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Recursive Robust Filtering with Finite-Step Correlated Process Noises and Missing Measurements

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Abstract In this paper, the robust filter design problem is studied for a class of uncertain dynamical systems with finite-step correlated process noises and missing measurements. The dynamical system under consideration is subject to both deterministic norm-bounded uncertainties in the measurement output and stochastic uncertainties on the system states. The process noises are assumed to be finite-step correlated. The missing measurement phenomenon is modeled as a binary switching sequence. Based on the min-max game theory, a recursive robust filter is designed that is suitable for online application. A particular feature is that, as the proposed robust filters work in a recursive fashion, there is no need to investigate the existence issue of the filters. A simulation example is presented to illustrate the usefulness of the proposed filter.

Keywords Recursive filters · Robust filters · Finite-step correlated noises · Min–max game theory · Multiplicative noises · Norm-bound uncertainty · Missing measurements

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1 Introduction

The renowned Kalman filter is the optimal linear state-space filter in the least mean squares sense. As is well recognized now, the performance of the Kalman filter is highly sensitive to the model uncertainties since one of the key assumptions in Kalman filter theory is that a dynamical model has to be exactly known. However, such a key assumption does not always hold in practical applications due to a variety of reasons such as model reduction, parameter variation, and unmodeled dynamics. In order to cope with this problem, over the past decades, considerable research attention has been devoted to develop *robust* state-space filters including the H_∞ filters [2, 7, 8, 10, 24, 29], mixed H_2/H_∞ robust filters [2, 6, 10, 29], set-valued filters [1], and robust mean-square filters [23]. It is worth mentioning that, in [18], a regularized robust filter was proposed. Compared with other robust filters, the regularized robust filter developed in [18] has a nice property of circumventing existence conditions for the filters and is therefore convenient for online operation. Later, an extended version of the regularized robust filter was designed in [5] where an iterative strategy was included.

Multiplicative noise, also called state-dependent noise or stochastic uncertainty, has already received significant research attention in recent years [26, 27]. The main reason is that models with multiplicative noise are commonly encountered in many areas of application, such as aerospace systems [14], communication systems [22], and image processing systems [12, 13, 21], etc. So far, several approaches have been proposed to deal with the analysis and synthesis problems for systems with multiplicative noise, such as the linear matrix inequality approach [9], the game theoretic method [3, 17], the Riccati equation approach [11, 28], and B-spline expansion [15], etc. On the other hand, in most of the literature concerning filtering algorithms, the process noises have been assumed to be uncorrelated across time, which is fairly unrealistic, as pointed out in [20]. For example, if the system under consideration is a discretized version of a continuous dynamical system, then the process noises with discretization errors are inherently correlated across time. Up to now, much attention has been focused on dynamical systems with time-correlated process noises; see, e.g., [19, 20].

In many practical applications, the phenomenon of missing measurements widely exists. A typical example is the network control system where, due to the probabilistic network transmission delay or packet loss, the measurement outputs may contain noises only. Another example is that of the ground target tracking problem. In this case, if the range rate of the target drops below a special threshold in magnitude, the measurements will be deliberately suppressed [4]. Due to the practical significance, both filtering and control problems with missing measurements have been extensively investigated in the past decade; see [16, 25] for some representative publications. Summarizing the above discussion, despite its clear engineering importance, the recursive robust filter design problem has not been fully studied for uncertain dynamical systems with finite-step correlated process noises and/or missing measurements. It is, therefore, the purpose of this paper to shorten this gap.

In this paper, we aim to study the recursive robust filter design problem for uncertain dynamical system with time-correlated process noises and missing measurements. The process noises are assumed to be finite-step correlated. The dynamical system under consideration is subjected to both deterministic norm-bounded uncertainties and stochastic uncertainties, where the latter are actually multiplicative noises. Similar to [16, 25], the missing measurements phenomenon is characterized as a binary switching sequence satisfying a conditional probability distribution. Based on the min-max game theory, a recursive robust filter is proposed. Compared with most of the existing robust filters, the proposed robust filter is not required to have the existence conditions because of its recursive nature and is therefore very convenient for online operation. The main contribution of this paper is twofold: (1) a new filtering problem is formulated that takes into account the multiplicative perturbations on the system states, the missing measurements, and the parameter uncertainties with finite-step correlated process noises; and (2) a new recursive algorithm is developed to solve the addressed problem completely. We present a simulation example to illustrate the usefulness of the proposed filter.

The remainder of the paper is organized as follows. In Sect. 2, the recursive robust filter design problem is formulated for a class of uncertain dynamical systems with finite-correlated process noises and missing measurements. The main results of the paper are derived in Sect. 3. In Sect. 4, a simulation example is presented to illustrate the proposed method. We end the paper with some concluding remarks in Sect. 5.

