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Positive Hybrid Real-Trigonometric Polynomials and Applications to Adjustable Filter Design and Absolute Stability Analysis

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Abstract The present paper points out that a class of positive polynomials deserves special attention due to several interesting applications in signal processing, system analysis and control. We consider positive hybrid polynomials with two variables, one real, the other complex, belonging to the unit circle. We present several theoretical results regarding the sum-of-squares representations of such polynomials, treating the cases where positivity occurs globally or on domains. We also give a specific Bounded Real Lemma. All the characterizations of positive hybrid polynomials are expressed in terms of positive semidefinite matrices and can be extended to polynomials with more than two variables. On the applicative side, we show how several problems are numerically tractable via semidefinite programming (SDP) algorithms. The first problem is the minimax design of adjustable FIR filters, using a modified Farrow structure. We discuss linear-phase and approximately linear-phase designs. The second is the absolute stability of time-delay feedback systems with unknown delay, for which we treat the cases of bounded and unbounded delay. Finally, we discuss the application of our methods to checking the stability of parameter-dependent systems. The design procedures are illustrated with numerical examples.

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1 Introduction

The current decade has seen an increasing interest in formulating problems in a positive polynomial setting, leading to a significant number of applications in optimization [11, 14], control [10] and signal processing [6]. This large variety of applications is due to the connection between sum-of-squares polynomials and positive semidefinite matrices, which allows the transformation of optimization problems with positive polynomials into semidefinite programming (SDP) problems, which are numerically more easily tractable.

The theory of positive polynomials typically involved polynomials with real variables and so the first applications [10, 14] addressed feedback control problems dealing with real polynomials. The particular case of trigonometric polynomials [13] was treated later, with applications in signal processing. In this paper, we concentrate on positive polynomials with both real and complex variables, the latter lying on the unit circle. We call such polynomials *hybrid*, since they inherit features from both real and trigonometric polynomials. They appear genuinely in some applications and so deserve a special treatment. To the best of our knowledge, there is no work dedicated to the properties and applications of positive hybrid polynomials.

For the sake of simplicity, we present all our results for polynomials with two variables, one real and the other complex. Appendix C shows the modifications that are necessary in the situation where more variables are involved. The hybrid polynomial which is the object of our study has the form

$$R(t,z)=\sum_{k_1=0}^{n_1}\sum_{k_2=-n_2}^{n_2} r_{k_1,k_2}t^{k_1}z^{-k_2}, \tag{1}$$

with $t\in\mathbb{R}$, $z\in\mathbb{C}$. The symmetry relation

$$r_{k_1,-k_2}=r^*_{k_1,k_2}, \tag{2}$$

which will be supposed to always hold, and implies that the polynomial (1) takes real values on $\mathbb{R}\times\mathbb{T}$, where $\mathbb{T}$ is the unit circle.

The paper is organized as follows. First, we review some of the properties of positive hybrid polynomials, starting with the Gram matrix parameterization of hybrid sum-of-squares described in Sect. 2. We give then, in Sect. 3, conditions for positivity on semialgebraic domains and also a specific version of the Bounded Real Lemma. Further, we present two applications in detail and hint to other possible ones. The first application, discussed in Sect. 4, is the design of adjustable FIR filters using a Farrow structure; we transform the specifications of a standard lowpass adjustable filter design problem into positivity conditions on hybrid polynomials, which have an equivalent SDP form; we treat both the cases of linear-phase and nonlinear-phase filters and solve the respective design problems without recurring to discretization. In

Sect. 5, we present the second application, on the absolute stability of systems with uncertain delay; we tackle two cases, where the delay is unbounded and bounded; in both cases, we are able to transform (or to approximate) Popov's absolute stability criterion into a condition solvable via an SDP approach. Suggestive for the hybrid character, the first application is in (discrete-time) signal processing and the second in the stability analysis of (continuous-time) feedback systems. Finally, Sect. 6 suggests other stability applications without a detailed investigation.

The notation is standard. Multivariate entities (vectors, matrices) are denoted by bold characters. $\boldsymbol{X}^{\mathrm{T}}$ is the transposed of matrix $\boldsymbol{X}$ and $\boldsymbol{X}^{\mathrm{H}}$ is the transposed and complex conjugated of $\boldsymbol{X}$. The Kronecker product is denoted by $\otimes$. For $a \in \mathbb{R}$, $\lfloor a \rfloor$ is the greatest integer smaller than a. For $a \in \mathbb{C}$, $\Re(a)$ denotes the real part of a. If $H(z)$ is a polynomial, by $H^*(z)$ we denote the polynomial whose coefficients are the complex conjugated of those of $H(z)$.

2 Hybrid Sum-of-Squares

A hybrid polynomial (1) is sum-of-squares if it can be written as

$$R(t,z) = \sum_{\ell=1}^{\nu} H_\ell(t,z) H_\ell^*\big(t, z^{-1}\big). \tag{3}$$

In this case, the polynomial (1) has an even degree in t; we denote $n_1 = 2m_1$. In (3), each polynomial $H_\ell(t,z)$ is causal in z (and named simply causal), i.e. it has the expression (for clarity, we omit the index ℓ)

$$H(t,z) = \sum_{k_1=0}^{m_1} \sum_{k_2=0}^{n_2} h_{k_1,k_2} t^{k_1} z^{-k_2}. \tag{4}$$

We denote

$$\boldsymbol{\psi}_n(t) = \begin{bmatrix} 1 & t & t^2 & \dots & t^n \end{bmatrix}^{\mathrm{T}} \tag{5}$$

the standard basis for degree n polynomials and

$$\boldsymbol{\psi}_{m_1,n_2}(t,z) = \boldsymbol{\psi}_{n_2}(z) \otimes \boldsymbol{\psi}_{m_1}(t) \tag{6}$$

the standard basis for polynomials of two variables. The index (m_1, n_2) will be omitted when clear from the context. We denote $N = (m_1 + 1)(n_2 + 1)$ the number of monomials in the basis. A causal hybrid polynomial (4) can be written as

$$H(t,z) = \boldsymbol{\psi}^{\mathrm{T}}\big(t, z^{-1}\big)\boldsymbol{h}, \tag{7}$$

where $\boldsymbol{h} \in \mathbb{C}^N$ is a vector containing the coefficients of $H(t,z)$ ordered as corresponding to the basis (6).

