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Existence and Global Asymptotic Stability of Fuzzy Cellular Neural Networks with Time Delay in the Leakage Term and Unbounded Distributed Delays

P. Balasubramaniam · M. Kalpana · R. Rakkiyappan

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Abstract Fuzzy cellular neural network (FCNN) structures are based on the uncertainties in human cognitive processes and in modeling neural systems, and they provide an interface between a human expert and classical cellular neural networks (CNNs). In this paper, an existence and global asymptotic stability analysis of the equilibrium point of FCNNs with time delay in the leakage term and unbounded distributed delays is investigated. Based on the Lyapunov-Krasovskii functional with free-weighting matrix, and using the homeomorphism mapping principle and linear matrix inequalities (LMIs), a new set of stability criteria for FCNNs is obtained with time delay in the leakage term, time-varying delays and unbounded distributed delays. The proposed results can be easily checked via the LMI Control Toolbox in MATLAB. Moreover, it is well known that the stability behavior of FCNNs is very sensitive to the time delay in the leakage term. In the absence of the leakage term, a new stability criterion is also derived by employing a Lyapunov-Krasovskii functional using an LMI approach. Numerical examples are provided to illustrate the effectiveness and reduced conservativeness of the developed techniques.

Keywords Global asymptotic stability · Fuzzy cellular neural networks · Leakage delay · Unbounded distributed delays · Linear matrix inequality · Homeomorphism mapping

1 Introduction

Since the rebirth of interest in artificial neural networks (ANNs) in 1982 [2, 8, 13], many different implementations for this concept have been developed. ANNs have

P. Balasubramaniam (✉) · M. Kalpana · R. Rakkiyappan
Department of Mathematics, Gandhigram Rural Institute, Deemed University, Gandhigram 624 302, Tamilnadu, India
e-mail: balugru@gmail.com

found extensive applications in pattern recognition, real time image processing, signal processing, and control, etc. The major disadvantage of most neural network implementations however is the number of interconnections between neurons. In order to reduce the number of interconnections by keeping the advantages of parallel processing, Chua and Yang proposed a cellular neural network (CNN) in 1988 [6, 7] where neurons were only connected to other neurons within a certain neighborhood. Also, a CNN can easily be extended without having to re-adjust the entire network because a cell is not connected to every other cell in the network but rather to cells within a certain neighborhood. So far, there are two basic CNN structures being proposed. The first one is the traditional CNN. The second one is the fuzzy cellular neural network (FCNN) [28, 29], which integrates fuzzy logic into the structure of a traditional CNN and maintains local connectedness among cells. Unlike previous CNN structures, the FCNN has fuzzy logic between its template and input and/or output besides the "sum of product" operation.

The existence of time delays may lead to instability or bad performance of systems [23]. So, it is of prime importance to consider the delay effects on the dynamical behavior of systems. Recently, FCNNs with various types of delay have been widely investigated by many authors; see [5, 9, 14, 15, 20–22, 25, 26, 30]. For instance, Liu et al. [22] investigated the novel stability criteria of a new FCNN with time-varying delays. In [25], Tan investigated the global asymptotic stability of FCNNs with unbounded distributed delays. However, so far, there is very little existing work on neural networks (NNs) with time delay in the leakage (or "forgetting") term [1, 10, 11, 16–19, 24]. This is due to some theoretical and technical difficulties [10]. In fact, time delay in the leakage term also has great impact on the dynamics of NNs.

Almost all the models of the FCNNs are variations of the following system of differential equations:

$$\begin{aligned}\dot{x}_i(t) = {} & -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j\big(x_j(t)\big) + \sum_{j=1}^{n} b_{ij} u_j + I_i + \bigwedge_{j=1}^{n} \alpha_{ij} f_j\big(x_j(t-\tau)\big) \\ & + \bigvee_{j=1}^{n} \beta_{ij} f_j\big(x_j(t-\tau)\big) + \bigwedge_{j=1}^{n} T_{ij} u_j + \bigvee_{j=1}^{n} H_{ij} u_j, \quad i = 1, 2, \ldots, n,\end{aligned}$$

in which the first term of the right side of the above equation corresponds to a stabilizing negative feedback of the system which acts instantaneously without time delay; this kind of term is variously known as a "forgetting" or leakage term. As pointed out by Gopalsamy [11], time delay in the stabilizing negative feedback term has a tendency to destabilize a system. Moreover the effects of leakage delay on FCNNs cannot be ignored.

To the best of the authors' knowledge, there are only a few results on FCNNs with time delay in the leakage term. Our results indicate that the stability behavior of FCNNs is very sensitive to the time delay in the leakage term. This implies that the effects of leakage delay on FCNNs cannot be ignored. Three numerical examples are given to show the effectiveness of our result.

2 Model Description

Notation $\mathcal{R}^n$ denotes the n-dimensional Euclidean space. For any matrix $A = [a_{ij}]_{n\times n}$, let A^T and A^{-1} denote the transpose and the inverse of A, respectively. $|A| = [|a_{ij}|]_{n\times n}$. Let $A > 0$ ($A < 0$) denote the positive definite (negative-definite) symmetric matrix, respectively. I denotes the identity matrix of appropriate dimension, and $\Lambda = \{1, 2, \dots, n\}$. $*$ denotes the symmetric terms in a symmetric matrix.

Consider the following FCNNs with leakage delay and unbounded distributed delays:

$$\begin{aligned}\dot{x}_i(t) &= -d_i x_i(t-\sigma) + \sum_{j=1}^{n} a_{ij} f_j\big(x_j(t)\big) + \sum_{j=1}^{n} b_{ij} f_j\big(x_j\big(t-\tau(t)\big)\big) + \sum_{j=1}^{n} c_{ij} u_j + I_i \\ &\quad + \bigwedge_{j=1}^{n} \alpha_{ij} \int_{-\infty}^{t} k_j(t-s) f_j\big(x_j(s)\big)\,ds + \bigvee_{j=1}^{n} \beta_{ij} \int_{-\infty}^{t} k_j(t-s) f_j\big(x_j(s)\big)\,ds \\ &\quad + \bigwedge_{j=1}^{n} \zeta_{ij} u_j + \bigvee_{j=1}^{n} \delta_{ij} u_j, \quad i \epsilon \Lambda, \end{aligned} \tag{1}$$

where α_{ij}, β_{ij}, ζ_{ij} and δ_{ij} are the elements of the fuzzy feedback MIN template, fuzzy feedback MAX template, fuzzy feedforward MIN template and fuzzy feedforward MAX template, respectively; a_{ij}, b_{ij} are the elements of the feedback template and c_{ij} is the element of the feedforward template; $\bigwedge$, $\bigvee$ denote the fuzzy AND and fuzzy OR operation, respectively; x_i, u_i and I_i denote the state, input and bias of the ith neuron, respectively; d_i is a diagonal matrix, d_i represents the rates with which the i-th neuron will reset its potential to the resting state in isolation when disconnected from the networks and external inputs; $f_i(\cdot)$ is the activation function; $k_i(s) \geq 0$ is the feedback kernel and satisfies

$$\int_0^{\infty} k_i(s)\,ds = 1, \quad i \epsilon \Lambda. \tag{2}$$

In this paper, we make the following assumptions.

(A_1) The neuron activation function $f_j(\cdot)$ is Lipschitz continuous; that is, there exist constants $l_j > 0$ such that

$$\big|f_j(\xi_1) - f_j(\xi_2)\big| \leq l_j|\xi_1 - \xi_2|, \quad \text{for all } \xi_1, \xi_2 \in \mathcal{R}, \xi_1 \neq \xi_2.$$

(A_2) The transmission delay $\tau(t)$ is a time-varying delay, and it satisfies $0 \leq \tau(t) \leq \tau$, where τ is a positive constant.

(A_3) The leakage delay satisfies $\sigma \geq 0$.

