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A Convolution and Correlation Theorem for the Linear Canonical Transform and Its Application

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Abstract As a generalization of the fractional Fourier transform (FRFT), the linear canonical transform (LCT) plays an important role in many fields of optics and signal processing. Many properties for this transform are already known, but the correlation theorem, similar to the version of the Fourier transform (FT), is still to be determined. In this paper, firstly, we introduce a new convolution structure for the LCT, which is expressed by a one dimensional integral and easy to implement in filter design. The convolution theorem in FT domain is shown to be a special case of our achieved results. Then, based on the new convolution structure, the correlation theorem is derived, which is also a one dimensional integral expression. Last, as an application, utilizing the new convolution theorem, we investigate the sampling theorem for the band limited signal in the LCT domain. In particular, the formulas of uniform sampling and low pass reconstruction are obtained.

Keywords Convolution theorem · Correlation theorem · Linear canonical transform · Sampling

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1 Introduction

The linear canonical transform (LCT) [6, 13, 15, 16, 25] is an integral transform with three free parameters. It was introduced in 1970s and many transforms such as the Fourier transform (FT), the fractional Fourier transform (FRFT), the Fresnel transform (FST) and the scaling operations are all special cases of the LCT [3, 15]. In some papers, the LCT is also known as the affine Fourier transform [1], the generalized Fresnel transform [9], the ABCD transform [6] and Moshinsky-queue-transform [25], among other things.

Recently, along with applications of the FRFT in the signal processing community, the rote of the LCT for signal processing has also been noticed. Comparing to the FRFT with one extra degree of freedom and FT without a parameter, the LCT is more flexible for extra three degree of freedom. It has found many applications in optics, phase retrieval, radar system analysis, signal separation, filter design, pattern recognition and many other areas [5, 7, 15, 16, 18, 23].

With intensive research of the LCT, many properties have been found including time shift, phase shift, scaling, differentiation, integration, energy conservation and so on [15]. Simultaneously, as the generalization of FT, the relevant theory of LCT has been developed including the convolution theorem, uncertainty principle, sampling theory and so on [7, 8, 10, 19–21, 23, 28], which are generalizations of the corresponding properties of the FT and FRFT [4, 14, 17, 24, 26, 27]. However, the correlation theorem for the LCT still yet remains unknown.

Convolution and correlation operations are fundamental in the theory of linear time-invariant (LTI) system [15]. The output of any continuous time LTI system is found via the convolution of the input signal with the system impulse response. Correlation, which is similar to convolution, is another important operation in signal processing, as well as in optics, in pattern recognition, especially in detection applications [2, 11, 12, 22]. As the LCT has found wide applications in signal processing fields, it is theoretically interesting and practically useful to consider the convolution and correlation theory in the LCT domain. In this paper, firstly, we introduce a new convolution structure for the LCT, which is expressed by a simple integral and easy to implement in filter design. Then, based on the new convolution structure, we obtain a one dimensional integral expression of the correlation for LCT. Since the correlation of two functions is no more than their convolution after one of the two functions has been axis-reversed and complex conjugated, the property of the new convolution results in the property of the correlation. Last, as an application, utilizing the new convolution theorem, we study the sampling theorem for the band limited signal in the LCT domain. The formulas of uniform sampling and reconstruction are obtained.

2 Preliminaries

2.1 The Linear Canonical Transform

The LCT of a signal $f(t)$ with parameter $A=(a,b,c,d)$ is defined [6, 13, 15, 16, 25] as

$$F_A(u)=L_A[f(t)](u)=\begin{cases}\int_{-\infty}^{+\infty} f(t)K_A(u,t)\,dt, & b\neq 0,\\ \sqrt{d}e^{j(cd/2)u^2}, & b=0,\end{cases} \tag{1}$$

where

$$K_A(u,t) = B_A e^{j\frac{1}{2}[\frac{a}{b}t^2 - (\frac{2}{b})tu + \frac{d}{b}u^2]}, \tag{2}$$

where $B_A = \sqrt{1/(j2\pi b)}$, a, b, c, d are real numbers satisfying $ad - bc = 1$. The inverse of the LCT is given by

$$f(t) = \int_{-\infty}^{+\infty} F_A(u) K_A^*(u,t)\, du, \tag{3}$$

where superscript "$*$" denote complex conjugation.