A Hermitian matrix $\boldsymbol{Q} \in \mathbb{C}^{N\times N}$ is called a *Gram* matrix associated with the hybrid polynomial (1) if

$$R(t,z) = \boldsymbol{\psi}^{\mathrm{T}}\big(t, z^{-1}\big) \cdot \boldsymbol{Q} \cdot \boldsymbol{\psi}(t,z). \tag{8}$$

Theorem 1 *The relation between the coefficients of the hybrid polynomial* (1) *and the elements of the associated Gram matrix is*

$$r_{k_1,k_2} = trace[\boldsymbol{T}_{k_1,k_2} \cdot \boldsymbol{Q}], \tag{9}$$

with $\boldsymbol{T}_{k_1,k_2} = \boldsymbol{\Theta}_{k_2} \otimes \boldsymbol{\Upsilon}_{k_1}$, *where* $\boldsymbol{\Theta}_{k_2} \in \mathbb{R}^{(n_2+1)\times(n_2+1)}$ *is the elementary Toeplitz matrix with ones on diagonal* k_2 *and zeros elsewhere, and* $\boldsymbol{\Upsilon}_{k_1} \in \mathbb{R}^{(m_1+1)\times(m_1+1)}$ *is the elementary Hankel matrix with ones on antidiagonal* k_1 *and zeros elsewhere.*

Proof The relation (8) is equivalent to

$$\begin{aligned}
R(t,z) &= \mathrm{trace}\big[\boldsymbol{\psi}(t,z)\cdot\boldsymbol{\psi}^{\mathrm{T}}(t,z^{-1})\cdot\boldsymbol{Q}\big] \\
&= \mathrm{trace}\big[\big(\big(\boldsymbol{\psi}(z)\boldsymbol{\psi}^{\mathrm{T}}\big(z^{-1}\big)\big)\otimes\big(\boldsymbol{\psi}(t)\boldsymbol{\psi}^{\mathrm{T}}(t)\big)\big)\cdot\boldsymbol{Q}\big] \\
&= \sum_{k_1=0}^{n_1}\sum_{k_2=-n_2}^{n_2}\mathrm{trace}\big[(\boldsymbol{\Theta}_{k_2}\otimes\boldsymbol{\Upsilon}_{k_1})\cdot\boldsymbol{Q}\big]t^{k_1}z^{-k_2}.
\end{aligned}$$

By identifying this expression with (1), the equality (9) results. □

Theorem 2 *A hybrid polynomial* (1) *is sum-of-squares if and only if there exists a positive semidefinite matrix* $\boldsymbol{Q} \in \mathbb{C}^{N\times N}$ *such that* (8) *holds.*

Proof Using the eigenvalue decomposition of $\boldsymbol{Q}$, we can write $\boldsymbol{Q} = \sum_{\ell=1}^{\nu} \boldsymbol{h}_\ell \boldsymbol{h}_\ell^{\mathrm{H}}$. Inserting this in (8) and using (7) for each vector $\boldsymbol{h}_\ell$, we obtain (3). The reverse implication is now obvious. □

Remark 1 Since the real part of a positive semidefinite matrix is also positive semidefinite, if the sum-of-squares (1) has real coefficients, then the matrix $\boldsymbol{Q} \succeq 0$ from the above theorem has also real coefficients.

Theorems 1 and 2 show that sum-of-squares polynomials can be parameterized in terms of positive semidefinite matrices. The linearity of the relation (9) allows the transformation of optimization problems involving sum-of-squares into SDP problems.

Finally, we recall that not all positive hybrid polynomials are sum-of-squares. Although this result is not contained directly in [18], it is clear from there that once a variable is unbounded (t, in our case), there cannot be equivalence between positivity and sum-of-squares. This equivalence holds in general only for trigonometric polynomials [3], where all variables are bounded. For real polynomials it holds only in three cases: univariate polynomials of any degree, quadratic polynomials of any number of variables and quartic polynomials of two variables. The first example of positive polynomial that is not sum-of-squares was given by Motzkin [6, 17]; it has two variables (three variables in the homogeneous form that mathematicians favor for presentation) and degree equal to six.

3 Positivity on Domains

In this section, we study hybrid polynomials that are positive on domains. We define these domains by the positivity of some polynomials, i.e.

$$\mathcal{D} = \big\{(t, z) \in \mathbb{R} \times \mathbb{T} \mid D_\ell(t, z) \geq 0,\ \ell = 1 : L\big\}, \tag{10}$$

where $D_\ell(t, z)$ are hybrid polynomials defined as in (1). We assume that $\mathcal{D}$ is bounded and thus we have $\mathcal{D} \subset [a, b] \times \mathbb{T}$ for some constants a and b. We also assume that among the polynomials defining $\mathcal{D}$ is

$$D_L(t, z) = (t - a)(b - t). \tag{11}$$

In practice, this polynomial can be explicitly added to those defining (10), if not already present, so this is not a serious restriction.

Theorem 3 *If a polynomial* (1) *is positive on* $\mathcal{D}$, *i.e.* $R(t, z) > 0$, $\forall (t, z) \in \mathcal{D}$, *then there exist sum-of-squares* $S_\ell(t, z)$, $\ell = 0 : L$, *such that*

$$R(t, z) = S_0(t, z) + \sum_{\ell=1}^{L} D_\ell(t, z) \cdot S_\ell(t, z). \tag{12}$$

If the polynomials $R(t, z)$ *and* $D_\ell(t, z)$ *have real coefficients, then the sum-of-squares* $S_\ell(t, z)$ *have also real coefficients.*

Proof See Appendix A. □

Remark 2 Conversely, if (12) holds, then $R(t, z)$ is obviously nonnegative on $\mathcal{D}$.

Remark 3 Similarly to the case of real [15] or trigonometric [5] polynomials, the degrees of the sum-of-squares may be larger than the degree of $R(t, z)$.

Remark 4 In the particular case where $\mathcal{D} = [a, b] \times \mathbb{T}$, the relation (12) has the form

$$R(t, z) = S_0(t, z) + (t - a)(b - t)S_1(t, z). \tag{13}$$

Remark 5 Using Theorem 2, we express the sum-of-squares appearing in (12) using the parameterization (9), in terms of positive semidefinite matrices $\boldsymbol{Q}_\ell$, $\ell = 0 : L$. Accordingly, the relation (12) can be rewritten

$$r_{k1,k2} = \operatorname{trace}[\boldsymbol{T}_{k_1,k_2} \boldsymbol{Q}_0] + \sum_{\ell=1}^{L} \operatorname{trace}[\boldsymbol{\Psi}_{\ell,k_1,k_2} \boldsymbol{Q}_\ell], \tag{14}$$

where

$$\boldsymbol{\Psi}_{\ell,k_1,k_2} = \sum_{i_1+j_1=k_1} \sum_{i_2+j_2=k_2} (d_\ell)_{i_1,i_2} \boldsymbol{T}_{j_1,j_2}. \tag{15}$$

By $(d_\ell)_{i_1,i_2}$ we have denoted the coefficients of $D_\ell(t, z)$.

Relation (14) can be used to obtain SDP problems when optimization on polynomials that are positive on domains is involved. At implementation, the sizes of the matrices $\boldsymbol{Q}_\ell$, $\ell = 0 : L$, are determined by the degrees chosen for the sum-of-squares $S_\ell(t, z)$.

Another useful result has a typical Bounded Real Lemma (BRL) form.