We shall consider system (1) with the initial condition

$$x(s) = \varphi(s), \quad s \in (-\infty, 0].$$

Assume that $x^* = [x_1^*, x_2^*, \ldots, x_n^*]^T$ is an equilibrium point of FCNNs (1). By the transformation $y_i = x_i - x_i^*$ one can transform (1) into the following system:

$$\begin{aligned}
\dot{y}_i(t) &= -d_i y_i(t-\sigma) + \sum_{j=1}^{n} a_{ij} g_j\big(y_j(t)\big) + \sum_{j=1}^{n} b_{ij} g_j\big(y_j\big(t-\tau(t)\big)\big) \\
&\quad + \bigwedge_{j=1}^{n} \alpha_{ij} \int_{-\infty}^{t} k_j(t-s) f_j\big(y_j(s)+x_j^*\big)\,ds \\
&\quad - \bigwedge_{j=1}^{n} \alpha_{ij} \int_{-\infty}^{t} k_j(t-s) f_j\big(x_j^*\big)\,ds \\
&\quad + \bigvee_{j=1}^{n} \beta_{ij} \int_{-\infty}^{t} k_j(t-s) f_j\big(y_j(s)+x_j^*\big)\,ds \\
&\quad - \bigvee_{j=1}^{n} \beta_{ij} \int_{-\infty}^{t} k_j(t-s) f_j\big(x_j^*\big)\,ds, \\
y(s) &= \varphi(s) - x^*, \quad s \in (-\infty, 0],
\end{aligned} \tag{3}$$

where $g_j(y_j(s)) = f_j(y_j(s)+x_j^*) - f_j(x_j^*)$.

Using a simple transformation, system (3) has an equivalent form as follows:

$$\begin{aligned}
&\frac{d}{dt}\left[y_i(t) - d_i \int_{t-\sigma}^{t} y_i(s)\,ds\right] \\
&\quad = -d_i y_i(t) + \sum_{j=1}^{n} a_{ij} g_j\big(y_j(t)\big) + \sum_{j=1}^{n} b_{ij} g_j\big(y_j\big(t-\tau(t)\big)\big) \\
&\qquad + \bigwedge_{j=1}^{n} \alpha_{ij} \int_{-\infty}^{t} k_j(t-s) f_j\big(y_j(s)+x_j^*\big)\,ds \\
&\qquad - \bigwedge_{j=1}^{n} \alpha_{ij} \int_{-\infty}^{t} k_j(t-s) f_j\big(x_j^*\big)\,ds \\
&\qquad + \bigvee_{j=1}^{n} \beta_{ij} \int_{-\infty}^{t} k_j(t-s) f_j\big(y_j(s)+x_j^*\big)\,ds \\
&\qquad - \bigvee_{j=1}^{n} \beta_{ij} \int_{-\infty}^{t} k_j(t-s) f_j\big(x_j^*\big)\,ds, \\
&y(s) = \varphi(s) - x^*, \quad s \in (-\infty, 0].
\end{aligned} \tag{4}$$

In the following, we use the notation: $y(t) = [y_1(t), y_2(t), \dots, y_n(t)]^T$, $K = \operatorname{diag}\{k_1, k_2, \dots, k_n\}$, $D = \operatorname{diag}\{d_1, d_2, \dots, d_n\}$, $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$, $\alpha = [\alpha_{ij}]_{n\times n}$, $\beta = [\beta_{ij}]_{n\times n}$, $g(y(t)) = [g_1(y_1(t)), g_2(y_2(t)), \dots, g_n(y_n(t))]^T$.

Definition 2.1 [4] A map $H : \mathcal{R}^n \to \mathcal{R}^n$ is a homeomorphism of $\mathcal{R}^n$ onto itself if H is continuous and one-to-one and its inverse map H^{-1} is also continuous.

Lemma 2.1 (Schur Complement [3]) *The linear matrix inequality* (*LMI*) $\left[\begin{smallmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{smallmatrix}\right] > 0$, *where* $Q(x) = Q^T(x)$, $R(x) = R^T(x)$, *is equivalent to*

$$R(x) > 0 \quad \text{and} \quad Q(x) - S(x)R^{-1}(x)S^T(x) > 0.$$

Lemma 2.2 *For any* $x, y \in \mathcal{R}^n, \epsilon > 0$ *and positive definite matrix* $Q \in \mathcal{R}^{n\times n}$, *the following matrix inequality holds*:

$$2x^T y \le \epsilon x^T Q x + \epsilon^{-1} y^T Q^{-1} y.$$

Lemma 2.3 [27] *Let* z, z' *be two states of system* (1); *then we have*

$$\left|\bigwedge_{j=1}^{n} \alpha_{ij} f_j(z) - \bigwedge_{j=1}^{n} \alpha_{ij} f_j(z')\right| \le \sum_{j=1}^{n} |\alpha_{ij}| \left|f_j(z) - f_j(z')\right|,$$
$$\left|\bigvee_{j=1}^{n} \beta_{ij} f_j(z) - \bigvee_{j=1}^{n} \beta_{ij} f_j(z')\right| \le \sum_{j=1}^{n} |\beta_{ij}| \left|f_j(z) - f_j(z')\right|.$$

Lemma 2.4 [22] *For any* $x \in \mathcal{R}^n$, *for any constant matrix* $A = [a_{ij}]_{n\times n}$ *with* $a_{ij} \ge 0$, *the following matrix inequality holds*:

$$x^T A^T A x \le n x^T A_s^T A_s x,$$

where $A_s = \operatorname{diag}\{\sum_{i=1}^n a_{i1}, \sum_{i=1}^n a_{i2}, \dots, \sum_{i=1}^n a_{in}\}$.

Lemma 2.5 [12] *Given any real matrix* $M = M^T > 0$ *of appropriate dimension*, *a scalar* $\eta > 0$, *and a vector function* $\omega(\cdot) : [a, b] \to \mathcal{R}^n$ *such that the integrations concerned are well defined*, *we have*

$$\left[\int_a^b \omega(s)\, ds\right]^T M \left[\int_a^b \omega(s)\, ds\right] \le (b-a) \int_a^b \omega^T(s) M \omega(s)\, ds.$$

Lemma 2.6 [4] *Let* $H : \mathcal{R}^n \to \mathcal{R}^n$ *be continuous*. *If* H *satisfies the following conditions*:

(1) $H(u)$ *is injective on* $\mathcal{R}^n$,
(2) $\|H(u)\| \to \infty$, *as* $\|u\| \to \infty$, *then* H *is a homeomorphism*.

3 Existence and Uniqueness of the Equilibrium Point

In order to study the existence and uniqueness of the equilibrium point, we consider the following equations associated with system (1):

$$-d_i x_i + \sum_{j=1}^{n} a_{ij} f_j(x_j) + \sum_{j=1}^{n} b_{ij} f_j(x_j) + \sum_{j=1}^{n} c_{ij} u_j + I_i + \bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j) + \bigvee_{j=1}^{n} \beta_{ij} f_j(x_j) + \bigwedge_{j=1}^{n} \zeta_{ij} u_j + \bigvee_{j=1}^{n} \delta_{ij} u_j = 0.$$

Define the maps H and H^* respectively as follows:

$$H_i(x) = -d_i x_i + \sum_{j=1}^{n} a_{ij} f_j(x_j) + \sum_{j=1}^{n} b_{ij} f_j(x_j) + \sum_{j=1}^{n} c_{ij} u_j + I_i + \bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j) + \bigvee_{j=1}^{n} \beta_{ij} f_j(x_j) + \bigwedge_{j=1}^{n} \zeta_{ij} u_j + \bigvee_{j=1}^{n} \delta_{ij} u_j$$

and

$$H_i^*(x) = -d_i x_i + \sum_{j=1}^{n} a_{ij}\big(f_j(x_j) - f_j(0)\big) + \sum_{j=1}^{n} b_{ij}\big(f_j(x_j) - f_j(0)\big) + \bigwedge_{j=1}^{n} \alpha_{ij} f_j(x_j) - \bigwedge_{j=1}^{n} \alpha_{ij} f_j(0) + \bigvee_{j=1}^{n} \beta_{ij} f_j(x_j) - \bigvee_{j=1}^{n} \beta_{ij} f_j(0).$$