It should be noted that, when $b = 0$, the LCT of a signal is essentially a chirp multiplication and it is of no particular interest to our objective in this work. Therefore, from now on, we shall confine our attention to LCT for $b \neq 0$. When $A = (\cos\alpha, \sin\alpha, -\sin\alpha, \cos\alpha)$, the LCT reduces to the FRFT; when $\alpha = \pi/2$, it reduces to FT.

The LCT has the following important space shift and phase shift properties [15], which are used to derive the new convolution and correlation theorems for LCT in this paper.

Before presenting the properties of LCT, we define some operators. We denote S_k $(k \in R)$ a translation operator, defined by

$$S_k f(t) = S_k[f](t) = f(t-k), \tag{4}$$

P_k $(k \in R)$ is a linear phase shift operator, defined by

$$P_k f(t) = e^{jkt} f(t). \tag{5}$$

Let τ and v be two real numbers and consider the function $P_v S_\tau f(t)$, such that

$$P_v S_\tau f(t) = e^{jtv} f(t-\tau). \tag{6}$$

In order to express simple, we use (4) and (5) as the notation expression. Then, based on the definition of the LCT, we present the following important space shift and phase shift properties of LCT [6, 13, 15, 16, 25].

Property 1 The space shift property

$$L_A\big[S_\tau f(t)\big](u) = L_A\big[f(t-\tau)\big](u) = e^{-jac\tau^2/2 + jc\tau u} F_A(u - a\tau). \tag{7}$$

Property 2 The phase shift property

$$L_A\big[P_v f(t)\big](u) = L_A\big[e^{jtv} f(t)\big](u) = e^{-jbdv^2/2 + jdvu} F_A(u - bv). \tag{8}$$

Property 3 The space and phase shift property

$$\begin{aligned} L_A\big[P_v S_\tau f(t)\big](u) &= L_A\big[e^{jtv} f(t-\tau)\big](u) \\ &= e^{-j[ac\tau^2 + bdv^2]/2 + j(c\tau + dv)u - bc\tau v} F_A(u - a\tau - bv), \end{aligned} \tag{9}$$

where τ, v represent the space and phase shift parameters, respectively. For further details of the definition and properties of the LCT, the reader may refer to [15].

2.2 The Convolution and Correlation Theory

Convolution and correlation operations are fundamental in the theory of LTI system. Moreover, convolution and correlation are widely used in signal processing, as well as in optics, in pattern recognition or in the description of image formation with incoherent illumination [2, 4, 11, 12, 14, 15, 22, 27]. The convolution and correlation theorems in FT domain are defined as

$$(f \otimes g)(t) = \int_{-\infty}^{+\infty} f(\tau) g(t-\tau)\, d\tau, \tag{10}$$

$$f(t) \otimes g(t) \xleftrightarrow{\text{FT}} F(u)G(u), \tag{11}$$

$$f(t) \odot g(t) = \int_{-\infty}^{+\infty} f(\tau) g^{*}(\tau - t)\, d\tau, \tag{12}$$

where '$\otimes$' and '$\odot$' denote the conventional convolution and correlation operations, respectively.

Based on the expression for the generalized translation in the LCT domain, the generalized convolution theorem has been derived in the LCT domain [23], which preserves the elegance and simplicity comparable to that of the FT:

$$(f \Theta g)(t) = \int_{-\infty}^{+\infty} f(\tau) g(t\theta\tau)\, d\tau, \tag{13}$$

$$f(t)\Theta g(t) \xleftrightarrow{\text{LCT}} F_A(u)G_A(u), \tag{14}$$

where θ in the argument of the function $g(t\theta\tau)$ is the generalized delay operator for the generalized translation. This shows that the generalized convolution of two signals in time domain is equivalent to simple multiplication of their LCTs in the LCT domains. However, the result is a triple integral. So, it is complicated to reduce the expression of the generalized convolution to a single integral form as in the ordinary convolution expression.

3 Convolution and Correlation Theorems for the LCT

In this section, firstly, we propose a convolution structure for the LCT, which is different from the generalized convolution structure [23]. Moreover, it can be expressed by a simple one dimensional integral. This result is an extension of the convolution theorem from the FT to the LCT domain, and can be more useful in practical analog filtering in the LCT domain. Then, a natural definition of a correlation would be derived using the new convolution theorem. And more important, we can also obtain a one dimensional integral expression of the linear canonical correlation. In Sect. 4, the derived convolution structure will be applied to analyze the sampling in the LCT domain.

