

Effect of Initialization on a Class of Fractional Order Systems: Experimental Verification and Dependence on Nature of Past History and System Parameters

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Received: 15 June 2012 / Revised: 4 December 2012 / Published online: 20 December 2012
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Abstract The objectives of this work are (i) to verify experimentally the theoretical claim that neither Riemann–Liouville nor Caputo fractional derivative can be used to predict the time response of fractional order systems without proper correction for the system’s past history in terms of an initialization function and (ii) to study quantitatively how the error incurred due to ignoring initialization depends on the nature of the past history and the system parameters. The entire analysis is restricted to a special class of single input single output linear time invariant fractional order system which can be realized by a simple electrical circuit consisting of a resistance and a fractance in series. This work involves two different realizations of fractances or constant phase elements whose characteristic parameters are first determined based on their respective impedance frequency responses and then used to simulate the time responses of the circuit with the input same as the one used for experimentation using a numerical method for two cases: (i) taking past history into account and (ii) without taking past history into account. Thereafter, an integral square error criterion is presented and variation of the same is studied with respect to system parameters and nature of the history function to have a relative idea of how much partial past history in the absence of a complete one should suffice in practical applications.

Keywords Fractional order systems · Riemann–Liouville derivative · Caputo derivative · Fractance · Constant phase element · Initialization function

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1 Introduction

The birth of fractional calculus is attributed to some intuitions of L'Hospital and Leibnitz in the year 1695 about the generalization of the order of derivative of a function from the domain of integers to that of real numbers. Having grown up as a purely mathematical discipline with its inherent complexities in the eighteenth and nineteenth centuries with contributions from mathematicians like Euler, Laplace, Fourier, Abel, Liouville, Riemann, Grunwald, Letnikov and others [22, 28], fractional calculus started finding widespread applications in engineering and applied sciences only in the twentieth century [10]. Among the applications, a major thrust is on the use of differential equations involving fractional derivatives and integrals to model a number of physical phenomena; examples include viscoelasticity [30, 36], diffusion [5, 12, 25], transmission lines [17], chaos [16, 18], biological cells, tissues, diseases and medication [3, 11, 14, 24], and many others. Systems modeled as such belong to the general category of 'Fractional Order Systems' (FOSs). In control theory, the idea of using fractional order controllers in feedback loops emerged as Bode's ideal transfer function [6], followed by several related works [4, 8, 9, 15, 23, 31].

Modeling as fractional order proves to be useful particularly for systems where memory or hereditary properties play a significant role. This is due to the fact that an integer order derivative at a given instant is a local operator which considers the nature of the function only at that instant and its neighborhood, whereas a fractional derivative takes into account the past history of the function from some earlier point in time, called the 'lower terminal' up to the instant at which the derivative is to be computed. Therefore, if we have a Fractional Differential Equation (FDE) involving one or more fractional derivatives of the output of an FOS, say $y(t)$, an appropriate choice of such a lower terminal becomes important. There are two choices: (i) time $t = p$ such that $y(t) = 0$ for all $t \leq p$ where 'p' can be termed as the 'Process Starting Time (PST)', (ii) time $t = q \geq p$ so that $y(t) = 0$ in $[p, q]$ is not necessary, and we are interested in analyzing the system behavior by computing fractional derivative(s) of $y(t)$ for $t > q$ where 'q' can be termed as the 'Computation Starting Time (CST)'. In general, we may want to look into the system dynamics long after the system has been put to action, which means that PST 'p' and CST 'q' may be widely apart. At the same time, we note that taking 'q' as the lower terminal indicates that history of $y(t)$ during the time interval $[p, q]$ is not taken into consideration.

Now the question is: In order to predict accurately $y(t)$ for time $t > q$, is it at all required to capture the history of $y(t)$ during the interval $[p, q]$ and if so, how should one go for the same? As far as integer order systems are concerned, we have 'initial condition(s)' as an essential component. For an integer order system of order n , where the input and output are related by an n th order differential equation in time domain, a total of n initial conditions at time 'q', or in particular, values of the output and its first $(n - 1)$ derivatives at $t = q$ is sufficient. However, as stated earlier, all those $(n - 1)$ integer derivatives are local operators which depend on the output in only a small neighborhood of $t = q$. But for fractional order systems, the scenario is

quite different, because any fractional derivative of the output $y(t)$ evaluated at $t = q$ will consider the entire history of $y(t)$ in $[p, q)$.

Therefore, for an FOS with $q > p$ one may tend to ignore $y(t)$ in $[p, q)$ and compute the fractional derivative(s) of $y(t)$ using any one of the two most common definitions of fractional derivatives, viz. either the Riemann–Liouville or the Caputo definition with $t = q$ as the lower terminal. It was first claimed in the work of Hartley and Lorenzo [19] that in such a situation a correction term is to be added with the uninitialized derivative which takes ‘ q ’ as the lower terminal so that the computed response matches the actual one. Opposed to the initial conditions which are some numbers in case of integer order systems, this correction term is a function of time and involves an integral known as the ‘Initialization Function’ which takes into account the history of $y(t)$ in $[p, q)$. Thus came into existence the concept of initialization. Several issues pertaining to the initialization problem in the above mentioned approach have been addressed by Hartley and Lorenzo [1, 13, 19–21]. Other approaches to capture such history effects have been studied by Ortigueira [29] and Sabatier et al. [35].

Major contributions of this paper can be summarized as follows. First, obtaining closed form analytical solutions of FDEs, especially when initialization is augmented, is an involved task in terms of mathematical complexity. In this work, we have proposed an extension of a numerical method discussed in Podlubny [33] to include initialization and the solutions thus yielded are shown to be close approximations to the analytical ones derived in Hartley and Lorenzo [21] for the type of FDE considered for experimental verification. Moreover, the final numerical expression for the solution is shown to have clear physical interpretations and can easily be extended for other types of FDEs, e.g., FDEs containing more than one fractional derivative term. Second, experimental evidence to justify the need of an initialization function so far has been found only for a heat conduction problem [13]. In this work, we have concentrated on electrical circuits and used two different realizations of constant phase elements (CPEs)—one being a single component CPE which is a capacitive type probe coated with a thin film of polymethyl-methacrylate (PMMA), dipped into a polarizable solution; and the other being a domino ladder network which is an array of standard resistors and capacitors. For both the realizations, experimental results are shown to coincide with theory. Third, a system following fractional order dynamics may, in general, have infinite history and satisfying the theory demands an external storage element with infinite memory. Therefore, for practical applications, it is necessary to check how much of the past should be looked into so as to keep the error between the modeled and actual response within certain bounds. As a first attempt, the current work provides an error criterion to quantify how prominent the effect of initialization is and also analyzes how it depends on system parameters and the nature of past history.

The work is presented in several sections. Section 2 briefly discusses the concept of initialization, formulates the problem and suggests an intuitive approach. Section 3 establishes the method to be used for all subsequent simulations. Comparisons between experimental and simulation results are covered in Section 4. In Section 5, we study an integral square error criterion related to approximation of the initialization function. Achievements and future scope of work are summarized in Section 6.

2 Mathematical Preliminaries and the Problem Statement

We start with the definitions of fractional derivatives and their relationships which will be used extensively throughout this work. Detailed derivations can be found in [32, 33].

- (i.a) Riemann–Liouville (RL) integral of order ‘ α ’ (or derivative of order ‘ $-\alpha$ ’) of a function $y(t)$ is defined as:

$${}^{\text{RL}}_q d_t^{-\alpha} y(t) \triangleq \frac{1}{\Gamma(\alpha)} \int_q^t (t-\tau)^{\alpha-1} y(\tau) d\tau, \quad (1)$$

where α is any positive real number and $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ is the gamma function.

