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Adaptive Variable Structure Control for Uncertain Switched Delay Systems

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Abstract This paper investigates the robust H_∞ control problem for a class of uncertain switched delay systems with parameter uncertainties, unknown nonlinear perturbations, and external disturbance. Based on the multiple Lyapunov functions method, a sufficient condition for the solvability of the robust H_∞ control problem is derived by employing a hysteresis switching law and variable structure controllers. When the upper bounds of the nonlinear perturbations are unknown, an adaptive variable structure control strategy is developed. The use of the adaptive technique is to adapt the unknown upper bounds of the nonlinear disturbances so that the objective of asymptotic stabilization with an H_∞-norm bound is achieved under the hysteresis switching law. A numerical example illustrates the effectiveness of the proposed design methods.

Keywords Uncertain switched delay systems · Hysteresis switching law · Adaptive control · Robust H_∞ control · Multiple Lyapunov functions

1 Introduction

Switched systems constitute an important class of hybrid systems. A typical switched system consists of a family of subsystems and a switching law specifying which subsystem will be activated at each instant of time. The motivation for studying switched

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systems comes from the fact that many practical systems are inherently multi-models in the sense that several dynamical subsystems are required to describe their behavior depending on various changing environmental factors. In the study of stability analysis for switched systems, the multiple Lyapunov functions method has been considered as an important analysis tool in [1, 20, 21]. Robust H_∞ control and stabilization of uncertain switched linear systems are considered by utilizing a state-dependent switching strategy with the multiple Lyapunov functions method in [5]. Reference [22] addresses the problem of stability and L_2-gain analysis for nonlinear switched systems via the multiple Lyapunov functions method. The stability result is generalized by defining more general weak multiple Lyapunov functions. Reference [13] designs a hysteresis switching law to avoid the sliding mode that often occurs in the state-dependent switching strategy. The value of the hysteresis switching signal is not determined by the current value of the state alone, but depends also on the previous value of the switching signal.

Time delay commonly exists in various industrial systems, and its existence is frequently a source of instability. Switched systems with time delay are one of the most useful models and have a strong engineering background as power systems [10] and networked control systems [6]. This draws attention to the analysis and synthesis of switched delay systems. Sufficient conditions of asymptotical stability are established for switched linear delay systems under arbitrary and constructed switching signals in [19]. Reference [17] investigates the problem of delay-dependent common Lyapunov functions for switched linear delay systems, which establish the relationship between the delay-dependent common Lyapunov functions and the common Lyapunov functions for corresponding switched systems without delays. The stabilization problem of arbitrary switched linear systems with unknown time-varying delays is considered in [4]. For uncertain linear discrete-time switched systems with state delays, sufficient conditions of robust stability and stabilizability in term of matrix inequalities and Riccati-like inequalities are given in [14]. The stability of a class of switched delay systems is shown in [7] by using a common Lyapunov functional method. Reference [16] studies stability and L_2-gain analysis of switched delay systems based on the average time technique.

Another critical issue for switched delay systems is to enhance the robustness against system uncertainties and perturbations. Variable structure control with sliding mode or without sliding mode is an effective robust scheme for uncertain systems, which employs a discontinuous control law to overcome uncertainties and improve performance and stability. Due to the simultaneous existence of switching and delay in switched delay systems, it is very difficult to utilize variable structure control techniques to synthesize such systems and a design switching law. In [15], the control input is used as a switching signal based on the sign function, which is also a kind of variable structure controller. Reference [18] considers the sliding mode control of switched delay systems without uncertainties by using the average dwell approach. In [8], the common Lyapunov function method and integral sliding mode control are adopted for uncertain switched delay systems. For uncertain switched delay systems, robust H_∞ sliding mode control is studied based on the single Lyapunov function method in [9].

In this paper, we consider the problem of robust H_∞ control for a class of uncertain switched delay systems with parameter uncertainties and unknown nonlinear perturbations and external disturbance. Based on the multiple Lyapunov functions method, a sufficient condition for robust stabilization with H_∞ disturbance attenuation level γ is derived and a hysteresis switching law is designed. For the case of known upper bounds of the nonlinear perturbations, variable structure controllers are developed such that the uncertain switched delay system is asymptotically stabilizable with H_∞ disturbance attenuation level γ under the hysteresis switching law. For the case of unknown upper bounds of the nonlinear perturbations, adaptive variable structure controllers are developed so that the objective of asymptotic stabilization with disturbance attenuation level γ is achieved under the hysteresis switching law. A numerical example illustrates the effectiveness of the proposed design method.

In this paper, $\|\cdot\|$ denotes the Euclidean norm for a vector or the matrix induced norm for a matrix; Z^+ denotes the set of all nonnegative integers, N denotes the set of all positive integers.

