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Closed-Form Analytical Expression of Fractional Order Differentiation in Fractional Fourier Transform Domain

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Abstract In this paper, a closed-form analytical expression for fractional order differentiation in the fractional Fourier transform (FrFT) domain is derived by utilizing the basic principles of fractional order calculus. The reported work is a generalization of the differentiation property to fractional (noninteger or real) orders in the FrFT domain. The proposed closed-form analytical expression is derived in terms of the well-known confluent hypergeometric function. An efficient computation method has also been derived for the proposed algorithm in the discrete-time domain, utilizing the principles of the discrete fractional Fourier transform algorithm. An application example of a low-pass finite impulse response (FIR) fractional order differentiator in the FrFT domain has also been investigated to show the practicality of the proposed method in signal processing applications.

Keywords Fractional order calculus · Fractional order differentiation · Fractional Fourier transform · Grünwald–Letnikov fractional derivative · Kummer confluent hypergeometric function · Riemann–Liouville fractional derivative

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1 Introduction

The differentiation operation in signal processing has proven to be a very useful mathematical operator to determine and estimate the time derivative of a given signal [3]. Digital differentiators form an integral part of many physical systems. For example, digital low-pass differentiation is often required in processing various biological or biomechanical data [20], in image processing, the edge of an image can be detected by using the differential operation [5], and in radar and sonar, the velocity and acceleration can be computed from the position measurements using differentiators [17]. An excellent survey on the state of the art of the differentiation operation in signal processing and its applications was given in [3, 5, 10, 17–20].

In recent years, the concepts of fractional order operators have been investigated extensively in science and engineering applications [10], including the design of fractional order digital differentiators, which have received much attention in the research community. Also, there has been a surge of research in signal processing following the introduction of the fractional Fourier transform (FrFT) by Namias [9].

In the research area of fractional order calculus (FOC), the integer order n of the derivative $D^n p(x) = d^n p(x)/dx^n$ of the function $p(x)$ is generalized to the fractional order '$D^\mu p(x)$', where μ is a real number [10]. One of the important research issues in FOC is to implement the fractional order operator D^μ in continuous and discrete-time domains.

Since fractional order differentiation is the theme of this paper, we can emphasize that on two occasions including the work of [7] and [18], the differentiation property was independently extended to the class of the Fourier transform (FT) and FrFT respectively, but it was not extended to the noninteger orders.

In this paper, the fractional order differentiation of a given signal in the FrFT for different fractional orders is proposed by utilizing the inherent approach of fractional operators of FOC. The concept behind the study is that it involves two different variable parameters: the fractional order parameter μ and the fractional Fourier parameter φ. These two parameters have not been involved in any of the literature so far. In the context of [10], the FOC generalizes the derivative operator D^μ by encompassing real and complex values for the exponent μ, which is ordinarily integer valued. Derivatives of noninteger order have been considered in physics, engineering, and in the signal processing area [6, 11], following the work of Liouville and Riemann at the beginning of the nineteenth century.

The idea of this study was motivated by the work of Pei et al. [18] and McBride et al. [7]. The main differences between the proposed method and the work of Pei et al. [18] are as follows. (i) Pei et al. used the Cauchy integral formula and generalized it to define the fractional derivative of the functions, whereas the Riemann–Liouville (RL) definition for the general fractional differintegral is used in the proposed method. (ii) The aim of Pei et al. [18] was to obtain the fractional derivative using the FT, whereas in the proposed method the aim is to obtain the fractional derivative using the FrFT. Similarly, McBride et al. [7] derive the differentiation property in the FrFT to integer order only, whereas the proposed method generalizes it to obtain the differentiation property in the FrFT to noninteger orders. Therefore, the outcome of this study, "*establishing a closed-form expression for fractional differentiation in the FrFT domain*," is novel and unique.

The rest of this paper is organized as follows. Section 2 presents the proposed method of computing the fractional derivative in the FrFT domain. Section 3 presents an efficient calculation of the proposed algorithm along with a new fractional order differentiating filter model in the FrFT domain. An application example along with the simulation results have been presented in Sect. 4. Some concluding remarks are given in Sect. 5.

2 Computation of Fractional Derivative in Fractional Fourier Transform

Fractional order calculus (FOC), which is an extension of noninteger order derivatives and integrals, has received great attention in the last few decades, because of its ability to model systems more accurately than integer order calculus [10, 14].

FOC is a generalization of integration and differentiation to a fractional, or noninteger order fundamental operator ${}_lD_t^{\mu}$, where l and t are the lower/upper bounds of integration and μ is the order of the operation.

$$
{}_lD_t^{\mu} = \begin{cases} \frac{d^{\mu}}{dt^{\mu}} & \mathbb{R}(\mu) > 0 \\ 1 & \mathbb{R}(\mu) = 0 \\ \int_l^t (d\tau)^{(-\mu)} & \mathbb{R}(\mu) < 0 \end{cases} \tag{1}
$$

where $\mathbb{R}(\mu)$ is the real part of μ. The most frequently used equivalent definitions for the general fractional differintegral are the Riemann–Liouville (RL), the Grünwald–Letnikov (GL), and the Caputo definitions [2, 10].

