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Controller Failure Analysis for Systems with Interval Time-Varying Delay: A Switched Method

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Abstract This paper studies the problem of controller failure analysis for a class of interval time-varying delay systems with a predesigned state feedback controller. Our objective is to establish conditions concerning controller failure time, under which the systems still keep exponentially stable. For this purpose, the systems with controller failures are first formulated as a class of switched delay systems. Next, based on a piecewise Lyapunov functional method, the exponential stability of such systems is guaranteed by restricting the unavailable rate and failure frequency of the controller. All the results are presented in terms of linear matrix inequalities. Two examples are provided to demonstrate the effectiveness of the proposed technique.

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1 Introduction

Controller failures are frequently encountered in practical control systems due to various factors [21–24]. One reason is that the data is not transmitted smoothly over the unreliable links, such as in packet dropout in networked control systems [25]. Failure can also occur when we suspend the controller purposefully in a positive way. In [21], exponential stability of feedback control systems with occasional controller failures is guaranteed by using a piecewise Lyapunov functional method. The corresponding result was extended to linear time-invariant discrete-time systems [22]. The corresponding method was also used to deal with L_2 gain analysis [23] and dynamical output feedback [24]. However, in all the above results, time delay is not taken into account, and it is often a source of instability and performance deterioration [1, 4–7, 10, 11, 18–20]. In particular, systems with interval time-varying delay are important, where the lower bound of the time-varying delay can vary in a time interval [6, 7, 16, 17]. The interval time-varying delay contains a time-invariant delay and a time-varying delay, where the lower bound of the time-varying delay is restricted to be zero. Nevertheless, controller failure problems for systems with interval time-varying delay are obviously more challenging and important issues, because unstable subsystems occur in engineering applications and in practical systems; there are many systems with interval time-varying delay such as networked control systems [20]. For example, a controller is designed to stabilize a networked control system with interval time-varying delay, which is an open-loop unstable system. Due to certain factors, the controller fails for some time such that the system is open loop and unstable until the controller resumes working. A system with such a process can be looked upon as a switched system, consisting of a stable subsystem (the closed-loop system with stabilizing controller) and an unstable subsystem (the open-loop system without the stabilizing controller). To date, controller failure problems for systems with interval time-varying delay have not been fully investigated yet due to some difficulties. In this paper, we overcome two main difficulties in dealing with controller failure problems for systems with interval time-varying delay. One is the system model description. It is assumed that the designed controller can stabilize the system and that the complete breakdown of the controller causes the system to be unstable in a time interval. Using the idea of switched control, the systems with control failures are modeled as one kind of switched system (see, for example, the survey papers [3, 8, 13], recent books [9, 14] and the references [12, 15, 26]. Then, the stability problems of the considered systems can be reduced to the corresponding problems for switched systems, for which many results are available in the literature. The other difficulty is stability analysis for the proposed new model. Two lemmas are first given to reflect the change property of Lyapunov functionals. Then, based on a piecewise Lyapunov functional method, exponential stability criteria for switched delay systems are developed under certain conditions of unavailability rate and failure frequency of the controller.

The paper is organized as follows. In Sect. 2, the model description and preliminaries are given. Section 3 presents two lemmas and the main results in this paper. Two examples are provided in Sect. 4 to show the effectiveness of the proposed methods. The conclusions are drawn in Sect. 5.

Notation The notation used in this paper is fairly standard. $P < 0$, $\lambda_{\max}(P)$ ($\lambda_{\min}(P)$) and $*$ denote a negative definite matrix P, the maximum (minimum) eigenvalues of P and the symmetric terms in symmetric matrices, respectively.