3.1 New Convolution Structure for the LCT

Theorem 1 *For any function* $f(t)$, $g(t)$, *let* F_A, G_A *denote the LCT of* $f(t)$, $g(t)$, *respectively. Then*

$$\big[f \overset{A}{\otimes} g\big](t) = B_A L_{A^{-1}}\big[F_A(u)G_A(u)e^{-jdu^2/(2b)}\big](t), \tag{15}$$

where operator '$\overset{A}{\otimes}$' *is defined by*

$$\big[f \overset{A}{\otimes} g\big](t) = B_A^2 \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)e^{-ja\tau(t-\tau)/b}\, d\tau. \tag{16}$$

Proof Let τ and v be two real numbers; based on (6) we have

$$P_v S_\tau f(t) = e^{jtv} f(t-\tau). \tag{17}$$

Then, using the space shift and phase shift property of LCT (9), we obtain

$$L_A\big[P_v S_\tau f(t)\big](u) = e^{-j[ac\tau^2+bdv^2]/2+j(c\tau+dv)u-bc\tau v} F_A(u-a\tau-bv). \tag{18}$$

From (18), we see that both $L_A[P_v S_\tau f(t)]$ and $L_A[f(t)]$ depend on the same parameter if we choose τ and v such that

$$a\tau + bv = 0. \tag{19}$$

According to (19), we get $v = -a\tau/b$. Substituting $v = -a\tau/b$ into (18), equation (18) then reduces to

$$L_A\big[P_{-\tau a/b} S_\tau f(t)\big](u) = e^{-ja\tau^2/(2b)-j\tau u/b} L_A\big[f(t)\big](u). \tag{20}$$

Making use of the generalized convolution of the LCT, we have

$$L_A[f \Theta g](u) = L_A[f](u)L_A[g](u) = F_A(u)G_A(u). \tag{21}$$

Then, using the definition of the LCT, we have

$$\begin{aligned} F_A(u)G_A(u) &= L_A[g](u)B_A e^{j\frac{du^2}{2b}} \int_{-\infty}^{+\infty} e^{j\frac{a\tau^2}{2b}-j\frac{u\tau}{b}} f(\tau)\, d\tau \\ &= B_A e^{j\frac{du^2}{2b}} \int_{-\infty}^{+\infty} e^{j\frac{a\tau^2}{2b}-j\frac{u\tau}{b}} L_A[g](u) f(\tau)\, d\tau. \end{aligned} \tag{22}$$

According to (20), we get

$$L_A\big[f(t)\big](u)e^{-ju\tau/b} = e^{ja\tau^2/(2b)} L_A\big[P_{-\tau a/b} S_\tau f(t)\big](u). \tag{23}$$

Similarly, we can get

$$\begin{aligned} e^{-ju\tau/b} L_A\big[g(t)\big](u) &= e^{ja\tau^2/(2b)} L_A\big[P_{-\tau a/b} S_\tau g(t)\big](u) \\ &= e^{ja\tau^2/(2b)} L_A\big[e^{-jat\tau/b} g(t-\tau)\big](u). \end{aligned} \tag{24}$$

Substituting (24) into (22), we can obtain

$$F_A(u)G_A(u) = B_A e^{jdu^2/(2b)} \int_{-\infty}^{+\infty} e^{ja\tau^2/(2b)} \times \left(e^{ja\tau^2/(2b)} L_A\left[e^{-jat\tau/b} g(t-\tau)\right](u)\right) f(\tau)\,d\tau. \tag{25}$$

Using the integral form of $L_A[e^{-jat\tau/b} g(t-\tau)](u)$, we rewrite (25) as

$$F_A(u)G_A(u) = B_A^2 e^{jdu^2/(2b)} \int_{-\infty}^{+\infty} f(\tau) e^{ja\tau^2/b} e^{jdu^2/(2b)} \times \int_{-\infty}^{+\infty} e^{jat^2/(2b)} e^{-jtu/b} \left(e^{-jat\tau/b} g(t-\tau)\right) dt\,d\tau. \tag{26}$$

Let h be such that

$$h(t) = B_A \int_{-\infty}^{+\infty} f(\tau) g(t-\tau) e^{-ja\tau(t-\tau)/b}\,d\tau. \tag{27}$$