Theorem 4 *Let $H(t, z)$ be a hybrid polynomial that is causal in z, i.e. has the form* (4) (*written also as* (7)). *If the inequality $|H(t, z)| < \gamma$, $\forall(t, z) \in \mathcal{D}$, holds for $\mathcal{D}$ defined in* (10), *then there exist matrices $\boldsymbol{Q}_\ell \succeq 0$, $\ell = 0 : L$, such that*

$$\gamma^2 \delta_{k_1 k_2} = trace[\boldsymbol{T}_{k_1,k_2} \boldsymbol{Q}_0] + \sum_{\ell=1}^{L} trace[\boldsymbol{\Psi}_{\ell,k_1,k_2} \boldsymbol{Q}_\ell], \tag{16}$$

where $\delta_{k_1 k_2}$ is the Kronecker symbol, and

$$\begin{bmatrix} \boldsymbol{Q}_0 & \boldsymbol{h} \\ \boldsymbol{h}^{\mathrm{H}} & 1 \end{bmatrix} \succeq 0, \tag{17}$$

where $\boldsymbol{Q}_0$ is a Gram matrix associated with $S_0(t, z)$, as in (9). *Conversely,* (16) *and* (17) *imply $|H(t, z)| \le \gamma$, $\forall(t, z) \in \mathcal{D}$.*

Proof See Appendix B. □

Remark 6 Similarly to the equivalence between (12) and (14), the relation (16) is equivalent to the polynomial equality

$$\gamma^2 = S_0(t, z) + \sum_{\ell=1}^{L} D_\ell(t, z) \cdot S_\ell(t, z). \tag{18}$$

The size of the matrices $\boldsymbol{Q}_\ell$, $\ell = 0 : L$, from (16) depends on the degrees of the sum-of-squares polynomials that appear in (18). Note that the minimal degree of $S_0(t, z)$ is $(2m_1, n_2)$.

4 Minimax Design of Adjustable FIR Filters

As a first application of optimization with hybrid polynomials, we discuss the design of adjustable FIR filters with the transfer function

$$H(p, z) = \sum_{k=0}^{K} (p - p_0)^k H_k(z), \tag{19}$$

where $H_k(z)$, $k = 0 : K$, are FIR filters, $p_0 \in \mathbb{R}$ is a constant and $p \in \mathbb{R}$ is variable. The implementation of the adjustable filter (19) is made with the Farrow structure [8] shown in Fig. 1.

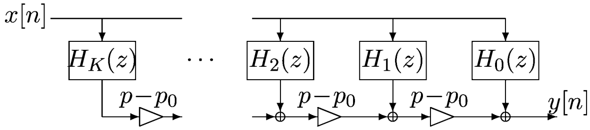

Fig. 1 Farrow structure for the implementation of adjustable filters

The optimization of adjustable filters has been typically performed using a least-squares criterion, see e.g. [2, 22]. Minimax optimization was employed in [12], using linear programming, and [24], using SDP. In both the latter papers, the optimization problem is convex, but the formulations are obtained through discretization. Here, we present solutions that do not appeal at all to discretization.

4.1 Linear-Phase Designs

We first examine the case where the filters $H_k(z)$ are all symmetric and have the same length, the same setup as in e.g. [12]. We want to design lowpass filters (19) whose bandpass width is continuously adjustable via the parameter p. Since only the magnitude response is optimized, we can assume without losing generality that the filters $H_k(z)$ are actually zero-phase, i.e.

$$H_k(z) = \sum_{i=-N}^{N} h_{k,i} z^{-i}, \quad h_{k,i} = h_{k,-i}, \tag{20}$$

and thus the transfer function (19) is a hybrid polynomial in the variables p and z.

A standard way to present the minimax design problem is as follows. We re-denote $p = \theta$, as the parameter will represent a frequency. The parameter θ takes values in a given interval $[\theta_l, \theta_u]$. The parameter p_0 can have any fixed value, e.g. $p_0 = (\theta_l + \theta_u)/2$; as advocated in [12], this value of p_0 makes the coefficients of the filters (20) have much smaller range of values than with the standard choice $p_0 = 0$, easing implementation and roundoff error concerns; however, the difficulty of the design does not depend on the value of p_0. The (adjustable) passband of the filter (19) is $[0, \theta - \Delta]$, where Δ is a constant, while the stopband is $[\theta + \Delta, \pi]$ and so the transition band has a fixed width of 2Δ. Setting a prescribed passband error bound γ_b, our goal is to minimize the stopband error γ_s and we obtain the minimax problem

$$\begin{aligned} \min \quad & \gamma_s \\ \text{s.t.} \quad & 1 - \gamma_p \le H\big(\theta, e^{j\omega}\big) \le 1 + \gamma_p, \quad \forall \omega \in [0, \theta - \Delta], \\ & -\gamma_s \le H\big(\theta, e^{j\omega}\big) \le \gamma_s, \quad \forall \omega \in [\theta + \Delta, \pi]. \end{aligned} \tag{21}$$

This problem has been solved in [12] via linear programming. We have recently proposed a discretization-free method using 2-D trigonometric polynomials [7].

We discuss here a modification of the problem (21) that: (i) for similar design specifications, it allows us to obtain filters with lower degrees than those resulting from (21), and (ii) can be solved using 2-D hybrid polynomials. The new problem is

$$\begin{aligned} \min \quad & \gamma_s \\ \text{s.t.} \quad & 1-\gamma_p \le H\bigl(p, e^{j\omega}\bigr) \le 1+\gamma_p, \quad \forall \cos\omega \in \bigl[p+\tilde{\Delta}, 1\bigr], \\ & -\gamma_s \le H\bigl(p, e^{j\omega}\bigr) \le \gamma_s, \quad \forall \cos\omega \in \bigl[-1, p-\tilde{\Delta}\bigr]. \end{aligned} \tag{22}$$

Now the parameter has a different significance, namely $p = \cos\tilde{\theta}$. (A somewhat similar construction was proposed in [23], but in a different context.) As the parameter p takes values in the interval $[p_l, p_u]$, the corresponding frequency $\tilde{\theta}$ takes values in the interval $[\tilde{\theta}_l, \tilde{\theta}_u]$ with $p_l = \cos\tilde{\theta}_u$, $p_u = \cos\tilde{\theta}_l$. The passband is now $[0, \operatorname{acos}(p+\tilde{\Delta})]$ and the stopband is $[\operatorname{acos}(p-\tilde{\Delta}), \pi]$. The width of the transition band is $\operatorname{acos}(p-\tilde{\Delta}) - \operatorname{acos}(p+\tilde{\Delta})$ and is no longer constant.