2 Model Description and Preliminaries

Consider the following delay system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Ex\big(t - d_1(t)\big) + Fu(t), \\ x(\theta) &= \phi(\theta), \quad \theta \in [-h_2, 0], \end{aligned} \tag{1}$$

where A, E and F are constant matrices with the appropriate dimensions. $x(t) \in R^n$ is the system state and $d_1(t)$ denotes an interval time-varying delay satisfying

$$h_1 \le d_1(t) \le h_2, \qquad \dot{d}_1(t) \le d_1, \tag{2}$$

where $h_2 > h_1 \ge 0$ are scalars representing the upper and the lower bounds of the time delay, respectively. $\phi(\theta)$ is the initial vector function, which is continuous on the segment $[-h_2, 0]$.

For system (1), the following assumptions are made throughout the paper:

(A1) When $u(t) = 0$, system (1) is unstable.

(A2) A state feedback controller $u(t) = Kx(t - d_2(t))$ has been designed to stabilize system (1), where $d_2(t)$ denotes the interval time-varying delay satisfying

$$h_3 \le d_2(t) \le h_4, \qquad \dot{d}_2(t) \le d_2 \tag{3}$$

where $h_4 > h_3 \ge 0$ are scalars representing the upper and the lower bounds of the time delay, respectively. Then, the resulting closed-loop system is

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Ex\big(t - d_1(t)\big) + Bx\big(t - d_2(t)\big), \\ x(\theta) &= \phi(\theta), \quad \theta \in [-\tau, 0], \end{aligned} \tag{4}$$

where $B = FK$ and $\tau = \max\{h_2, h_4\}$.

(A3) Let $t_0 < t_1 < \cdots < t_{2k}$, $k = 0, 1, \ldots$ denote the switching points during $[0, t)$. The designed controller $u(t)$ is available during $[t_{2k}, t_{2k+1})$, $k = 0, 1, 2, \ldots$, while it fails during $[t_{2k+1}, t_{2k+2})$. Moreover, it finally resumes working when $t \in [t_{2k+2}, \infty)$.

Remark 1 Assumption (A1) shows that the unforced system (1) is unstable. Assumption (A2) means that system (1) can be stabilized by state feedback controllers. In Assumption (A3), it is shown that the controller may fail over some time interval until it resumes working. For example, in networked control systems over Internet-type or wireless networks, the controller is prone to failure due to unreliable communication links, that is $u(t) = 0$.

Using the above assumptions, we formulate system (1) with controller failures as a class of switched delay system described by

$$\begin{aligned}
&\dot{x}(t) = Ax(t) + Ex\bigl(t - d_1(t)\bigr) + B_{\sigma(t)}x\bigl(t - d_2(t)\bigr), \\
&x(\theta) = \phi(\theta), \quad \theta \in [-\tau, 0],
\end{aligned} \tag{5}$$

where $\sigma(t) : [0, \infty) \to N = \{1, 2\}$ is the switching law which is a piecewise constant function; subsystem 1 denotes system (1) with the controller available while subsystem 2 denotes it without the controller, that is, $B_1 = FK$, $B_2 = 0$. Then, such switched delay systems are composed of a stable subsystem and an unstable one.

Definition 1 [15] The equilibrium point $x^* = 0$ of switched delay system (5) is exponentially stable if there exist $\gamma \geq 1$ and $\lambda > 0$ such that the solution $x(t)$ of system (5) satisfies

$$\bigl\|x(t)\bigr\| \leq \gamma \bigl\|x(t_0)\bigr\|_\theta \mathrm{e}^{-\lambda(t-t_0)}, \quad \forall t \geq t_0, \tag{6}$$

where $\| \bullet \|$ denotes the Euclidean norm and $\|x(t)\|_\theta = \max_{-\tau \leq \theta \leq 0}\{\|x(t + \theta)\|, \|\dot{x}(t + \theta)\|\}$.

Definition 2 Let $N_c(t_0, t)$ and $T_c(t_0, t)$ denote the number and the total time interval of controller failure during $[t_0, t)$ while $f_N(t_0, t) = N_c(t_0, t)/(t - t_0)$ and $f_T = T_c(t_0, t)/(t - t_0)$ are referred to as the failure frequency and the unavailable rate of the controller.