Theorem 3.1 *Under assumptions* (A_1)–(A_3), *the FCNN* (1) *has a unique equilibrium point if there exist* $n \times n$ *positive diagonal matrices* P, R, U_1, U_2, *some* $n \times n$ *positive definite symmetric matrices* N, Q_1, Q_2, Z_1, Z_2, W, *two constants* $\mu_1 > 0$, $\mu_2 > 0$, *and a* $2n \times 2n$ *matrix* $\begin{pmatrix} T_{11} & T_{12} \\ \star & T_{22} \end{pmatrix} > 0$ *such that the following LMI has a feasible solution*:

$$\Omega = \begin{bmatrix} \Omega_{i,j} & \Gamma_1^T & \Gamma_2^T \\ \star & -\mu_1 n^{-1} I & 0 \\ \star & \star & -\mu_2 n^{-1} I \end{bmatrix} < 0, \tag{5}$$

where $i, j = 1, 2, \ldots, 9$ *with*

$$\Omega_{1,1} = -2PD + P + Q_1 + Q_2 - Z_1 - Z_2 + \sigma^2 N + LU_1L, \quad \Omega_{1,2} = Z_1,$$

$$\Omega_{1,3} = -WD + Z_2, \quad \Omega_{1,4} = T_{12}^T, \quad \Omega_{1,5} = -W, \quad \Omega_{1,6} = D^T PD,$$

$$\Omega_{1,7} = PA + WA, \quad \Omega_{1,8} = PB + WB, \quad \Omega_{1,9} = 0,$$

$$\Omega_{2,2} = -Q_1 - Z_1, \quad \Omega_{2,3} = 0, \quad \Omega_{2,4} = 0,$$

$$\Omega_{2,5}=0,\qquad \Omega_{2,6}=0,\qquad \Omega_{2,7}=0,\qquad \Omega_{2,8}=0,$$
$$\Omega_{2,9}=0,\qquad \Omega_{3,3}=-Q_2-Z_2,\qquad \Omega_{3,4}=0,$$
$$\Omega_{3,5}=-D^TW^T,\qquad \Omega_{3,6}=0,\qquad \Omega_{3,7}=0,\qquad \Omega_{3,8}=0,\qquad \Omega_{3,9}=0,$$
$$\Omega_{4,4}=\tau T_{11}-2T_{12}^T+LU_2L,\qquad \Omega_{4,5}=0,\qquad \Omega_{4,6}=0,$$
$$\Omega_{4,7}=0,\qquad \Omega_{4,8}=0,\qquad \Omega_{4,9}=0,$$
$$\Omega_{5,5}=\tau T_{22}-2W+\tau^2Z_1+\sigma^2Z_2,\qquad \Omega_{5,6}=0,$$
$$\Omega_{5,7}=WA,\qquad \Omega_{5,8}=WB,\qquad \Omega_{5,9}=0,$$
$$\Omega_{6,6}=-N+D^TPD,\qquad \Omega_{6,7}=-D^TPA,$$
$$\Omega_{6,8}=-D^TPB,\qquad \Omega_{6,9}=0,\qquad \Omega_{7,7}=R-U_1,$$
$$\Omega_{7,8}=0,\qquad \Omega_{7,9}=0,\qquad \Omega_{8,8}=-U_2,\qquad \Omega_{8,9}=0,$$
$$\Omega_{9,9}=2nS^TPS+\mu_1I+\mu_2I-R,$$
$$|\alpha|_s=\operatorname{diag}\left\{\sum_{i=1}^n|\alpha_{i1}|,\sum_{i=1}^n|\alpha_{i2}|,\dots,\sum_{i=1}^n|\alpha_{in}|\right\},$$
$$|\beta|_s=\operatorname{diag}\left\{\sum_{i=1}^n|\beta_{i1}|,\sum_{i=1}^n|\beta_{i2}|,\dots,\sum_{i=1}^n|\beta_{in}|\right\},$$
$$S=|\alpha|_s+|\beta|_s,\qquad \Gamma_1^T=\left[(WS)^T\ 0\,0\,0\,0\,0\,0\,0\,0\right]^T,$$
$$\Gamma_2^T=\left[0\,0\,0\,0\,(WS)^T\ 0\,0\,0\,0\right]^T.$$

Proof We will first prove that if $x\neq\bar{x}$ then $H(x)\neq H(\bar{x})$ holds for any $x,\bar{x}\in\mathcal{R}^n$. Now suppose $H(x)=H(\bar{x})$, that is,

$$-d_i(x_i-\bar{x}_i)+\sum_{j=1}^n a_{ij}\big(f_j(x_j)-f_j(\bar{x}_j)\big)+\sum_{j=1}^n b_{ij}\big(f_j(x_j)-f_j(\bar{x}_j)\big)$$
$$+\bigwedge_{j=1}^n\alpha_{ij}f_j(x_j)-\bigwedge_{j=1}^n\alpha_{ij}f_j(\bar{x}_j)+\bigvee_{j=1}^n\beta_{ij}f_j(x_j)-\bigvee_{j=1}^n\beta_{ij}f_j(\bar{x}_j)=0.$$

From Lemma 2.3, we obtain

$$-D(x-\bar{x})+(A+B)\big(f(x)-f(\bar{x})\big)+\big(|\alpha|+|\beta|\big)\big(f(x)-f(\bar{x})\big)\geq 0,$$
$$(x-\bar{x})^T\big[2(P+W)\big]\times\big\{-D(x-\bar{x})+(A+B)\big(f(x)-f(\bar{x})\big)$$
$$+\big(|\alpha|+|\beta|\big)\big(f(x)-f(\bar{x})\big)\big\}\geq 0,$$

By using Lemmas 2.2 and 2.4, we obtain

$$(x-\bar{x})^T[-2PD-2WD](x-\bar{x})+(x-\bar{x})^T\big[2(PA+PB+WA+WB)\big]$$

$$\times \big(f(x)-f(\bar{x})\big)+(x-\bar{x})^T P(x-\bar{x})+2n\big(f(x)-f(\bar{x})\big)^T\big(|\alpha|_s+|\beta|_s\big)^T P$$
$$\times \big(|\alpha|_s+|\beta|_s\big)\big(f(x)-f(\bar{x})\big)+n(x-\bar{x})^T W\big(|\alpha|_s+|\beta|_s\big)\mu_1^{-1}\big(|\alpha|_s+|\beta|_s\big)^T$$
$$\times W^T(x-\bar{x})+\big(f(x)-f(\bar{x})\big)^T\mu_1\big(f(x)-f(\bar{x})\big)\geq 0. \tag{6}$$

By assumption (A_1), we get

$$0\leq -\big(f(x)-f(\bar{x})\big)^T U_1\big(f(x)-f(\bar{x})\big)+(x-\bar{x})^T LU_1L(x-\bar{x}),$$
$$0\leq -\big(f(x)-f(\bar{x})\big)^T U_2\big(f(x)-f(\bar{x})\big)+(x-\bar{x})^T LU_2L(x-\bar{x}).$$

This, together with inequality (6), gives us

$$\xi^T(x,\bar{x})\Theta\xi(x,\bar{x})\geq 0, \tag{7}$$

where

$$\xi(x,\bar{x})=\big\{x-\bar{x},\, f(x)-f(\bar{x})\big\}^T,\qquad \Theta=[\Theta_{ij}]_{3\times 3},$$
$$\Theta_{11}=-2PD-2WD+P+LU_1L+LU_2L,$$
$$\Theta_{12}=PA+PB+WA+WB,\qquad \Theta_{13}=WS,$$
$$\Theta_{22}=-U_1-U2+2nS^TPS+\mu_1 I,\qquad \Theta_{23}=0,\qquad \Theta_{33}=-\mu_1 n^{-1}I.$$

On the other hand, pre- and post-multiplying Ω by

$$\begin{bmatrix} I & I & I & I & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & I & I & I & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \end{bmatrix}_{3\times 11}$$

and its transpose, respectively, we obtain

$$\Theta+\operatorname{diag}\big\{\sigma^2N+\tau T_{11},\mu_2 I,0\big\}<0.$$

Note that $\sigma\geq 0$, $\tau\geq 0$, $N>0$, $T_{11}>0$, $\mu_2>0$, thus $\Theta<0$. Obviously, this contradicts with (7). The contradiction implies that $H(x)\neq H(\bar{x})$. Hence, the map H is injective.