The FrFT is a generalization of the ordinary Fourier transform with a fractional Fourier order parameter a, which corresponds to the ath fractional power of the Fourier transform operator, $\mathcal{F}$. The ath-order FrFT of $x(t)$ is defined as

$$
Ŧ_{\mathcal{F}}^{\varphi}\big(x(t)\big) = X(u_{\varphi}) = \int_{-\infty}^{\infty} x(t) K_{\varphi}(t, u_{\varphi})\, dt \tag{2}
$$

where $0 < |a| < 2$, the transformation kernel,

$$
K_{\varphi}(t, u_{\varphi}) = \sqrt{\frac{1 - i\cot\varphi}{2\pi}} \exp\left[i\left(\frac{t^2 + u_{\varphi}^2}{2}\right)\cot\varphi - i u_{\varphi} t \csc\varphi \right] \tag{3}
$$

with the transform angle $\varphi = a\pi/2$ [12], and $Ŧ_{\mathcal{F}}^{\varphi}$ denotes the CFrFT operator. The FrFT has several applications in the areas of signal processing [1, 4, 7, 9, 12, 15, 16, 21]. The ath fractional Fourier domain makes an angle $\varphi = a\pi/2$ with the time domain in the time–frequency plane, as shown in Fig. 1.

2.1 Derivation of Closed-Form Analytical Expression for Computing Fractional Derivative of a Signal in FrFT Domain

To derive the closed-form analytical expression of the fractional derivative of the signal in the FrFT domain, the inherent approach of the FOC has been utilized. Our FOC approach is confined to the RL definition for the general fractional differintegral [14].

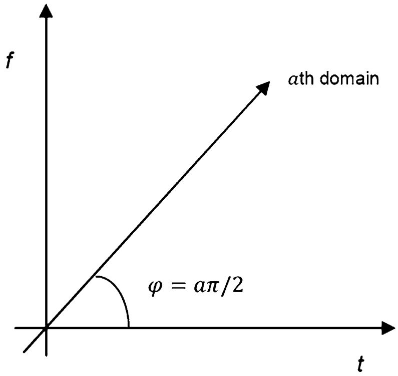


Fig. 1 The ath fractional Fourier domain

Let D_+^μ and D_-^μ be the left and right RL fractional derivatives of order μ on the real axis, defined by

$$D_+^\mu x(t) = \frac{d}{dt}\left\{I_+^{1-\mu} x(t)\right\} \tag{4}$$

where I_+^μ is the RL fractional integral operator,

$$I_+^\mu x(t) = \frac{1}{\Gamma(\mu)} \int_{-\infty}^{t} (t-\tau)^{\mu-1} x(\tau)\, d\tau = \frac{t^{\mu-1}}{\Gamma(\mu)} * x(t) \tag{5}$$

Here, $\Gamma(\cdot)$ is the well-known Euler's gamma function, and $\mu \in \mathbb{R}$ $(0 < \mu < 1)$. The operator "$*$" represents the convolution operation between the two signals of interest, here $\frac{t^{\mu-1}}{\Gamma(\mu)}$ and $x(t)$, respectively.

$$D_-^\mu x(t) = \left(-\frac{d}{dt}\right)\left\{I_-^{1-\mu} x(t)\right\} \tag{6}$$

where I_-^μ is the RL fractional integral operator,

$$I_-^\mu x(t) = \frac{1}{\Gamma(\mu)} \int_{t}^{+\infty} (t-\tau)^{\mu-1} x(\tau)\, d\tau \tag{7}$$

In this paper, by considering D_+^μ the RL fractional derivative operator and $0 < \mu < 1$, the following expression is obtained from (4) and (5):

$$D_+^\mu x(t) = \frac{d}{dt}\left\{\frac{t^{-\mu}}{\Gamma(1-\mu)} * x(t)\right\} \tag{8}$$

Therefore, taking the FrFT of the fractional derivative of (8) results in the following expression:

$$\mathfrak{F}_{\mathcal{F}}^{\varphi}\left\{D_+^\mu x(t)\right\} = \mathfrak{F}_{\mathcal{F}}^{\varphi}\left[\frac{d}{dt}\left\{\frac{t^{-\mu}}{\Gamma(1-\mu)} * x(t)\right\}\right] \tag{9}$$

By letting $b(t) = \frac{t^{-\mu}}{\Gamma(1-\mu)} * x(t)$, (9) becomes

$$\mathfrak{F}_{\mathcal{F}}^{\varphi}\left\{D_+^\mu x(t)\right\} = \mathfrak{F}_{\mathcal{F}}^{\varphi}\left[\frac{db(t)}{dt}\right] \tag{10}$$

Now, according to the differentiation property of the FrFT [1],

$$\mathfrak{F}_{\mathcal{F}}^{\varphi}\left[\frac{db(t)}{dt}\right]=\left[ju_\varphi\sin\varphi+\cos\varphi\cdot\frac{d}{du_\varphi}\right]B(u_\varphi) \tag{11}$$

where $B(u_\varphi)=\mathfrak{F}_{\mathcal{F}}^{\varphi}(b(t))$ is the FrFT of the signal $b(t)$.

Therefore,

$$B(u_\varphi)=\mathfrak{F}_{\mathcal{F}}^{\varphi}\big[b(t)\big]=\mathfrak{F}_{\mathcal{F}}^{\varphi}\left[\left\{\frac{t^{-\mu}}{\Gamma(1-\mu)}*x(t)\right\}\right] \tag{12}$$

Now, from the convolution property of the FrFT [15] and [21], the above expression reduces to

$$B(u_\varphi)=\sqrt{\frac{2\pi}{1-j\cot\varphi}}\cdot\exp\left[-\frac{j}{2}u_\varphi^2\cot\varphi\right]\cdot\mathfrak{F}_{\mathcal{F}}^{\varphi}\big[x(t)\big]\cdot\mathfrak{F}_{\mathcal{F}}^{\varphi}\left[\frac{t^{-\mu}}{\Gamma(1-\mu)}\right]$$

Thus,

$$B(u_\varphi)=\sqrt{\frac{2\pi}{1-j\cot\varphi}}\cdot\exp\left[-\frac{j}{2}u_\varphi^2\cot\varphi\right]\cdot X(u_\varphi)\cdot\mathfrak{F}_{\mathcal{F}}^{\varphi}\left[\frac{t^{-\mu}}{\Gamma(1-\mu)}\right] \tag{13}$$

where $X(u_\varphi)$ is the FrFT of the signal $x(t)$, i.e., $X(u_\varphi)=\mathfrak{F}_{\mathcal{F}}^{\varphi}[x(t)]$.