The objective of this paper is to find conditions concerning f_N and f_T, under which system (1) with occasional controller breakdown ($u = 0$) is still exponentially stable in the sense of (6).

3 Stability Analysis

In this section, the exponential stability criterion of system (5) is presented. First, based on the Lyapunov functional method, we address exponential estimations of the Lyapunov functional of subsystems 1 and 2, respectively. Then, delay-dependent exponential stability of the entire system is guaranteed by using the piecewise Lyapunov functional method.

For subsystem 1 of system (5), which is also denoted by (4), we select the Lyapunov functional candidate as

$$\begin{aligned}
V_1(t) = {} & x^{\mathrm{T}}(t)P_1x(t) + \sum_{i=1}^{4} \int_{t-h_i}^{t} x^{\mathrm{T}}(s)\mathrm{e}^{\alpha(s-t)}Q_{1i}x(s)\,\mathrm{d}s \\
& + \int_{-h_2}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)\mathrm{e}^{\alpha(s-t)}Z_{11}\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta \\
& + \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)\mathrm{e}^{\alpha(s-t)}Z_{12}\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta
\end{aligned}$$

$$+\int_{-h_4}^{0}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)\mathrm{e}^{\alpha(s-t)}Z_{13}\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta$$
$$+\int_{-h_4}^{-h_3}\int_{t+\theta}^{t}\dot{x}^{\mathrm{T}}(s)\mathrm{e}^{\alpha(s-t)}Z_{14}\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta$$
$$+\int_{t-d_1(t)}^{t}x^{\mathrm{T}}(s)\mathrm{e}^{\alpha(s-t)}Q_{15}x(s)\,\mathrm{d}s$$
$$+\int_{t-d_2(t)}^{t}x^{\mathrm{T}}(s)\mathrm{e}^{\alpha(s-t)}Q_{16}x(s)\,\mathrm{d}s, \tag{7}$$

where $P_1 > 0$, $Z_{1k} > 0$, $Q_{1j} > 0$, $j = 1, 2, \dots, 6$, $k = 1, 2, 3, 4$ are symmetric matrices.

Lemma 1 *Under* (2) *and* (3), *for some positive scalars* α, h_k, d_1 *and* d_2, *system* (4) *is exponentially stable if there exist symmetric matrices* $P_1 > 0$, $Z_{1k} > 0$, $Q_{1j} > 0$, $j = 1, 2, \dots, 6$ *and matrices* Y_{1k}, T_{1k}, L_{1k}, $k = 1, 2, 3, 4$ *such that*

$$\begin{bmatrix} \Omega_1 & R_1 \\ * & -S_1 \end{bmatrix} < 0. \tag{8}$$

Then, we have

$$V_1(t) \le \mathrm{e}^{-\alpha(t-t_0)}V_1(t_0), \tag{9}$$

where

$$\Omega_1 = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} \\ * & \varphi_{14} & 0 \\ * & * & \varphi_{15} \end{bmatrix},$$

$$R_1 = \begin{bmatrix} L_{11} & -T_{11} & L_{13} & -T_{13} & Y_{11} & T_{11} & L_{11} & Y_{13} & T_{13} & L_{13} & A^{\mathrm{T}}U_1 \\ L_{12} & -T_{12} & 0 & 0 & Y_{12} & T_{12} & L_{12} & 0 & 0 & 0 & E^{\mathrm{T}}U_1 \\ 0 & 0 & L_{14} & -T_{14} & 0 & 0 & 0 & Y_{14} & T_{14} & L_{14} & B^{\mathrm{T}}U_1 \end{bmatrix},$$

$$S_1 = \mathrm{diag}\big\{\mathrm{e}^{-\alpha h_1}Q_{11}, \mathrm{e}^{-\alpha h_2}Q_{12}, \mathrm{e}^{-\alpha h_3}Q_{13}, \mathrm{e}^{-\alpha h_4}Q_{14}, H_{10a}^{-1}Z_{11},$$
$$H_{11a}^{-1}Z_3, H_{11a}^{-1}Z_{12}, H_{10b}^{-1}Z_{13}, H_{11b}^{-1}Z_4, H_{11b}^{-1}Z_{14}, U_1\big\},$$