Let us comment on the extent of the passband and stopband with the help of Fig. 2. We assume that the passband edge ω_b has the same values for the problems (21) and (22), which means that $\theta - \Delta = \operatorname{acos}(p+\tilde{\Delta})$ and, in particular, the extreme values of ω_b are

$$\begin{aligned} \theta_l - \Delta &= \operatorname{acos}\bigl(p_u + \tilde{\Delta}\bigr), \\ \theta_u - \Delta &= \operatorname{acos}\bigl(p_l + \tilde{\Delta}\bigr). \end{aligned} \tag{23}$$

The stopband edge for (21) is $\omega_s = \omega_b + 2\Delta$. The solid line segments from Fig. 2 can be used to determine the values of ω_b and of the stopband edge ω_s. A horizontal line cutting the vertical axis at θ, cuts the two solid line segments in two points whose abscissas are a passband edge $\omega_b = \theta - \Delta$ and the corresponding stopband edge $\omega_s = \theta + \Delta$. The distance between the two points is 2Δ, the width of the transition band.

The stopband edge is different for problem (22) and is given by the intersection of the same horizontal line with the dashed line curve, giving $\omega_s = \operatorname{acos}(p-\tilde{\Delta})$ (recall that $\omega_b = \theta - \Delta = \operatorname{acos}(p+\tilde{\Delta})$). The problem (22) has the distinctive feature that the transition band is larger when the passband is narrow (and a good stopband attenuation is more difficult to obtain). As θ grows (and p decreases), the transition band becomes narrower. It is easy to choose the constants θ_l, θ_u, p_l, p_u, Δ, $\tilde{\Delta}$ such that (23) holds and also the average width of the transition band is the same for problems (21) and (22), i.e.

$$\frac{1}{p_u - p_l} \int_{p_l}^{p_u} \bigl[\operatorname{acos}\bigl(p-\tilde{\Delta}\bigr) - \operatorname{acos}\bigl(p+\tilde{\Delta}\bigr)\bigr]\, dp = 2\Delta. \tag{24}$$

An appealing feature of problem (22) is that it can be written in terms of hybrid polynomials that are positive on domains, as

$$\begin{aligned} \min \quad & \gamma_s \\ \text{s.t.} \quad & R_1(p,z) = H(p,z) - \gamma_p + 1 \ge 0, \quad \forall (p,z) \in \mathcal{D}_p, \\ & R_2(p,z) = \gamma_p + 1 - H(p,z) \ge 0, \quad \forall (p,z) \in \mathcal{D}_p, \\ & R_3(p,z) = H(p,z) + \gamma_s \ge 0, \quad \forall (p,z) \in \mathcal{D}_s, \\ & R_4(p,z) = \gamma_s - H(p,z) \ge 0, \quad \forall (p,z) \in \mathcal{D}_s \end{aligned} \tag{25}$$

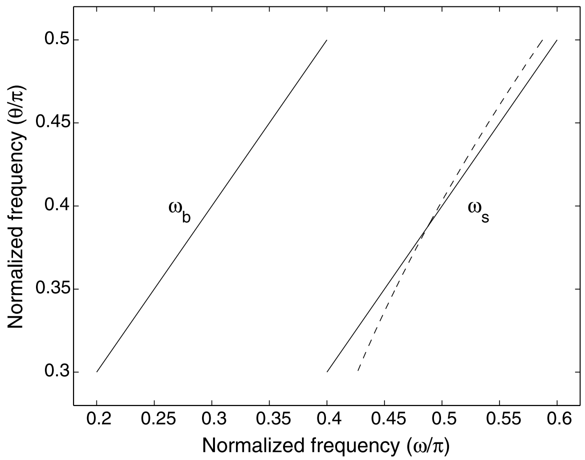

Fig. 2 Example of passband edge ω_b and stopband edge ω_s for the design problems (21), with constant transition band width (*solid line*), and (22), with variable transition band width (*dashed line*)

with domains $\mathcal{D}_p, \mathcal{D}_s$ defined as in (10),

$$\begin{aligned} \mathcal{D}_p &= \{(t,z) \in \mathbb{R} \times \mathbb{T} \mid D_{p\ell}(t,z) \geq 0,\ \ell = 1:2\}, \\ \mathcal{D}_s &= \{(t,z) \in \mathbb{R} \times \mathbb{T} \mid D_{s\ell}(t,z) \geq 0,\ \ell = 1:2\}, \end{aligned} \tag{26}$$

by the polynomials (recall that $\cos\omega = (z + z^{-1})/2$)

$$\begin{aligned} D_{p1}(p,z) &= \frac{1}{2}(z + z^{-1}) - p - \tilde{\Delta}, \\ D_{s1}(p,z) &= -\frac{1}{2}(z + z^{-1}) + p - \tilde{\Delta}, \\ D_{p2}(p,z) &= D_{s2}(p,z) = (p - p_l)(p_u - p). \end{aligned} \tag{27}$$

Each of the constraints of (25) can be expressed via (14) as linear equalities involving positive semidefinite matrices. Hence, the problem (25) becomes an SDP problem.

Example 1 We consider the design data used in [12] for problem (21): $\theta_l = 0.3\pi$, $\theta_u = 0.5\pi$, $\Delta = 0.1\pi$, $\gamma_p = 0.01$. For a fair comparison, we force the parameters of problem (22) to respect (23) and (24), obtaining $p_l = 0.019$, $p_u = 0.518$, $\tilde{\Delta} = 0.29$. The passband and stopband edges have the values given in Fig. 2 (see explanations above); in particular, the stopband edge varies between $\operatorname{acos}(p_u - \tilde{\Delta}) = 0.4268\pi$ and $\operatorname{acos}(p_l - \tilde{\Delta}) = 0.5874\pi$. The SDP version of the problem (25) has been implemented using SeDuMi [21]. There are three sum-of-squares polynomials in each relation (12) corresponding to a constraint of (25); denoting $\tilde{K} = 2\lfloor K/2 + 1 \rfloor$, we have used the degrees $(\tilde{K}, N)$ for $S_0(p,z)$, $(\tilde{K} - 2, N - 1)$ for $S_1(p,z)$ and $(\tilde{K} - 2, N)$ for $S_2(p,z)$; note that the degree in z is minimum; also, the degree in p is minimum for odd K.

For each value K (there are $K + 1$ filters in the adjustable filter (19)), we find the minimal orders N for which the optimal stopband attenuation resulted by solving (25) is $\gamma_s \leq 0.00316 = -50$ dB. The results are shown in Table 1; the upper half of the

Table 1 Minimal orders satisfying the design data from Example 1

	K	N	$(K+1)(N+1)$
Problem (21)	2	50	153
solved in [12]	3	18	76
(linear programming)	4	13	70
Problem (22)	2	24	75
solved via (25)	3	15	64
(SDP)	4	13	70