From (21) and (26), we get

$$L_A[f \Theta g](u) = e^{jdu^2/(2b)} L_A[h](u). \tag{28}$$

From (27) and (28), we see that the function $h(t)$ could be a good candidate to be a new convolution of f and g for the LCT. For that reason, we denote by $\overset{A}{\otimes}$ a new convolution, defined by (15) as

$$\left[f \overset{A}{\otimes} g\right](t) = B_A L_{A^{-1}}\left[F_A(u)G_A(u)e^{-jdu^2/(2b)}\right](t). \tag{29}$$

Let $y(t) = B_A h(t)$, we have

$$Y_A(u) = L_A\left[y(t)\right](u) = B_A H_A(u) = B_A F_A(u)G_A(u)e^{-jdu^2/(2b)}. \tag{30}$$

Therefore

$$H_A(u) = F_A(u)G_A(u)e^{-jdu^2/(2b)}. \tag{31}$$

Moreover, (27) and (28) can lead to

$$\left[f \overset{A}{\otimes} g\right](t) = B_A^2 \int_{-\infty}^{+\infty} f(\tau) g(t-\tau) e^{-ja\tau(t-\tau)/b}\,d\tau. \tag{32}$$

□

Further, based on the conventional convolution operator, we can express the new convolution as

$$\left[f \overset{A}{\otimes} g\right](t) = B_A^2 e^{-jat^2/(2b)} \left(f(t)e^{jat^2/(2b)}\right) * \left(g(t)e^{jat^2/(2b)}\right). \tag{33}$$

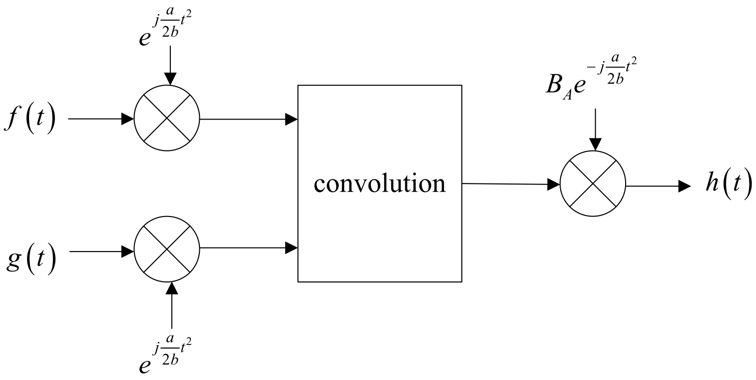

Fig. 1 Convolution for the LCT

See Fig. 1 for realization of the convolution operation '$\overset{A}{\otimes}$'.

When $(a, b, c, d) = (0, 1, -1, 0)$, (33) reduces to the ordinary convolution theorem of the FT. From the convolution theorem for the LCT defined by (29) and (32), we see that (29) is similar to (14), up to a quadratic phase factor and a multiplicative factor. However, the new convolution structure is different from the generalized convolution introduced in [23]. The generalized convolution is a triple integral. It is complicated to reduce the expression of the generalized convolution to a single integral form as in the ordinary convolution expression. Indeed, the advantage of the new convolution structure is that it can be expressed by a simple one dimensional integral. Moreover, it can be more useful in practical analog filtering in the LCT domain.

We also notice that the definition of the new convolution is identical to the one proposed by Tao [7], up to a multiplicative factor. However, we came to such a definition in a different way, and more important, we obtain a one dimensional integral expression of the linear canonical convolution, (32) that does not explicitly appear in Tao's paper and cannot be so easy deduced from Tao's reasoning. Such an integral expression is useful for actual derivations and applications as will be shown later on.

The convolution theory is widely used in signal processing, as well as in optics. The new convolution can be useful in practical analog filtering in the LCT domain as discussed in [7, 23]. Equation (31) is particularly useful in filter design. For example, if we are interested only in the frequency spectrum of the LCT in the region $[u_1, u_2]$ of a signal f, we choose the filter impulse response, g, so that G_A is constant over $[u_1, u_2]$, and zero or of rapid decay outside that region. Passing the output of the filter through the chirp multiplier, $e^{-jdu^2/(2b)}$, yields that part of the spectrum of f over $[u_1, u_2]$. In Sect. 4, as another application of new convolution for LCT, it is applied to reconstruct a uniform sampling for bandlimited signal in LCT domain.