2 Problem Formulation

Consider the following uncertain dynamical system with finite-step correlated process noises and missing measurements.

$$\text{System model:}\quad x_{k+1} = \left(A_k + \sum_{i=1}^{m_1} A_{i,k}\varepsilon_{i,k}\right)x_k + B_k\omega_k, \tag{1}$$

$$\text{Measurement model:}\quad y_k = \lambda_k(H_k + \Delta H_k)x_k + v_k, \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the state of the system to be estimated, $y_k \in \mathbb{R}^m$ is the measured output, $\omega_k \in \mathbb{R}^q$ is the finite-step correlated process noise, $v_k \in \mathbb{R}^m$ is the measurement noise, $\varepsilon_{i,k} \in \mathbb{R}$ is the multiplicative noise, A_k, $A_{i,k}$, B_k and H_k are known real time-varying matrices with appropriate dimensions, and ΔH_k is a norm-bounded uncertain term of the structure

$$\Delta H_k = E_k \varDelta_k M_k. \tag{3}$$

Here, M_k, E_k are known real time-varying matrices of appropriate dimensions and $\varDelta_k$ is a time-varying unknown contraction bounded by $\|\varDelta_k\| \le 1$. $\lambda_k \in \mathbb{R}$ is a binary switching sequence with the known conditional probability distribution given below:

$$\text{Prob}\{\lambda_k = 1\} = \mathcal{E}\{\lambda_k\} = \beta_k, \tag{4}$$

$$\text{Prob}\{\lambda_k = 0\} = 1 - \mathcal{E}\{\lambda_k\} = 1 - \beta_k. \tag{5}$$

The initial state x_0 and the noise signals ω_k, v_k and $\varepsilon_{i,k}$ are uncorrelated with each other while having the following statistical properties:

$$
\begin{aligned}
&\mathcal{E}(\omega_k) = 0, \qquad \mathcal{E}(v_k) = 0, \qquad \mathcal{E}(\varepsilon_{i,k}) = 0, \qquad \mathcal{E}(x_0) = \bar{x}_0, \\
&\mathcal{E}\big(\varepsilon_{i,k}\varepsilon_{j,l}^T\big) = \delta_{i-j}\delta_{k-l}, \qquad \mathcal{E}(v_k v_j) = R_k\delta_{k-j}, \\
&\mathcal{E}\big[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)^T\big] = P_{0/0}, \\
&\mathcal{E}\big(\omega_k\omega_j^T\big) = Q_k\delta_{k-j} + \sum_{t=1}^{e_k} Q_{k,j}\delta_{k-j+t} + \sum_{t=1}^{f_k} Q_{k,j}\delta_{k-j-t},
\end{aligned}
\tag{6}
$$

where $\mathcal{E}$ stands for the mathematical expectation operator, and e_k and f_k are, respectively, the numbers of steps forward and backward correlated with $Q_{k,j} = Q_{k,j}^T$. δ_{i-j} is the Kronecker delta function, which is equal to unity for $i = j$ and zero for $i \neq j$.

Remark 1 From the last equality in (6), one can see that the process noise at time k is correlated with the process noises at time steps $k-1, k-2, \ldots, k-f_k$ as well as $k+1, k+2, \ldots, k+e_k$ with covariance $Q_{k,k-1}, Q_{k,k-2}, \ldots, Q_{k,k-f_k}$ and $Q_{k,k+1}$, $Q_{k,k+2}, \ldots, Q_{k,k+e_k}$, respectively.

Most optimal filter recursive algorithms can be generally described in two stages, that is, time update and measurement update. In the time update stage, the optimal state prediction at $k+1$ is obtained using the system dynamics and the optimal state estimate at time k. In the measurement update stage, the newly obtained measurements are used to improve the accuracy of the state prediction. We follow the same method to develop a filter that is robust against the system uncertainties, correlated process noise, and missing measurements. In particular, the min-max game theory is intensively utilized to design the desired robust filters ensuring the recursion and optimality for the benefits of online operation. For convenience of later development, it can be calculated that