$$\therefore\ B(u_\varphi)=X(u_\varphi)\cdot\int_{-\infty}^{\infty}\frac{t^{-\mu}}{\Gamma(1-\mu)}\cdot\exp\left[\frac{j}{2}t^2\cot\varphi-ju_\varphi t\csc\varphi\right]\cdot dt \tag{14}$$

From [8, Eq. (A.1.55)],

$$\int_{-\infty}^{\infty}t^\gamma\exp\big[\pm jbt-c^2t^2\big]dt=\frac{\pi j^{-\gamma}}{c^{\gamma+1}}\left[\frac{{}_1F_1(\frac{\gamma+1}{2};\frac{1}{2};\frac{-b^2}{4c^2})}{\Gamma(\frac{1-\gamma}{2})}\pm\frac{b}{c}j\frac{{}_1F_1(\frac{\gamma+2}{2};\frac{3}{2};\frac{-b^2}{4c^2})}{\Gamma(\frac{-\gamma}{2})}\right] \tag{15}$$

The expression on the right-hand side of (15) involves the function ${}_1F_1$, which is known as the Kummer confluent hypergeometric function (CHF) of the first kind [2], which is an infinite power series. To compute the Kummer CHF using a computing machine, the series must be truncated to some finite number of terms. So, if the series truncation is used, a computation error must exist. Abramowitz and Stegun [2] provide the methodology for determining the truncation error of an infinite power series. Figure 2 shows the variation of the relative error (in percentages) after truncating an infinite power series for different CHF functions (for different a's and b's); clearly, the truncation error decreases to zero pointwise, as the number of terms increases.

Solving (14) and (15) step by step, and letting $\gamma=-\mu$, $b=u_\varphi\csc\varphi$, $c^2=\frac{-j}{2}\cot\varphi$, i.e., $c=\frac{(1-j)}{2}\sqrt{\cot\varphi}$ and $\frac{-b^2}{4c^2}=-ju_\varphi^2\csc(2\varphi)$, (15) becomes

$$\int_{-\infty}^{\infty}t^{-\mu}\exp\left[-ju_\varphi t\csc\varphi+\frac{j}{2}t^2\cot\varphi\right]dt=\frac{\pi j^\mu(1+j)^{1-\mu}}{(\cot\varphi)^{(1-\mu)/2}}\left[\frac{{}_1F_1\big(\frac{1-\mu}{2};\frac{1}{2};-ju_\varphi^2\csc(2\varphi)\big)}{\Gamma\big(\frac{1+\mu}{2}\big)}\right.$$

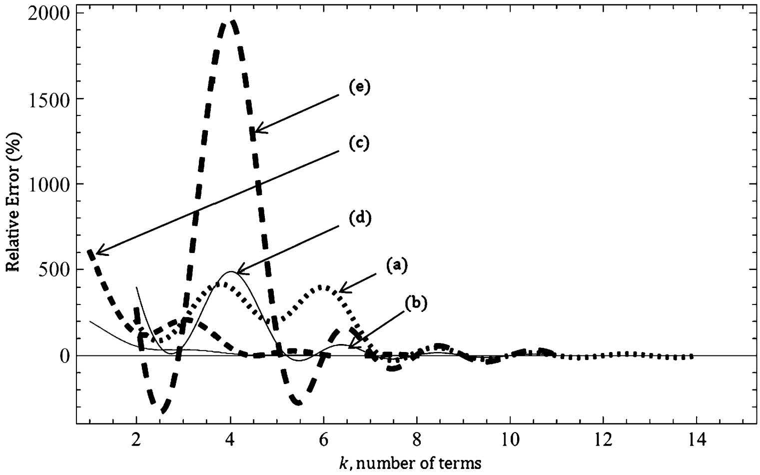


Fig. 2 Variation of the relative error (in percentages) with the number of terms, k of the following CHF functions: (a) ${}_1F_1(a;a;-x) = \exp(-x)$, (b) ${}_1F_1(\frac{1}{2};\frac{3}{2};-x) = \frac{1}{2}\sqrt{\frac{\pi}{x}}\,\mathrm{erf}(\sqrt{x})$, (c) ${}_1F_1(\frac{3}{2};\frac{5}{2};-x) = \frac{3}{2}[\frac{1}{2}\sqrt{\frac{\pi}{x}}\,\mathrm{erf}(\sqrt{x}) - e^{-x}]$, (d) ${}_1F_1(1;\frac{3}{2};-x) = \frac{1}{2}\sqrt{\frac{\pi}{x}}\,\mathrm{erfi}(\sqrt{x})e^{-x}$, (e) ${}_1F_1\ (2;\frac{5}{2};-x) = \frac{3\sqrt{\pi}\cdot e^{-x}\cdot(1+2x)\cdot\mathrm{erfi}\sqrt{x}}{8x^{3/2}} - \frac{3}{4x}$, by letting $x = 2$

$$+\, ju_{\varphi}(1+j)\sqrt{2\csc(2\varphi)}\frac{{}_1F_1\left(\frac{2-\mu}{2};\frac{3}{2};-ju_{\varphi}^2\csc(2\varphi)\right)}{\Gamma\left(\frac{\mu}{2}\right)}\Bigg] \tag{16}$$

Thus, it can be seen that the integral representation (16) is a generalized expression in terms of the fractional order parameter μ, and hence the closed-form expression for the integral representation (16) can be obtained by considering different values of the parameter μ, respectively.