2 Problem Formulation and Preliminaries

Consider the uncertain switched delay system of the form

$$\begin{aligned} \dot{x}(t) &= (A_\sigma + \Delta A_\sigma)x(t) + (A_{d\sigma} + \Delta A_{d\sigma})x(t-\tau) \\ &\quad + B_\sigma\big(u_\sigma + f_\sigma(x,t)\big) + G_\sigma\omega(t), \\ z(t) &= C_\sigma x(t), \\ x(\theta) &= \varphi(\theta), \quad \theta \in [-\bar{\tau}, 0], \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the system state, $\sigma(t) : [0,\infty) \to \Xi = \{1, 2, \ldots, l\}$ is the piecewise constant switching signal, $u_i \in R^m$ is the control input of the i-th subsystem, $\omega(t) \in R^h$ denotes the external disturbance input which belongs to $L_2[0,\infty)$, $z(t) \in R^q$ is the controlled output, A_i, A_{di}, B_i, G_i and C_i are constant matrices with appropriate dimensions, τ denotes unknown constant time delay which is bounded by the known constant $\bar{\tau}$, $\varphi(\theta)$ is a differentiable vector-valued initial function on $[-\bar{\tau}, 0]$, ΔA_i, ΔA_{di} represent the system uncertainties, and $f_i(x,t)$ is an unknown nonlinear perturbation. The following assumptions are introduced.

Assumption 1 The uncertainties can be represented and emulated as

$$\Delta A_i = D_i \Sigma_i(t) E_i, \qquad \Delta A_{di} = B_i \Delta M_{di}(t), \quad i \in \Xi,$$

where D_i and E_i are constant matrices with appropriate dimensions, $\Sigma_i(t)$ are unknown matrix functions that satisfy $\Sigma_i^{\mathrm{T}}(t)\Sigma_i(t) \le I$, and $\Delta M_{di}(t)$ are unknown but bounded as $\|\Delta M_{di}(t)\| \le a_i$ with known nonnegative constants a_i.

Assumption 2 There exist known nonnegative scalar-valued functions $\phi_i(x,t)$, $i \in \Xi$ such that $\|f_i(x,t)\| \le \phi_i(x,t)$.

We adopt the following notation from [1]. A switching sequence is expressed by

$$\Psi = \big\{x_0; (i_0, t_0), (i_1, t_1), \ldots, (i_j, t_j), \ldots \,|\, i_j \in \Xi, j \in Z^+\big\}, \tag{2}$$

where t_0 is the initial time, x_0 is the initial state, and (i_k, t_k) means that the i_k-th subsystem is activated for $[t_k, t_{k+1})$. Therefore, when $t \in [t_k, t_{k+1})$, the trajectory of the switched system (1) is produced by the i_k-th subsystem. For any $j \in \Xi$,

$$\Psi_t(j) = \{[t_{j_1}, t_{j_1+1}), [t_{j_2}, t_{j_2+1}), \ldots, [t_{j_n}, t_{j_n+1}), \ldots, \\ \sigma(t) = j,\ t_{j_k} \le t < t_{j_k+1},\ k \in N\} \tag{3}$$

denotes the sequence of switching time of the j-th subsystem, in which the j-th subsystem is switched on at t_{j_k} and switched off at t_{j_k+1}.

Lemma 1 [12] *Given real matrices R_1 and R_2 with appropriate dimensions and an unknown matrix $\Sigma(t)$ satisfying $\Sigma^{\mathrm{T}}(t)\Sigma(t) \le I$, there exists $\beta > 0$ such that*

$$R_1 \Sigma R_2 + R_1^{\mathrm{T}} \Sigma^{\mathrm{T}} R_2^{\mathrm{T}} \le \beta R_1 R_1^{\mathrm{T}} + \beta^{-1} R_2^{\mathrm{T}} R_2.$$

3 Main Results

The objective in this paper is to design a switching law $\sigma(t)$ and an associated controller u_σ such that the system (1) is stabilizable with an H_∞-norm bound. To formulate the problem clearly, we give the following definition.

Definition 1 Given a constant $\gamma > 0$, the uncertain switched delay system (1) is said to be robust stabilizable with H_∞ disturbance attenuation level γ if there exists a switching law $\sigma(t)$ and an associated controller u_σ such that

(i) The resulting closed-loop system of the system (1) with $\omega(t) = 0$ is stabilizable for all admissible uncertainties.
(ii) With zero initial condition $\varphi(\theta) = 0$, $\theta \in [-\bar{\tau}, 0]$, $\|z(t)\|_2 < \gamma \|\omega(t)\|_2$ is guaranteed for all admissible uncertainties and all nonzero $\omega \in L_2[0, \infty)$.

First, we design a switching law $\sigma(t)$ and an associated controller u_σ such that the system (1) with $\omega(t) = 0$ is stabilizable via the multiple Lyapunov functions method.

The following lemma is important for developing our results.