Next, we show that $\|H(x)\|\to\infty$ as $\|x\|\to\infty$. To prove this, it suffices to show that $\|H^*(x)\|\to\infty$ as $\|x\|\to\infty$, where $H^*(x)=H(x)-H(0)$. Similarly to the aforementioned proof, noticing that $\lambda_M(\Theta)<0$, we obtain

$$2\big[x^T(P+W)\big]H^*(x)\leq \xi^T(x,0)\Theta\xi(x,0)\leq \lambda_M(\Theta)\xi^T(x,0)\xi(x,0)\leq \lambda_M(\Theta)\|x\|^2.$$

By the Schwarz inequality, we have

$$\begin{aligned}-\lambda_M(\Theta)\|x\|^2 &\leq 2\big\|x^T(P+W)\big\|\times\big\|H^*(x)\big\|\\ &\leq 2\big(\|x\|\cdot\|P\|+\|x\|\cdot\|W\|\big)\big\|H^*(x)\big\|\\ &\leq 2\big(\|P\|+\|W\|\big)\|x\|\cdot\big\|H^*(x)\big\|.\end{aligned}$$

That is,

$$\|H^*(x)\| \geq -\frac{1}{2}\frac{\lambda_M(\Theta)}{\|P\|+\|W\|}\|x\|,$$

for $\|x\| \neq 0$. It is easy to see that $\|H^*(x)\| \to \infty$ as $\|x\| \to \infty$, which directly implies that $\|H(x)\| \to \infty$ as $\|x\| \to \infty$. By Lemma 2.6, H is a homeomorphism on $\mathcal{R}^n$, which implies that FCNN (1) has a unique equilibrium point x^* such that $H(x^*) = 0$. This completes the proof. □

4 Main Results

Theorem 4.1 *The unique equilibrium point of FCNN* (1) *is globally asymptotically stable provided the conditions of Theorem* 3.1 *are satisfied.*

Proof Consider the following Lyapunov-Krasovskii functional

$$V(t) = \sum_{i=1}^{8} V_i(t), \tag{8}$$

where

$$\begin{aligned}
V_1(t) &= \left[y(t) - D\int_{t-\sigma}^{t} y(s)\,ds\right]^T P\left[y(t) - D\int_{t-\sigma}^{t} y(s)\,ds\right] \\
&= \sum_{i=1}^{n} p_i\left(y_i(t) - d_i\int_{t-\sigma}^{t} y_i(s)\,ds\right)^2, \\
V_2(t) &= \int_{t-\tau}^{t} y^T(s)Q_1y(s)\,ds + \int_{t-\sigma}^{t} y^T(s)Q_2y(s)\,ds, \\
V_3(t) &= \tau\int_{-\tau}^{0}\int_{t+\theta}^{t} \dot{y}^T(s)Z_1\dot{y}(s)\,ds\,d\theta, \\
V_4(t) &= \sigma\int_{-\sigma}^{0}\int_{t+\theta}^{t} \dot{y}^T(s)Z_2\dot{y}(s)\,ds\,d\theta, \\
V_5(t) &= \sum_{j=1}^{n} r_j\int_{0}^{\infty} k_j(\theta)\int_{t-\theta}^{t} g_j^2\big(y_j(s)\big)\,ds\,d\theta, \\
V_6(t) &= \sigma\int_{t-\sigma}^{t}\int_{\theta}^{t} y^T(s)Ny(s)\,ds\,d\theta, \\
V_7(t) &= \int_{0}^{t}\int_{u-\tau(u)}^{u} \begin{bmatrix} y(u-\tau(u)) \\ \dot{y}(s) \end{bmatrix}^T \begin{bmatrix} T_{11} & T_{12} \\ \star & T_{22} \end{bmatrix} \begin{bmatrix} y(u-\tau(u)) \\ \dot{y}(s) \end{bmatrix} ds\,du, \\
V_8(t) &= \int_{-\tau}^{0}\int_{t+u}^{t} \dot{y}^T(s)T_{22}\dot{y}(s)\,ds\,du.
\end{aligned}$$

From Lemma 2.3, we obtain

$$\begin{aligned}
&\left|\bigwedge_{j=1}^{n}\alpha_{ij}\int_{-\infty}^{t}k_j(t-s)f_j\big(y_j(s)+x_j^*\big)\,ds-\bigwedge_{j=1}^{n}\alpha_{ij}\int_{-\infty}^{t}k_j(t-s)f_j\big(x_j^*\big)\,ds\right|\\
&\quad\le\sum_{j=1}^{n}|\alpha_{ij}|\left|\int_{-\infty}^{t}k_j(t-s)\big(f_j\big(y_j(s)+x_j^*\big)-f_j\big(x_j^*\big)\big)\,ds\right|\\
&\quad=\sum_{j=1}^{n}|\alpha_{ij}|\left|\int_{-\infty}^{t}k_j(t-s)g_j\big(y_j(s)\big)\,ds\right|.
\end{aligned}$$

By calculating the time derivation of V_i along the trajectory of system (1), we get

$$\begin{aligned}
\dot V_1(t)&=2\sum_{i=1}^{n}p_i\left(y_i(t)-d_i\int_{t-\sigma}^{t}y_i(s)\,ds\right)\Bigg\{-d_iy_i(t)+\sum_{j=1}^{n}a_{ij}g_j\big(y_j(t)\big)\\
&\quad+\sum_{j=1}^{n}b_{ij}g_j\big(y_j\big(t-\tau(t)\big)\big)+\bigwedge_{j=1}^{n}\alpha_{ij}\int_{-\infty}^{t}k_j(t-s)f_j\big(y_j(s)+x_j^*\big)\,ds\\
&\quad-\bigwedge_{j=1}^{n}\alpha_{ij}\int_{-\infty}^{t}k_j(t-s)f_j\big(x_j^*\big)\,ds\\
&\quad+\bigvee_{j=1}^{n}\beta_{ij}\int_{-\infty}^{t}k_j(t-s)f_j\big(y_j(s)+x_j^*\big)\,ds\\
&\quad-\bigvee_{j=1}^{n}\beta_{ij}\int_{-\infty}^{t}k_j(t-s)f_j\big(x_j^*\big)\,ds\Bigg\}\\
&\le2\left[y(t)-D\int_{t-\sigma}^{t}y(s)\,ds\right]^TP\big[-Dy(t)+Ag\big(y(t)\big)+Bg\big(y\big(t-\tau(t)\big)\big)\big]\\
&\quad+2\left|y(t)-D\int_{t-\sigma}^{t}y(s)\,ds\right|^TP\big(|\alpha|+|\beta|\big)\left|\int_{-\infty}^{t}K(t-s)g\big(y(s)\big)\,ds\right|\\
&\le-2y^T(t)PDy(t)+2y^T(t)PAg\big(y(t)\big)+2y^T(t)PBg\big(y\big(t-\tau(t)\big)\big)\\
&\quad+2y^T(t)D^TPD\int_{t-\sigma}^{t}y(s)\,ds-2\left[\int_{t-\sigma}^{t}y(s)\,ds\right]^TD^TPAg\big(y(t)\big)\\
&\quad-2\left[\int_{t-\sigma}^{t}y(s)\,ds\right]^TD^TPBg\big(y\big(t-\tau(t)\big)\big)+y^T(t)Py(t)\\
&\quad+\left[\int_{t-\sigma}^{t}y(s)\,ds\right]^TD^TPD\left[\int_{t-\sigma}^{t}y(s)\,ds\right]
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2\big[y(t) + \dot{y}(t)\big]^T W\big[-\dot{y}(t) + \dot{y}(t)\big] + 2n\bigg(\int_{-\infty}^{t} K(t-s)g\big(y(s)\big)\,ds\bigg)^T \\
&\quad \times \big(|\alpha|_s + |\beta|_s\big)^T P\big(|\alpha|_s + |\beta|_s\big)\bigg(\int_{-\infty}^{t} K(t-s)g\big(y(s)\big)\,ds\bigg) \\
&\le -2y^T(t)PDy(t) + 2y^T(t)PAg\big(y(t)\big) + 2y^T(t)PBg\big(y\big(t-\tau(t)\big)\big) \\
&\quad + 2y^T(t)D^T PD\int_{t-\sigma}^{t} y(s)\,ds - 2\bigg[\int_{t-\sigma}^{t} y(s)\,ds\bigg]^T D^T PAg\big(y(t)\big) \\
&\quad - 2\bigg[\int_{t-\sigma}^{t} y(s)\,ds\bigg]^T D^T PBg\big(y\big(t-\tau(t)\big)\big) + y^T(t)Py(t) \\
&\quad + \bigg[\int_{t-\sigma}^{t} y(s)\,ds\bigg]^T D^T PD\bigg[\int_{t-\sigma}^{t} y(s)\,ds\bigg] \\
&\quad + 2n\bigg(\int_{-\infty}^{t} K(t-s)g\big(y(s)\big)\,ds\bigg)^T \big(|\alpha|_s + |\beta|_s\big)^T P\big(|\alpha|_s + |\beta|_s\big) \\
&\quad \times \bigg(\int_{-\infty}^{t} K(t-s)g\big(y(s)\big)\,ds\bigg) - 2y^T(t)W\dot{y}(t) - 2y^T(t)WDy(t-\sigma) \\
&\quad + 2y^T(t)WAg\big(y(t)\big) + 2y^T(t)WBg\big(y\big(t-\tau(t)\big)\big) \\
&\quad + \mu_1^{-1} n y^T(t)W\big(|\alpha|_s + |\beta|_s\big)\big(|\alpha|_s + |\beta|_s\big)^T W^T y(t) \\
&\quad + \mu_1\bigg(\int_{-\infty}^{t} K(t-s)g\big(y(s)\big)\,ds\bigg)^T \bigg(\int_{-\infty}^{t} K(t-s)g\big(y(s)\big)\,ds\bigg) \\
&\quad - 2\dot{y}^T(t)W\dot{y}(t) - 2\dot{y}^T(t)WDy(t-\sigma) + 2\dot{y}^T(t)WAg\big(y(t)\big) \\
&\quad + 2\dot{y}^T(t)WBg\big(y\big(t-\tau(t)\big)\big) + \mu_2^{-1} n\dot{y}^T(t) \\
&\quad \times W\big(|\alpha|_s + |\beta|_s\big)\big(|\alpha|_s + |\beta|_s\big)^T W^T \dot{y}(t) \\
&\quad + \mu_2\bigg(\int_{-\infty}^{t} K(t-s)g\big(y(s)\big)\,ds\bigg)^T \bigg(\int_{-\infty}^{t} K(t-s)g\big(y(s)\big)\,ds\bigg),
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\dot{V}_2(t) &= y^T(t)Q_1 y(t) - y^T(t-\tau)Q_1 y(t-\tau) \\
&\quad + y^T(t)Q_2 y(t) - y^T(t-\sigma)Q_2 y(t-\sigma),
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\dot{V}_3(t) &= \tau\int_{-\tau}^{0} \dot{y}^T(t)Z_1\dot{y}(t)\,d\theta - \tau\int_{-\tau}^{0} \dot{y}^T(t+\theta)Z_1\dot{y}(t+\theta)\,d\theta \\
&\le \tau^2\dot{y}^T(t)Z_1\dot{y}(t) - \int_{t-\tau}^{t} \dot{y}^T(s)\,ds\,Z_1\int_{t-\tau}^{t} \dot{y}(s)\,ds \\
&\le \tau^2\dot{y}^T(t)Z_1\dot{y}(t) - y^T(t)Z_1 y(t) \\
&\quad + 2y^T(t)Z_1 y(t-\tau) - y^T(t-\tau)Z_1 y(t-\tau),
\end{aligned} \tag{11}
$$