- (i.b) Riemann–Liouville (RL) derivative of order ‘ α ’ of a function $y(t)$ is defined as:

$${}^{\text{RL}}_q d_t^\alpha y(t) \triangleq \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_q^t (t-\tau)^{n-\alpha-1} y(\tau) d\tau, \quad (2)$$

where $n-1 \leq \alpha < n$; α being any positive real number, n is an integer.

- (ii) The Caputo derivative of order ‘ α ’ of a function $y(t)$ is defined as:

$${}^{\text{C}}_q d_t^\alpha y(t) \triangleq \frac{1}{\Gamma(m-\alpha)} \int_q^t (t-\tau)^{m-\alpha-1} y^{(m)}(\tau) d\tau \quad \text{with } m-1 \leq \alpha < m, \quad (3)$$

where m is a positive integer and $y^{(m)}$ is the conventional m th order derivative of y .

- (iii) The Grunwald–Letnikov (GL) derivative of order ‘ α ’ of a function $y(t)$ is defined as:

$${}^{\text{GL}}_q d_t^\alpha y(t) \triangleq \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{j=0}^{\frac{t-q}{h}} (-1)^j \frac{\alpha(\alpha-1) \cdots (\alpha-j+1)}{j!} y(t-jh), \quad (4)$$

where α is any real number. This definition unifies derivatives and integrals; with α positive signifying derivatives and α negative signifying integrals.

The RL and GL derivatives defined by (2) and (4), respectively, are equivalent [33] if $y(t)$ is continuous in $[q, t]$.

For $0 < \alpha < 1$, RL and Caputo derivatives defined by (2) and (3), respectively, are related as [32]:

$${}^{\text{RL}}_q d_t^\alpha y(t) = {}^{\text{C}}_q d_t^\alpha y(t) + \frac{(t-q)^{-\alpha}}{\Gamma(1-\alpha)} y(q). \quad (5)$$

We note that in both RL (equivalently, GL) and Caputo definitions, there is a term ‘ q ’ such that the behavior of $y(t)$ is considered only from $t = q$ onwards. In fractional calculus this ‘ q ’ is called the ‘Lower Terminal’. From a physical point of view, ‘ q ’ can be termed as Computation Starting Time (CST) such that variation of the system variable $y(t)$ after $t = q$ is of interest. The concept of initialization (Lorenzo and

Hartley [19] follows from the claim that ideally the lower terminal should be chosen as the Process Starting Time (PST) ‘ p ’ such that we have $y(t) = 0$ for all $t \leq p$. If the CST $q > p$ happens to be chosen as the lower terminal, then for the RL fractional integral as defined by (1), we should have

$${}_q D_t^{-\alpha} y(t) \triangleq {}^{\text{RL}}_q d_t^{-\alpha} y(t) + \psi(y, -\alpha, p, q, t) = {}^{\text{RL}}_p d_t^{-\alpha} y(t). \tag{6}$$

Here ‘ D ’ stands for the initialized derivative (of order ‘ $-\alpha$ ’ in this case) and ψ is the correction term to be added with the uninitialized derivative (of order ‘ $-\alpha$ ’) with lower terminal ‘ q ’ such that ‘ p ’ becomes the effective lower terminal.

Using definition (1) of RL integral, we get

$$\frac{1}{\Gamma(\alpha)} \int_q^t (t - \tau)^{\alpha-1} y(\tau) d\tau + \psi(y, -\alpha, p, q, t) = \frac{1}{\Gamma(\alpha)} \int_p^t (t - \tau)^{\alpha-1} y(\tau) d\tau.$$

Simplifying, we get

$$\psi(y, -\alpha, p, q, t) = \frac{1}{\Gamma(\alpha)} \int_p^q (t - \tau)^{\alpha-1} y(\tau) d\tau. \tag{7}$$

The factor ψ is known as the “Initialization Function”. It is a time varying quantity which takes into account the entire past history of the system output $y(t)$ from PST ‘ p ’ to CST ‘ q ’.

For the RL derivative defined by (2) with $0 < \alpha < 1$, correction should be as follows (Hartley and Lorenzo, [19]):

$${}_q D_t^\alpha y(t) = {}^{\text{RL}}_q d_t^\alpha y(t) + \frac{d}{dt} \psi(y, -(1 - \alpha), p, q, t) = {}^{\text{RL}}_p d_t^\alpha y(t), \tag{8}$$

where ψ is as defined by (7).

For the Caputo fractional derivative defined by (3) with $0 < \alpha < 1$, we have (Achar et al., [1]):

$${}_q D_t^\alpha y(t) = {}^{\text{C}}_q d_t^\alpha y(t) + \frac{(t - q)^{-\alpha} f(q)}{\Gamma(1 - \alpha)} + \frac{d}{dt} \psi(y, -(1 - \alpha), p, q, t) = {}^{\text{C}}_p d_t^\alpha y(t). \tag{9}$$

Here we note that

$${}_q D_t^\alpha y(t) = {}^{\text{RL}}_p d_t^\alpha y(t) = {}^{\text{C}}_p d_t^\alpha y(t). \tag{10}$$

This is because ‘ p ’ is the instant before which the system was throughout at rest, and thus, by (5), RL and Caputo derivatives become identical. Therefore, after initialization RL and Caputo derivatives yield the same result though uninitialized RL and Caputo derivatives in general yield different results.

Now, the problem can be formulated as follows:

Let us consider a linear time invariant FOS of order ‘ α ’ with a single input $u(t)$ and a single output $y(t)$ modeled by the FDE; where ‘ d ’ represents fractional derivative, in general.

$${}_q d_t^\alpha y(t) + ky(t) = u(t) \quad 0 \leq \alpha \leq 1 \text{ and } k \text{ is a positive constant,} \tag{11}$$

with $y(t) = 0$ for all $t \leq p$ and $y(q) = y_0$ is a constant; where $q > p$.

At the extremities, i.e., $\alpha = 0$ and $\alpha = 1$, we can use the theory of integer order calculus to find the response of the system.

For $0 < \alpha < 1$, the objective is to find out which one among the following three should be used in place of ‘ d ’ to make model (11) the most accurate one:

- (i) The uninitialized RL derivative defined by (2);
- (ii) The uninitialized Caputo derivative defined by (3);
- (iii) The initialized derivative defined by (8).

One way to accomplish the task is to first find out the solution of the FDE (11) for all the three cases (i), (ii), (iii) by some suitable method and then perform an experiment with a set up which realizes an FOS of this type to ascertain which of the three solutions yields the closest response to the experimental one.

3 Solution of FDE with and without Initialization

Considering initialization as explained in case (iii) above, we are to solve:

$${}_q D_t^\alpha y(t) + ky(t) = u(t) \quad 0 \leq \alpha \leq 1 \text{ and } k \text{ is a positive constant.} \quad (12)$$

Using the definition of initialized derivative as per (8), we have

$${}^{\text{RL}}_q d_t^\alpha y(t) + \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_p^q (t-\tau)^{-\alpha} y(\tau) d\tau + ky(t) = u(t). \quad (12.a)$$

The Leibnitz Rule for differentiation of integral is:

$$\frac{d}{dt} \int_{a(t)}^{b(t)} y(t, \tau) d\tau = \int_{a(t)}^{b(t)} \frac{\partial y(t, \tau)}{\partial t} d\tau + y(t, b) \frac{db}{dt} - y(t, a) \frac{da}{dt}.$$

We note here that the last two terms on the right hand side (RHS) will not appear if limits of integration are constants. Therefore, applying this rule to the differentiation of integral on the left hand side (LHS) of FDE (12.a), we get

$${}^{\text{RL}}_q d_t^\alpha y(t) + \frac{-\alpha}{\Gamma(1-\alpha)} \int_p^q (t-\tau)^{-\alpha-1} y(\tau) d\tau + ky(t) = u(t). \quad (13)$$

The uninitialized RL derivative, i.e., the first term in the LHS of (13) is equivalent to the GL derivative defined by (4), which can be numerically approximated as [33]:

$$\begin{aligned} {}^{\text{RL}}_q d_t^\alpha y(t = mh) &= {}^{\text{GL}}_q d_t^\alpha y(t = mh) \\ &\approx \frac{1}{h^\alpha} \sum_{j=0}^{m-Q} (-1)^j \frac{\alpha(\alpha-1) \cdots (\alpha-j+1)}{j!} y((m-j)h), \end{aligned} \quad (14)$$

where $q = Qh$; with h as the time step used for discretization.