Lemma 2 *Under Assumptions 1 and 2, the system (1) with $\omega(t) = 0$ is stabilizable if there exist matrices $P_i > 0$, $Q_i < 0$, $Q > 0$ and scalars $\eta_i > 0$ $\alpha_{ij} < 0$ $(i, j \in \Xi)$, $\varepsilon > 0$, such that the following matrix inequalities*:

$$\begin{bmatrix} \Gamma_i + Q + \sum_{j=1}^{l} \alpha_{ij}(P_i - P_j + \eta_i Q_i) & P_i A_{di} \\ A_{di}^{\mathrm{T}} P_i & -Q \end{bmatrix} < 0 \tag{4}$$

hold with $\Gamma_i = A_i^{\mathrm{T}} P_i + P_i A_i - P_i B_i B_i^{\mathrm{T}} P_i + \varepsilon^{-1} P_i D_i D_i^{\mathrm{T}} P_i + \varepsilon E_i^{\mathrm{T}} E_i$. The stabilizing variable structure controller for the i-th subsystem is given by

$$u_i(t) = -\frac{1}{2} B_i^{\mathrm{T}} P_i x(t) - \hat{u}_i(t), \tag{5}$$

where $\hat{u}_i(t) = (\lambda a_i \|x(t)\| + \phi_i(x,t) + \mu)\,\mathrm{sign}(s_i(t))$, $s_i(t) = B_i^{\mathrm{T}} P_i x(t)$, *and* μ *is positive constant.*

Proof We define regions

$$\Phi_i = \left\{x \,|\, x^{\mathrm{T}}(P_i - P_j + \eta_i Q_i)x \le 0,\ j = 1, 2, \ldots, l\right\}, \quad i = 1, 2, \ldots, l. \tag{6}$$

Obviously, $\bigcup_{i\in\Xi} \Phi_i = R^n$.

Then, we design a hysteresis switching law by

$$\begin{aligned} &\sigma(0) = \min\arg\left\{\Phi_i \,|\, x(0) \in \Phi_i\right\},\\ &\text{for } t > 0, \quad \sigma(t) = \begin{cases} i, & \text{if } x(t) \in \Phi_i \text{ and } \sigma(t^-) = i,\\ \min\arg\{\Phi_k \,|\, x(t) \in \Phi_k\}, & \text{if } x(t) \notin \Phi_i \text{ and } \sigma(t^-) = i. \end{cases} \end{aligned} \tag{7}$$

Choose the Lyapunov functional candidate

$$V_i(t) = x^{\mathrm{T}}(t) P_i x(t) + \int_{t-\tau}^{t} x^{\mathrm{T}}(\theta) Q x(\theta)\, d\theta. \tag{8}$$

Differentiating (8) with respect to t, we obtain

$$\dot{V}_i(t) = 2x^{\mathrm{T}}(t) P_i \dot{x}(t) + x^{\mathrm{T}}(t) Q x(t) - x^{\mathrm{T}}(t-\tau) Q x(t-\tau). \tag{9}$$

Using (5) in the system (1) with $\omega(t) = 0$, we have

$$\dot{x}(t) = \left(A_i - \frac{1}{2} B_i B_i^{\mathrm{T}} P_i + \Delta A_i\right) x(t) + (A_{di} + \Delta A_{di}) x(t-\tau) - B_i \hat{u}_i(t) + B_i f_i(x,t). \tag{10}$$

Let $\xi(t) = \left[\begin{smallmatrix} x(t) \\ x(t-\tau) \end{smallmatrix}\right]$. Substituting (10) into (9), we have

$$\begin{aligned} \dot{V}_i(t) = {} & \xi^{\mathrm{T}}(t) \Pi_i \xi(t) + 2x^{\mathrm{T}}(t) P_i B_i \Delta M_{di} x(t-\tau) \\ & - 2x^{\mathrm{T}}(t) P_i B_i \hat{u}_i(t) + 2x^{\mathrm{T}}(t) P_i B_i f_i(x,t), \end{aligned} \tag{11}$$

where

$$\Pi_i = \begin{bmatrix} A_i^{\mathrm{T}} P_i + P_i A_i - P_i B_i B_i^{\mathrm{T}} P_i + Q + \Delta A_i^{\mathrm{T}} P_i + P_i \Delta A_i & P_i A_{di} \\ A_{di}^{\mathrm{T}} P_i & -Q \end{bmatrix}.$$

It follows from Lemma 1 that

$$\Delta A_i^{\mathrm{T}} P_i + P_i \Delta A_i \le \varepsilon^{-1} P_i D_i D_i^{\mathrm{T}} P_i + \varepsilon E_i^{\mathrm{T}} E_i.$$

Applying the Razumikin theorem [3], we know that for any solution $x(t+\tau)$ of the system (1), there exists a constant $\lambda > 1$ such that

$$\left\|x(t+\tau)\right\| < \lambda \left\|x(t)\right\|, \quad 0 \le \tau < \bar{\tau}. \tag{12}$$

Thus, it holds that

$$\dot{V}_i(t) < 0, \quad [t_{i-1}, t_i).$$

Let

$$\tilde{\Phi}_{ij} = \left\{x | x^{\mathrm{T}}(P_i - P_j + \eta_i Q_i)x = 0\right\}, \quad j \neq i.$$

Obviously, $\tilde{\Phi}_i = \bigcup_{j=1, j\neq i}^{l} \tilde{\Phi}_{ij}$ is the boundary of Φ_i.