$$\begin{aligned}\dot{V}_4(t) &\le \sigma^2 \dot{y}^T(t)Z_2\dot{y}(t) - y^T(t)Z_2y(t)\\ &\quad + 2y^T(t)Z_2y(t-\sigma) - y^T(t-\sigma)Z_2y(t-\sigma), \end{aligned}\tag{12}$$

$$\begin{aligned}\dot{V}_5(t) &= \sum_{j=1}^{n} r_j \int_0^{\infty} k_j(\theta)g_j^2\big(y_j(t)\big)\,d\theta - \sum_{j=1}^{n} r_j \int_0^{\infty} k_j(\theta)g_j^2\big(y_j(t-\theta)\big)\,d\theta\\ &= g^T\big(y(t)\big)Rg\big(y(t)\big) - \sum_{j=1}^{n} r_j \int_0^{\infty} k_j(\theta)\,d\theta \int_0^{\infty} k_j(\theta)g_j^2\big(y_j(t-\theta)\big)\,d\theta\\ &\le g^T\big(y(t)\big)Rg\big(y(t)\big) - \bigg(\int_{-\infty}^{t} K(t-s)g\big(y(s)\big)\,ds\bigg)^T\\ &\quad \times R\bigg(\int_{-\infty}^{t} K(t-s)g\big(y(s)\big)\,ds\bigg), \end{aligned}\tag{13}$$

$$\begin{aligned}\dot{V}_6(t) &= \sigma^2 y^T(t)Ny(t) - \sigma \int_{t-\sigma}^{t} y^T(s)Ny(s)\,ds\\ &\le \sigma^2 y^T(t)Ny(t) - \bigg[\int_{t-\sigma}^{t} y(s)\,ds\bigg]^T N \bigg[\int_{t-\sigma}^{t} y(s)\,ds\bigg], \end{aligned}\tag{14}$$

$$\begin{aligned}\dot{V}_7(t) &= \int_{t-\tau(t)}^{t} \begin{bmatrix} y(t-\tau(t)) \\ \dot{y}(s)\end{bmatrix}^T \begin{bmatrix} T_{11} & T_{12} \\ \star & T_{22}\end{bmatrix}\begin{bmatrix} y(t-\tau(t)) \\ \dot{y}(s)\end{bmatrix} ds\\ &= \tau(t)y^T\big(t-\tau(t)\big)T_{11}y\big(t-\tau(t)\big) + 2y^T(t)T_{12}^Ty\big(t-\tau(t)\big)\\ &\quad - 2y^T\big(t-\tau(t)\big)T_{12}^Ty\big(t-\tau(t)\big) + \int_{t-\tau(t)}^{t} \dot{y}^T(s)T_{22}\dot{y}(s)\,ds\\ &\le y^T\big(t-\tau(t)\big)\big[\tau T_{11} - 2T_{12}^T\big]y\big(t-\tau(t)\big) + 2y^T(t)T_{12}^Ty\big(t-\tau(t)\big)\\ &\quad + \int_{t-\tau}^{t} \dot{y}^T(s)T_{22}\dot{y}(s)\,ds, \end{aligned}\tag{15}$$

$$\begin{aligned}\dot{V}_8(t) &= \tau\dot{y}^T(t)T_{22}\dot{y}(t) - \int_{-\tau}^{0} \dot{y}^T(t+u)T_{22}\dot{y}(t+u)\,du\\ &= \tau\dot{y}^T(t)T_{22}\dot{y}(t) - \int_{t-\tau}^{t} \dot{y}^T(s)T_{22}\dot{y}(s)\,ds. \end{aligned}\tag{16}$$

In addition, for any $n \times n$ diagonal matrices $U_1 > 0$, $U_2 > 0$, we can get from assumption (A_1) that

$$0 \le -g^T\big(y(t)\big)U_1g\big(y(t)\big) + y^T(t)LU_1Ly(t), \tag{17}$$

$$0 \le -g^T\big(y\big(t-\tau(t)\big)\big)U_2g\big(y\big(t-\tau(t)\big)\big) + y^T\big(t-\tau(t)\big)LU_2Ly\big(t-\tau(t)\big). \tag{18}$$

Hence, from (8)–(18) we have

$$\dot{V}(t) \le \xi^T(t)\big[\Omega_{i,j} + \Gamma_1^T \mu_1^{-1} n \Gamma_1 + \Gamma_2^T \mu_2^{-1} n \Gamma_2\big]\xi(t) = \xi^T(t)\Omega\xi(t), \tag{19}$$

where

$$\xi(t) = \Big[y^T(t), y^T(t-\tau), y^T(t-\sigma), y^T\big(t-\tau(t)\big), \dot{y}^T(t), \int_{t-\sigma}^t y^T(s)\,ds, g^T\big(y(t)\big),$$
$$g^T\big(y\big(t-\tau(t)\big)\big), \int_{-\infty}^t K(t-s) g^T\big(y(s)\big)\,ds\Big]^T,$$

$\Omega = \Omega_{i,j} + \Gamma_1^T \mu_1^{-1} n \Gamma_1 + \Gamma_2^T \mu_2^{-1} n \Gamma_2$.