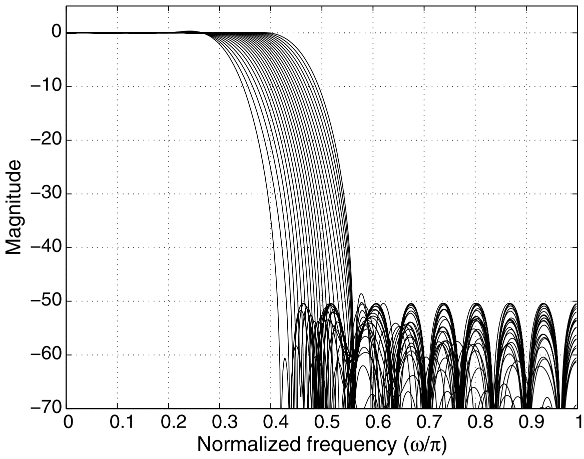

Fig. 3 Frequency responses of adjustable filters designed in Example 1 by solving (22), with $K = 3$, for 25 values of the parameter $p \in [0.019, 0.518]$

table is taken from [12] and gives the results of solving (21); the lower half shows our results for (22). The number of fixed multipliers needed to implement the adjustable filter as in Fig. 1 is $(K+1)(N+1)$, shown in the last column of Table 1. We note that the modified problem (22), with variable transition width, gives a solution with lower complexity (both in terms of fixed, 64 vs. 70, and adjustable, 3 vs. 4, multipliers) than the fixed transition width problem (21). The magnitude responses of the family of filters obtained for $K = 3$ is given in Fig. 3; their optimal stopband attenuation is $\gamma_s = 0.00303 = -50.37$ dB. (Note that the ripples slightly higher than -50 dB are inside the transition band.)

4.2 Approximately Linear-Phase Designs

We turn now to the case where the filters from (19) have no imposed symmetry and so they are

$$H_k(z) = \sum_{i=0}^{M} h_{k,i} z^{-i}. \tag{28}$$

An interesting problem in this case is to design approximately linear-phase low-delay filters. Given a desired group delay τ, such a problem with the same frequency band

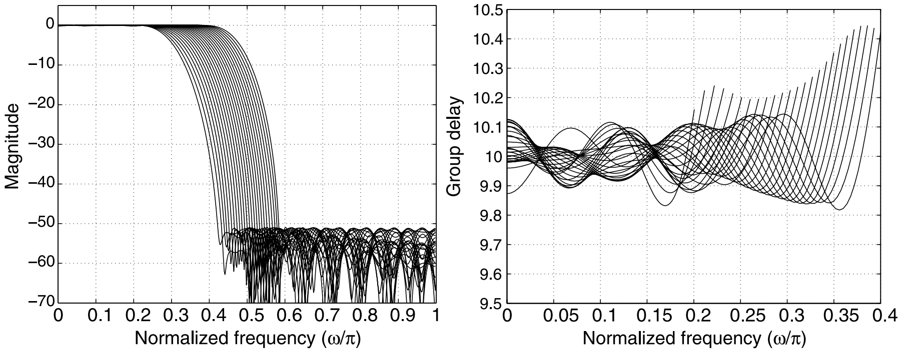

Fig. 4 Frequency responses (*left*) and passband group delays (*right*) of adjustable filters designed in Example 2 by solving (29), with $K = 4$, $M = 26$, $\tau = 10$, for 25 values of the parameter $p \in [0.019, 0.518]$

characteristics as (22) has the form

$$\begin{aligned} &\min \quad \gamma_s \\ &\text{s.t.} \quad \left|H\left(p, e^{j\omega}\right) - e^{-j\omega\tau}\right| \le \gamma_p, \quad \forall \cos\omega \in \left[p + \tilde{\Delta}, 1\right], \\ &\qquad \left|H\left(p, e^{j\omega}\right)\right| \le \gamma_s, \quad \forall \cos\omega \in \left[-1, p - \tilde{\Delta}\right]. \end{aligned} \tag{29}$$

Such a problem can be solved by discretization [24], using SDP. If τ is an integer then $H(p, z) - z^{-\tau}$ is a hybrid polynomial, and so the problem (29) can be solved using properties of hybrid polynomials, precisely Theorem 4; this kind of solution involves no discretization. Each of the two constraints from (29) can be expressed via (16) and (17) and thus (29) is transformed into an SDP problem.

Example 2 We solve (29) using the same data and setup as in Example 1. The order of the filters (28) are $M = 2N$, where N has the optimal values determined in Example 1; hence, the number of coefficients of the filters (19) and (28) is the same. In the implementation of (16), we take the overall degree of the equivalent polynomial equality (18) to be $(2(K + 1), N)$. With $M = 26$, the best results are now obtained with $K = 4$. For $\tau = 12$, we obtain $\gamma_s = 0.00269$ and for $\tau = 10$ we get $\gamma_s = 0.00285$; in both cases, the optimal stopband attenuation is better than for linear-phase filters. The magnitude responses and passband group delays of the optimal family of filters for $\tau = 10$ are shown in Fig. 4.

5 Absolute Stability of Systems with Delays

Positive hybrid real-trigonometric polynomials appear naturally in frequency-domain absolute stability conditions involving time-delay systems. For illustration, we consider the feedback system (see [16] and Problem 6.6 in [1])

$$\begin{aligned} \dot{x}(t) &= -ax(t) + \phi\bigl(y(t)\bigr), \\ y(t) &= x(t) + cx(t - \tau) \end{aligned} \tag{30}$$

where $a > 0$, $c \in \mathbb{R}$, $\tau > 0$ and ϕ is a sector-type nonlinearity,

$$0 \le \frac{\phi(\sigma)}{\sigma} \le k \le \infty. \tag{31}$$

The developments presented in this section can be easily extended to systems with linear parts of order larger than one and with multiple delays, but the form (30) allows a better understanding of the main ideas.

Since the linear part is stable ($a > 0$), according to the Popov's absolute stability criterion, the system (30) is asymptotically stable for every nonlinearity ϕ satisfying the sector-type inequality (31) if there exists $q \ge 0$ such that Popov's *frequency-domain condition* is verified:

$$\frac{1}{k} + \Re\big[(1 + j\omega q)G(j\omega)\big] > 0, \quad \forall \omega \in \mathbb{R}. \tag{32}$$

Here $G(s)$ is the transfer function of the linear part of the system and is given by

$$G(s) = \frac{1 + ce^{-s\tau}}{s + a}. \tag{33}$$

Our aim is to present SDP methods for verifying (32) and deciding on the stability of the system (30) in the case where the delay τ is unknown.

5.1 Delay-Independent Stability

We study first the conditions in which (32) holds for *all* $\tau > 0$, i.e. the absolute stability is delay independent. After elementary algebraic manipulations, the frequency condition (32) rewrites as

$$\begin{aligned} &2(a + j\omega)(a - j\omega) + k(1 + j\omega q)(a - j\omega)\big(1 + ce^{-j\omega\tau}\big) \\ &\quad + k(1 - j\omega q)(a + j\omega)\big(1 + ce^{j\omega\tau}\big) > 0, \quad \forall \omega \in \mathbb{R},\ \tau \ge 0. \end{aligned}$$

We denote $z = e^{j\omega\tau}$; since τ can have any value, the variables z and ω are decoupled; hence (32) is equivalent to

$$\begin{aligned} &R(\omega, z) = 2\big(a^2 + \omega^2\big) + H(\omega, z) + H\big(-\omega, z^{-1}\big) > 0, \\ &\forall \omega \in \mathbb{R},\ z \in \mathbb{T}, \end{aligned} \tag{34}$$

where $H(\omega, z) = k[a + j(aq - 1)\omega + q\omega^2](1 + cz^{-1})$. Thus, we have obtained a positivity condition on a hybrid polynomial.

