &\le \alpha\rho^2\breve{x}^T(t_k)\big\{H_1^T(u)H_1(u) + H_3^T(u)H_3(u)\big\}\breve{x}(t_k).\end{aligned}$$

Then, we have

$$\breve{v}^T(t_k)\breve{\mathcal{R}}^{-1}v(t_k) \le \breve{x}^T(t_k)C_{cl}^T(u)C_{cl}(u)\breve{x}(t_k)$$

where

$$C_{cl}(u) = \rho^2\begin{bmatrix} H_1(u) & 0\\ H_3(u) & 0\end{bmatrix}.$$

Adding and subtracting $\breve{v}^T(t_k)\breve{\mathcal{R}}^{-1}\breve{v}(t_k)$ to and from (68), we obtain the following inequality:

$$\delta V\big(\breve{x}(t_k)\big) \le \bar{x}^{\mathrm{T}}(t_k)\begin{bmatrix} (\mathrm{T}-2)P & PA_{cl}(u) & P\breve{B}_{cl}(u) \\ * & \Pi_1 & \Pi_2 \\ * & * & -(\breve{\mathcal{R}} - \mathcal{O}^T(u)\mathcal{O}(u)) \end{bmatrix}\bar{x}(t_k)$$
$$- \big(y(t_k) - \hat{y}(t_k)\big)^T\big(y(t_k) - \hat{y}(t_k)\big)$$

where

$$\bar{x}(t_k) = \begin{bmatrix} \delta(\breve{x}(t_k)) \\ \breve{x}(t_k) \\ \breve{\mathcal{R}}^{-1}\breve{v}(t_k) \end{bmatrix},$$

$$\Pi_1 = A_{cl}^T(u)P + PA_{cl}(u) + \begin{bmatrix} C(u) & -\hat{C}(u) \end{bmatrix}^T\begin{bmatrix} C(u) & -\hat{C}(u) \end{bmatrix} + C_{cl}^{\mathrm{T}}(u)C_{cl}(u),$$
$$\Pi_2 = P\breve{B}_{cl}(u) + \begin{bmatrix} C(u) & -\hat{C}(u) \end{bmatrix}^T\mathcal{O}(u).$$

Following [4], without loss of generality, we partition P as

$$P = \begin{bmatrix} X & Y-X \\ Y-X & X-Y \end{bmatrix}. \tag{60}$$

Utilizing (60) and letting

$$\breve{x}(t_k) = \begin{bmatrix} \begin{bmatrix} 0 & I \\ I & -I \end{bmatrix}\delta\breve{x}(t_k) \\ \begin{bmatrix} 0 & I \\ I & -I \end{bmatrix}\breve{x}(t_k) \\ \breve{\mathcal{R}}^{-1}\breve{v}(t_k) \end{bmatrix},$$

we have the inequality

$$\delta V\big(\breve{x}(t_k)\big) \le \breve{x}^T(t_k)\begin{bmatrix} \breve{\Phi}_1(u) & \breve{\Phi}_2(u) & \breve{\Phi}_3(u) \\ * & \Phi_4(u) & \breve{\Phi}_5(u) \\ * & * & \breve{\Phi}_6(u) \end{bmatrix}\breve{x}(t_k)$$
$$- \big(y(t_k) - \hat{y}(t_k)\big)^T\big(y(t_k) - \hat{y}(t_k)\big)$$

where

$$\breve{\Phi}_1(u) = (\mathrm{T}-2)\begin{bmatrix} Y & Y \\ Y & X \end{bmatrix},$$

$$\breve{\Phi}_2(u) = \begin{bmatrix} YA(u) & YA(u) \\ XA(u) + \mathcal{B}(u)C(u) + (Y-X)\hat{A}(u) & XA(u) + \mathcal{B}(u)C(u) \end{bmatrix},$$

$$\breve{\Phi}_3(u) = \begin{bmatrix} YE_1(u) & 0 \\ XE_1(u) & \mathcal{B}(u)E_3(u) \end{bmatrix},$$

$$\breve{\Phi}_4(u) = \begin{bmatrix} YA(u) + A^T(u)Y + \rho^2[H_1^T(u)H_1(u) + H_3(u)^T H_3(u)] & (*)^T \\ \mathcal{A}(u) & \Phi_4(2,2) \end{bmatrix},$$