$$H_{11a} = \frac{\mathrm{e}^{\alpha h_2} - \mathrm{e}^{\alpha h_1}}{\alpha}, \qquad H_{10a} = \frac{\mathrm{e}^{\alpha h_2} - 1}{\alpha},$$

$$H_{10b} = \frac{\mathrm{e}^{\alpha h_4} - 1}{\alpha}, \qquad H_{11b} = \frac{\mathrm{e}^{\alpha h_4} - \mathrm{e}^{\alpha h_3}}{\alpha},$$

$$U_1 = h_2 Z_{11} + (h_2 - h_1)Z_{12} + h_4 Z_{13} + (h_4 - h_3)Z_{14},$$

$$Z_3 = Z_{11} + Z_{12}, \qquad Z_4 = Z_{13} + Z_{14},$$

$$\varphi_{11} = P_1 A + A^{\mathrm{T}}P_1 + \sum_{i=1}^{6} Q_{1i} + Y_{11} + Y_{11}^{\mathrm{T}} + Y_{13} + Y_{13}^{\mathrm{T}} + \alpha P_1,$$

$$\varphi_{12} = P_1 E - Y_{11} + Y_{12}^{\mathrm{T}} + T_{11} - L_{11}, \qquad \varphi_{13} = P_1 B - Y_{13} + Y_{14}^{\mathrm{T}} + T_{13} - L_{13},$$
$$\varphi_{14} = (d_1 - 1)\mathrm{e}^{-\alpha h_1} Q_{15} - Y_{12} - Y_{12}^{\mathrm{T}} + T_{12} + T_{12}^{\mathrm{T}} - L_{12} - L_{12}^{\mathrm{T}},$$
$$\varphi_{15} = (d_2 - 1)\mathrm{e}^{-\alpha h_3} Q_{16} - Y_{14} - Y_{14}^{\mathrm{T}} + T_{14} + T_{14}^{\mathrm{T}} - L_{14} - L_{14}^{\mathrm{T}}.$$

Proof Along the trajectory of system (4), taking the derivative of (7), we have

$$\begin{aligned}
\dot{V}_1(t) = {} & 2x^{\mathrm{T}}(t) P_1 \dot{x}(t) + x^{\mathrm{T}}(t)\alpha P_1 x(t) - \alpha V_1(t) \\
& + x^{\mathrm{T}}(t) Q_{11} x(t) - x^{\mathrm{T}}(t-h_1)\mathrm{e}^{-\alpha h_1} Q_{11} x(t-h_1) \\
& + x^{\mathrm{T}}(t) Q_{12} x(t) - x^{\mathrm{T}}(t-h_2)\mathrm{e}^{-\alpha h_2} Q_{12} x(t-h_2) \\
& + x^{\mathrm{T}}(t) Q_{13} x(t) - x^{\mathrm{T}}(t-h_3)\mathrm{e}^{-\alpha h_3} Q_{13} x(t-h_3) \\
& + x^{\mathrm{T}}(t) Q_{14} x(t) - x^{\mathrm{T}}(t-h_4)\mathrm{e}^{-\alpha h_4} Q_{14} x(t-h_4) \\
& + x^{\mathrm{T}}(t) Q_{15} x(t) - \big(1 - \dot{d}_1(t)\big) x^{\mathrm{T}}\big(t - d_1(t)\big)\mathrm{e}^{-\alpha d_1(t)} Q_{15} x\big(t - d_1(t)\big) \\
& + x^{\mathrm{T}}(t) Q_{16} x(t) - \big(1 - \dot{d}_2(t)\big) x^{\mathrm{T}}\big(t - d_2(t)\big)\mathrm{e}^{-\alpha d_2(t)} Q_{16} x\big(t - d_2(t)\big) \\
& + h_2 \dot{x}^{\mathrm{T}}(t) Z_{11} \dot{x}(t) - \int_{t-h_2}^{t} \dot{x}^{\mathrm{T}}(s)\mathrm{e}^{\alpha(s-t)} Z_{11} \dot{x}(s)\,\mathrm{d}s \\
& + (h_2 - h_1)\dot{x}^{\mathrm{T}}(t) Z_{12} \dot{x}(t) - \int_{t-h_2}^{t-h_1} \dot{x}^{\mathrm{T}}(s)\mathrm{e}^{\alpha(s-t)} Z_{12} \dot{x}(s)\,\mathrm{d}s \\
& + h_4 \dot{x}^{\mathrm{T}}(t) Z_{13} \dot{x}(t) - \int_{t-h_4}^{t} \dot{x}^{\mathrm{T}}(s)\mathrm{e}^{\alpha(s-t)} Z_{13} \dot{x}(s)\,\mathrm{d}s \\
& + (h_4 - h_3)\dot{x}^{\mathrm{T}}(t) Z_{14} \dot{x}(t) - \int_{t-h_4}^{t-h_3} \dot{x}^{\mathrm{T}}(s)\mathrm{e}^{\alpha(s-t)} Z_{14} \dot{x}(s)\,\mathrm{d}s.
\end{aligned} \tag{10}$$