According to the switching law (7), if $\sigma(t^-) = i$ and $x(t) \in \operatorname{int}\Phi_i$, then the trajectory will remain in Φ_i until it hits the boundary $\tilde{\Phi}_{ij}$, $j \in \Xi$. This means that switching only takes place on $\tilde{\Phi}_i$. In fact, $x(t_k) \in \tilde{\Phi}_{ij}$ means that $x^{\mathrm{T}}(t_k)(P_i - P_j + \eta_i Q_i)x(t_k) = 0$ and $x(t_k) \in \Phi_j$. Thus,

$$x^{\mathrm{T}}(t_k)P_i x(t_k) = x^{\mathrm{T}}(t_k)(P_j - \eta_i Q_i)x(t_k) > x^{\mathrm{T}}(t_k)P_j x(t_k).$$

Then according to the switching law (7), at each switching time t_j,

$$V_{i_{j+1}}(t_j) < V_{i_j}(t_j)$$

is true. In view of the multiple Lyapunov functions method, the system (1) with $\omega(t) = 0$ is asymptotically stabilizable. This completes the proof. □

Remark 1 The conventional state-dependent switching law has appeared in [5, 11]; it may result in sliding mode in the switching surface. From the proof of Lemma 2, it is obvious that regions Φ_{i_j} and $\Phi_{i_{j-1}}$ overlap, thus we introduce hysteresis to avoid sliding motions by designing a hysteresis switching law.

Next, based on the previous arguments, we consider stabilization with H_∞ disturbance attenuation lever γ of the uncertain switched delay system (1).

Theorem 1 *Under Assumptions 1 and 2, given a constant $\gamma > 0$, the uncertain switched delay system (1) is stabilizable with H_∞ disturbance attenuation γ if there exist matrices $P_i > 0$, $Q_i < 0$, $Q > 0$ and scalars $\eta_i > 0$, $\alpha_{ij} < 0$ $(i, j = 1, 2, \ldots, l)$, $\varepsilon > 0$ such that the matrix inequalities*

$$\begin{bmatrix} W_i + Q + \sum_{j=1}^{l} \alpha_{ij}(P_i - P_j + \eta_i Q_i) & P_i A_{di} \\ A_{di}^{\mathrm{T}} P_i & -Q \end{bmatrix} < 0 \tag{13}$$

hold with $W_i = A_i^{\mathrm{T}} P_i + P_i A_i - P_i B_i B_i^{\mathrm{T}} P_i + C_i^{\mathrm{T}} C_i + \varepsilon^{-1} P_i D_i D_i^{\mathrm{T}} P_i + \varepsilon E_i^{\mathrm{T}} E_i + \gamma^{-2} P_i G_i G_i^{\mathrm{T}} P_i$. In this case, the switching law $\sigma(t)$ and the controller u_σ are taken as (7) and (5), respectively.

Proof First, take the multiple Lyapunov functions $V_i(t)$ as (8). By Lemma 2, the system (1) is asymptotically stabilizable with $\omega(t) = 0$ for all admissible uncertainties.

In the following, we show that the overall L_2-gain from ω to z is less than or equal to γ. Under the zero initial condition, without loss of generality, for $\forall T \geq t_0 = 0$, assume $T \in [t_k, t_{k+1})$ for some k. Now we introduce

$$J = \int_0^T \left(z^{\mathrm{T}} z - \gamma^2 \omega^{\mathrm{T}} \omega\right) dt. \tag{14}$$

According to the switching sequence (2), we have

$$J=\sum_{j=0}^{k-1}\Big(\int_{t_j}^{t_{j+1}}\big(\|z\|^2-\gamma^2\|\omega\|^2+\dot V_{i_j}(t)\big)\,dt-\big(V_{i_j}(t_{j+1})-V_{i_j}(t_j)\big)\Big)$$
$$+\int_{t_k}^{T}\big(\|z\|^2-\gamma^2\|\omega\|^2+\dot V_{i_j}(t)\big)\,dt-\big(V_{i_k}(T)-V_{i_k}(t_j)\big),$$

where $V_i(t)$ is as in (8).

Note that

$$\begin{aligned}
&\|z\|^2-\gamma^2\|\omega\|^2+\dot V_{i_j}(t)\\
&\le\begin{bmatrix}x(t)\\ x(t-\tau)\end{bmatrix}^{\mathrm T}\begin{bmatrix}\Gamma_i+Q & P_iA_{di}\\ A_{di}^{\mathrm T}P_i & -Q\end{bmatrix}\begin{bmatrix}x(t)\\ x(t-\tau)\end{bmatrix}+2x^{\mathrm T}(t)P_iB_i\Delta M_{di}x(t-\tau)\\
&\quad-2x^{\mathrm T}(t)P_iB_i\hat u_i(t)+2x^{\mathrm T}(t)P_iB_if_i(x,t)\\
&\quad-\big(\gamma^{-1}G_i^{\mathrm T}P_ix(t)-\gamma\omega\big)^{\mathrm T}\big(\gamma^{-1}G_i^{\mathrm T}P_ix(t)-\gamma\omega\big)
\end{aligned}\tag{15}$$

where $\Gamma_i=A_i^{\mathrm T}P_i+P_iA_i-P_iB_iB_i^{\mathrm T}P_i+C_i^{\mathrm T}C_i+\gamma^{-2}P_iG_iG_i^{\mathrm T}P+\Delta A_i^{\mathrm T}P_i+P_i\Delta A_i$.