$$\breve{\Phi}_5(u) = \begin{bmatrix} YE_1(u) & 0 \\ XE_1(u) & \mathcal{B}(u)E_3(u) + C^T(u)E_3(u) \end{bmatrix},$$

$$\breve{\Phi}_6(u) = -\big(\breve{\mathcal{R}} - \mathcal{O}^{\mathbf{T}}(u)\mathcal{O}(u)\big)$$

with

$$\begin{aligned}\breve{\Phi}_4(2,2) = {} & A^T(u)X + XA(u) + \mathcal{B}(u)C(u) + C^T(u)\mathcal{B}^T(u) + C^T(u)C(u) \\ & + \alpha\rho^2\big[H_1^T(u)H_1(u) + H_3^T(u)H_3(u)\big],\end{aligned}$$

and

$$\begin{aligned}\mathcal{A}(u) &= XA(u) + \mathcal{B}(u)C(u) + (Y - X)\hat{A}(u) + A^T(u)Y \\ &\quad + \alpha\rho^2\big[H_1^T(u)H_1(u) + H_3^T(u)H_3(u)\big], \\ \mathcal{B}(u) &= (Y - X)\hat{B}(u), \qquad \mathcal{C}(u) = \hat{C}(u).\end{aligned}$$

Note that

$$\begin{bmatrix} \Phi_1(u) & \breve{\Phi}_2(u) & \breve{\Phi}_3(u) \\ * & \breve{\Phi}_4(u) & \breve{\Phi}_5(u) \\ * & * & \breve{\Phi}_6(u) \end{bmatrix} = \sum_{i=1}^{r}\sum_{j=1}^{r} u_i u_j \begin{bmatrix} \breve{\Phi}_{1ij} & \breve{\Phi}_{2ij} & \breve{\Phi}_{3ij} \\ * & \breve{\Phi}_{4ij} & \breve{\Phi}_{5ij} \\ * & * & \breve{\Phi}_{6ij} \end{bmatrix}.$$

Considering (52)–(54), we have that

$$\delta V\big(\breve{x}(t_k)\big) \le 0.$$

Hence, (51) is asymptotically stable. □

5 Filter Algorithm

The value γ is very useful for threshold selection in detection decision-making. The ratio β/γ indicates how good a designed fault detection filter is, and therefore can be used for evaluation of fault detection filters. As will be shown, the fault detection is equivalent to a constrained H_∞ estimation problem, the latter can be further reformulated as a standard problem of constrained optimization. Thus, we give the following algorithm:

Algorithm Given a scalar β, search for the lowest possible value of γ making the error delta operator dynamic system (13)–(14) asymptotically stable and formulate the following convex optimization problem:

$$\min\ \gamma$$
$$\text{s.t.}\quad \begin{cases} \text{(i)} & (20)\text{–}(22); \\ \text{(ii)} & (38)\text{–}(40); \\ \text{(iii)} & (52)\text{–}(54) \end{cases}$$

which can be effectively solved by the existing Matlab LMI toolbox.

6 Numerical Example

In the following, we will provide a numerical example to demonstrate the effectiveness of the proposed methods in this paper.

Example: Consider the following two-rules T-S fuzzy model.

Rule 1: IF $x_1(t_k)$ is M_1, THEN

$$\delta x(t_k) = [A_1 + \Delta A_1]x(t_k) + Bw(t_k) + Gf(t_k),$$
$$y(t_k) = Cx(t_k) + Dw(t_k) + Jf(t_k).$$

Rule 2: IF $x_2(t_k)$ is M_2, THEN

$$\delta x(t_k) = [A_2 + \Delta A_2]x(t_k) + Bw(t_k) + Gf(t_k),$$
$$y(t_k) = Cx(t_k) + Dw(t_k) + Jf(t_k)$$

where

$$A_1 = \begin{bmatrix} -0.134 & 0.006 \\ 0.007 & -0.121 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -0.162 & 0.008 \\ 0.005 & -0.148 \end{bmatrix},$$
$$B = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

with $D = 0.1$, $G = 0.5$, $J = 0.3$, $\alpha = 1.4$, $\mathrm{T} = 0.02$, $\Delta A_1 = E_{1_1}F(x(t_k), t_k)H_{1_1}$, $\Delta A_2 = E_{1_2}F(x(t_k), t_k)H_{1_2}$. Assume $\|F(x(t_k), t_k)\| \le \rho = 1$ and

$$E_{1_1} = E_{1_2} = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0 \end{bmatrix}, \qquad H_{1_1} = H_{1_2} = \begin{bmatrix} 0 & 0.2 \\ 0 & 0 \end{bmatrix},$$

and that the membership functions for rules 1 and 2 are

$$M_1\big(x_1(t_k)\big) = \frac{1}{1 + \exp(-2x_1(t_k))}, \qquad M_2\big(x_1(t_k)\big) = 1 - M_1\big(x_1(t_k)\big).$$