Now, from (14) and (16), the following expression results:

$$\begin{aligned} B(u_{\varphi}) = X(u_{\varphi})\cdot\frac{\pi j^{\mu}(1+j)^{1-\mu}}{(\cot\varphi)^{(1-\mu)/2}}\Bigg[&\frac{{}_1F_1\left(\frac{1-\mu}{2};\frac{1}{2};-ju_{\varphi}^2\csc(2\varphi)\right)}{\Gamma\left(\frac{1+\mu}{2}\right)} \\ &+ ju_{\varphi}(1+j)\sqrt{2\csc(2\varphi)}\cdot\frac{{}_1F_1\left(\frac{2-\mu}{2};\frac{3}{2};-ju_{\varphi}^2\csc(2\varphi)\right)}{\Gamma\left(\frac{\mu}{2}\right)}\Bigg] \end{aligned} \tag{17}$$

Now, by letting $K(\mu,\varphi) = \frac{\pi(j^{\mu})(1+j)^{1-\mu}}{(\cot\varphi)^{(1-\mu)/2}}$ and $M(\varphi) = j(1+j)\sqrt{2\csc(2\varphi)}$, (17) becomes

$$\begin{aligned} B(u_{\varphi}) = X(u_{\varphi})\cdot K(\mu,\varphi)\Bigg[&\frac{{}_1F_1\left(\frac{1-\mu}{2};\frac{1}{2};-ju_{\varphi}^2\csc(2\varphi)\right)}{\Gamma\left(\frac{1+\mu}{2}\right)} \\ &+ u_{\varphi}\cdot M(\varphi)\cdot\frac{{}_1F_1\left(\frac{2-\mu}{2};\frac{3}{2};-ju_{\varphi}^2\csc(2\varphi)\right)}{\Gamma\left(\frac{\mu}{2}\right)}\Bigg] \end{aligned} \tag{18}$$

Now, from (10) and (11),

$$\begin{aligned}\mathfrak{F}_{\mathcal{F}}^{\varphi}\left\{D_{+}^{\mu}x(t)\right\} &= \left(ju_{\varphi}\sin\varphi + \cos\varphi\cdot\frac{d}{du_{\varphi}}\right)B(u_{\varphi}) \\ &= ju_{\varphi}\sin\varphi\cdot B(u_{\varphi}) + \cos\varphi\cdot\frac{dB(u_{\varphi})}{du_{\varphi}}\end{aligned} \tag{19}$$

Therefore, to solve (19), one has to determine $\frac{dB(u_{\varphi})}{du_{\varphi}}$. By knowing the fact that [2]

$$\frac{d}{dz}{}_1F_1(a;b;z) = \left(\frac{a}{b}\right){}_1F_1(a+1;b+1;z) \tag{20}$$

and solving for $\frac{dB(u_{\varphi})}{du_{\varphi}}$, the following expression results:

$$\begin{aligned}\frac{dB(u_{\varphi})}{du_{\varphi}} = {}& \frac{K(\mu,\varphi)}{\Gamma(\frac{1+\mu}{2})}\left[{}_1F_1\left(\frac{1-\mu}{2};\frac{1}{2};-ju_{\varphi}^2\csc(2\varphi)\right)\cdot\frac{dX(u_{\varphi})}{du_{\varphi}}\right. \\ & \left.+(1-\mu)\cdot X(u_{\varphi})\cdot{}_1F_1\left(\frac{3-\mu}{2};\frac{3}{2};-ju_{\varphi}^2\csc(2\varphi)\right)\right] \\ & +\frac{K(\mu,\varphi)\cdot M(\varphi)}{\Gamma(\frac{\mu}{2})}\left[u_{\varphi}\cdot X(u_{\varphi})\cdot\left(\frac{2-\mu}{3}\right)\right. \\ & \times{}_1F_1\left(\frac{4-\mu}{2};\frac{5}{2};-ju_{\varphi}^2\csc(2\varphi)\right) \\ & +X(u_{\varphi})\cdot{}_1F_1\left(\frac{2-\mu}{2};\frac{3}{2};-ju_{\varphi}^2\csc(2\varphi)\right) \\ & \left.+u_{\varphi}\cdot{}_1F_1\left(\frac{2-\mu}{2};\frac{3}{2};-ju_{\varphi}^2\csc(2\varphi)\right)\cdot\frac{dX(u_{\varphi})}{du_{\varphi}}\right]\end{aligned} \tag{21}$$

Now, by expressing the following CHFs by the corresponding functions as

$${}_1F_1\left(\frac{1-\mu}{2};\frac{1}{2};-ju_{\varphi}^2\csc(2\varphi)\right) = H_1(\mu,\varphi,u_{\varphi}) \tag{22a}$$

$${}_1F_1\left(\frac{2-\mu}{2};\frac{3}{2};-ju_{\varphi}^2\csc(2\varphi)\right) = H_2(\mu,\varphi,u_{\varphi}) \tag{22b}$$

$${}_1F_1\left(\frac{3-\mu}{2};\frac{3}{2};-ju_{\varphi}^2\csc(2\varphi)\right) = H_3(\mu,\varphi,u_{\varphi}) \tag{22c}$$

$${}_1F_1\left(\frac{4-\mu}{2};\frac{5}{2};-ju_{\varphi}^2\csc(2\varphi)\right) = H_4(\mu,\varphi,u_{\varphi}) \tag{22d}$$