$$
\begin{aligned}
\mathcal{E}\big(x_k\omega_k^T\big) &= \mathcal{E}\left[\left(A_{k-1}x_{k-1} + \sum_{i=1}^{m_1} A_{i,k-1}\varepsilon_{i,k-1}x_{k-1} + B_{k-1}\omega_{k-1}\right)\omega_k^T\right] \\
&= A_{k-1}\mathcal{E}\big(x_{k-1}\omega_k^T\big) + \sum_{i=1}^{m_1} A_{i,k-1}\mathcal{E}(\varepsilon_{i,k-1})\mathcal{E}\big(x_{k-1}\omega_k^T\big) + B_{k-1}Q_{k-1,k} \\
&= A_{k-1}\mathcal{E}\left[\left(A_{k-2}x_{k-2} + \sum_{i=1}^{m_1} A_{i,k-2}\varepsilon_{i,k-2}x_{k-2} + B_{k-2}\omega_{k-2}\right)\omega_k^T\right] \\
&\quad + B_{k-1}Q_{k-1,k} \\
&= A_{k-1}A_{k-2}\mathcal{E}\big(x_{k-2}w_k^T\big) + A_{k-1}B_{k-2}Q_{k-2,k} + B_{k-1}Q_{k-1,k} \\
&\;\;\vdots \\
&= B_{k-1}Q_{k-1,k} + \sum_{t=2}^{f_k}\left(\prod_{j=2}^{t} A_{k+1-j}\right)B_{k-t}Q_{k-t,k}.
\end{aligned}
\tag{7}
$$

Remark 2 In the classical Kalman filtering, a standard assumption is that the process noise sequence is uncorrelated across time and, therefore, $Q_{k,j}$ is equal to zero and the states $x_k, x_{k-1}, \dots, x_{k-f_k+1}$ are uncorrelated with the process noise ω_k. Unfortunately, in many practical applications, such a standard assumption does not hold. As can be seen in (7), the states $x_k, x_{k-1}, \dots, x_{k-f_k+1}$ are correlated with the process noise ω_k and $Q_{k,j}$ is no longer zero. In our later filter performance analysis, the relationship (7) will be widely used in our computation.

3 Main Results

3.1 Time Update Stage

Noticing that the multiplicative noise $\varepsilon_{i,k}$ and process noise ω_k are both zero-mean random process sequences, the state prediction can be obtained as follows:

$$\hat{x}_{k+1/k} = A_k \hat{x}_{k/k}, \tag{8}$$

where $\hat{x}_{k/k}$ is the state estimate and $\hat{x}_{k+1/k}$ is the state prediction. Thus, the prediction error covariance $P_{k+1/k}$ can be described as follows:

$$\begin{aligned} P_{k+1/k} &= \mathcal{E}\big[(x_{k+1} - \hat{x}_{k+1/k})(x_{k+1} - \hat{x}_{k+1/k})^T\big] \\ &= \mathcal{E}\left\{\left[A_k(x_k - \hat{x}_{k/k}) + \sum_{i=1}^{m_1} A_{i,k}\varepsilon_{i,k}x_k + B_k\omega_k\right]\right. \\ &\qquad \left. \times \left[A_k(x_k - \hat{x}_{k/k}) + \sum_{i=1}^{m_1} A_{i,k}\varepsilon_{i,k}x_k + B_k\omega_k\right]^T\right\} \\ &= A_k P_{k/k} A_k^T + B_k Q_k B_k^T + \sum_{i=1}^{m_1} A_{i,k} X_k A_{i,k}^T + A_k L_k B_k^T + B_k L_k^T A_k^T, \end{aligned} \tag{9}$$

where $P_{k/k}$ is the state estimate error covariance at time k, $X_k = \mathcal{E}(x_k x_k^T)$, and $L_k = \mathcal{E}[(x_k - \hat{x}_{k/k})\omega_k^T]$. Note that, when deriving the last equality in (9), we have used the facts that (i) x_k and $\hat{x}_{k/k}$ are correlated with ω_k; and (ii) $\varepsilon_{i,k}$ and ω_k are zero-mean random processes uncorrelated with each other.

It follows from (1) and (7) that

$$\begin{aligned} X_{k+1} &= \mathcal{E}\big(x_{k+1}x_{k+1}^T\big) \\ &= \mathcal{E}\left[\left(A_k x_k + \sum_{i=1}^{m_1} A_{i,k}\varepsilon_{i,k}x_k + B_k\omega_k\right)\left(A_k x_k + \sum_{i=1}^{m_1} A_{i,k}\varepsilon_{i,k}x_k + B_k\omega_k\right)^T\right] \\ &= A_k X_k A_k^T + \sum_{i=1}^{m_1} A_{i,k} X_k A_{i,k}^T + B_k Q_k B_k^T + A_k\mathcal{E}\big(x_k\omega_k^T\big)B_k^T + B_k\mathcal{E}\big(\omega_k x_k^T\big)A_k^T \end{aligned}$$