Again, the integral present in second term on the left hand side (LHS) of (13) can be approximated as the discrete sum:

$$\int_p^q (t - \tau)^{-\alpha-1} y(\tau) d\tau \approx \sum_{l=P}^{Q-1} h(mh - lh)^{-\alpha-1} y(lh), \tag{15}$$

where $p = Ph$ is the Process Starting Time (PST).

Applying approximations (14) and (15) to FDE (13), we get

$$\frac{1}{h^\alpha} \sum_{j=0}^{m-Q} w_j y((m-j)h) + \frac{-\alpha}{\Gamma(1-\alpha)} \sum_{l=P}^{Q-1} h(mh - lh)^{-\alpha-1} y(nh) + ky(mh) = u(mh), \tag{16}$$

where $w_0 = 1, w_j = (1 - \frac{\alpha+1}{j})w_{j-1}$ for $j = 1, 2, 3, \dots$

On simplification, we obtain

$$y(mh) = \frac{1}{k + h^{-\alpha}} u(mh) + \frac{-h^{-\alpha}}{k + h^{-\alpha}} \sum_{j=1}^{m-Q} w_j y((m-j)h) + \frac{\alpha/\Gamma(1-\alpha)}{k + h^{-\alpha}} \sum_{l=P}^{Q-1} h(mh - lh)^{-\alpha-1} y(lh). \tag{17}$$

A close scrutiny of (17) reveals that the output y at any instant depends explicitly on three factors: the present input u ; weighted past outputs from CST prior to the current instant and weighted past outputs from PST prior to CST. It follows that if we are to solve an FDE which is a generalization of (12) with the LHS being a linear combination of a number of fractional derivatives, for each of the derivative orders there will be one term coming for outputs from PST to CST and another term for outputs from CST to current instant.

We also note here that the third term on the right hand side (RHS) of (17) is introduced because of including initialization; and therefore should be omitted if initialization is ignored. Thus, considering uninitialized RL derivative in (11), the solution can be directly written as:

$$y(mh) = \frac{1}{k + h^{-\alpha}} u(mh) + \frac{-h^{-\alpha}}{k + h^{-\alpha}} \sum_{j=1}^{m-Q} w_j y((m-j)h), \tag{18}$$

where the weights w_j are the same as defined by (16).

If we consider uninitialized Caputo derivative in (11), using relation (5) between the RL and Caputo derivatives, the solution can be found as:

$$y(mh) = \frac{1}{k + h^{-\alpha}} u(mh) + \frac{-h^{-\alpha}}{k + h^{-\alpha}} \sum_{j=1}^{m-Q} w_j y((m-j)h) + \frac{(mh-Qh)^{-\alpha}}{\Gamma(1-\alpha)} y(Qh). \tag{19}$$

We note that (18) and (19) are derived from (17) which is the solution of FDE (12) which includes initialization. In order to verify that the numerical solution given by (17) is correct, let us compare it with the analytical solution of the same FDE (12) with the following modifications corresponding to ‘constant history function’ (Hartley and Lorenzo [21]) case with the following: $k = 1$ (i.e., time constant of 1 s for $\alpha = 1$); $p = -a$; $q = 0$ (i.e., computation starts from time 0, past history is of duration ‘ a ’); $y(t) = 1$ for $-a \leq t \leq 0$ (i.e., output was of constant magnitude 1 during the history period); $u(t) = 0$ for $t \geq 0$ (i.e., response due to initialization only is desired).

The corresponding analytical solution is given by [21]:

(A) For small values of time ‘ t ’:

$$y(t) = \Lambda_{\alpha,\alpha-1}(-1, t) - \sum_{n=0}^{\infty} \frac{\Lambda_{\alpha,-n-1}(-1, t)}{a^{n+\alpha} \Gamma(1-n-\alpha)}, \tag{20}$$

where $\Lambda_{x,y}(z, w) = \sum_{n=0}^{\infty} \frac{z^n w^{(n+1)x-1-y}}{\Gamma((n+1)x-y)}$ and $\Gamma(\cdot)$ stands for the gamma function.

(B) For large values of time ‘ t ’:

$$y(t) = \Lambda_{\alpha,\alpha-1}(-1, t) - \Lambda_{\alpha,\alpha-1}(-1, t+a)U(t) + \sum_{n=0}^{\infty} \frac{a^{n+1-\alpha} \Lambda_{\alpha,n}(-1, t)}{\Gamma(n+2-\alpha)}, \tag{21}$$

where $U(\cdot)$ is the unit step function, and $\Lambda(\cdot)$ and $\Gamma(\cdot)$ are as defined earlier.

The analytical solutions given by (20) and (21) are not very straightforward and, at the same time, the discrimination between ‘small’ and ‘large’ values of time is not very clear. On the other hand, the numerical solution given by (17) is easy to comprehend and also free from any such ambiguity in notion of time.

The simulation results for $\alpha = 0.25, 0.50, 0.75$ with history duration ‘ a ’ = 1, 10, 100, and 1000 seconds are presented in Figs. 1, 2 and 3. The numerical solution