By (5), this yields

$$\dot{V}(t) \le -\xi^T(t)\Omega^*\xi(t), \quad t > 0,$$

where $\Omega^* = -\Omega > 0$.

Thus, it can be deduced that

$$V(t) + \int_0^t \xi^T(s)\Omega^*\xi(s)\,ds \le V(0) < \infty, \quad t \ge 0, \tag{20}$$

where

$$\begin{aligned}
V(0) \le{}& \Big[y(0) - D\int_{-\sigma}^0 y(s)\,ds\Big]^T P\Big[y(0) - D\int_{-\sigma}^0 y(s)\,ds\Big] + \int_{-\tau}^0 y^T(s) Q_1 y(s)\,ds \\
&+ \int_{-\sigma}^0 y^T(s) Q_2 y(s)\,ds + \tau\int_{-\tau}^0\int_\theta^0 \dot{y}^T(s) Z_1 \dot{y}(s)\,ds\,d\theta \\
&+ \sigma\int_{-\sigma}^0\int_\theta^0 \dot{y}^T(s) Z_2 \dot{y}(s)\,ds\,d\theta + \sum_{j=1}^n r_j\int_0^\infty k_j(\theta)\int_{-\theta}^0 g_j^2\big(y_j(s)\big)\,ds\,d\theta \\
&+ \sigma\int_{-\sigma}^0\int_\theta^0 y^T(s) N y(s)\,ds\,d\theta + \int_{-\tau}^0\int_u^0 \dot{y}^T(s) T_{22}\dot{y}(s)\,ds\,du \\
\le{}& \Big\{2\lambda_{\max}(P)\Big(1 + \sigma^2\max_{i\in\Lambda} d_i\Big) + \tau\lambda_{\max}(Q_1) + \sigma\lambda_{\max}(Q_2) + \tau^3\lambda_{\max}(Z_1) \\
&+ \sigma^3\lambda_{\max}(Z_2) + \sum_{j=1}^n r_j k_j \max_{j\in\Lambda} l_j^2\int_0^\infty \theta k_j(\theta)\,d\theta \\
&+ \sigma^3\lambda_{\max} N + \tau^2\lambda_{\max}(T_{22})\Big\}\|\varphi_y\|^2 < \infty.
\end{aligned}$$

From the definition of $V_2(t)$ and Lemma 2.5, we have

$$\Big\|\int_{t-\sigma}^t y(s)\,ds\Big\|^2 = \Big[\int_{t-\sigma}^t y(s)\,ds\Big]^T\Big[\int_{t-\sigma}^t y(s)\,ds\Big]$$

$$
\begin{aligned}
&\le \sigma \int_{t-\sigma}^{t} y^T(s)y(s)\,ds \\
&\le \frac{\sigma}{\lambda_{\min}(Q_2)} \int_{t-\sigma}^{t} y^T(s)Q_2 y(s)\,ds \\
&\le \frac{\sigma}{\lambda_{\min}(Q_2)} V(t) \le \frac{\sigma}{\lambda_{\min}(Q_2)} V(0),
\end{aligned}
$$

which, together with the definition of $V_1(t)$, yields

$$
\begin{aligned}
\|y(t)\| &\le \left\| D \int_{t-\sigma}^{t} y(s)\,ds \right\| + \sqrt{\frac{V_1(t)}{\lambda_{\min}(P)}} \\
&\le \left\| D \int_{t-\sigma}^{t} y(s)\,ds \right\| + \sqrt{\frac{V(0)}{\lambda_{\min}(P)}} \\
&\le \left\{ \sqrt{\sum_{i=1}^{n} d_i \frac{\sigma}{\lambda_{\min}(Q_2)}} + \sqrt{\frac{1}{\lambda_{\min}(P)}} \right\} \sqrt{V(0)}.
\end{aligned}
$$

This implies that the equilibrium point of model (1) is locally stable. Next we shall prove that $\|y(t)\| \to 0$ as $t \to \infty$.

First, for any constant $\theta \in [0, 1]$, it follows from (8) and Lemma 2.5 that

$$
\begin{aligned}
\|y(t+\theta) - y(t)\|^2 &= \left[\int_{t}^{t+\theta} \dot{y}(s)\,ds \right]^T \left[\int_{t}^{t+\theta} \dot{y}(s)\,ds \right] \\
&\le \theta \int_{t}^{t+\theta} \dot{y}^T(s)\dot{y}(s)\,ds \\
&\le \int_{t}^{t+1} \dot{y}^T(s)\dot{y}(s)\,ds \\
&\le \frac{1}{\lambda_{\min}(\Omega^*)} \int_{t}^{t+1} \xi^T(s)\Omega^*\xi(s)\,ds \to 0 \quad \text{as } t \to \infty,
\end{aligned}
$$

which implies that for any $\epsilon > 0$, $\theta \in [0, 1]$, there exists a $T_1 = T_1(\epsilon) > 0$ such that

$$
\|y(t+\theta) - y(t)\| < \frac{\epsilon}{2}, \qquad t > T_1. \tag{21}
$$

On the other hand, from (8) we have

$$
\begin{aligned}
\left\| \int_{t}^{t+1} y(s)\,ds \right\|^2 &= \left[\int_{t}^{t+1} y(s)\,ds \right]^T \left[\int_{t}^{t+1} y(s)\,ds \right] \\
&\le \int_{t}^{t+1} y^T(s)y(s)\,ds \\
&\le \frac{1}{\lambda_{\min}(\Omega^*)} \int_{t}^{t+1} \xi^T(s)\Omega^*\xi(s)\,ds \to 0 \quad \text{as } t \to \infty,
\end{aligned}
$$

which implies that, for any $\epsilon > 0$, there exists a $T_2 = T_2(\epsilon) > 0$ such that

$$\left\| \int_t^{t+1} y(s)\, ds \right\| < \frac{\epsilon}{2}, \quad t > T_2.$$

Note that $y(s)$ is continuous on $[t, t+1]$, $t > 0$. Applying the integral mean value theorem, there exists a vector $\delta_t = (\delta_{t1}, \delta_{t2}, \dots, \delta_{tn})^T \in \mathcal{R}^n$, $\delta_{tj} \in [t, t+1]$, such that

$$\left\| y(\delta_t) \right\| = \left\| \int_t^{t+1} y(s)\, ds \right\| < \frac{\epsilon}{2}, \quad t > T_2. \tag{22}$$

By (21) and (22), we obtain that for any $\epsilon > 0$, there exists a $T = \max\{T_1, T_2\} > 0$ such that $t > T$ implies

$$\left\| y(t) \right\| \le \left\| y(t) - y(\delta_t) \right\| + \left\| y(\delta_t) \right\| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This proves that $\|y(t)\| \to 0$ as $t \to \infty$. Therefore, we can conclude that the model (1) has a unique equilibrium point which is globally asymptotically stable. This completes the proof.

When there is no time delay in the leakage term, that is $\sigma = 0$, FCNN (1) becomes the following:

$$\begin{aligned} \dot{x}_i(t) = {} & -d_i x_i(t) + \sum_{j=1}^n a_{ij} f_j\big(x_j(t)\big) + \sum_{j=1}^n b_{ij} f_j\big(x_j\big(t - \tau(t)\big)\big) + \sum_{j=1}^n c_{ij} u_j + I_i \\ & + \bigwedge_{j=1}^n \alpha_{ij} \int_{-\infty}^t k_j(t-s) f_j\big(x_j(s)\big)\, ds + \bigvee_{j=1}^n \beta_{ij} \int_{-\infty}^t k_j(t-s) f_j\big(x_j(s)\big)\, ds \\ & + \bigwedge_{j=1}^n \zeta_{ij} u_j + \bigvee_{j=1}^n \delta_{ij} u_j. \end{aligned} \tag{23}$$

In the following corollary, we will discuss the global asymptotic stability criteria for FCNNs (23). □