Since each of q and k enter linearly in the coefficients of $R(\omega, z)$, one can solve several types of problems. For instance, one problem is to compute, for given a and c, the maximum value of k for which there exists $q \ge 0$ such that (34) holds. We can approach this problem in two ways. The first is to take several values of q on a grid

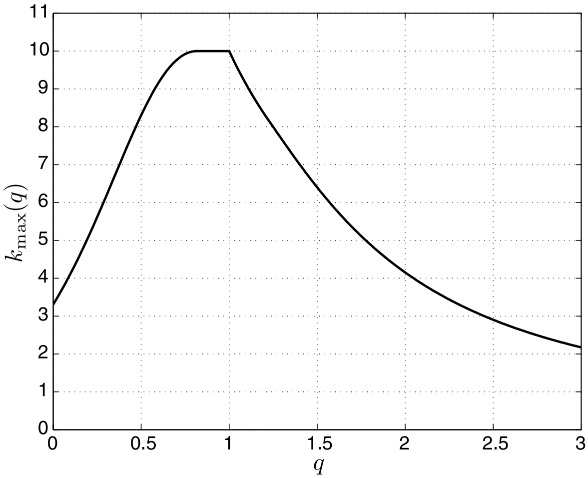

Fig. 5 Maximal values of absolute stability sector for Example 3

$\mathcal{G}$ and, for each of them, to compute

$$\begin{aligned} k_{\max}(q) = \max \quad & k \\ \text{s.t.} \quad & R(\omega, z) - \varepsilon \geq 0, \\ & \forall \omega \in \mathbb{R},\ z \in \mathbb{T} \quad (q \text{ given}) \end{aligned} \tag{35}$$

for a small given $\varepsilon \geq 0$. We replace the positivity constraint from (35) with the condition that $R(\omega, z) - \varepsilon$ is sum-of-squares and then appeal to the parameterization (9) of sum-of-squares hybrid polynomials; thus, we transform (35) to an SDP problem whose solution is possibly smaller than $k_{\max}(q)$. We have then a (conservative) estimation of the maximum sector value in $k_{\max} = \max_{q \in \mathcal{G}} k_{\max}(q)$.

A second approach is to solve, for given k, the feasibility problem

$$\begin{aligned} \text{find} \quad & q \geq 0 \\ \text{s.t.} \quad & R(\omega, z) - \varepsilon \geq 0, \quad \forall \omega \in \mathbb{R},\ z \in \mathbb{T} \quad (k \text{ given}). \end{aligned} \tag{36}$$

Again, by replacing positivity with a sum-of-squares condition, this can be transformed into a more conservative SDP problem. An estimation of the maximum sector $k_{\max}$ can be found by a bisection process, in which the value of k is increased or decreased as the relaxed problem (36) is found feasible or not, respectively.

Example 3 It can be proved [1, Problem 6.6] that the condition (32) holds for any k if $|c| < 1$. However, for $|c| > 1$, the maximal sector of absolute stability $k_{\max}$ has a finite value. We take $c = 1.1$ and $a = 1$. By solving (35) for various values of q (with $\varepsilon = 10^{-6}$), we obtain the curve $k_{\max}(q)$ from Fig. 5, which suggests that $k_{\max} = 10$. Indeed, by solving (36) in a bisection process, we obtain the value $k_{\max} = 10$ with an accuracy comparable to the tolerance used for stopping the bisection process (we used values between 10^{-3} and 10^{-6}).

5.2 Robust Stability with Unknown Bounded Delay

We tackle now the case where the delay is still unknown but is upper bounded, i.e. $\tau \in [0, \tilde{\tau}]$, with given $\tilde{\tau}$. The simple substitution $z = e^{j\omega\tau}$ used in the previous subsection is no longer useful. Instead, we use the Padé approximation of an exponential. The m-th order Padé approximation of e^{-s} is

$$P_m(s) = \frac{Q_m(s)}{Q_m(-s)}, \quad \text{with } Q_m(s) = \sum_{k=0}^{m} \frac{(2m-k)!m!(-s)^k}{(2m)!k!(m-k)!}. \tag{37}$$

Lemma 1 [25] *Given* $\tilde{\tau} > 0$ *and* $\omega \ge 0$, *we define the sets*

$$\begin{aligned} \boldsymbol{\Omega}(\omega, \tilde{\tau}) &= \left\{ e^{-j\omega\tau} \mid \tau \in [0, \tilde{\tau}] \right\}, \\ \boldsymbol{\Omega}_{\mathrm{o}}(\omega, \tilde{\tau}) &= \left\{ P_m(j\alpha_m\omega\tau) \mid \tau \in [0, \tilde{\tau}] \right\}, \\ \boldsymbol{\Omega}_{\mathrm{i}}(\omega, \tilde{\tau}) &= \left\{ P_m(j\omega\tau) \mid \tau \in [0, \tilde{\tau}] \right\}, \end{aligned} \tag{38}$$

where α_m *is a constant whose computation is detailed in* [25] (*for* $m = 3, 4, 5$, *the values of* α_m *are* 1.2329, 1.0315, 1.00363, *respectively*). *As* $|P_m(j\omega)| = 1$, *the three sets in* (38) *are arcs on the unit circle. With the above definitions*, *the following inclusions hold*:

$$\boldsymbol{\Omega}_{\mathrm{i}}(\omega, \tilde{\tau}) \subset \boldsymbol{\Omega}(\omega, \tilde{\tau}) \subset \boldsymbol{\Omega}_{\mathrm{o}}(\omega, \tilde{\tau}). \tag{39}$$

(*The subscripts i and o stand for* inner *and* outer *approximation*, *respectively*; *these names are justified by* (39).)

For a small given $\varepsilon > 0$, we replace the absolute stability condition (32) with

$$\frac{1}{k} + \Re\left[(1 + j\omega q) \frac{1 + c e^{-j\omega\tau}}{j\omega + a} \right] \ge \varepsilon, \quad \forall \omega \in \mathbb{R}, \ \forall \tau \in [0, \tilde{\tau}]. \tag{40}$$

Using Lemma 1, we substitute $e^{-j\omega\tau}$ with $P_m(j\alpha_m\omega\tau)$, to obtain a more conservative condition (we name it "outer", in the style of [25]), and with $P_m(j\omega\tau)$, to obtain a more relaxed condition (named "inner"). In both cases, we end up with polynomial conditions. To reduce the degree of the polynomials, we substitute $t = \omega\tau$ and eliminate ω. After some computation (including the elimination of the positive denominator), the "outer" condition can be written as

$$\begin{aligned} R_1(t, \tau) = {} & 2(1 - k\varepsilon)\left(a^2\tau^2 + t^2\right) Q_m(j\alpha_m t) Q_m(-j\alpha_m t) \\ & + H_1(t, \tau) + H_1(-t, \tau) \ge 0, \quad \forall t \in \mathbb{R}, \ \tau \in [0, \tilde{\tau}], \end{aligned} \tag{41}$$

where

$$H_1(t, \tau) = k(\tau + jqt)(a\tau - jt)\left[Q_m(-j\alpha_m t) + c Q_m(j\alpha_m t) \right] Q_m(j\alpha_m t). \tag{42}$$