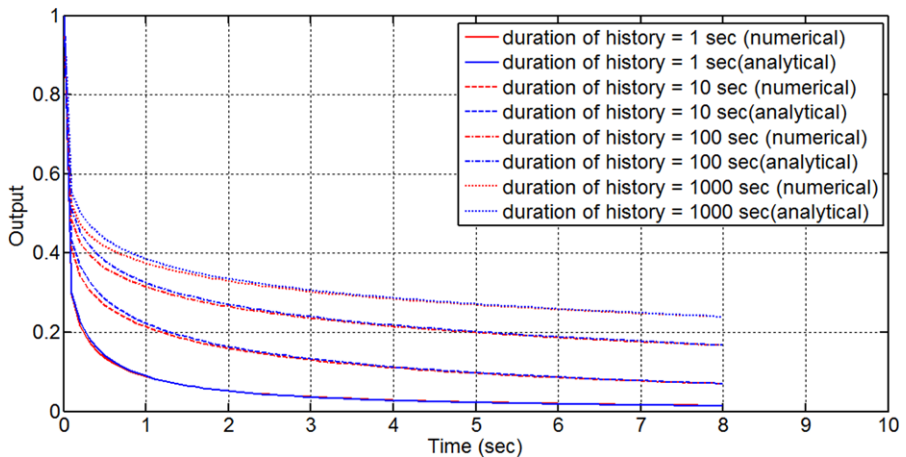


Fig. 1 Comparison of numerical and analytical responses for $\alpha = 0.25$

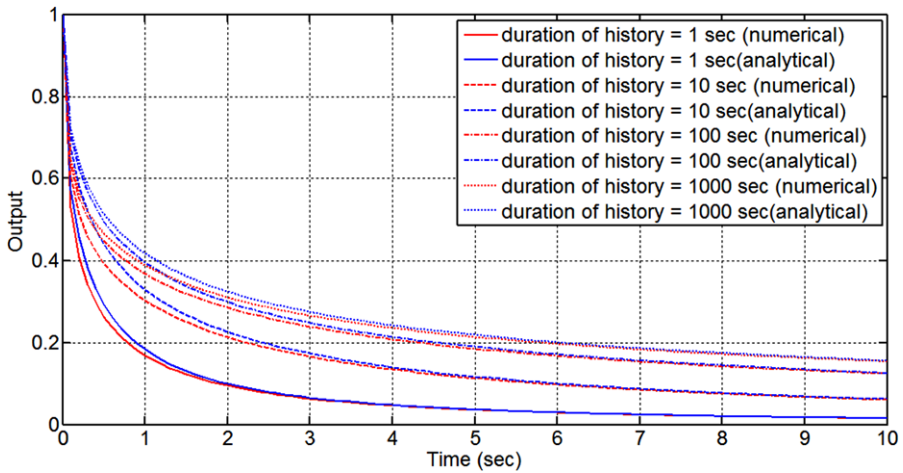


Fig. 2 Comparison of numerical and analytical responses for $\alpha = 0.50$

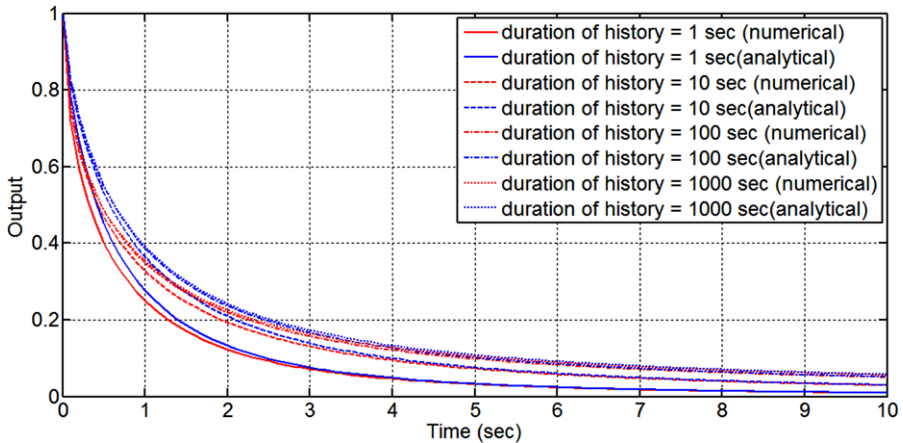


Fig. 3 Comparison of numerical and analytical responses for $\alpha = 0.75$

assumes a time step of 0.1 s for each case. As far as analytical solutions are concerned, small time solution given by (20) have been plotted for cases with $a = 10, 100,$ and 1000 seconds for all 3 values of α . For $a = 1$ s, the small time solution has been plotted up to $t = 1$ s, after which the large time solution given by (21) has been plotted for all the three values of α .

Moreover, numerical responses for the extreme cases, i.e., $\alpha = 0$ and $\alpha = 1$ yielded by (17) are also plotted in Fig. 4. It is expected that, for $\alpha = 0$, since the system is a memoryless one, the output will die down immediately to zero as the input is zero; and for $\alpha = 1$, since the system is a conventional first order system, the output will decay exponentially to zero with a time constant of 1 s; therefore, expected value of output after 1 s is about 0.37 and that after 4 s is about 0.02. Also, outputs for different durations of past history should be identical.

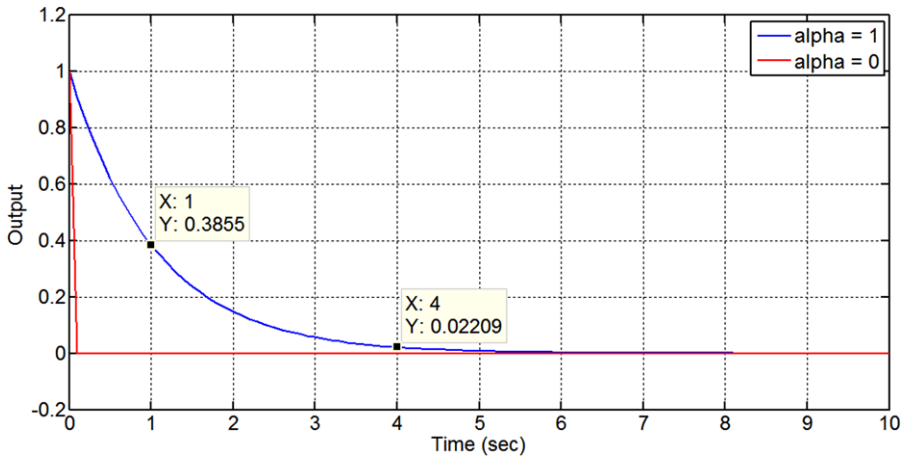


Fig. 4 Numerical responses closely match expected ones for $\alpha = 0$ and $\alpha = 1$

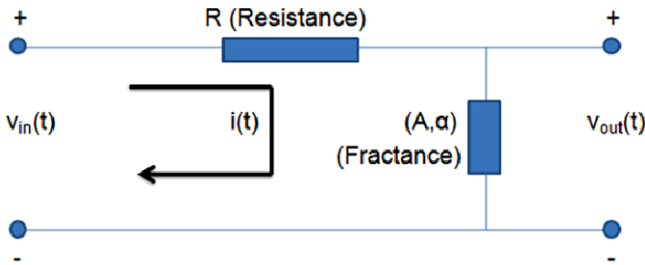


Fig. 5 The experimental circuit

We observe that the numerical solutions for $\alpha = 0.25, 0.50, 0.75$ match exactly with the analytical ones [21] after a small initial deviation for some time. Moreover, the solutions yielded by the numerical method have been found to match very closely with standard solutions for the extreme cases $\alpha = 0$ and $\alpha = 1$. We would therefore like to proceed with this method towards the experimental verification mentioned in the earlier section, using (17) as a closely accurate numerical solution to FDE (11) with initialization.

4 Experimental Verification

So far experimental evidence on the effect of initialization has been furnished for a heat conduction process which follows the dynamics of the type of (11) with $\alpha = 0.5$ (half order) (Gambone et al. [13]). For a parallel verification with values of α other than 0.5 as well, we can realize an FOS of this type by a simple electrical circuit as shown in Fig. 5.

The fractance has the time domain characteristics: $i(t) = \frac{1}{A} q d_t^\alpha v_{out}(t)$.

Here ‘ A ’ is a characteristic constant of the fractance, ‘ d ’ is the fractional derivative in general (of order ‘ α ’) as in (11).

Therefore, the dynamics of the circuit shown in Fig. 5 is given by:

$${}_q d_t^\alpha v_{\text{out}}(t) + \frac{A}{R} v_{\text{out}}(t) = \frac{A}{R} v_{\text{in}}(t). \quad (22)$$

Equation (22) is of the same form as (11) with $y(t) = v_{\text{out}}(t)$, $k = \frac{A}{R}$, and $u(t) = \frac{A}{R} v_{\text{in}}(t)$, and the job is therefore to determine by experiment and simulation whether the uninitialized RL derivative defined by (2), uninitialized Caputo derivative defined by (3), or the initialized derivative defined by (8) should be used in place of the general fractional derivative operator ‘ d ’. For a particular fractance, the parameters ‘ A ’ and ‘ α ’ can be easily obtained from its impedance frequency response as follows.