Substituting (5) into (15) gives

$$\begin{aligned}
&\|z\|^2-\gamma^2\|\omega\|^2+\dot V_{i_j}(t)\\
&\le-\big(\gamma^{-1}G_i^{\mathrm T}P_ix(t)-\gamma\omega\big)^{\mathrm T}\big(\gamma^{-1}G_i^{\mathrm T}P_ix(t)-\gamma\omega\big)\\
&\le 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
J&\le-\sum_{j=0}^{k-1}\big\{V_{i_j}\big(x(t_{j+1})\big)-V_{i_j}\big(x(t_j)\big)\big\}-\big(V_{i_k}\big(x(T)\big)-V_{i_k}\big(x(t_k)\big)\big)\\
&=-\big(V_{i_0}\big(x(t_1)\big)-V_{i_0}\big(x(t_0)\big)+V_{i_1}\big(x(t_2)\big)-V_{i_1}\big(x(t_1)\big)+\cdots+V_{i_{k-1}}\big(x(t_k)\big)\\
&\quad-V_{i_{k-1}}\big(x(t_{k-1})\big)\big)-V_{i_k}\big(x(T)\big)+V_{i_k}\big(x(t_k)\big).
\end{aligned}$$

In view of $V_{\sigma(t_{k+1})}(t_k)<V_{\sigma(t_k)}(t_k)$, we have

$$\begin{aligned}
J&<V_{i_0}\big(x(t_0)\big)-V_{i_k}\big(x(T)\big)=-V_{i_k}\big(x(T)\big)\\
&<0
\end{aligned}$$

for $\forall\omega\in L_2[0,\infty)$ and all admissible uncertainties. That is, $\|z(t)\|_2<\gamma\|\omega(t)\|_2$. This completes the proof. □

From a practical point of view, the bounds $\phi_i(x,t)$ of the disturbances $f_i(x,t)$ are not easy to know. In order to handle this problem, we introduce the following assumption.

Assumption 3 There exist unknown positive constants g_{i1} and g_{i2}, $i \in \Xi$ such that

$$\|f_i(x,t)\| \le g_{i1} + g_{i2}\|x(t)\|. \tag{16}$$

We now have to deal with g_{i1} and g_{i2}. To this end, we employ parameter estimates $\hat{g}_{i1}$ and $\hat{g}_{i2}$ to adapt the unknown constant parameters g_{i1} and g_{i2}, respectively. The adaptation error of each parameter estimated is defined as $\tilde{g}_{i1} = \hat{g}_{i1} - g_{i1}$ and $\tilde{g}_{i2} = \hat{g}_{i2} - g_{i2}$.

Theorem 2 *Under Assumptions 1 and 3 and given a constant $\gamma > 0$, the uncertain switched delay system (1) is stabilizable with H_∞ disturbance attenuation γ if the matrix inequalities (13) are feasible. The switching law is taken as (7). The stabilizing adaptive variable structure controller for the subsystem is given by*

$$u_i(t) = -\frac{1}{2}B_i^{\mathrm{T}} P_i x(t) - \tilde{u}_i(t) \tag{17}$$

where $\tilde{u}_i(t) = (\lambda a_i\|x(t)\| + \hat{g}_{i1} + \hat{g}_{i2}\|x(t)\| + \mu)\,\mathrm{sign}(s_i(t))$, $s_i(t) = B_i^{\mathrm{T}} P_i x(t)$, λ, μ are positive constants, and P_i, Q satisfy (13). The parameter update laws of the i-th subsystem are chosen as

$$\begin{aligned} \dot{\hat{g}}_{i1} &= \begin{cases} 0, & \sigma \ne i, \\ \beta_1\|s_i(t)\|, & \sigma = i, \end{cases} \\ \dot{\hat{g}}_{i2} &= \begin{cases} 0, & \sigma \ne i, \\ \beta_2\|s_i(t)\|\|x(t)\|, & \sigma = i, \end{cases} \end{aligned} \tag{18}$$

where β_1 and β_2 are positive constants specified by the designer.

Proof We choose the Lyapunov functional candidate

$$V_i(t) = x^{\mathrm{T}}(t)P_i x(t) + \int_{t-\tau}^{t} x^{\mathrm{T}}(\theta)Qx(\theta)\,d\theta + \beta_1^{-1}\sum_{j\in\Xi}\tilde{g}_{j1}^2 + \beta_2^{-1}\sum_{j\in\Xi}\tilde{g}_{j2}^2. \tag{19}$$

In subsequent arguments, we shall first verify the stabilization of the system (1) with $\omega(t) = 0$.