Since τ belongs to $[0, \tilde{\tau}]$, we can substitute

$$\tau = \left(1 + \frac{z + z^{-1}}{2} \right) \frac{\tilde{\tau}}{2}, \quad z \in \mathbb{T}. \tag{43}$$

Hence $R_1(t,\tau)$ is transformed into an hybrid polynomial, denoted here $R_o(t,z)$. Similarly, for the "inner" condition we obtain a polynomial $R_\mathrm{i}(t,z)$ (note that in (41) and (42) we only have to replace α_m with 1). The degree of these polynomials is $(2(m+1),2)$.

We can now solve the same problems as in the delay-independent case, for instance to compute the maximum sector $\tilde{k}_{\max}$ for which the system (30) is absolutely stable $\forall\tau\in[0,\tilde{\tau}]$. The same two approaches valid in the delay-independent case can be used, but only for computing approximations of $\tilde{k}_{\max}$. For example, for given q, we can find an "outer" approximation by solving

$$\begin{aligned}\tilde{k}^{\mathrm{o}}_{\max}(q) = \max \quad & k \\ \text{s.t.} \quad & R_\mathrm{o}(t,z)-\varepsilon \quad \text{is sum-of-squares}, \\ & \forall t\in\mathbb{R},\ z\in\mathbb{T}\ (q \text{ given}).\end{aligned} \tag{44}$$

The "inner" approximation $\tilde{k}^{\mathrm{i}}_{\max}(q)$ is obtained similarly by replacing $R_\mathrm{o}(t,z)$ with $R_\mathrm{i}(t,z)$. We always have $\tilde{k}^{\mathrm{o}}_{\max}(q)\le\tilde{k}_{\max}(q)$, since both Lemma 1 the sum-of-squares approximation of positivity contribute to the decrease of the computed value. We probably have $\tilde{k}_{\max}(q)\le\tilde{k}^{\mathrm{i}}_{\max}(q)$, since the effect of Lemma 1 should be greater than the effect of sum-of-squares approximation (which is typically negligible).

The feasibility problem (36) can be treated in the same way and "outer" and "inner" approximations can be computed by bisection.

Example 4 We take again $a=1$, $c=1.1$ and consider several values of the delay bound $\tilde{\tau}$, namely 0.2, 0.5 and 1. The "outer" (thick lines) and "inner" (thin lines) approximations obtained by solving (44) (and its "inner" version) for $m=4$ are shown in Fig. 6 (the solid line curve for $\tilde{\tau}=\infty$ is copied from Fig. 5). It is visible that, for the same value q, the distance between the two approximations is very small. For example, for $\tilde{\tau}=1$ we obtain $\max_q \tilde{k}^{\mathrm{o}}_{\max}(q)=12.90$ and $\max_q \tilde{k}^{\mathrm{i}}_{\max}(q)=13.05$. We conclude that we obtain a good estimate of the maximum sector given by Popov's absolute stability criterion.

6 Other Applications

We mention just in passing other possible applications of our results on positive hybrid polynomials. Let us consider the stability of a 2-D continuous-discrete-time system whose transfer function has the denominator $A(s,z)$. For the system to be stable, the denominator must be Hurwitz–Schur, i.e. $A(s,z)\ne 0$, $\forall\Re(s)\ge 0$, $|z|\le 1$. Similarly to the DeCarlo–Strintzis [20] conditions for 2-D discrete-time systems, the test can be reduced to some 1-D conditions and the 2-D "border" condition $A(jt,z)\ne 0$, $\forall t\in\mathbb{R}$, $z\in\mathbb{T}$. This condition can be transformed into

$$R(t,z)=A(jt,z)A\big(-jt,z^{-1}\big)>0, \quad \forall t\in\mathbb{R},\ z\in\mathbb{T}, \tag{45}$$

where $R(t,z)$ has the form (1). Although we cannot test exactly this positivity condition, we can change it into a sufficient condition by requiring that $R(t,z)$ is a

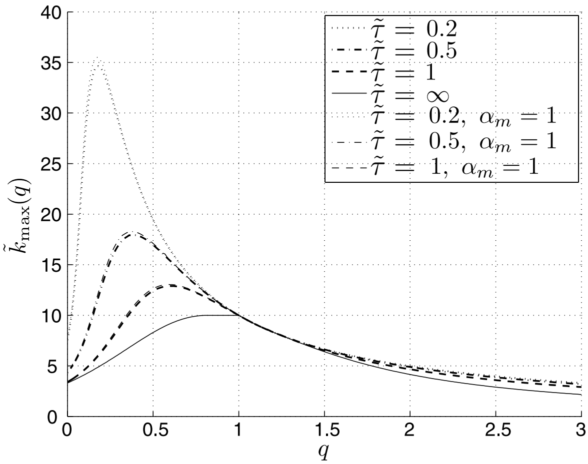

Fig. 6 Maximal values of absolute stability sector for Example 4 ($m = 4$). *Thick lines*: "outer" approximation; thin lines: "inner" approximation

strictly positive sum-of-squares. Similarly to the multidimensional discrete systems case treated in [4], we expect that the sum-of-squares condition is practically necessary.

A second problem is that of robust stability. Let us consider the (1-D) discrete-time system whose transfer function has the denominator

$$A(\tau, z) = \sum_{k=0}^{n} p_k(\tau) z^{-k}, \tag{46}$$

where $p_k(\tau)$ are polynomials in the unknown parameter $\tau \in [a, b]$. We want to test if the polynomial is Schur for all admissible values of the parameter, i.e. $A(\tau, z) \neq 0$, $\forall |z| \leq 1$, $\forall \tau \in [a, b]$. Different algorithms for this problem have been proposed in [9, 19]. As above, we can transform this into a positivity problem, by requiring that the hybrid polynomial $R(\tau, z) = A(\tau, z)A(\tau, z^{-1})$ has the form (13) (which is a sufficient condition). Using Remark 5, this condition can be transformed into a feasibility SDP problem.

The above problems can be easily generalized to more than two variables, for example in the case where the polynomial coefficients of (46) depend on more than one parameter.

7 Conclusion

We have presented basic properties of hybrid real-trigonometric polynomials that are positive globally or on certain domains defined as in (10). The relations with sum-of-squares polynomials allow the relaxation of optimization problems with positive hybrid polynomials to SDP problems. Using these properties, we have transformed adjustable FIR filter design and delay-independent absolute stability problems into SDP form, thus enjoying the benefits of reliable solutions. Further work will be devoted to enlarge the area of applications and to solve problems with higher complexity.