Then, from (19)–(22d),

$$\begin{aligned}&\mathfrak{F}_{\mathcal{F}}^{\varphi}\left\{D_{+}^{\mu}x(t)\right\} \\ &\quad= K(\mu,\varphi)\cdot M(\varphi)\left\{\frac{1}{M(\varphi)}\cdot\frac{1}{\Gamma\left(\frac{(1+\mu)}{2}\right)}\cdot H_1(\mu,\varphi,u_{\varphi})\right. \\ &\qquad\times\left[ju_{\mu}\sin\varphi + \cos\varphi\cdot\frac{d}{du_{\varphi}}\right] \\ &\qquad+u_{\varphi}\cdot\frac{1}{\Gamma\left(\frac{\mu}{2}\right)}\cdot H_2(\mu,\varphi,u_{\varphi})\cdot\left[ju_{\varphi}\sin\varphi + \cos\varphi\cdot\frac{d}{du_{\varphi}}\right]\end{aligned}$$

$$+\frac{1}{\Gamma(\frac{\mu}{2})}\cdot\cos\varphi\cdot H_2(\mu,\varphi,u_\varphi)+\frac{1}{M(\varphi)}\cdot\frac{(1-\mu)\cdot\cos\varphi\cdot}{\Gamma((1+\mu)/2)}\cdot H_3(\mu,\varphi,u_\varphi)$$
$$+\left(\frac{2-\mu}{3}\right)\cdot\frac{1}{\Gamma(\mu/2)}\cdot\cos\varphi\cdot H_4(\mu,\varphi,u_\varphi)\cdot u_\varphi\biggr\}\cdot X(u_\varphi) \tag{23}$$

Thus, the above expression gives the fractional derivative of the input signal $x(t)$ for fractional orders varying from 0 to 1 and for different rotation angles (φ) in the time–frequency plane of the FrFT.

3 Efficient Calculation of the Proposed Algorithm

In this section, we have derived an efficient algorithm to compute a discrete counterpart of the proposed relation (23). There exist various fast discrete-time versions of the continuous FrFT, namely, the direct form of DFrFT, improved sampling-type DFrFT, linear combination-type DFrFT, eigenvectors decomposition-type DFrFT, group theory-type DFrFT, and impulse train-type DFrFT [12].

The discrete FrFT algorithm proposed in [12] has a very important advantage that it is efficient to calculate and implement. Because there are two chirp multiplications and one FFT, the total number of multiplications required is $\{2P+(P/2)\cdot\log_2 P\}$, where $P=2M+1$ is the length of the output. The DFrFT introduced in [12] has the lowest complexity among all the types of DFrFT that still work similarly to the continuous FrFT. Thus, utilizing the DFrFT proposed in [12], a discrete-time calculation of the fractional order derivative of a discrete-time signal can be realized. In this method, the fractional order derivative of a continuous-time input signal $x(t)$ is evaluated in discrete time by using the following steps.

First, uniformly sample the input function $x(t)$ and the output function $\mathfrak{T}^{\varphi}_{\mathcal{F}}(x(t))=X(u_\varphi)$ by the interval Δt, Δu_φ respectively as

$$g(n)=x(n\cdot\Delta t)\qquad G_\varphi(m)=X(m\cdot\Delta u_\varphi) \tag{24}$$

where $n=-N,-N+1,\dots,N-1,N$, and $m=-M,-M+1,\dots,M-1,M$.

Additionally, the constraints $M\ge N$ ($2N+1$, $2M+1$ are the number of points in the time, frequency domain), and

$$\Delta u_\varphi\cdot\Delta t=S\cdot 2\pi\cdot\sin\varphi/(2M+1)$$

must also be satisfied, where $|S|$ is some integer prime to $2M+1$.

For simplicity, choose $S=\mathrm{sgn}(\sin\varphi)=1$ and obtain the transformation matrix as

$$R_\varphi(m,n)=\sqrt{\frac{\mathrm{sgn}(\sin\varphi)\cdot(\sin\varphi-j\cos\varphi)}{2M+1}}\cdot e^{\frac{j}{2}\cdot\cot\varphi\cdot m^2\cdot(\Delta u_\varphi)^2}\cdot e^{-j\cdot\frac{\mathrm{sgn}(\sin\varphi)\cdot 2\pi\cdot n\cdot m}{2M+1}}$$
$$\cdot e^{\frac{j}{2}\cdot\cot\varphi\cdot n^2\cdot(\Delta t)^2} \tag{25}$$

Considering only the case for $\sin\varphi>0$, the following formula for the DFrFT is obtained:

$$G_\varphi(m)=\sqrt{\frac{\sin\varphi-j\cos\varphi}{2M+1}}\cdot e^{\frac{j}{2}\cdot\cot\varphi\cdot m^2\cdot(\Delta u_\varphi)^2}\cdot\sum_{n=-N}^{N}e^{-j\cdot\frac{2\pi\cdot n\cdot m}{2M+1}}$$
$$\cdot e^{\frac{j}{2}\cdot\cot\varphi\cdot n^2\cdot(\Delta t)^2}\cdot g(n) \tag{26}$$

when $\varphi \in 2P\pi + (0, \pi)$, P is an integer.

Now, the evaluation of the fractional derivative of the discrete-time signal in the DFrFT domain is described in the following paragraph.

We consider here the GL definition [10, 14] of computing the fractional derivative, based on the generalization of the backward difference as

$$D^{\mu}x(t) = \frac{d^{\mu}x(t)}{dt^{\mu}} = \lim_{\Delta t \to 0} \sum_{k=0}^{\infty} \frac{(-1)^k \tilde{A}_k^{\mu}}{(\Delta t)^{\mu}} x(t - k \cdot \Delta t) \tag{27}$$

where coefficient $\tilde{A}_k^{\mu}$ is given by

$$\tilde{A}_k^{\mu} = \frac{\Gamma(\mu + 1)}{\Gamma(k+1)\Gamma(\mu - k + 1)} = \begin{cases} 1 & k = 0 \\ \frac{\mu(\mu-1(\mu-2)\cdots(\mu-k+1))}{1\cdot 2\cdot 3\cdots k} & k \geq 1 \end{cases} \tag{28}$$