Corollary 4.1 *Under assumption* (A$_1$), *the equilibrium point of* (23) *is globally asymptotically stable if there exist* $n \times n$ *positive diagonal matrices* P, R, U_1, U_2, *some* $n \times n$ *positive definite symmetric matrices* Q, Z, W, *two constants* $\mu_1 > 0$, $\mu_2 > 0$, *and a* $2n \times 2n$ *matrix* $\begin{pmatrix} T_{11} & T_{12} \\ \star & T_{22} \end{pmatrix} > 0$ *such that the following LMI has a feasible solution*:

$$\begin{bmatrix} \Omega_{i,j} & \Gamma_1^T & \Gamma_2^T \\ \star & -\mu_1 n^{-1} I & 0 \\ \star & \star & -\mu_2 n^{-1} I \end{bmatrix} < 0, \tag{24}$$

where $i, j = 1, 2, \dots, 7$ *with*

$$\Omega_{1,1} = -2PD + P - 2WD + Q - Z + LU_1L, \qquad \Omega_{1,2} = Z, \qquad \Omega_{1,3} = T_{12}^T,$$

$\Omega_{1,4} = -W - D^T W^T, \qquad \Omega_{1,5} = PA + WA,$

$\Omega_{1,6} = PB + WB, \qquad \Omega_{1,7} = 0,$

$\Omega_{2,2} = -Q - Z, \qquad \Omega_{2,3} = 0, \qquad \Omega_{2,4} = 0,$

$\Omega_{2,5} = 0, \qquad \Omega_{2,6} = 0, \qquad \Omega_{2,7} = 0,$

$\Omega_{3,3} = \tau T_{11} - 2T_{12}^T + LU_2L, \qquad \Omega_{3,4} = 0,$

$\Omega_{3,5} = 0, \qquad \Omega_{3,6} = 0, \qquad \Omega_{3,7} = 0,$

$\Omega_{4,4} = \tau T_{22} - 2W + \tau^2 Z, \qquad \Omega_{4,5} = WA, \qquad \Omega_{4,6} = WB,$

$\Omega_{4,7} = 0, \qquad \Omega_{5,5} = R - U_1, \qquad \Omega_{5,6} = 0, \qquad \Omega_{5,7} = 0, \qquad \Omega_{6,6} = -U_2,$

$\Omega_{6,7} = 0, \qquad \Omega_{7,7} = nS^T PS + \mu_1 I + \mu_2 I - R,$

$|\alpha_s|, |\beta_s|, S,$ *are defined in Theorem* 4.1,

$\Gamma_1^T = \big[(WS)^T \; 0\,0\,0\,0\,0\,0\big]^T, \qquad \Gamma_2^T = \big[0\,0\,0\,(WS)^T \; 0\,0\,0\,0\big]^T.$

Proof Consider the following Lyapunov-Krasovskii functional:

$$V(t) = \sum_{i=1}^{6} V_i(t), \tag{25}$$

where

$$V_1(t) = y^T(t)Py(t) = \sum_{i=1}^{n} p_i y_i^2(t),$$

$$V_2(t) = \int_{t-\tau}^{t} y^T(s)Qy(s)\,ds,$$

$$V_3(t) = \tau \int_{-\tau}^{0} \int_{t+\theta}^{t} \dot{y}^T(s)Z\dot{y}(s)\,ds\,d\theta,$$

$$V_4(t) = \sum_{j=1}^{n} r_j \int_0^{\infty} k_j(\theta) \int_{t-\theta}^{t} g_j^2\big(y_j(s)\big)\,ds\,d\theta,$$

$$V_5(t) = \int_0^t \int_{u-\tau(u)}^{u} \begin{bmatrix} y(u-\tau(u)) \\ \dot{y}(s) \end{bmatrix}^T \begin{bmatrix} T_{11} & T_{12} \\ \star & T_{22} \end{bmatrix} \begin{bmatrix} y(u-\tau(u)) \\ \dot{y}(s) \end{bmatrix} ds\,du,$$

$$V_6(t) = \int_{-\tau}^{0} \int_{t+u}^{t} \dot{y}^T(s)T_{22}\dot{y}(s)\,ds\,du.$$

The proof of this corollary follows immediately from Theorem 4.1. □

Table 1 The MAUB τ for different values of σ

Theorem 4.1	$\sigma = 0.05$	$\sigma = 0.1$	$\sigma = 0.15$	$\sigma \geq 0.17$
τ	2.2945	1.5148	0.7606	–

5 Numerical Examples

Example 5.1 Consider the FCNN (1) with parameters defined as

$$A = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.25 & 0.125 \\ -0.15 & -0.25 \end{bmatrix}, \qquad D = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix},$$
$$\alpha = \begin{bmatrix} 1/32 & -1/32 \\ 1/32 & 1/32 \end{bmatrix}, \qquad \beta = \begin{bmatrix} 1/32 & 1/32 \\ -1/32 & 1/32 \end{bmatrix}.$$

Letting $f_j(x_j) = \frac{1}{2}(|x+1| - |x-1|)$, $j = 1, 2$, which satisfies assumption (A_1), we get $l_j = 1$ and then $L = I$. The time-varying delays are chosen as $\tau(t) = 1.5148|\sin(t)|$, which means that the maximum allowable upper bound (MAUB) is $\tau = 1.5148$ when $\sigma = 0.1$. By using the MATLAB LMI Toolbox to solve the LMI (5), we obtain the MAUB τ for different values of σ as given in Table 1.

Also, it can be found that the LMI (5) is feasible for $\tau = 1.5148$ when $\sigma = 0.1$, and

$$P = \begin{bmatrix} 86.7515 & 0 \\ 0 & 67.3143 \end{bmatrix}, \qquad R = \begin{bmatrix} 17.0341 & 0 \\ 0 & 15.8206 \end{bmatrix},$$
$$N = 10^4 \times \begin{bmatrix} 1.4775 & 0.0074 \\ 0.0074 & 0.9382 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} 0.0068 & -0.0054 \\ -0.0054 & 0.0111 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 140.9593 & -0.6705 \\ -0.6705 & 62.7850 \end{bmatrix},$$
$$W = \begin{bmatrix} 11.0698 & -0.0148 \\ -0.0148 & 6.6178 \end{bmatrix},$$
$$Z_1 = 10^{-3} \times \begin{bmatrix} 0.0443 & -0.0447 \\ -0.0447 & 0.1132 \end{bmatrix}, \qquad Z_2 = \begin{bmatrix} 283.5831 & 0.7636 \\ 0.7636 & 175.7859 \end{bmatrix},$$
$$T_{11} = \begin{bmatrix} 1.8820 & -0.0420 \\ -0.0420 & 0.9653 \end{bmatrix}, \qquad T_{12} = \begin{bmatrix} 3.1226 & 0.0287 \\ -0.0952 & 1.6859 \end{bmatrix},$$
$$T_{22} = \begin{bmatrix} 5.1818 & 0.0030 \\ 0.0030 & 2.9502 \end{bmatrix},$$
$$U_1 = \begin{bmatrix} 87.7862 & 0 \\ 0 & 106.1375 \end{bmatrix}, \qquad U_2 = \begin{bmatrix} 3.0895 & 0 \\ 0 & 1.7005 \end{bmatrix},$$
$$\mu_1 = 1.7008, \qquad \mu_2 = 9.9092.$$

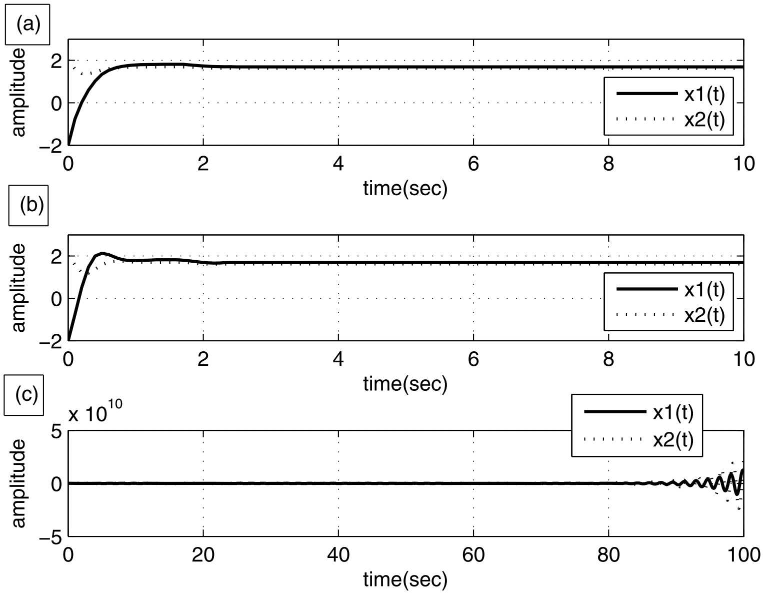

Fig. 1 (**a**) State trajectory for system (1) converges to unique equilibrium point (1.6933, 1.6245) when $\sigma = 0.0$, (**b**) state trajectory for system (1) converges to unique equilibrium point (1.6933, 1.6245) when $\sigma = 0.1$, (**c**) state trajectory for system (1) is unstable when $\sigma = 0.4$

Therefore, it follows from Theorem 4.1 that the FCNN with leakage delay (1) is globally asymptotically stable. The response of the state dynamics for the delayed FCNN (1) is shown in Fig. 1.