$$
\begin{aligned}
&= A_k X_k A_k^T + \sum_{i=1}^{m_1} A_{i,k} X_k A_{i,k}^T + B_k Q_k B_k^T \\
&\quad + A_k \left[B_{k-1} Q_{k-1,k} + \sum_{t=2}^{f_k} \left(\prod_{j=2}^{t} A_{k+1-j} \right) B_{k-t} Q_{k-t,k} \right] B_k^T \\
&\quad + B_k \left[B_{k-1} Q_{k-1,k} + \sum_{t=2}^{f_k} \left(\prod_{j=2}^{t} A_{k+1-j} \right) B_{k-t} Q_{k-t,k} \right]^T A_k^T
\end{aligned}
\tag{10}
$$

with initial value

$$
X_0 = \bar{x}_0 \bar{x}_0^T + P_{0/0}. \tag{11}
$$

Remark 3 The last three terms in (9) result from the inclusion of multiplicative noises and finite-step correlated process noises in the filter design. Indeed, these three terms make the difference between the prediction error covariance in this paper and that in the standard Kalman filtering algorithm. Furthermore, the summation (third) term in (9) caused by multiplicative noises constitutes the main difference of the prediction error covariance between our work and the work of [20].

3.2 Measurement Update Stage

At this stage, let us use the min-max game theory, the aim of which is to minimize the estimate error under maximum uncertainties, to design the measurement update equations. For the addressed system and measurement model, by using the min-max game theory, the problem of measurement update can be described as follows:

$$
\begin{aligned}
\hat{x}_{k+1/k+1} = \arg \min_{x_{k+1}} \max_{\|m_{k+1}\| \le \alpha_{k+1}} &\Big\{ \|x_{k+1} - \hat{x}_{k+1/k}\|_{P_{k+1/k}}^2 \\
&+ \big\| y_{k+1} - \beta_{k+1} \hat{y}_{k+1}(m_{k+1}) \big\|_{\tilde{S}_{k+1}}^2 \Big\},
\end{aligned}
\tag{12}
$$

where $\hat{y}_{k+1}(m_{k+1}) = H_{k+1}\hat{x}_{k+1/k} + E_{k+1}m_{k+1}$, $P_{k+1/k}$ is the prediction error covariance, α_{k+1} is a known time-varying scalar, $\tilde{S}_{k+1}$ is a weighting matrix that will be defined later, and m_{k+1} has the following form:

$$
m_{k+1} = \Delta_{k+1} M_{k+1} \hat{x}_{k+1/k} \tag{13}
$$

with the special norm function being defined as

$$
\| \cdot \|_P^2 = (\cdot)^T P^{-1} (\cdot). \tag{14}
$$

Remark 4 In [5], the parameter uncertainties in the measurement model have been considered. In our work, we have taken one step further to consider both the multiplicative noises and the probabilistic missing measurements phenomenon.

The solution of (12) relies on solving the constrained maximization problem of the form

$$u^* = \arg \max_{\|u\| \le \alpha} \|x - Bu\|_P^2, \tag{15}$$

where P is a positive definite weighting matrix and α is a known scalar.

The following two lemmas are useful in deriving our main results.

Lemma 1 [5] *When $x \neq 0$, the solution to (15) is*

$$u^* = \alpha\big(\beta^* I + \alpha B^T P^{-1} B\big)^{-1} B^T P^{-1} x, \tag{16}$$

where $\beta^* \in M \triangleq \{\beta_+, -\beta_-\}$ and β^* satisfies

$$\beta^* = \arg \max_{\beta \in M} \left\| P\left(P + \frac{\alpha}{\beta} B B^T\right)^{-1} x \right\|_P^2. \tag{17}$$

The maximum value of the objective function is

$$\max_{\|u\| \le \alpha} \|x - Bu\|_P^2 = \|x - Bu^*\|_P^2 = \left\| P\left(P + \frac{\alpha}{\beta^*} B B^T\right)^{-1} x \right\|_P^2. \tag{18}$$

In the above equations, β_+ and β_- satisfy, respectively,

$$\big\| \big(\beta_+ I + \alpha B^T P^{-1} B\big)^{-1} B^T P^{-1} x \big\| = 1, \tag{19}$$

$$\det\big(\beta_+ I + \alpha B^T P^{-1} B\big) \neq 0 \tag{20}$$

and

$$\big\| \big(\beta_- I - \alpha B^T P^{-1} B\big)^{-1} B^T P^{-1} x \big\| = 1, \tag{21}$$

$$\det\big(\beta_- I - \alpha B^T P^{-1} B\big) \neq 0. \tag{22}$$

Lemma 2 [5] *When $x = 0$, u^* is simply an eigenvector of $B^T P^{-1} B$ which generates the maximum error.*