3.2 Correlation Theory for the LCT

Correlation, which is similar to convolution, is another important operation in signal processing. Since the correlation of two functions is no more than their convolution after one of the two functions has been axis-reversed and complex conjugated, the property of the new convolution results in the property of the correlation. On the basis of (14), a natural definition of a correlation in the LCT domain could be

$$f \overset{A}{\Theta} g = L_{A^{-1}}\big[L_A[f]L_A^*[g]\big]. \tag{34}$$

From (34), we see that the correlation in the LCT domain is also triple integral. Nevertheless, we are looking for a correlation that is expressible by a simple integral. Using the new convolution structure, we introduce a new correlation in the LCT domain defined by

$$\big[f \overset{A}{\odot} g\big](t) = B_A L_{A^{-1}}\big[L_A[f](u)L_A^*[g](u)e^{jdu^2/(2b)}\big](t). \tag{35}$$

We can introduce the integral form of the correlation as follows

Theorem 2 *For any function $f(t)$, $g(t)$, the correlation operator '$\overset{A}{\odot}$' can also be defined by*

$$\big[f \overset{A}{\odot} g\big](t) = B_A B_A^* \int_{-\infty}^{+\infty} f(\tau)\, g^*(\tau - t)e^{jat(\tau-t)/b}\, d\tau. \tag{36}$$

Equation (36) *is equivalent to* (35).

Proof Now, we prove that (36) is equivalent to (35). From (36), using the definition of LCT, we get

$$\begin{aligned}
&L_A\big[f \overset{A}{\odot} g\big](u)\\
&\quad= B_A^2 B_A^* e^{jdu^2/(2b)} \int_{-\infty}^{+\infty} e^{jat^2/(2b)-jtu/b} \int_{-\infty}^{+\infty} f(\tau)g^*(\tau - t)e^{jat(\tau-t)/b}\, d\tau\, dt\\
&\quad= B_A^2 B_A^* e^{jdu^2/(2b)} \int_{-\infty}^{+\infty} f(\tau) \int_{-\infty}^{+\infty} e^{-jat^2/(2b)} e^{-jt(u-a\tau)/b} g^*(\tau - t)\, d\tau\, dt\\
&\quad= B_A^2 B_A^* e^{jdu^2/(2b)} \int_{-\infty}^{+\infty} f(\tau)e^{jd(u-a\tau)^2/(2b)} e^{-jd(u-a\tau)^2/(2b)}\\
&\qquad\times \int_{-\infty}^{+\infty} e^{-jat^2/(2b)} e^{jt(a\tau-u)/b} g^*(\tau - t)\, d\tau\, dt\\
&\quad= B_A^2 e^{jdu^2/(2b)} \int_{-\infty}^{+\infty} f(\tau)e^{jd(u-a\tau)^2/b} L_A^*[S_\tau \tilde{g}]\, d\tau,
\end{aligned} \tag{37}$$

where $\tilde{g}$ is the axis-reversed version of g. According to the space shift and phase shift property of LCT (9), we get

$$L_A[S_\tau \tilde{g}](x) = e^{-jac\tau^2/2+jxc\tau} L_A[\tilde{g}](x - a\tau). \tag{38}$$

Then, we have

$$L_A^*[S_\tau \tilde{g}](a\tau - u) = e^{jac\tau^2/2-j(a\tau-u)c\tau} L_A^*[\tilde{g}](-u) \tag{39}$$

and using $L_A[\tilde{g}](-u) = L_A[g](u)$, (37) becomes

$$\begin{aligned} L_A\big[f \overset{A}{\odot} g\big](u) &= B_A^2 e^{jdu^2/(2b)} \int_{-\infty}^{+\infty} f(\tau) e^{jd(u-a\tau)^2/b} e^{jac\tau^2/2} e^{-jc\tau(a\tau-u)} L_A^*[g](u)\, d\tau \\ &= B_A^2 L_A^*[g](u) e^{jdu^2/(2b)} \int_{-\infty}^{+\infty} f(\tau) e^{jdu^2/(2b)} e^{ja\tau^2/(2b)} e^{-ju\tau/b}\, d\tau \\ &= B_A L_A[f](u) L_A^*[g](u) e^{jdu^2/(2b)} \end{aligned} \tag{40}$$

which is once more (35). □

4 Analysis of Sampling in the LCT Domain Based on the Convolution Theorem

The sampling process is central in almost any domain and it explains how to sample continuous signals without aliasing. The sampling theorem expansions for the LCT have been derived in [8, 10, 20, 28], which provides the link between the continuous signals and the discrete signals, and can be used to reconstruct the original signal from their samples satisfying the Nyquist rate of that domain. Here, utilizing the new derived convolution theorem, sampling of bandlimited signals in the LCT domain is further investigated. In particularly, the formulas of uniform sampling and low pass reconstruction are obtained.