Appendix A: Proof of Theorem 3

The proof is inspired by a transformation method [5] for trigonometric polynomials and uses a basic result from [15]. Since z is on the unit circle $\mathbb{T}$, we put $z = x + jy$, with $x^2 + y^2 = 1$. The polynomials $R(t,z)$, $D_\ell(t,z)$ are changed into the real polynomials $R(t,x,y)$, $D_\ell(t,x,y)$, in three variables, and the set $\mathcal{D}$ into $\mathcal{D}' = \mathcal{D}_r \cap \mathcal{T}$, where $\mathcal{T} = \{(t,x,y) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ and

$$\mathcal{D}_r = \bigl\{(t,x,y) \in \mathbb{R}^3 \mid D_\ell(t,x,y) \geq 0,\ \ell = 1:L\bigr\}.$$

To define $\mathcal{D}'$ in the same style (by positivity of polynomials) we need two more polynomials:

$$D_{L+1}(t,x,y) = 1 - x^2 - y^2, \qquad D_{L+2}(t,x,y) = x^2 + y^2 - 1. \tag{47}$$

We also modify (11) into

$$D_L(t,x,y) = (t-a)(b-t)\bigl(2 - x^2 - y^2\bigr), \tag{48}$$

a transformation which leaves $\mathcal{D}'$ unchanged.

We want now to prove that all polynomials $R(t,x,y)$ that are positive on $\mathcal{D}'$ can be written as

$$R(t,x,y) = S_0(t,x,y) + \sum_{\ell=1}^{L+2} D_\ell(t,x,y) S_\ell(t,x,y). \tag{49}$$

A theorem from [15] states that this is true if there exists a polynomial $R_0(t,x,y)$ defined as in the right hand side of (49) such that the set $\{(t,x,y) \in \mathbb{R}^3 \mid R_0(t,x,y) \geq 0\}$ is bounded. (The theorem holds in the general multivariate case, not only in $\mathbb{R}^3$.) In our case, we simply take $R_0(t,x,y)$ equal to (48); this polynomial has the form (49), with $S_L(t,x,y) = 1$, $S_\ell(t,x,y) = 0$ for $\ell \neq L$; the polynomial is positive only for $t \in [a,b]$, $x^2 + y^2 \leq 2$, which is clearly a bounded set.

Transforming back $x + jy = z$ (this is a one-to-one transformation), the polynomials (47) disappear from (49), the polynomial (48) becomes (11) (these happen because $x^2 + y^2 = 1$) and the sum-of-squares $S_\ell(t,x,y)$ are transformed into sum-of-squares $S_\ell(t,z)$. Hence, we obtain (12).

Appendix B: Proof of Theorem 4

The proof is similar to that of Theorem 3 from [5]. We give here a short version. We prove first the reverse implication.

Using (7), for $z \in \mathbb{T}$ we write

$$\bigl|H(t,z)\bigr|^2 = \boldsymbol{\psi}^T\bigl(t, z^{-1}\bigr)\boldsymbol{h}\boldsymbol{h}^H\boldsymbol{\psi}(t,z).$$

Using (18), the above equality and the Gram matrix form (8) associated to $S_0(t,z)$ we obtain

$$\gamma^2 - \left|H(t,z)\right|^2 = \boldsymbol{\psi}^{\mathrm{T}}\left(t,z^{-1}\right)\left(\boldsymbol{Q}_0 - \boldsymbol{h}\boldsymbol{h}^{\mathrm{H}}\right)\boldsymbol{\psi}(t,z) + \sum_{\ell=1}^{L} D_\ell(t,z)\cdot S_\ell(t,z). \tag{50}$$

From (17) it results that $\boldsymbol{Q}_0 - \boldsymbol{h}\boldsymbol{h}^{\mathrm{H}} \succeq 0$ and so all the polynomials on the right hand side of (50) are nonnegative on $\mathcal{D}$, which implies that $\gamma^2 - |H(t,z)|^2 \geq 0$, $\forall (t,z) \in \mathcal{D}$.

The direct implication follows the backward way. However, according to Theorem 3 and (12) we can write

$$\gamma^2 - \left|H(t,z)\right|^2 = \boldsymbol{\psi}^{\mathrm{T}}\left(t,z^{-1}\right)\tilde{\boldsymbol{Q}}_0\boldsymbol{\psi}(t,z) + \sum_{\ell=1}^{L} D_\ell(t,z)\cdot S_\ell(t,z), \tag{51}$$

with $\tilde{\boldsymbol{Q}}_0 \succeq 0$, only if the left hand term is strictly positive, i.e. only if $\gamma > |H(t,z)|$. This explains the asymmetry between the direct and reverse implications, coming from the difference between (51) and (50). We put now $\boldsymbol{Q}_0 = \tilde{\boldsymbol{Q}}_0 + \boldsymbol{h}\boldsymbol{h}^{\mathrm{H}}$ and (18) results, etc.

Appendix C: The General Multivariate Case

We list here the modifications that are necessary in the general multivariate case, where there are d_1 real variables and d_2 trigonometric variables. In (1), we understand now that $k_1 \in \mathbb{N}^{d_1}$, $k_2 \in \mathbb{Z}^{d_2}$ and a monomial is e.g. $t^{k_1} = t_1^{k_{1,1}} \cdots t_{d_1}^{k_{1,d_1}}$; the sums are taken for all possible values, e.g. in the first sum we take all $k_1 \in \mathbb{N}^{d_1}$ for which $0 \leq k_1 \leq n_1$; note that now $n_1 \in \mathbb{N}^{d_1}$. The base (6) becomes accordingly

$$\boldsymbol{\psi}_{m_1,n_2}(t,z) = \boldsymbol{\psi}_{n_{2,d_2}}(z) \otimes \cdots \otimes \boldsymbol{\psi}_{n_{2,1}}(z) \otimes \boldsymbol{\psi}_{m_{1,d_1}}(t) \otimes \cdots \otimes \boldsymbol{\psi}_{m_{1,1}}(t). \tag{52}$$

In Theorem 1, the constant matrix appearing in (9) becomes

$$\boldsymbol{T}_{k_1,k_2} = \boldsymbol{\Theta}_{k_{2,d_2}} \otimes \cdots \otimes \boldsymbol{\Theta}_{k_{2,1}} \otimes \boldsymbol{\Upsilon}_{k_{1,d_1}} \cdots \otimes \boldsymbol{\Upsilon}_{k_{1,1}}. \tag{53}$$

The proof goes along the same line.

The results from Sect. 3 remain valid in the multivariate case, with a modification of the assumptions on the set (10). We assume that the set where one of the polynomials defining (10) is nonnegative is bounded. For example, this polynomial can be $D_L(t,z) = \rho^2 - t_1^2 - \cdots - t_{d_1}^2$. This polynomial replaces $D_L(t,z) = (t-a)(b-t)$ from the bivariate case and ensures that the conditions required in [15] hold.

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