Ideally, impedance transfer function of a fractance is:

$$\frac{V(s)}{I(s)} = Z(s) = \frac{A}{s^\alpha}. \quad (23)$$

Therefore, frequency response of impedance is given by:

$$Z(j\omega) = \frac{A}{(j\omega)^\alpha}. \quad (24)$$

Thus the phase response is $\angle Z(j\omega) = -\frac{\pi}{2}\alpha$ radians. For a particular fractance, the phase angle is ideally constant at all frequencies; hence a fractance is also called a ‘Constant Phase Element (CPE)’. The value of phase angle gives the parameter ‘ α ’. The magnitude response $20 \log |z(j\omega)| = 20 \log(A) - 20\alpha \log(\omega)$ has a slope of -20α dB/decade. Once ‘ α ’ is known, ‘ A ’ can be found out.

There exist several possible ways to realize a CPE to a close approximation, in the sense that they do exhibit more or less flat phase response but only in limited frequency ranges [26]. The current work deals with the following two realizations.

4.1 Case I: Experiment with Single Component CPEs

Fabrication and performance study of a single component CPE was reported in [7, 26]. It is a copper plated epoxy glass with a PMMA coating and dipped into a polarizable medium. Two such single component fractances, say F_1 and F_2 , were taken for experimentation. Their impedance characteristics as measured with an LCR meter and the corresponding CPE models are shown in Figs. 6 and 7, respectively. For each case, the magnitude response, within a certain frequency range, has a slope other than the conventional $+/-20$ dB/decade or their multiples, and in the same frequency range the phase response is more or less flat, however, at an angle different from $+/-90^\circ$ or its multiples. Thus modeling with an integer order transfer function is not convenient; however, allowing the exponent of the Laplace variable ‘ s ’ to be a fraction yields quite accurate models.

From experimental results, F_1 in the range [10 kHz, 100 kHz] can be modeled as:

$$Z_1(s) = \frac{A_1}{s^{\alpha_1}} = \frac{2.0421 \times 10^4}{s^{0.2213}}. \quad (25)$$

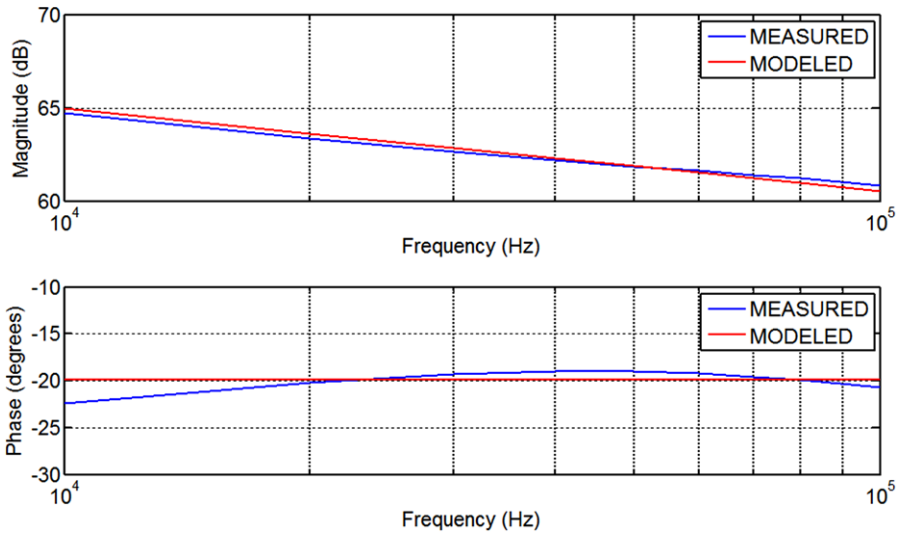


Fig. 6 Impedance characteristics of fractance F_1 from 10 kHz to 100 kHz

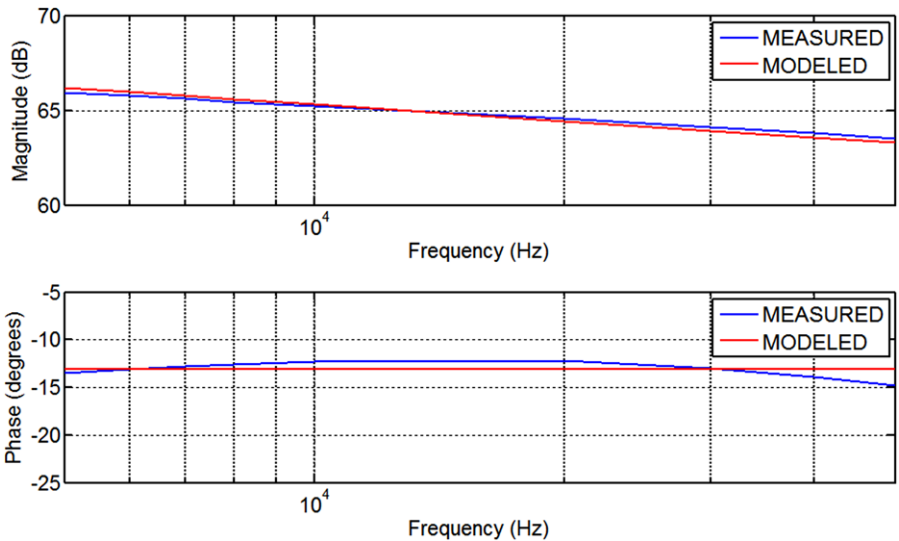


Fig. 7 Impedance characteristics of fractance F_2 from 5 kHz to 50 kHz

On the other hand, F_2 in the range [5 kHz, 50 kHz] can be modeled as:

$$Z_2(s) = \frac{A_2}{s^{\alpha_2}} = \frac{9.2194 \times 10^3}{s^{0.1456}}. \tag{26}$$

The experimental input as shown in Fig. 8 is given from a microcontroller. The fundamental frequency of the input pulse is chosen to be 10 kHz so that the CPE

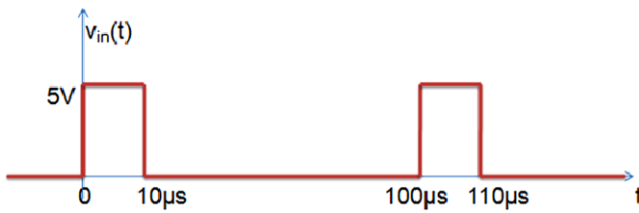


Fig. 8 The experimental input

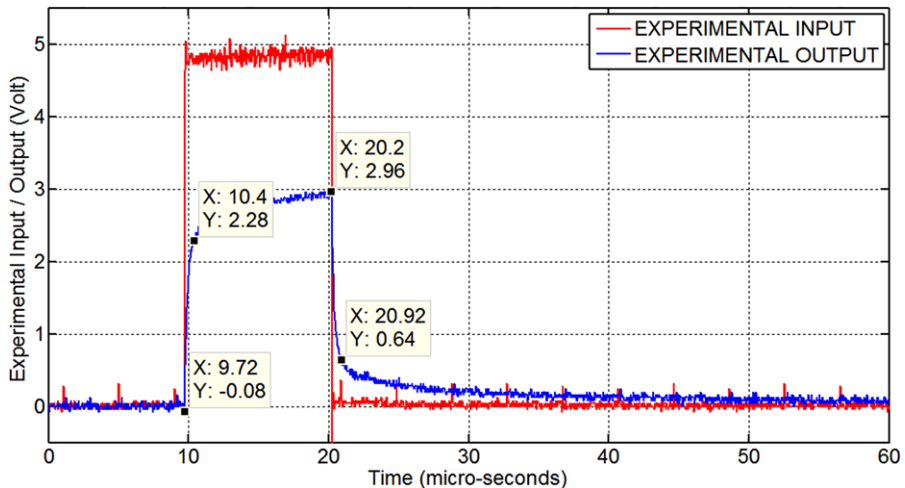


Fig. 9 Determination of simulation time step for fractance F_1

models (25) and (26) for the fractances F_1 and F_2 , respectively, can be used for simulation. The OFF time duration is chosen sufficiently (9 times) larger than ON time duration due to the fact that under such a condition, the output is found to die down to and stay at zero for quite some time, as a result of which we can effectively select the PST and CST for simulation as follows:

PST ' p ': Instant when the input goes ON, so output starts becoming non-zero.