The following Leibniz–Newton formula is introduced:

$$\begin{aligned}
& \big[x^{\mathrm{T}}(t) Y_{11} + x^{\mathrm{T}}\big(t - d_1(t)\big) Y_{12}\big] \\
& \quad \times \left[x(t) - x\big(t - d_1(t)\big) - \int_{t-d_1(t)}^{t} \dot{x}(s)\,\mathrm{d}s \right] = 0,
\end{aligned} \tag{11}$$

$$\begin{aligned}
& \big[x^{\mathrm{T}}(t) T_{11} + x^{\mathrm{T}}\big(t - d_1(t)\big) T_{12}\big] \\
& \quad \times \left[x\big(t - d_1(t)\big) - x(t - h_2) - \int_{t-h_2}^{t-d_1(t)} \dot{x}(s)\,\mathrm{d}s \right] = 0,
\end{aligned} \tag{12}$$

$$\begin{aligned}
& \big[x^{\mathrm{T}}(t) L_{11} + x^{\mathrm{T}}\big(t - d_1(t)\big) L_{12}\big] \\
& \quad \times \left[x(t - h_1) - x\big(t - d_1(t)\big) - \int_{t-d_1(t)}^{t-h_1} \dot{x}(s)\,\mathrm{d}s \right] = 0,
\end{aligned} \tag{13}$$

$$\begin{aligned}&\left[x^{\mathrm{T}}(t)Y_{13}+x^{\mathrm{T}}\big(t-d_2(t)\big)Y_{14}\right]\\&\quad\times\left[x(t)-x\big(t-d_2(t)\big)-\int_{t-d_2(t)}^{t}\dot{x}(s)\,\mathrm{d}s\right]=0,\end{aligned}\tag{14}$$

$$\begin{aligned}&\left[x^{\mathrm{T}}(t)T_{13}+x^{\mathrm{T}}\big(t-d_2(t)\big)T_{14}\right]\\&\quad\times\left[x\big(t-d_2(t)\big)-x(t-h_4)-\int_{t-h_4}^{t-d_2(t)}\dot{x}(s)\,\mathrm{d}s\right]=0,\end{aligned}\tag{15}$$

$$\begin{aligned}&\left[x^{\mathrm{T}}(t)L_{13}+x^{\mathrm{T}}\big(t-d_2(t)\big)L_{14}\right]\\&\quad\times\left[x(t-h_3)-x\big(t-d_2(t)\big)-\int_{t-d_2(t)}^{t-h_3}\dot{x}(s)\,\mathrm{d}s\right]=0,\end{aligned}\tag{16}$$

where Y_{1k}, T_{1k}, L_{1k}, $k=1,2,3,4$ are matrices with appropriate dimensions, called the free weighting matrices [19]. They are used to express the relationship between the terms in the Leibniz-Newton formula and can be determined by solving the corresponding linear matrix inequalities (LMIs).