Now, we are in a position to complete the measurement update. Based on Lemmas 1 and 2, it can be seen that

$$\begin{aligned}
&\max_{\|m_{k+1}\| \le \alpha_{k+1}} \big\| y_{k+1} - \beta_{k+1} \hat{y}_{k+1}(m_{k+1}) \big\|_{\tilde{S}_{k+1}}^2 \\
&\quad = \max_{\|m_{k+1}\| \le \alpha_{k+1}} \|\tilde{y}_{k+1} - \bar{E}_{k+1} m_{k+1}\|_{\tilde{S}_{k+1}}^2 \\
&\quad = \left\| \tilde{S}_{k+1}\left(\tilde{S}_{k+1} + \frac{\alpha_{k+1}}{\beta_m} \bar{E}_{k+1} \bar{E}_{k+1}^T\right)^{-1} \tilde{y}_{k+1} \right\|_{\tilde{S}_{k+1}}^2,
\end{aligned} \tag{23}$$

where β_m is defined and calculated as in Lemma 1, $\tilde{y}_{k+1} = y_{k+1} - \beta_{k+1} H_{k+1} \hat{x}_{k+1/k}$ is the nominal predicted output without considering the uncertainty in the model, $\bar{E}_{k+1} = \beta_{k+1} E_{k+1}$, and $\tilde{S}_{k+1}$ is defined as follows:

$$\tilde{S}_{k+1} = H_{k+1} P_{k+1/k} H_{k+1}^T + R_{k+1} + (1-\beta_{k+1})^2 H_{k+1} \Sigma_{k+1} H_{k+1}^T, \tag{24}$$

where $\Sigma_{k+1} = A_k \mathcal{E}(\hat{x}_{k/k} \hat{x}_{k/k}^T) A_k^T$.

Substituting (23) into (12), taking the derivative of (12) with respect to x_{k+1}, and then letting the derivative be zero, we can obtain the following measurement update equations:

$$\hat{x}_{k+1/k+1} = \hat{x}_{k+1/k} + W_{k+1} \tilde{y}_{k+1}, \tag{25}$$

where

$$W_{k+1} = P_{k+1/k} H_{k+1}^T S_{k+1}^{-1} \tag{26}$$

with

$$S_{k+1} = \tilde{S}_{k+1} + 2\frac{\alpha_{k+1}}{\beta_m} \bar{E}_{k+1} \bar{E}_{k+1}^T + \frac{\alpha_{k+1}^2}{\beta_m^2} \bar{E}_{k+1} \bar{E}_{k+1}^T \tilde{S}_{k+1}^{-1} \bar{E}_{k+1} \bar{E}_{k+1}^T. \tag{27}$$

Also, the estimation error covariance is calculated as

$$P_{k+1/k+1} = P_{k+1/k} - W_{k+1} S_{k+1} W_{k+1}^T. \tag{28}$$

Next, let us determine L_k. According to (25), we have

$$\begin{aligned} \hat{x}_{k/k} &= \hat{x}_{k-1/k} + W_k \tilde{y}_k = \hat{x}_{k/k-1} + W_k (y_k - \beta_k H_k \hat{x}_{k/k-1}) \\ &= G_k \hat{x}_{k/k-1} + W_k y_k, \end{aligned} \tag{29}$$

where $G_k = I - \beta_k W_k H_k$. Then, L_k can be determined as follows:

$$\begin{aligned} L_k &= \mathcal{E}\big[(x_k - \hat{x}_{k/k})\omega_k^T\big] \\ &= \mathcal{E}\big[(x_k - G_k \hat{x}_{k/k-1} - W_k y_k)\omega_k^T\big] \\ &= \mathcal{E}\big[(x_k - G_k \hat{x}_{k/k-1} - W_k H_k x_k - W_k v_k)\omega_k^T\big] \\ &= (I - W_k H_k)\mathcal{E}\big(x_k \omega_k^T\big) - G_k A_{k-1} \\ &\quad \times \mathcal{E}\big[(G_{k-1}\hat{x}_{k-1/k-2} + W_{k-1} H_{k-1} x_{k-1} + W_{k-1} v_{k-1})\omega_k^T\big] \\ &= (I - W_k H_k)\mathcal{E}\big(x_k \omega_k^T\big) - G_k A_{k-1} W_{k-1} H_{k-1} \mathcal{E}\big(x_{k-1}\omega_k^T\big) \\ &\quad - G_k A_{k-1} G_{k-1} \mathcal{E}\big(\hat{x}_{k-1/k-2}\omega_k^T\big) \\ &\ \ \vdots \\ &= (I - W_k H_k)\left[B_{k-1} Q_{k-1,k} + \sum_{t=2}^{f_k} \left(\prod_{j=2}^{t} A_{k+1-j}\right) B_{k-t,k} Q_{k-t,k}\right] \end{aligned}$$

$$
\begin{aligned}
&-\sum_{t=2}^{f_k}\left\{\left[\left(\prod_{j=2}^{t} G_{k+2-j}A_{k+1-j}\right)W_{k+1-t}H_{k+1-t}\right]\right.\\
&\left.\times\left[B_{k-t}Q_{k-t,k}+\sum_{j_1=t+1}^{f_k}\left(\prod_{j_2=t+1}^{j_1}A_{k+1-j_2}\right)B_{k-j_1}Q_{k-j_1,k}\right]\right\}.
\end{aligned}
\tag{30}
$$

To this end, we have finished the design of the desired recursive robust filters for the addressed system and measurement models.