The above notation $\Gamma(\cdot)$ is the gamma function. Based on this definition, it can be shown that the fractional derivatives of exponential, trigonometric, and power functions (assuming they are sufficiently large) are given by

$$D^{\mu}\big[\exp(\beta t)\big] = \beta^{\mu} \exp(\beta t) \tag{29a}$$

$$D^{\mu}\big[\hat{A}\sin(\omega t + \theta)\big] = \hat{A}\omega^{\mu} \sin\left(\omega t + \theta + \frac{\pi}{2}\mu\right) \tag{29b}$$

$$D^{\mu}\big[\hat{A}\cos(\omega t + \theta)\big] = \hat{A}\omega^{\mu} \cos\left(\omega t + \theta + \frac{\pi}{2}\mu\right) \tag{29c}$$

$$D^{\mu}\big(t^{\gamma}\big) = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma - \mu + 1)} \cdot t^{\gamma - \mu} \tag{29d}$$

Now, let us define the coefficient $\tilde{a}(k)$ as

$$\tilde{a}(k) = (-1)^k \tilde{A}_k^{\mu} \tag{30}$$

Then the fractional derivative in (27) can be rewritten as

$$D^{\mu}x(t) = \lim_{\Delta t \to 0} \sum_{k=0}^{\infty} \frac{\tilde{a}(k)}{(\Delta t)^{\mu}} x(t - k \cdot \Delta t) \tag{31}$$

The coefficient sequence $\tilde{a}(k)$ for various orders of fractional order parameter μ can be plotted as in Fig. 3. It can be seen from Fig. 3 that the sequence $\tilde{a}(k)$ is a rapidly decaying sequence for various orders of μ.

Thus, by truncation, (31) can be rewritten as

$$D^{\mu}x(t) \approx \lim_{\Delta t \to 0} \sum_{k=0}^{L} \frac{\tilde{a}(k)}{(\Delta t)^{\mu}} x(t - k \cdot \Delta t) \tag{32}$$

where L is the truncation length.

Furthermore, by removing the limit, (32) can be further approximated by

$$D^{\mu}x(t) \approx \sum_{k=0}^{L} \frac{\tilde{a}(k)}{(\Delta t)^{\mu}} x(t - k \cdot \Delta t) \tag{33}$$

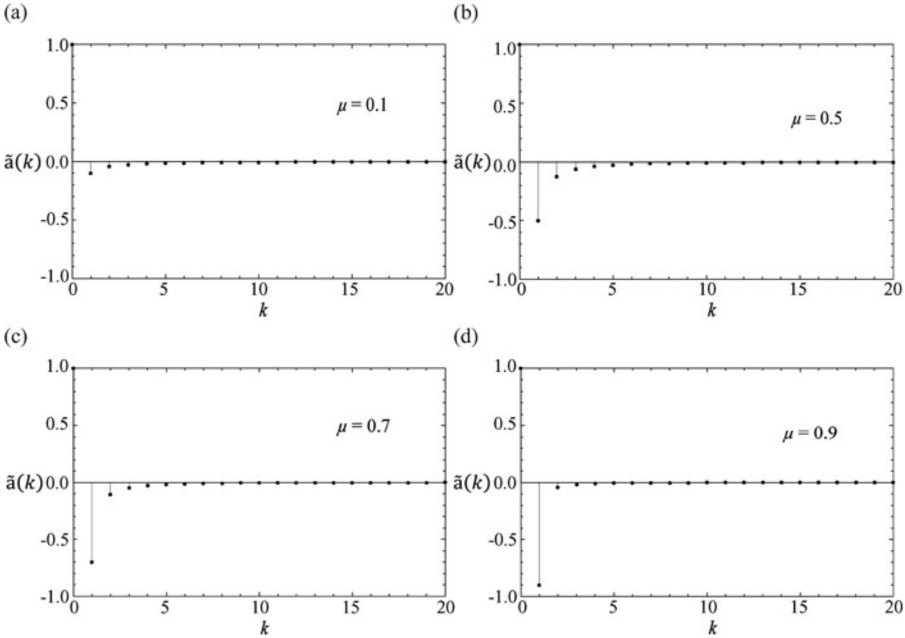


Fig. 3 The coefficient sequence $\tilde{a}(k)$ for varying fractional order parameter μ

Now, the operation $D^{\mu}x(t)$ at $t = n \cdot \Delta t$ is defined as

$$D^{\mu}x(t)|_{t=n\cdot\Delta t} = D^{\mu}x(n\cdot\Delta t) \approx \sum_{k=0}^{L} \frac{\tilde{a}(k)}{(\Delta t)^{\mu}} x(n\cdot\Delta t - k\cdot\Delta t) \tag{34}$$

where the fractional order parameter is $\mu \in R$ $(0 < \mu < 1)$ and $\tilde{a}(k)$ is given by (30).

As the proposed algorithm attempts to determine the closed-form fractional order differentiation in the FrFT domain, then taking the DFrFT of (34), according to [12],

$$\mathfrak{Q}_{\mathcal{F}}^{\varphi}\big[D^{\mu}x(n\cdot\Delta t)\big] \approx \mathfrak{Q}_{\mathcal{F}}^{\varphi}\left[\sum_{k=0}^{L} \frac{\tilde{a}(k)}{(\Delta t)^{\mu}} x(n-k)\right] \tag{35}$$

i.e.,

$$\mathfrak{Q}_{\mathcal{F}}^{\varphi}\big[D^{\mu}g(n)\big] = \sum_{k=0}^{L} \frac{\tilde{a}(k)}{(\Delta t)^{\mu}} \cdot \big(\mathfrak{Q}_{\mathcal{F}}^{\varphi}\big(x(n-k)\big)\big) \tag{36}$$

where $g(n)$ is given by (24) and the notation $\mathfrak{Q}_{\mathcal{F}}^{\varphi}$ represents the DFrFT operator.