Note that

$$\begin{aligned} \dot{\tilde{g}}_{i1} &= \dot{\hat{g}}_{i1}, \\ \dot{\tilde{g}}_{i2} &= \dot{\hat{g}}_{i2}. \end{aligned} \tag{20}$$

Differentiating (19) with respect to t, we obtain

$$\begin{aligned} \dot{V}_i(t) &= 2x^{\mathrm{T}}(t)P_i\dot{x}(t) + x^{\mathrm{T}}(t)Qx(t) - x^{\mathrm{T}}(t-\tau)Qx(t-\tau) \\ &\quad + 2\beta_1^{-1}\sum_{j\in\Xi}\tilde{g}_{j1}\dot{\hat{g}}_{j1} + 2\beta_2^{-1}\sum_{j\in\Xi}\tilde{g}_{j2}\dot{\hat{g}}_{j2}. \end{aligned} \tag{21}$$

Substituting (17) into (1) gives

$$\dot{x}(t) = \left(A_i - \frac{1}{2}B_i B_i^{\mathrm{T}} P_i + \Delta A_i\right)x(t) + (A_{di} + \Delta A_{di})x(t-\tau) - B_i\tilde{u}_i(t) + B_i f_i(x,t). \tag{22}$$

Substituting (22) and (18) into (21) and rearranging terms, we have

$$\begin{aligned}\dot{V}_i(t) &= \xi^{\mathrm{T}}(t)\Pi_i\xi(t) + 2x^{\mathrm{T}}(t)P_i B_i \Delta M_{di} x(t-\tau) - 2x^{\mathrm{T}}(t)P_i B_i \tilde{u}_i(t) \\ &\quad + 2x^{\mathrm{T}}(t)P_i B_i f_i(x,t) + 2\beta_1^{-1}(\hat{g}_{i1} - g_{i1})\dot{\hat{g}}_{i1} + 2\beta_2^{-1}(\hat{g}_{i2} - g_{i2})\dot{\hat{g}}_{i2},\end{aligned} \tag{23}$$

where $\xi^{\mathrm{T}}(t) = [x^{\mathrm{T}}(t)\ x^{\mathrm{T}}(t-\tau)]^{\mathrm{T}}$, and Π_i is defined as in (11).

Thus, it holds that

$$\dot{V}(t) < 0, \quad [t_{i-1}, t_i). \tag{24}$$

Note that $\sum_{j\in\Xi} \tilde{g}_{j1}^2$ and $\sum_{j\in\Xi} \tilde{g}_{j2}^2$ are continuous at all times. Similar to the proof of Lemma 1, at each switching time t_j, we have

$$V_{i_{j+1}}(t_j) < V_{i_j}(t_j).$$

In view of the multiple Lyapunov functions method, the system (1) with $\omega(t) = 0$ is asymptotically stabilizable.

Secondly, take

$$J = \int_0^T \left(z^{\mathrm{T}}z - \gamma^2\omega^{\mathrm{T}}\omega\right)dt. \tag{25}$$

Similar to the proof of Theorem 1, we have

$$J < 0,$$

for $\forall\omega \in L_2[0,\infty)$ and all admissible uncertainties. That is, $\|z(t)\|_2 < \gamma\|\omega(t)\|_2$. This completes the proof. □

Remark 2 From a practical viewpoint, the chattering phenomenon may occur around $\|x(t)\| = 0$. Therefore, $\|x(t)\|$ cannot be precisely guaranteed equal to zero. However, according to (18), $\hat{g}_{i0}$ and $\hat{g}_{i1}$ will be increasing all the time as long as $\|x(t)\| \neq 0$. Motivated by [2], to overcome this problem, we may apply the following modified adaptive laws:

$$\begin{aligned}\dot{\hat{g}}_{i1} &= \begin{cases} 0, & \text{if } \begin{cases}\sigma \neq i, \\ \|x(t)\| < d, \quad \sigma = i,\end{cases} \\ \beta_1\|s_i(t)\|, & \text{if } \|x(t)\| \geq d, \sigma = i,\end{cases} \\ \dot{\hat{g}}_{i2} &= \begin{cases} 0, & \text{if } \begin{cases}\sigma \neq i, \\ \|x(t)\| < d, \quad \sigma = i,\end{cases} \\ \beta_2\|s_i(t)\| \cdot \|x(t)\|, & \text{if } \|x(t)\| \geq d, \sigma = i.\end{cases}\end{aligned} \tag{26}$$

4 Example

In this section, we present a numerical example to demonstrate the effectiveness of the proposed design method. Consider the following uncertain switched delay system:

$$\begin{aligned} \dot{x}(t) &= (A_\sigma + \Delta A_\sigma)x(t) + (A_{d\sigma} + \Delta A_{d\sigma})x(t-\tau) \\ &\quad + B_\sigma\big(u_\sigma + f_\sigma(x,t)\big) + G_\sigma \omega(t), \\ z(t) &= C_\sigma x(t), \\ x(\theta) &= \varphi(\theta), \quad \theta \in [-\bar{\tau}, 0], \end{aligned} \tag{27}$$

where $\sigma(t) \in \Xi = \{1, 2\}$, $\bar{\tau} = 1$,

$$A_1 = \begin{bmatrix} -5 & 1 \\ 0.1 & -2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -2 & 1 \\ -5 & -4 \end{bmatrix}, \qquad A_{d1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix},$$

$$A_{d2} = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}^{\mathrm{T}},$$

$$C_2 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}^{\mathrm{T}}, \qquad G_1 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix},$$