For the integral terms of (10) and (11), we get

$$\begin{aligned}&2\left[x^{\mathrm{T}}(t)Y_{11}+x^{\mathrm{T}}\big(t-d_1(t)\big)Y_{12}\right]\int_{t-d_1(t)}^{t}\dot{x}(s)\,\mathrm{d}s+\int_{t-d_1(t)}^{t}\dot{x}^{\mathrm{T}}(s)\mathrm{e}^{\alpha(s-t)}Z_{11}\dot{x}(s)\,\mathrm{d}s\\&=\int_{t-d_1(t)}^{t}Y^{\mathrm{T}}\big(\mathrm{e}^{\alpha(s-t)}Z_{11}\big)^{-1}Y\,\mathrm{d}s-\int_{t-d_1(t)}^{t}\xi_1^{\mathrm{T}}(t)Y_1\big(\mathrm{e}^{\alpha(s-t)}Z_{11}\big)^{-1}Y_1^{\mathrm{T}}\xi_1(t)\,\mathrm{d}s\\&\geq -H_{10a}\xi_1^{\mathrm{T}}(t)Y_1Z_{11}^{-1}Y_1^{\mathrm{T}}\xi_1(t),\end{aligned}\tag{17}$$

where

$$\begin{aligned}\xi_1^{\mathrm{T}}(t)&=\left[x^{\mathrm{T}}(t)\quad x^{\mathrm{T}}(t-d_1(t))\right],\qquad Y_1^{\mathrm{T}}=\left[Y_{11}^{\mathrm{T}}\quad Y_{12}^{\mathrm{T}}\right],\\Y^{\mathrm{T}}&=\xi_1^{\mathrm{T}}(t)Y_1+\dot{x}^{\mathrm{T}}(s)\mathrm{e}^{\alpha(s-t)}Z_{11}.\end{aligned}$$

By using similar methods to those above, from (10), (12)–(16), we get

$$\begin{aligned}&\int_{t-h_2}^{t-d_1(t)}\dot{x}^{\mathrm{T}}(s)\mathrm{e}^{\alpha(s-t)}Z_{11}\dot{x}(s)\,\mathrm{d}s+\int_{t-h_2}^{t-d_1(t)}\dot{x}^{\mathrm{T}}(s)\mathrm{e}^{\alpha(s-t)}Z_{12}\dot{x}(s)\,\mathrm{d}s\\&\qquad+2\left[x^{\mathrm{T}}(t)T_{11}+x^{\mathrm{T}}\big(t-d_1(t)\big)T_{12}\right]\times\int_{t-h_2}^{t-d_1(t)}\dot{x}(s)\,\mathrm{d}s\\&\geq -H_{11a}\xi_1^{\mathrm{T}}(t)T_1(Z_{11}+Z_{12})^{-1}T_1^{\mathrm{T}}\xi_1(t),\end{aligned}\tag{18}$$

$$\begin{aligned}&2\left[x^{\mathrm{T}}(t)L_{11}+x^{\mathrm{T}}\big(t-d_1(t)\big)L_{12}\right]\int_{t-d_1(t)}^{t-h_1}\dot{x}(s)\,\mathrm{d}s+\int_{t-d_1(t)}^{t-h_1}\dot{x}^{\mathrm{T}}(s)\mathrm{e}^{\alpha(s-t)}Z_{12}\dot{x}(s)\,\mathrm{d}s\\&\geq -H_{11a}\xi_1^{\mathrm{T}}(t)L_1Z_{12}^{-1}L_1^{\mathrm{T}}\xi_1(t),\end{aligned}\tag{19}$$