Firstly, we define a linear canonical Dirac comb with period T and parameter A by

$$C_{T;A} = T \sum_{n=-\infty}^{n=+\infty} \delta_{nT} e^{-ja(nT)^2/(2b)}. \tag{41}$$

We will use the following result [15]:

$$L_A[\delta_{nT}](u) = B_A e^{j(du^2+a(nT)^2)/(2b)} e^{-junT/b}. \tag{42}$$

Based on the Poisson formula, we obtain

$$\begin{aligned} L_A[C_{T;A}](u) &= T B_A e^{jdu^2/(2b)} \sum_{n=-\infty}^{n=+\infty} e^{-junT/b} \\ &= B_A b\pi e^{jdu^2/(2b)} \sum_{n=-\infty}^{n=+\infty} \delta(u - n\pi b/T) \\ &= T B_A C_{b/T;A^{-1}}(u). \end{aligned} \tag{43}$$

Now, we consider a function whose LCT has a finite support $[-\Omega/2, \Omega/2]$. Let Γ be the function made up of translated replicas of $L_A[f]$, defined by

$$\Gamma = \frac{1}{\Omega} L_A[f] \overset{A^{-1}}{\otimes} C_{\Omega;A^{-1}}. \tag{44}$$

Then, according to (29) and (43), we obtain

$$L_{A^{-1}}[\Gamma](t) = B_A^* f(t) C_{b/\Omega;A}(t) e^{jdu^2/(2b)}$$
$$= B_A^* \frac{\pi b}{\Omega} \sum_{n=-\infty}^{n=+\infty} f\left(\frac{n\pi b}{\Omega}\right) \delta\left(t - \frac{n\pi b}{\Omega}\right). \tag{45}$$

From (45), we see that $L_{A^{-1}}[\Gamma]$ is a sampled version of f. Next, we can obtain $L_A[f]$ through a low pass filter H_Ω satisfying

$$H_\Omega = \begin{cases} 1, & |u| \le \Omega/2, \\ 0, & |u| > \Omega/2. \end{cases} \tag{46}$$

Then $L_A[f](u) = \Gamma H_\Omega(u)$ and

$$L_A[f](u) = \Gamma H_\Omega e^{jdu^2/(2b)} e^{-jdu^2/(2b)}. \tag{47}$$

Using the new convolution definition, (47) becomes

$$f = L_{A^{-1}}[\Gamma] \overset{A}{\otimes} L_{A^{-1}}[\Psi_\Omega], \tag{48}$$

where

$$\Psi_\Omega = H_\Omega(u) e^{jdu^2/(2b)}. \tag{49}$$

Using the LCT of function Ψ_Ω, we obtain

$$L_{A^{-1}}[\Psi_\Omega](t) = B_A \frac{\sin(t\Omega/b)}{t\Omega/b} e^{-jat^2/(2b)}. \tag{50}$$

Finally, using the definition of the new convolution and (48), we have

$$f(t) = e^{-jat^2/(2b)} \sum_{n=-\infty}^{n=+\infty} f(nT_A) e^{ja(nT_A)^2/(2b)} \frac{\sin[\Omega(t-nT_A)/b]}{\Omega(t-nT_A)/b}, \tag{51}$$

where T_A is sampling period and satisfies $T_A = \pi b/\Omega$. Equation (51) provides the values of $f(t)$, in terms of sampled values of f, and constitutes the sampling theorem for LCT.

5 Conclusion

In this paper, we have introduced expressions for the LCT of a convolution and a correlation of two functions. Firstly, we propose a new convolution structure for the LCT using the space shift and phase shift properties of the LCT. Moreover, it can be expressed by a simple one dimensional integral. This result is an extension of the convolution theorem from the FT to the LCT domain, and can be more useful in practical analog filtering in the LCT. Then, using the new convolution theorem, we also obtain a one dimensional integral expression of the correlation for LCT. Since the correlation of two functions is no more than their convolution after one of the two functions has been axis-reversed and complex conjugated, the property of the new convolution results in the property of the correlation. Last, utilizing the new convolution theorem derived in this paper, sampling of band limited signals in the LCT domain has been further investigated. The formulas of uniform sampling and low pass reconstruction are obtained.

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