The algorithm of the recursive robust filtering can be outlined as follows.

1. The time update stage

When $k=0$, ignoring the correlation between the process noises and using the given initial values, we have

$$
\begin{aligned}
&\hat{x}_{1/0}=A_0\hat{x}_{0/0},\\
&P_{1/0}=A_0P_{0/0}A_0^T+B_0Q_0B_0^T+\sum_{i=1}^{m_1}A_{i,0}X_0A_{i,0}^T.
\end{aligned}
\tag{31}
$$

When $k>0$, the time update is given by

$$
\begin{aligned}
&\hat{x}_{k+1/k}=A_k\hat{x}_{k/k},\\
&X_{k+1}=A_kX_kA_k^T+\sum_{i=1}^{m_1}A_{i,k}X_kA_{i,k}^T+B_kQ_kB_k^T\\
&\qquad+A_k\left[B_{k-1}Q_{k-1,k}+\sum_{t=2}^{f_k}\left(\prod_{j=2}^{t}A_{k+1-j}\right)B_{k-t}Q_{k-t,k}\right]B_k^T\\
&\qquad+B_k\left[B_{k-1}Q_{k-1,k}+\sum_{t=2}^{f_k}\left(\prod_{j=2}^{t}A_{k+1-j}\right)B_{k-t}Q_{k-t,k}\right]^TA_k^T,\\
&P_{k+1/k}=A_kP_{k/k}A_k^T+B_kQ_kB_k^T+\sum_{i=1}^{m_1}A_{i,k}X_kA_{i,k}^T+A_kL_kB_k^T+B_kL_k^TA_k^T,
\end{aligned}
\tag{32}
$$

where L_k is determined by (30).

2. Measurement update stage

$$
\begin{aligned}
&\hat{x}_{k+1/k+1}=\hat{x}_{k+1/k}+W_{k+1}\tilde{y}_{k+1},\\
&W_{k+1}=P_{k+1/k}H_{k+1}^TS_{k+1}^{-1},\\
&S_{k+1}=\tilde{S}_{k+1}+2\frac{\alpha_{k+1}}{\beta_m}\bar{E}_{k+1}\bar{E}_{k+1}^T+\frac{\alpha_{k+1}^2}{\beta_m^2}\bar{E}_{k+1}\bar{E}_{k+1}^T\tilde{S}_{k+1}^{-1}\bar{E}_{k+1}\bar{E}_{k+1}^T,\\
&P_{k+1/k+1}=P_{k+1/k}-W_{k+1}S_{k+1}W_{k+1}^T.
\end{aligned}
\tag{33}
$$

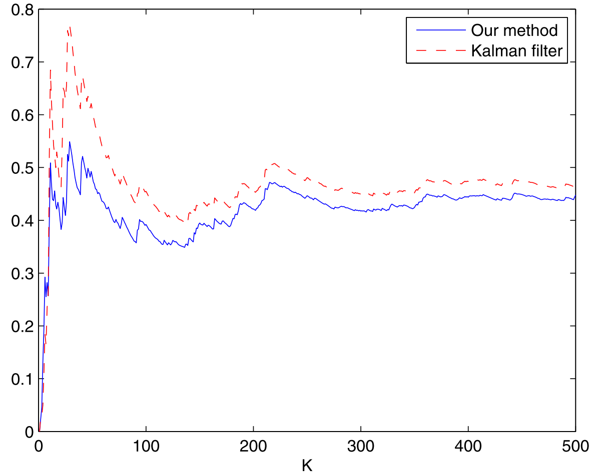


Fig. 1 MSE1 ($\beta = 1$)