To analyze the effects of fault and disturbance on the residual of the detection observer, consider the stuck fault, e.g.,

$$f(t_k) = \begin{cases} 0.05, & k > 100, \\ 0, & \text{elsewhere.} \end{cases}$$

Let the disturbance be

$$w(t_k) = 10 \cdot \begin{bmatrix} 1.6\cos(0.02k)e^{-0.05k} + 0.16\sin(0.02k) \\ 1.6\cos(0.02k)e^{-0.05k} + 0.16\sin(0.02k) \end{bmatrix}^T.$$

Using the LMI optimization Algorithm 1 and Theorems 1–3, we obtain

$$X = \begin{bmatrix} 4.1643 & 0.0573 \\ 0.0573 & 4.1451 \end{bmatrix}, \qquad Y = \begin{bmatrix} 1.4245 & -0.0362 \\ -0.0362 & 1.4067 \end{bmatrix},$$

$$\hat{A}_{11} = \begin{bmatrix} 0.2896 & 0.0489 \\ 0.0503 & 0.3023 \end{bmatrix}, \qquad \hat{A}_{22} = \begin{bmatrix} 0.2651 & 0.0570 \\ 0.0587 & 0.2712 \end{bmatrix},$$

$$\hat{A}_{12} = \begin{bmatrix} 0.2126 & 0.0320 \\ 0.0304 & 0.2103 \end{bmatrix}, \qquad \hat{A}_{21} = \begin{bmatrix} 0.1006 & 0.2319 \\ 0.2394 & 0.1212 \end{bmatrix},$$

$$\hat{B}_1 = \begin{bmatrix} 2.0965 & -0.6207 \\ -1.5345 & 3.0387 \end{bmatrix}, \qquad \hat{B}_2 = \begin{bmatrix} 0.7572 & 0.4597 \\ 1.0894 & 0.1570 \end{bmatrix},$$

$$\hat{C}_1 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \qquad \hat{C}_2 = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}.$$

The resulting fuzzy filter is

$$\delta\hat{x}(t_k) = \sum_{i=1}^{2}\sum_{j=1}^{2} u_i u_j \big[\hat{A}_{ij}\hat{x}(t_k) + \hat{B}_i y(t_k)\big],$$

$$\hat{y}(t_k) = \sum_{i=1}^{2} u_i \hat{C}_i \hat{x}(t_k)$$

where $u_1 = M_1(x_1(t_k))$ and $u_2 = M_2(x_1(t_k))$.

Considering the fact that a real state vector in the fuzzy system can be replaced by an estimated state vector using the fault detection observer obtained in Theorems 1–3, we first give the simulation results of the state estimate responses of the fuzzy system in this example for the initial conditions $\hat{x}_1(0) = \hat{x}_2(0) = 0$, shown in Fig. 1, where $\hat{x}_1(t_k)$ and $\hat{x}_2(t_k)$ are denoted by $xo_1(t_k)$ and $xo_2(t_k)$, respectively. For the initial condition $y(0) = 0$, the simulation result of the estimated output of fuzzy system in this example is shown in Fig. 2, where $\hat{y}(t_k)$ is denoted by $yo(t_k)$. Then, the residual outputs are shown in Fig. 3 with the initial condition $r(0) = 0$, from which we can see that the faults are well discriminated from disturbances. To detect the fault, we choose the residual evaluation function as stated in (15), and the residual evaluation output is shown in Fig. 4, where $Jr(n)$ and J_{th} are denoted by J_{rn} and J_{th}, respectively.