the state uncertainties are

$$\Delta A_1 = D_1 \Sigma_1(t) E_1, \qquad \Delta A_2 = D_2 \Sigma_2(t) E_2,$$

where

$$D_1 = \begin{bmatrix} -0.1 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}, \qquad \Sigma_1 = [0.5\nu_{11} \quad 0.5\nu_{12}], \quad \nu_{11}, \nu_{12} \in [-1, 1],$$

$$E_1 = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}^{\mathrm{T}}, \qquad D_2 = \begin{bmatrix} 0.2 & 0.1 \\ -0.1 & 0.1 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}^{\mathrm{T}},$$

$$\Sigma_2 = [0.5\nu_{21} \quad 0.5\nu_{22}], \quad \nu_{22} \in [-1, 1],\ \nu_{21} \in [-1, 1],$$

the delay state uncertainties are

$$\Delta A_{d1} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} 2\nu_1 [1 \quad 2], \quad \nu_1 \in [-1, 1],$$

$$\Delta A_{d2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} 2\nu_2 [1 \quad 2], \quad \nu_2 \in [-1, 1],$$

and the unknown nonlinear functions are

$$f_1 = 0.5x_1 \cos(x_2) - 0.5 \sin(x_1 + x_2), \qquad f_2 = 1.5x_1 x_2 + 1.5 \sin(x_1 + x_2).$$

We adopt adaptive control to estimate the upper bound of the disturbances. It is easy to verify that the conditions of Theorem 2 are satisfied. Using Theorem 2 and taking $\varepsilon = 0.1$, $\alpha_1 = \alpha_2 = -5$, and $\eta_1 = \eta_2 = 0.1$, we have the disturbance attenuation level $\gamma = \frac{1}{\sqrt{2}}$. Then solving (13) leads to

$$P_1 = \begin{bmatrix} 0.1183 & 0.1126 \\ 0.1126 & 0.3687 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0.1415 & 0.0313 \\ 0.0313 & 0.263 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.5483 & 0.4009 \\ 0.4009 & 0.6249 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} -0.0424 & 0.0363 \\ 0.0363 & -0.1016 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} -0.0266 & -0.0057 \\ -0.0057 & -0.0873 \end{bmatrix}.$$

Let $\mu = 2$. According to (17), the adaptive variable structure controllers are given by

$$\begin{aligned} u_1 &= -0.0873x_1 - 0.1485x_2 - \big(6\|x(t)\| + \hat{g}_{11} + \hat{g}_{12}\|x(t)\| + 2\big)\operatorname{sign}(s_1), \\ u_2 &= -0.0707x_1 - 0.0157x_2 - \big(8\|x(t)\| + \hat{g}_{21} + \hat{g}_{22}\|x(t)\| + 2\big)\operatorname{sign}(s_2). \end{aligned} \tag{28}$$

The parameter update laws are designed as (26), with $d = 0.04$, $\beta_1 = \beta_2 = 0.1$.

$$\begin{aligned} \dot{\hat{g}}_{11} &= \begin{cases} 0, & \text{if } \begin{cases} \sigma \neq 1, \\ \|x(t)\| < d, \quad \sigma = 1, \end{cases} \\ \beta_1\|s_1(t)\|, & \text{if } \|x(t)\| \geq d, \sigma = 1, \end{cases} \\ \dot{\hat{g}}_{12} &= \begin{cases} 0, & \text{if } \begin{cases} \sigma \neq 1, \\ \|x(t)\| < d, \quad \sigma = 1, \end{cases} \\ \beta_2\|s_1(t)\| \cdot \|x(t)\|, & \text{if } \|x(t)\| \geq d, \sigma = 1, \end{cases} \\ \dot{\hat{g}}_{21} &= \begin{cases} 0, & \text{if } \begin{cases} \sigma \neq 2, \\ \|x(t)\| < d, \quad \sigma = 2, \end{cases} \\ \beta_1\|s_2(t)\|, & \text{if } \|x(t)\| \geq d, \sigma = 2, \end{cases} \\ \dot{\hat{g}}_{22} &= \begin{cases} 0, & \text{if } \begin{cases} \sigma \neq 2, \\ \|x(t)\| < d, \quad \sigma = 2, \end{cases} \\ \beta_2\|s_i(t)\| \cdot \|x(t)\|, & \text{if } \|x(t)\| \geq d, \sigma = 2, \end{cases} \end{aligned} \tag{29}$$

where $s_1 = [0.1747, 0.297]x(t)$, $s_2 = [0.1415, 0.0313]x(t)$.

The simulation results for the state responses of two subsystems with the initial state vector $x_0 = [2, -2]^{\mathrm{T}}$ are shown in Fig. 1.