Thus,

$$\begin{aligned} &\mathfrak{Q}_{\mathcal{F}}^{\varphi}[D^{\mu}g(n)] \\ &\quad = \sum_{k=0}^{L} \frac{\tilde{a}(k)}{(\Delta t)^{\mu}} \cdot \exp\left[\frac{j}{2}(k\cdot\Delta t)^2 \cdot \sin\varphi \cdot \cos\varphi - j(m\cdot\Delta u_{\varphi})\cdot(k\cdot\Delta t)\cdot\sin\varphi\right] \\ &\qquad \cdot G_{\varphi}(m - k\cos\varphi) \end{aligned} \tag{37}$$

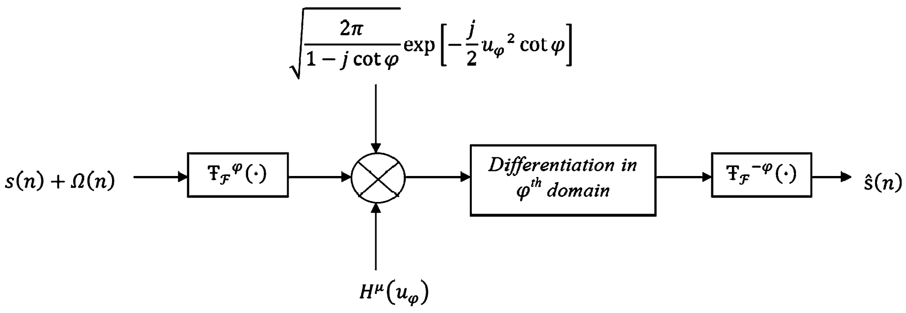


Fig. 4 Fractional order differentiating filter in fractional Fourier domain

where $G_\varphi(m)$ is given by (26).

Thus, (37) represents the DFrFT of the fractional order derivative of the discrete-time signal $g(n)$, respectively.

3.1 Fractional Order Differentiating Filter Model in Fractional Fourier Domain

The filtering scheme in the φth FrFT domain is shown in Fig. 4. In this configuration, first the φth domain of the FrFT of the input is obtained, and then the fractional order impulse response filter $H^\mu(u_\varphi)$ is applied in this domain. The weighted convolution theorem for the FrFT of [15] is used in the proposed filtering scheme. Finally, the resulting waveform is transformed with order '$-\varphi$' in order to obtain the output signal in the time domain.

4 Application Example and Simulation Results

The proposed model describing the fractional order differentiation in the fractional Fourier domain has been simulated on the platform of Wolfram Mathematica® software (version 8.0) on a system having configuration Pentium 4, with an Intel® CPU 1.8 GHz processor having 1 GB RAM.

The proposed model, which is described in Fig. 4, is used to simulate the fractional order differentiating filter in the fractional Fourier domain. The signal $s(n) = 2e^{18jn\pi/32} + e^{-8jn\pi/32}$ is corrupted by the chirp noise $\Omega(n) = 0.3e^{0.06j(n-1)^3-7jn}$, to obtain the input signal to the filter, as shown in Fig. 5(a) and (b), the real and imaginary parts, respectively. The signal $s(n) + \Omega(n)$ is applied to the proposed model of the filter shown in Fig. 4. The filtering is performed to compare the performance of time-domain ($\varphi = 0$, $\mu = 0.35$), frequency-domain ($\varphi = 1$, $\mu = 0.35$), and fractional Fourier domain filtering ($\varphi = 0.05\pi$, $\mu = 0.35$), as shown in Fig. 5(c)–(h). The criterion used for the optimal filtering is the root mean square error (RMSE) between the original signal and the filtered signal.

Therefore, it can be seen from Fig. 5(g) and (h) that the fractional Fourier domain filtered signal matches maximally with the original signal as compared with the time-domain and frequency-domain filtered signals. Finally, the RMSE between the original and the filtered signals is observed for different values of the fractional