7 Conclusion

This paper has presented a new approach to study the problem of fault detection for the T-S fuzzy systems in the delta domain. We have constructed a

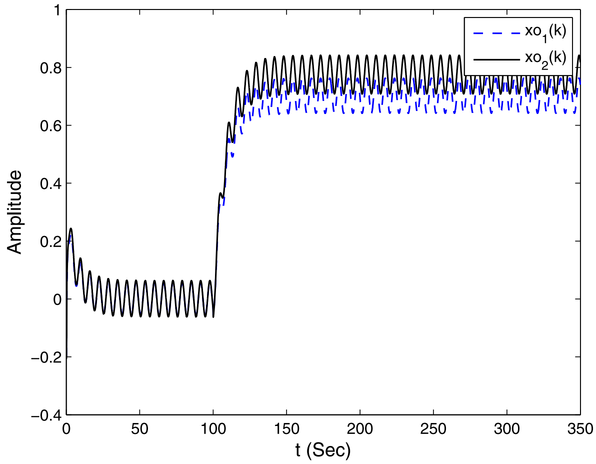

Fig. 1 State estimate response of $\hat{x}(t_k)$

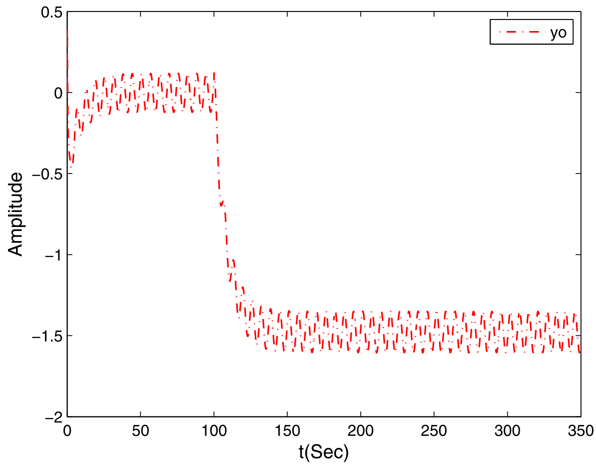

Fig. 2 Estimated output of $\hat{y}(t_k)$

fuzzy fault detection filter system and dynamics of filtering error generator by means of the T-S fuzzy model. The worst case fault sensitivity has been formulated in terms of LMIs, which can be effectively solved by an algorithm proposed. The existence of a robust fault detection system that guarantees (i) the H_--gain from a fault signal to a residual signal greater than a prescribed value and (ii) the H_--gain from an exogenous input to a residual signal less than a prescribed value is given in terms of the solvability of LMI. A numerical example has been given to illustrate the effectiveness and potential of the developed techniques.

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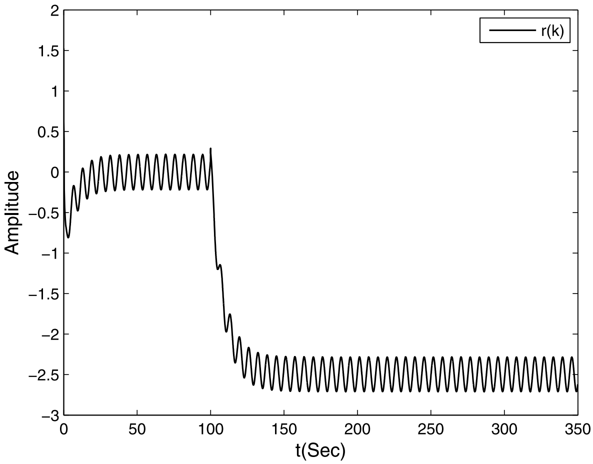

Fig. 3 Residual output $r(k)$

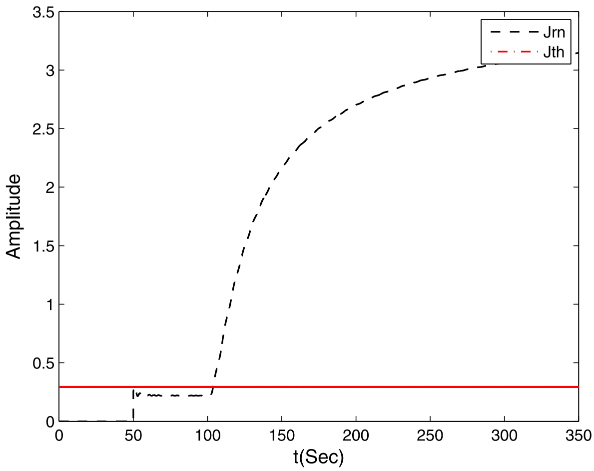

Fig. 4 Residual evaluation $Jr(n)$

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