By (7), we design the following switching law with initial state vector $x_0 = [2, -2]^{\mathrm{T}}$:

$$\begin{aligned} &\sigma(0) = 1, \\ &\sigma(t) = \begin{cases} 1, & \text{if } (x(t) \in \Phi_1 \text{ and } \sigma(t^-) = 1) \text{ or } (x(t) \notin \Phi_2 \text{ and } \sigma(t^-) \neq 1), \\ 2, & \text{if } (x(t) \in \Phi_2 \text{ and } \sigma(t^-) = 2) \text{ or } (x(t) \notin \Phi_1 \text{ and } \sigma(t^-) \neq 2). \end{cases} \end{aligned} \tag{30}$$

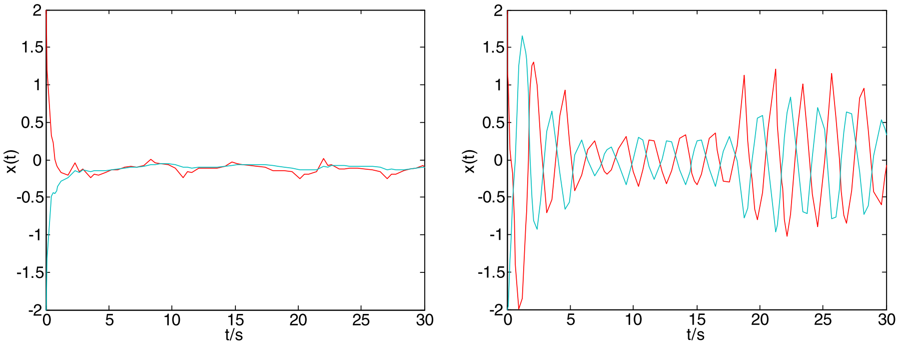
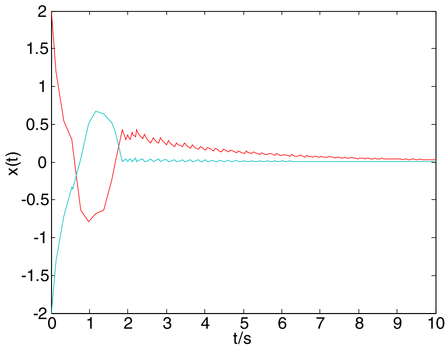


Fig. 1 The state responses of subsystems of the switched system (27)

Fig. 2 The state responses of the switched system (27)



Here

$$\Phi_1 = x^{\mathrm{T}}(t)\begin{bmatrix} -0.0274 & 0.085 \\ 0.085 & 0.0955 \end{bmatrix} x(t) < 0,$$

$$\Phi_2 = x^{\mathrm{T}}(t)\begin{bmatrix} 0.0205 & -0.0819 \\ -0.0819 & -0.1144 \end{bmatrix} x(t) < 0.$$

The simulation results are depicted in Fig. 2–Fig. 5.

The system states in the closed loop are shown in Fig. 2. It is clearly seen that the closed-loop system of the switched system (27) with the designed controllers (28), (29) and the switching law (30) is asymptotically stabilizable. Figure 3 is the input signal of the switched system (27). Figure 4 gives the estimations of the unknown nonlinear disturbances. The switching signal is given by Fig. 5.

5 Conclusion

In this paper, the problem of robust H_∞ control has been studied for a class of uncertain switched delay systems with parameter uncertainties and unknown nonlinear perturbations and external disturbance. Based on the multiple Lyapunov functions method, sufficient conditions have been derived for robust stabilization with

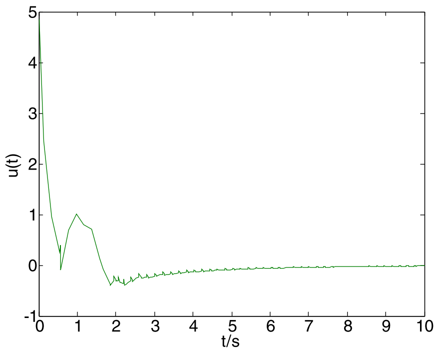


Fig. 3 The input signal of the switched system (27)

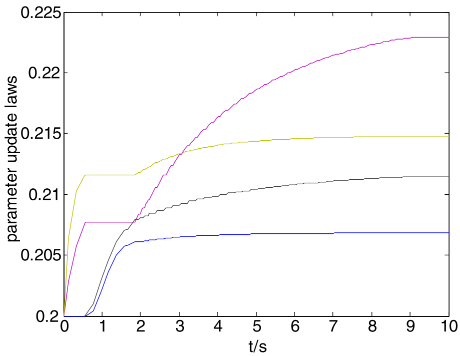


Fig. 4 The trajectories of the parameter update laws (29)

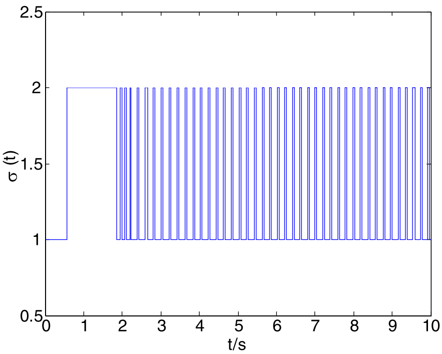


Fig. 5 The switching signal (30)

prescribed disturbance attenuation level γ. The hysteresis switching law has been designed. Variable structure control and adaptive variable structure control strategies have been developed to stabilize the uncertain switched delay system with H_∞ disturbance attenuation level γ under the hysteresis switching law for the cases of known and unknown upper bounds of the nonlinear disturbances, respectively.

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