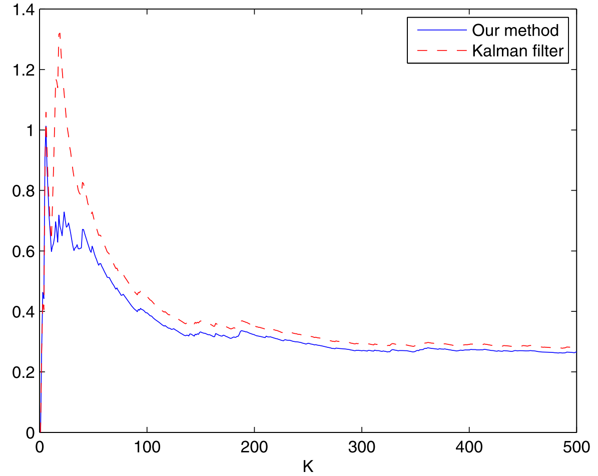


Fig. 2 MSE2 ($\beta = 1$)

Remark 5 When $k = 0$, for the time update stage, it is reasonable to ignore the correlation between the process noises simply because the process noises $\omega_{-1}, \omega_{-2}, \ldots, \omega_{-f_k}$ do not exist, and thus the process noise ω_0 is not correlated with x_0 and $\hat{x}_{0/0}$. When $k > 0$, also for the time update stage, if $k < f_k$, we can set $f_k = k$ because $\omega_{-1}, \omega_{-2}, \ldots, \omega_{-(f_k-k)}$ do not exist.

4 An Illustrative Example

As an illustrative example, let us apply the developed new recursive robust filter to the following target tracking system:

$$x_{k+1} = \left(\begin{bmatrix} 0.98 & T \\ 0 & 0.98 \end{bmatrix} + \varepsilon_k \begin{bmatrix} 0.005 & 0 \\ 0 & 0.01 \end{bmatrix} \right) x_k + \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix} \omega_k, \tag{34}$$

Fig. 3 MSE1 ($\beta = 0.9$)

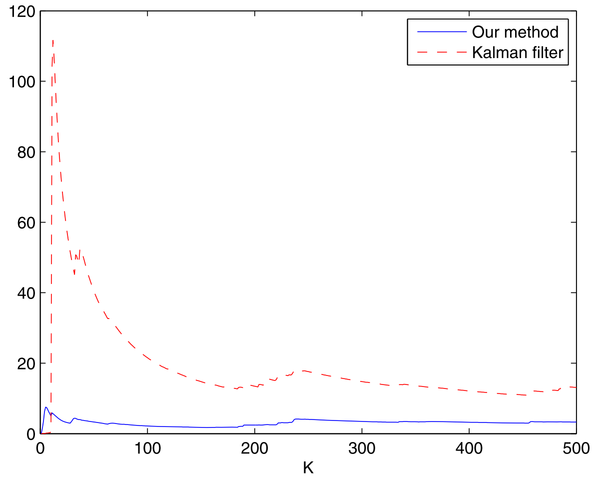
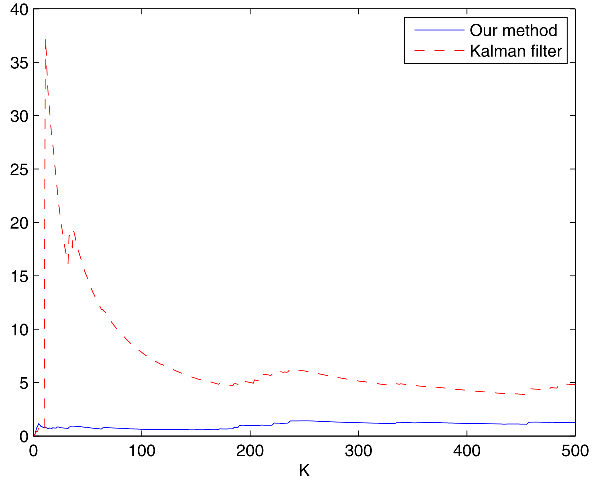


Fig. 4 MSE2 ($\beta = 0.9$)



$$y_k = \lambda_k \begin{bmatrix} 1 & 0 \\ 0 & 1 + 0.1\sin(k) \end{bmatrix} x_k + v_k, \tag{35}$$

$$\omega_k = \eta_k + \eta_{k-1}, \tag{36}$$

where T is the sampling period, the first component of x_k is the position of the target, and the second component of x_k is the velocity of the target. $v_k \in \mathbb{R}^2$, $\varepsilon_k \in \mathbb{R}$ and $\eta_k \in \mathbb{R}$ are zero-mean Gaussian noises with variance I_2, 1 and 0.5, respectively. Without loss of generality, the process noise ω_k is assumed to obey (36); that is, the process noise is one-step correlated with statistic properties as follows:

$$\mathcal{E}(\omega_k) = 0, \qquad \mathcal{E}\left(\omega_k \omega_j^T\right) = \delta_{k-j} + 0.5\delta_{k-j-1} + 0.5\delta_{k-j+1}. \tag{37}$$