CST ' q ': Any instant occurring later than ' p '; here we choose to work with two values of ' q ', viz. q_1 = instant when input goes OFF so that the full ON period of input (10 μ s) becomes the history period for simulation; $q_2 = q_1 - 5 \mu$ s so that half the ON period of input becomes the history period for simulation.

The series resistor R (as shown in Fig. 5) is kept at 1 $k\Omega$ such that rise and fall of the output voltage waveform should not become too fast to be captured. Now, choice of suitable time step for simulation for fractance F_1 is to be determined.

As seen in Fig. 9, steep rise or fall in experimental output occurs within $\sim 0.7 \mu$ s; thus for simulation, a time step $\sim 0.7 \mu$ s should be fine. Hence $h = 0.04 \mu$ s as used by the Digital Storage Oscilloscope to store the data is a good choice. With this time step, experimental and simulated responses for fractance F_1 are depicted in Figs. 10 and 11, respectively.

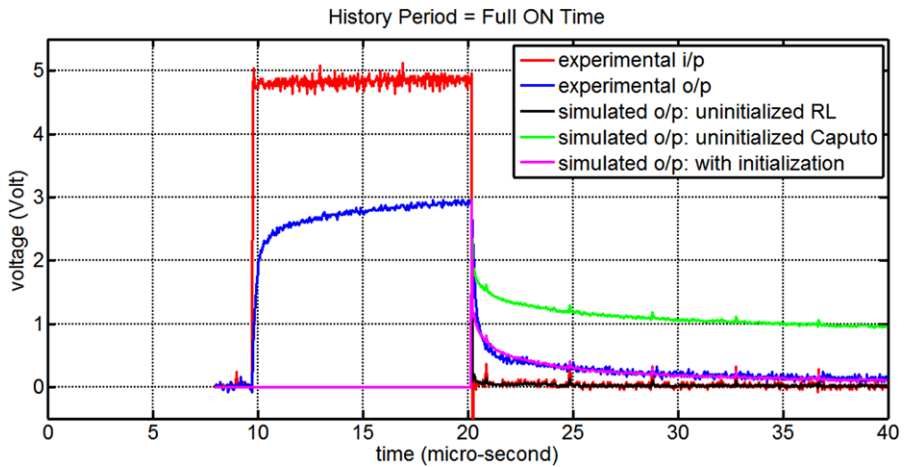


Fig. 10 Responses for F_1 when history period is the whole ON time of input

A brief explanation of Fig. 10 is as follows: For simulation, we are interested in predicting the output only after the CST ($\sim 20 \mu\text{s}$ in the time scale) and hence output before CST, which is not of interest, is deliberately kept at zero for all the simulations. In this case, the whole of ON time of input ($\sim 10 \mu\text{s}$ to $\sim 20 \mu\text{s}$) is the history period. Neglecting the values of the experimental output during this history period, the simulated output following uninitialized RL derivative as per (18) decays immediately to zero while the simulated output following uninitialized Caputo derivative as per (19) indicates a much slower decay. Initially both of them deviate from the experimental output, following which simulated RL output gets closer to the experimental one. On the other hand, the simulated output which includes initialization (i.e., experimental output data from $\sim 10 \mu\text{s}$ to $\sim 20 \mu\text{s}$) as per (17) remains close to experiment throughout after the CST. This provides evidence in support of including the past history in the form of initialization function as mentioned in Section 2.

We repeat the simulations for fractance F_1 with a different CST ($\sim 15 \mu\text{s}$ in the time scale) such that half of the input ON time ($\sim 10 \mu\text{s}$ to $\sim 15 \mu\text{s}$) becomes the history period, as shown in Fig. 11. The simulated uninitialized RL response which ignores the output data from $\sim 10 \mu\text{s}$ to $\sim 15 \mu\text{s}$ displays a deviation from the experimental output initially for a while, after which the former catches up with the latter. On the other hand, the initialized response once again remains closer to the experiment throughout after the CST (i.e., $15 \mu\text{s}$ onwards).

At the same time, it can also be seen that (a) for both the cases, the uninitialized RL response, after some initial deviation, eventually gets closer the experimental one, but (b) it does so in less time for the second case. A careful review of (17) and (18) reveals that the amount of instantaneous correction to the uninitialized response needed to get the initialized one is influenced by two factors: (i) it is less for an instant away from CST than one close to CST, which explains observation (a), and (ii) for the same instant it is more when the history output stays non-zero for a longer duration in the vicinity of CST, which explains observation (b). Thus both the need for a correction term for history effects and the suitability of the one involving the initialization func-

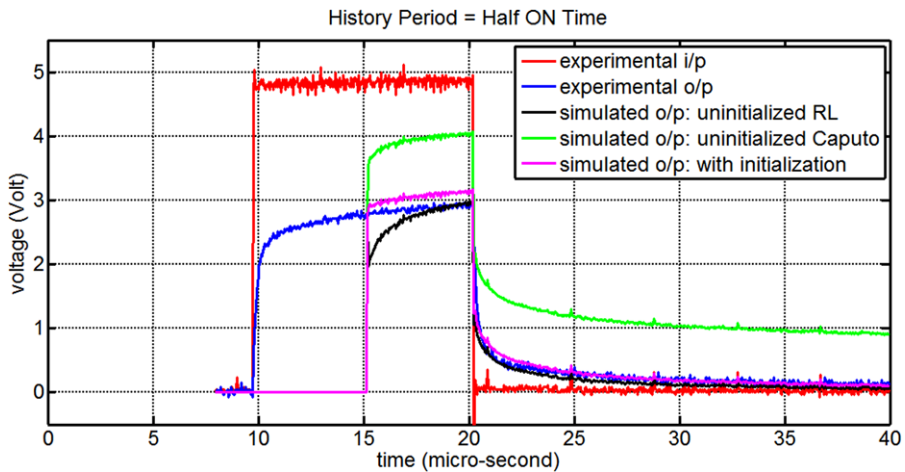


Fig. 11 Responses for F_1 when history period is half of the ON time of input

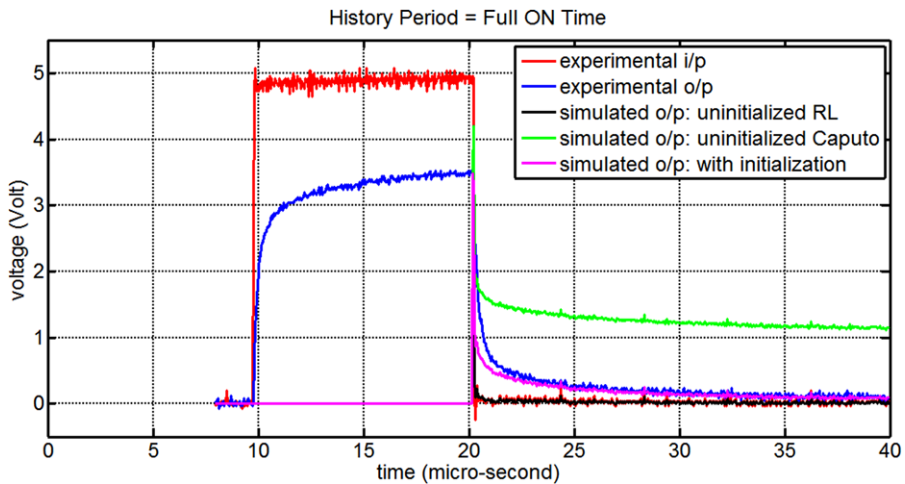


Fig. 12 Responses for F_2 when history period is whole of the ON time of input

tion are established. Similar are the results for both choices of CST for fractance F_2 as well. Observations with full ON time of input as the history period are presented in Fig. 12.