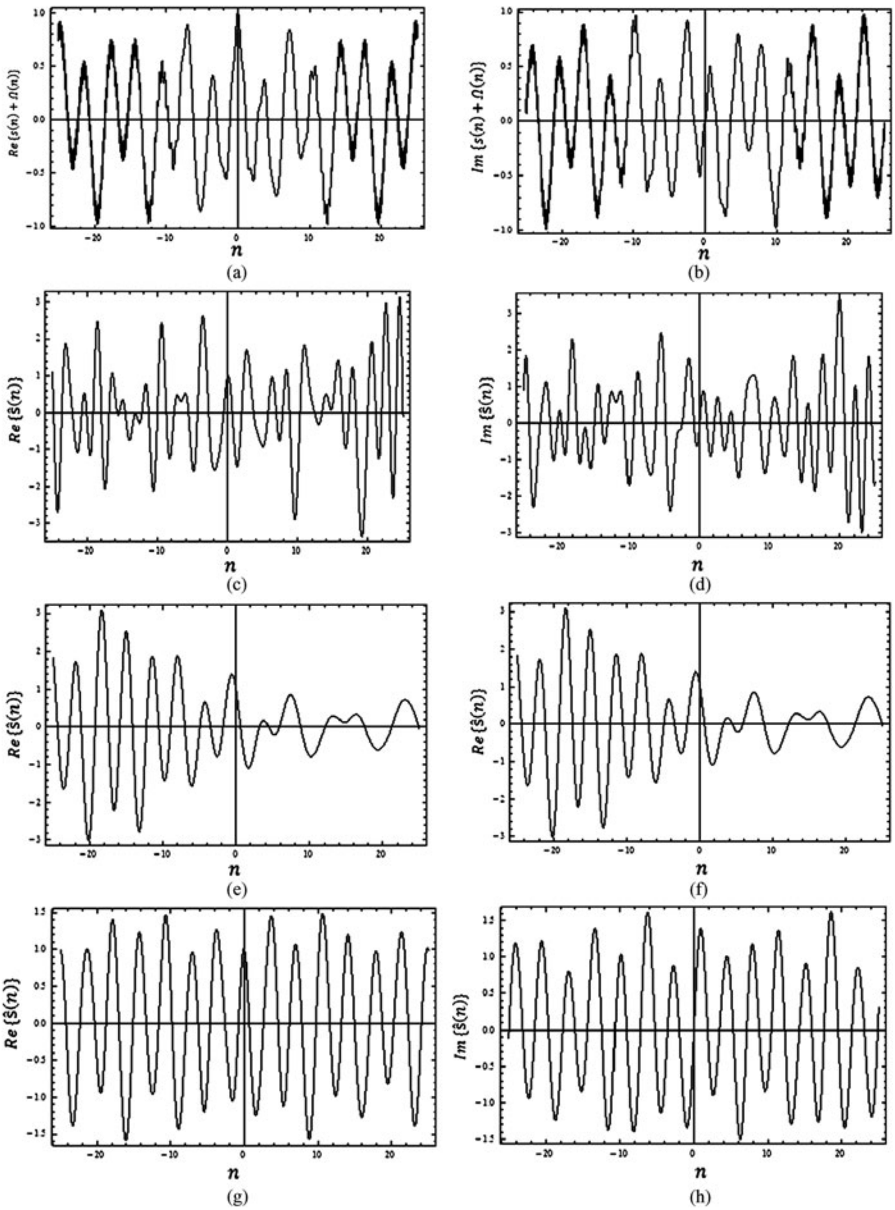
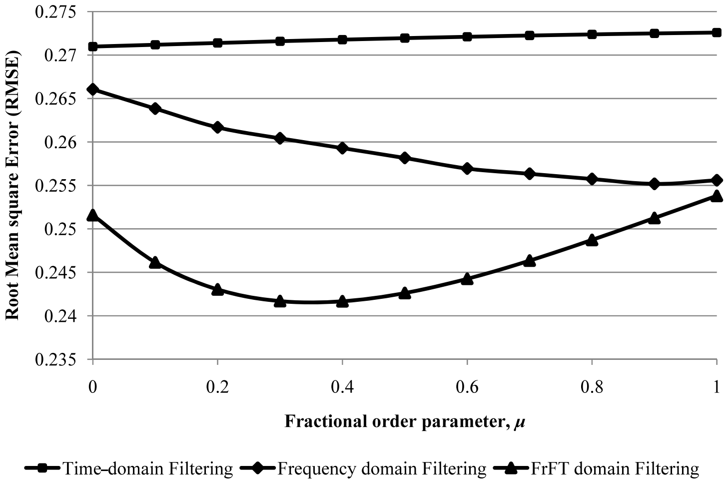


Fig. 5 Fractional order filtering results: (**a**), (**b**) corrupted signal, $s(n) + \Omega(n)$ (real (Re) and imaginary (Im) parts respectively) in time domain; (**c**), (**d**) time-domain filtered signal, $\hat{s}(n)$ (real (Re) and imaginary (Im) parts respectively); (**e**), (**f**) frequency-domain filtered signal, $\hat{s}(n)$ (real (Re) and imaginary (Im) parts respectively); (**g**), (**h**) fractional order FrFT-domain filtered signal $\hat{s}(n)$ (real (Re) and imaginary (Im) parts respectively), (**i**) RMSE vs. fractional order parameter, μ



(i)

Fig. 5 (*Continued*)

order parameter μ, which varies from 0 to 1 as shown in Fig. 5(i). This confirms that the FrFT domain filtering produces minimum RMSE for optimum FrFT order and fractional order parameter as compared with time-domain and frequency-domain filtering.

If one intends to implement the resulting systems of Fig. 4, a hardware implementation of the discrete FrFT using a field programmable gate array (FPGA) [13] can be utilized. Also, the DFrFT requires a computationally intensive trigonometric function, which can be accomplished using the well-known hardware efficient CORDIC (Coordinate Rotation Digital Computer) processor. If the input samples are complex values of the form $(a + jb)$, then the response of the system can be calculated separately for both the real and imaginary parts, as has been described in [13].

5 Conclusions

In this paper, a new closed-form analytical expression for fractional order differentiation in the FrFT domain has been presented. This work is the generalization of the differentiation property to fractional (noninteger) orders in the FrFT domain. It motivates the variation of two parameters: the fractional order parameter (μ) and the fractional Fourier parameter (φ), which has not been derived earlier. This closed-form analytical expression is obtained with the help of the Kummer confluent hypergeometric function.

The fractional order differentiation derived in this paper is a more generalized definition, since it achieves the flexibility of different rotation angles φ in the time–frequency plane of the FrFT with varying μ. Due to this variation of μ with φ in the FrFT domain, potential signal processing applications can be achieved, e.g., in filter design, radar system analysis, and edge detection in image processing, etc.

The application example of designing an FrFT-based low-pass finite impulse response fractional order differentiator (LP-FIR-FOD) has been simulated; the results have demonstrated its validity. The proposed LP-FIR-FOD includes the following advantages. First it is the first attempt at combining FOC with the FrFT, and it provides a new way of designing the digital fractional order differentiator. Second, it provides the flexibility of two different varying parameters, which could be beneficial in signal processing applications.

Thus, the freedom of utilizing a varying order of the derivative (fractional derivative) in the entire time–frequency plane of the FrFT domain can be utilized for different potential signal processing applications. Future works involve applying the proposed LP-FIR-FOD in image processing and radar signal processing applications.

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