In the simulation, we set the initial value as $\hat{x}_{0/0} = \bar{x}_0 = \mathcal{E}(x_0) = [100 \quad -2]$ and $P_{0/0} = \mathrm{diag}(3, 1)$. The sampling period is chosen as 0.1 and α_k is chosen as

Fig. 5 MSE1 ($\beta = 0.85$)

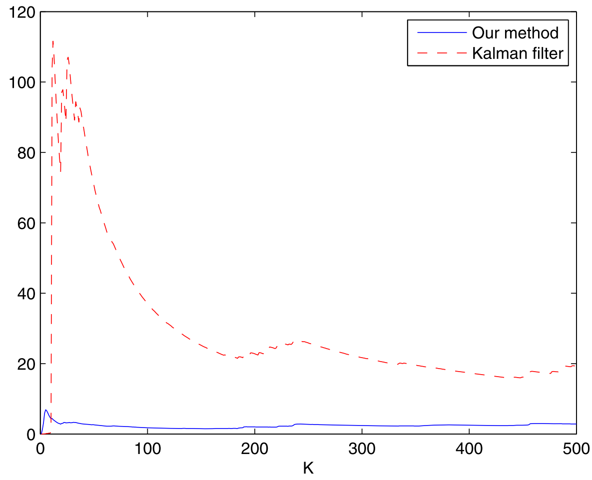
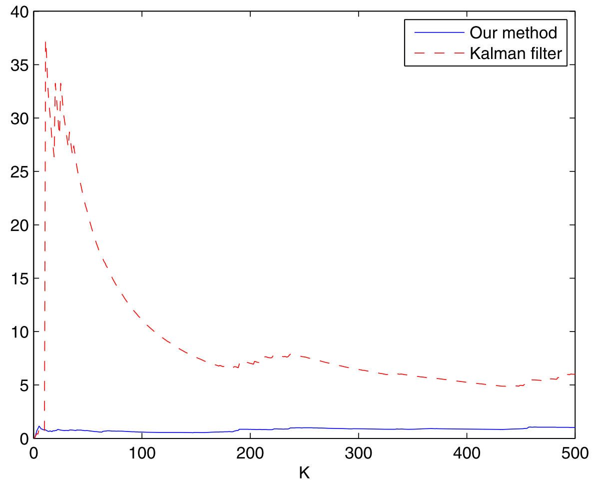


Fig. 6 MSE2 ($\beta = 0.85$)



0.005. The stochastic variable $\lambda_k \in \mathbb{R}$ is a binary switching sequence taking the value of 0 and 1 with $\mathrm{Prob}\{\lambda_k = 1\} = \mathcal{E}\{\lambda_k\} = \beta \in \{1, 0.9, 0.85\}$. The classical Kalman filter and the recursive robust filter developed in this paper are compared in the simulation. Let MSE1 denote the mean square error for the estimate of the position, i.e., $(1/K)\sum_{k=1}^{K}\{[1\ \ 0](x_k - \hat{x}_{k/k})\}^2$, where K is the number of the samples. Similarly, MSE2 is the mean square error for the estimation of the velocity, i.e., $(1/K)\sum_{k=1}^{K}\{[0\ \ 1](x_k - \hat{x}_{k/k})\}^2$.

When $\beta = 1$, i.e., there is no missing measurement phenomenon, the simulation results are given as Figs. 1 and 2. It can be seen that our method has better estimation accuracy than the classical Kalman filter. This is due to the fact that efforts have been made to compensate the uncertainties and correlated process noises.

When $\beta = 0.9$ and $\beta = 0.85$, that is, when the missing measurement phenomenon does exist, it can be seen from Figs. 3–6 that our method performs much better than the classical Kalman filter. This is not surprising, as our method has the mechanism to compensate the missing measurement phenomenon, uncertainties, as well as correlated process noises. Nevertheless, as can be observed from Figs. 3–6, our method is slightly worse than the classical Kalman filter in the initial period because our method is a bit more conservative in order to tolerate the possible missing measurement phenomenon, which causes a trade-off between the robustness and the accuracy.

5 Conclusions

In this paper, we have designed a novel recursive robust filter for uncertain dynamical systems with finite-step correlated process noise and missing measurements. The dynamical system under consideration is subjected to both deterministic norm-bounded uncertainties and stochastic uncertainties. The process noise is assumed to be finite-step correlated, and the missing measurements phenomenon is assumed to be a binary switching sequence satisfying a conditional probability distribution. Based on the min-max game theory, we have derived a recursive robust filter for the addressed uncertain dynamical system. Compared with most of the existing robust filters, the proposed recursive robust filter does not require existence conditions and is therefore convenient for online operation. An illustrative example has been presented to show the effectiveness of the proposed recursive robust filter.

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