Remark 5.1 Figures 1(a), 1(b) show that the state trajectory for system (1) converges to a unique equilibrium point (1.6933, 1.6245) when $\sigma = 0.0, 0.1$, Fig. 1(c) shows that the state trajectory for system (1) is unstable when $\sigma = 0.4$.

Example 5.2 [25] Consider the FCNN (23) with parameters defined as

$$A = B = \begin{bmatrix} 0.2 & 0.3 & 0.25 \\ -0.15 & 0.2 & -0.1 \\ 0.35 & -0.15 & 0.2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 1.2 & 0 \\ 0 & 0 & 1.2 \end{bmatrix},$$

$$\alpha = \frac{1}{25}\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}, \qquad \beta = \frac{1}{25}\begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

Letting $f_j(x_j) = \frac{1}{4}(|x+1| - |x-1|)$, $j = 1, 2$, which satisfies assumption (A_1), we get $l_j = 0.5$ and then $L = 0.5 \times I$. The time-varying delays are chosen as $\tau(t) = 0.3926|\sin(t)|$. By using the MATLAB LMI Toolbox to solve the LMI (24) in

Corollary 4.1, it can be found that the LMI is feasible and

$$P = 10^4 \times \begin{bmatrix} 1.8361 & 0 & 0 \\ 0 & 1.3461 & 0 \\ 0 & 0 & 1.6812 \end{bmatrix},$$

$$R = 10^3 \times \begin{bmatrix} 5.5651 & 0 & 0 \\ 0 & 4.7259 & 0 \\ 0 & 0 & 5.2976 \end{bmatrix},$$

$$W = 10^3 \times \begin{bmatrix} 1.7540 & 0.0043 & -0.3393 \\ 0.0043 & 2.0664 & 0.2218 \\ -0.3393 & 0.2218 & 1.8933 \end{bmatrix},$$

$$Q = 10^3 \times \begin{bmatrix} 0.8905 & -0.2216 & -1.0532 \\ -0.2216 & 2.7528 & 1.2400 \\ -1.0532 & 1.2400 & 1.6007 \end{bmatrix},$$

$$Z = 10^3 \times \begin{bmatrix} 1.0327 & 0.2568 & -1.0356 \\ 0.2568 & 2.2230 & 0.5242 \\ -1.0356 & 0.5242 & 1.3234 \end{bmatrix},$$

$$T_{11} = 10^3 \times \begin{bmatrix} 3.5432 & -0.1171 & -0.8151 \\ -0.1171 & 4.1398 & 1.0336 \\ -0.8151 & 1.0336 & 3.7237 \end{bmatrix},$$

$$T_{12} = 10^3 \times \begin{bmatrix} 3.1649 & -0.1570 & -0.1584 \\ 0.0318 & 2.9119 & 0.3764 \\ -0.3626 & 0.2042 & 3.1330 \end{bmatrix},$$

$$T_{22} = 10^3 \times \begin{bmatrix} 3.2241 & -0.1188 & -0.2254 \\ -0.1188 & 3.0971 & 0.1445 \\ -0.2254 & 0.1445 & 3.3571 \end{bmatrix},$$

$$U_1 = 10^4 \times \begin{bmatrix} 2.4278 & 0 & 0 \\ 0 & 1.8270 & 0 \\ 0 & 0 & 2.2376 \end{bmatrix},$$

$$U_2 = 10^4 \times \begin{bmatrix} 1.1927 & 0 & 0 \\ 0 & 0.8627 & 0 \\ 0 & 0 & 1.0886 \end{bmatrix},$$

$$\mu_1 = 976.0683, \qquad \mu_2 = 1415.4.$$

Therefore, it follows from Corollary 4.1 that the FCNN without leakage delay (23) is globally asymptotically stable. Also, the state trajectory for system (23) converges to a unique equilibrium point (5.0194, 4.3527, 4.7277).

Remark 5.2 In [25], the author studied delay-independent stability results of FCNNs with discrete and unbounded distributed delays. This paper proposes delay-dependent stability results of FCNNs with discrete and unbounded distributed delays. The results obtained in this paper are less conservative than the results presented in [25] based on the general theory of delay dependence. The maximum allowable upper bound $\tau = 0.3926$ is calculated for solving LMI (24) by using the MATLAB LMI Control Toolbox.

Example 5.3 [25] Consider the FCNN (23) with parameters defined as

$$A = \begin{bmatrix} 0.7 & 0.6 \\ -0.3 & 0.4 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\alpha = \begin{bmatrix} 0.05 & -0.05 \\ 0.05 & 0.05 \end{bmatrix}, \qquad \beta = \begin{bmatrix} 0.05 & 0.05 \\ -0.05 & 0.05 \end{bmatrix}.$$

Letting $f_j(x_j) = \frac{1}{4}(|x+1| - |x-1|)$, $j = 1, 2$, which satisfies assumption (A_1), we get $l_j = 0.5$ and then $L = 0.5 \times I$. The time-varying delays are chosen as $\tau(t) = |\sin(t)|$. By using the MATLAB LMI Toolbox to solve the LMI (24) in Corollary 4.1, it can be found that the LMI is feasible and

$$P = \begin{bmatrix} 4.3829 & 0 \\ 0 & 9.0476 \end{bmatrix}, \qquad R = \begin{bmatrix} 1.4207 & 0 \\ 0 & 2.4489 \end{bmatrix},$$
$$W = \begin{bmatrix} 0.2517 & 0.0828 \\ 0.0828 & 0.9194 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0.0901 & 0.1118 \\ 0.1118 & 0.8628 \end{bmatrix},$$
$$Z = \begin{bmatrix} 0.0642 & 0.0272 \\ 0.0272 & 0.2343 \end{bmatrix}, \qquad T_{11} = \begin{bmatrix} 0.0362 & 0.0446 \\ 0.0446 & 0.3449 \end{bmatrix},$$
$$T_{12} = \begin{bmatrix} 0.0406 & 0.0381 \\ 0.0302 & 0.3478 \end{bmatrix}, \qquad T_{22} = \begin{bmatrix} 0.0781 & 0.0483 \\ 0.0483 & 0.5416 \end{bmatrix},$$
$$U_1 = \begin{bmatrix} 8.0727 & 0 \\ 0 & 13.4899 \end{bmatrix}, \qquad U_2 = \begin{bmatrix} 0.0740 & 0 \\ 0 & 0.6251 \end{bmatrix},$$
$$\mu_1 = 0.4332, \qquad \mu_2 = 0.4819.$$

Therefore, it follows from Corollary 4.1 that the FCNN without leakage delay (23) is globally asymptotically stable. Also, the state trajectory for system (23) converges to the unique equilibrium point (5.5222, 4.9222).

Remark 5.3 The results proposed in this paper are less conservative than those discussed in [13] by considering the above Example 5.2.

6 Conclusion

In this paper, a class of FCNNs with time delay in the leakage term and unbounded distributed delays is considered. By using homeomorphism theory and an LMI approach, the existence and uniqueness of the equilibrium point of FCNNs with time-varying delays and unbounded distributed delays have been derived. To the best of the authors' knowledge, there are very few results in the literature on the existence, uniqueness and stability behavior of FCNNs with time delay in the leakage term. The proposed results indicate that the stability behavior of FCNNs is very sensitive to the time delay in the leakage term. Further, in the absence of leakage delay, the results are derived by employing a Lyapunov-Krasovskii functional and using an LMI approach. The effectiveness of the proposed results has been demonstrated through three numerical examples.

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