4.2 Case II: Experiment with a Domino Ladder Network

A domino ladder network is an array of resistors and capacitors with values so chosen that the resulting circuit exhibits constant phase behavior. In comparison to PMMA coated single component CPEs, it offers more design flexibility in the sense that the resistors and capacitors can be staged in pre-determined geometric common ratios in order to approximate a desired phase angle; however, the overall circuit becomes

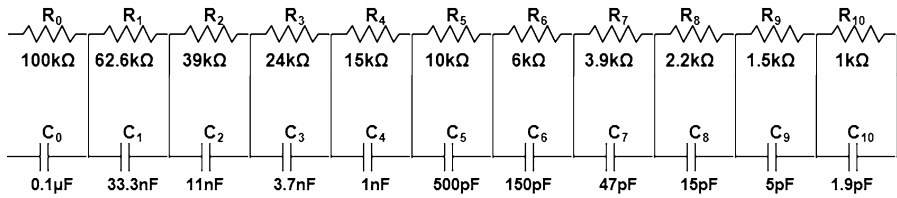


Fig. 13 The domino ladder network to realize a constant phase element

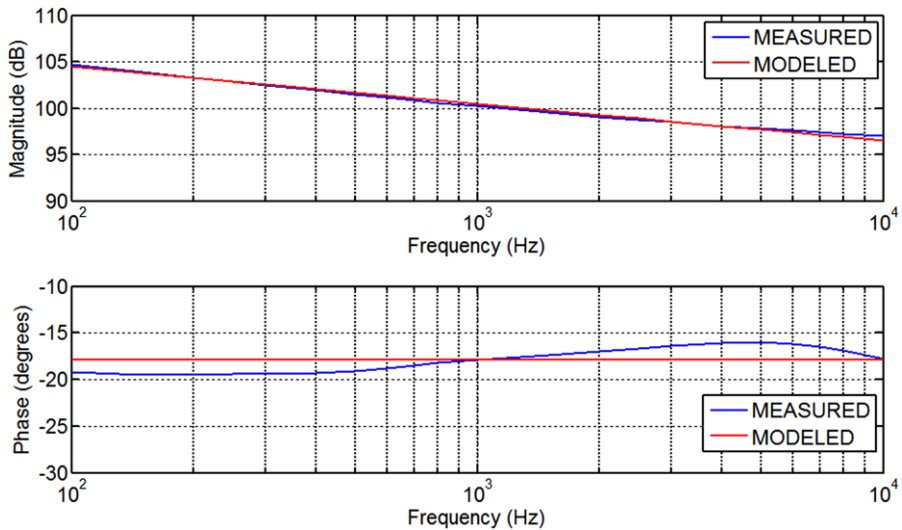


Fig. 14 Experimental and modeled frequency response of the ladder circuit

bulky and somewhat inconvenient to use. A detailed study of domino ladders can be found in [26, 27]. In the current work, we used a ladder circuit as shown in Fig. 13.

From frequency response, the ladder can be modeled in the range [100 Hz, 10 kHz] as:

$$Z(s) = \frac{6.0071 \times 10^5}{s^{0.1988}} \tag{27}$$

Experimental and modeled frequency responses are shown in Fig. 14.

In this case, a rectangular periodic input of amplitude 5 V, fundamental frequency 1 kHz, and duty cycle of about 16 % are provided from a signal generator; and a 100 kΩ resistor is put in series, justifications for all the choices being similar to the single component case.

Similar to the previous cases, analysis of experimental input and output waveforms lead to the conclusion that for simulation, a time step of the order of 1 μs would be sufficient. Simulated responses with this time step and history period equal to the whole of the input ON time, along with the experimental curves, are shown in Fig. 15. The procedure was repeated for cases where: (i) duration of history period is half the input ON time, (ii) value of the series resistor is different (e.g., 300 kΩ), (iii) input is

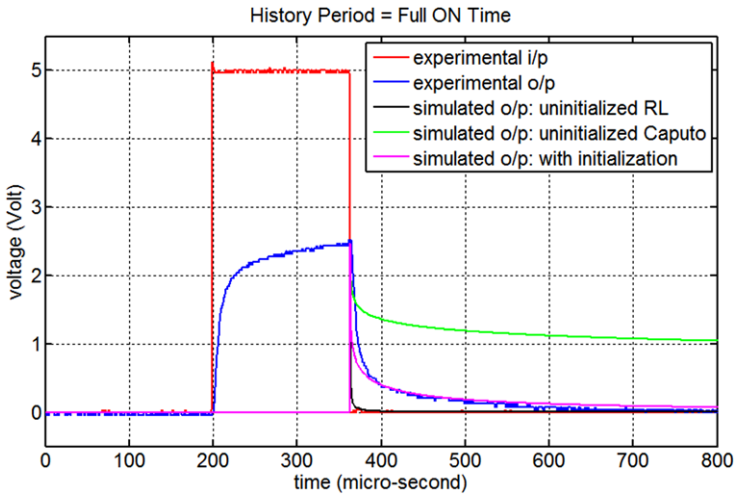


Fig. 15 Responses for domino ladder CPE with series resistor 100 kΩ and input as in Fig. 15, when history period is the whole of the ON time of the input

of same duty cycle but different fundamental frequency (e.g., 500 Hz). Observations are similar for all these combinations and all of them resemble the curves obtained for single component CPEs. The need for taking initialization into account is thus established for this type of CPE as well.

5 Quantitative Analysis in Terms of Integral Square Error

In practice, the entire past history may not be available, so for the type of FOS described by FDE (11) with PST ‘ p ’ and CST ‘ q ’, the objective now is to determine whether there exists a $H > q - p$ such that knowledge of $y(t)$ in $[q - H, q]$ instead of in $[p, q]$ can be used to construct the correction term without causing much error. Let $y_{init}(t) :=$ initialized response = solution (17) of FDE (12) with initialized derivative as per (8); and $y_{un}(t) :=$ uninitialized response = solution (18) of FDE (11) with uninitialized RL derivative as per (2).

A measure of the error between the initialized response and the uninitialized one can be Integral Square Error (ISE) which may be defined for the duration $[q, q + T]$ as:

$$ISE(T) = \int_q^{q+T} [y_{init}(t) - y_{un}(t)]^2 dt. \tag{28}$$

For convenience, let $p = -a$ and $q = 0$ such that duration of full history period is ‘ a ’ and

$$ISE(T) = \int_0^T [y_{init}(t) - y_{un}(t)]^2 dt. \tag{29}$$

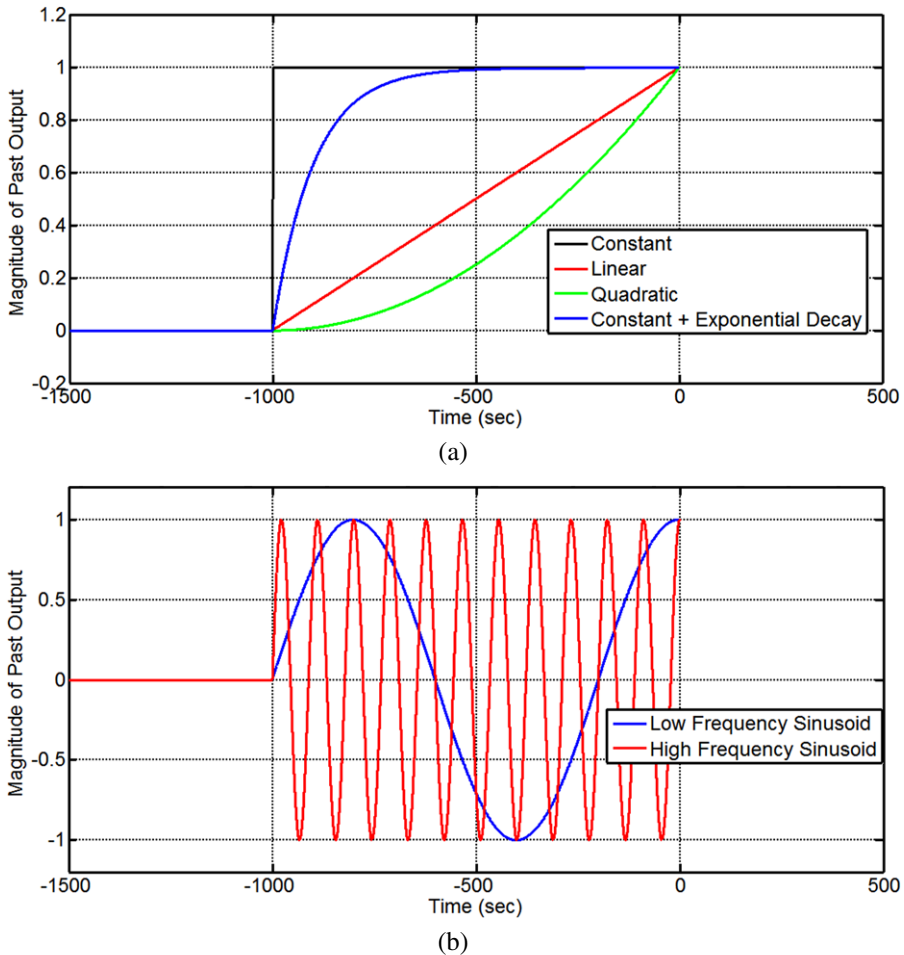


Fig. 16 Types of history functions

Let $a = 1000$ s, $T = 10$ s (meaning that the system has a long history period relative to the duration for which effect of history is studied) and let us determine ISE for the following cases:

- (A) $\alpha = 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$;
- (B) $k = 0.01, 0.1, 1, 10, 100$;
- (C) The history functions [i.e., $y(t)$ for $-a \leq t < 0$] as plotted in Figs. 16(a) and 16(b). All of them have value unity at $t = 0$.

Some key observations are provided below.

(a) *Dependence of ISE on Type of History Function:*

Though (i) and (iv) in Table 1 are different types of history functions, they lead to comparable ISE values, i.e., more or less similar effect as far as ignoring the history function is considered. We note that the exponential decay term had a time constant

Table 1 ISE for $T = 10$ s for different history functions for $\alpha = 0.3$, $k = 1$, $a = 1000$ s

Type of history function	Value of ISE for $T = 10$ seconds
(i) Constant	8.0635
(ii) Linear	6.7531
(iii) Quadratic	6.0636
(iv) Constant + exponential decay	7.9437
(v) Low frequency sinusoid	4.8466
(vi) High frequency sinusoid	1.5822

Table 2 ISE for $T = 10$ s for Constant Past History with ' α ' and ' k ' varying, $a = 1000$ s

' α ' (down)/' k ' (across)	0.01	0.1	1	10	100
0.05	5.7729	4.7924	1.3438	0.0413	0.0005
0.1	17.6679	14.4219	3.7357	0.1071	0.0012
0.2	42.2984	33.2464	7.2888	0.1838	0.0021
0.3	58.2389	43.8439	8.0635	0.1820	0.0020
0.4	64.0026	45.9013	7.0351	0.1449	0.0016
0.5	61.3046	41.6518	5.2969	0.1018	0.0011
0.6	52.0786	33.3295	3.5109	0.0644	0.0007
0.7	38.1509	22.8661	1.9978	0.0357	0.0004
0.8	21.8441	12.1915	0.8876	0.0157	0.0002
0.9	6.9736	3.6044	0.2204	0.0039	0.0000

of $(a/10)$, so the corresponding history function was almost constant at unity for the last 60 % duration of the history period. Moreover, in that duration, the rest of the history functions in descending order of average values are linear, quadratic, low frequency sinusoid and high frequency sinusoid, and the values of ISE follow exactly the same order. This is seen for other values of α and k as well.

Thus, for the general case with PST ' p ' and CST ' q ', within $[p, q)$, we should indeed be able to find a $H < q - p$ such that the average value of the history in $[q - H, q)$ decides how prominent the effect of initialization is going to be. The constant history function with the greatest average value in $[q - H, q)$ represents the worst case leading to maximum error.

(b) Dependence of ISE on System Parameters ' α ' and ' k ':

Table 2 shows the ISE values for different ' α ' and ' k ' for Constant Past History which corresponds to maximum ISE. Observations are similar in nature for the other types of history functions.

It is thus observed that irrespective of the type of history function and the value of k , the error due to neglecting initialization reaches a maximum roughly in the range of α from 0.2 to 0.4. However, the maximum error decreases by order of magnitude as k increases; which indicates that for a larger k ; the effect of initialization will die down at a quicker pace with time. Therefore, we can intuitively argue that we should consider history of $y(t)$ for the interval $[q - H, q)$ where H should be 0 at the

extremes $\alpha = 0$ and $\alpha = 1$ and maximum for $\alpha \sim 0.2$ to 0.4 and also that H should decrease as k increases.

6 Conclusions

In this work we have established the need for a suitable correction term to take into account the past history for a class of fractional order systems by experimenting with a simple electrical circuit and using a simple but accurate method for all simulations. The appropriateness of the initialization function as a possible means to construct the necessary correction term is also validated. Our experimentation entails two different realizations of constant phase element systems and the results are found to be valid for both the cases. We have also highlighted the fact that though a complete knowledge of the past history of the system is theoretically necessary, one can still perform a reasonably good analysis by considering only a partial knowledge of history, how much one should look into the past depends on parameters of the system and also on the nature of the history function. The parameter α ranging from 0.2 to 0.4 and a low value of k results in a more prominent effect of initialization and therefore the duration of history to be considered for accurate results is greater. If the exact nature of history function is not known, a worst case analysis can be performed by assuming the same to be constant at its maximum expected value.

Initialization is theoretically a fundamental phenomenon which is associated with all systems modeled as fractional order. However, for the theory to be universally accepted, experimental verification needs to be done for classes of FOS with $\alpha > 1$ as well. Moreover, an FOS as a part of a closed loop control system requires the issue of initialization to be handled so as to get an idea of how the loop stability and performance is likely to be affected. Nowadays fractional order controllers [15, 20, 23, 32, 34] are becoming popular because of extra freedom regarding fractional exponents which is not the case with integer order controllers. Apart from the fractional exponent, this initialization is another aid which can be used, particularly when the initialization of another fractional order plant in the loop is to be canceled [20]. Further, for a regulator problem pertaining to a FOS belonging to the class considered in this work, the control input to maintain the output once it reaches the set point, depends upon the initialization [2]. In-depth research is needed to address these issues.

Acknowledgements The authors are grateful to Prof. Anindya Chatterjee, Department of Mechanical Engineering, Indian Institute of Technology (IIT) Kanpur, Prof. Karabi Biswas, Department of Electrical Engineering, IIT Kharagpur, and Dr. Raj Kumar Biswas for their valuable suggestions and help during the work.

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