$$2\big[x^{\mathrm T}(t)Y_{13}+x^{\mathrm T}\big(t-d_2(t)\big)Y_{14}\big]\int_{t-d_2(t)}^{t}\dot x(s)\,\mathrm ds+\int_{t-d_2(t)}^{t}\dot x^{\mathrm T}(s)\mathrm e^{\alpha(s-t)}Z_{13}\dot x(s)\,\mathrm ds$$
$$\ge -H_{10b}\xi_2^{\mathrm T}(t)Y_2Z_{13}^{-1}Y_2^{\mathrm T}\xi_2(t), \tag{20}$$
$$\int_{t-h_4}^{t-d_2(t)}\dot x^{\mathrm T}(s)\mathrm e^{\alpha(s-t)}Z_{13}\dot x(s)\,\mathrm ds+\int_{t-h_4}^{t-d_2(t)}\dot x^{\mathrm T}(s)\mathrm e^{\alpha(s-t)}Z_{14}\dot x(s)\,\mathrm ds$$
$$+2\big[x^{\mathrm T}(t)T_{13}+x^{\mathrm T}\big(t-d_2(t)\big)T_{14}\big]\times\int_{t-h_4}^{t-d_2(t)}\dot x(s)\,\mathrm ds$$
$$\ge -H_{11b}\xi_2^{\mathrm T}(t)T_2(Z_{13}+Z_{14})^{-1}T_2^{\mathrm T}\xi_2(t), \tag{21}$$
$$2\big[x^{\mathrm T}(t)L_{13}+x^{\mathrm T}\big(t-d_2(t)\big)L_{14}\big]\int_{t-d_2(t)}^{t-h_3}\dot x(s)\,\mathrm ds+\int_{t-d_2(t)}^{t-h_3}\dot x^{\mathrm T}(s)\mathrm e^{\alpha(s-t)}Z_{14}\dot x(s)\,\mathrm ds$$
$$\ge -H_{11b}\xi_2^{\mathrm T}(t)L_2Z_{14}^{-1}L_2^{\mathrm T}\xi_2(t), \tag{22}$$

where

$$T_1^{\mathrm T}=\begin{bmatrix}T_{11}^{\mathrm T} & T_{12}^{\mathrm T}\end{bmatrix},\qquad L_1^{\mathrm T}=\begin{bmatrix}L_{11}^{\mathrm T} & L_{12}^{\mathrm T}\end{bmatrix},\qquad \xi_2^{\mathrm T}(t)=\begin{bmatrix}x^{\mathrm T}(t) & x^{\mathrm T}(t-d_2(t))\end{bmatrix},$$
$$Y_2^{\mathrm T}=\begin{bmatrix}Y_{13}^{\mathrm T} & Y_{14}^{\mathrm T}\end{bmatrix},\qquad T_2^{\mathrm T}=\begin{bmatrix}T_{13}^{\mathrm T} & T_{14}^{\mathrm T}\end{bmatrix},\qquad L_2^{\mathrm T}=\begin{bmatrix}L_{13}^{\mathrm T} & L_{14}^{\mathrm T}\end{bmatrix}.$$

Substituting (11)–(16) into (10) and using (17)–(22), we have

$$\dot V_1(t)+\alpha V_1(t)\le \varsigma^{\mathrm T}(t)\Theta_1\varsigma(t), \tag{23}$$

where

$$\Theta_1=\tilde\Omega_1+\bar A^{\mathrm T}U_1\bar A+H_{20a}Y_{1a}Z_{11}^{-1}Y_{1a}^{\mathrm T}+H_{21a}T_{1a}Z_3^{-1}T_{1a}^{\mathrm T}+H_{21a}L_{1a}Z_{12}^{-1}L_{1a}^{\mathrm T}$$
$$+H_{20b}Y_{1b}Z_{13}^{-1}Y_{1b}^{\mathrm T}+H_{21b}T_{1b}Z_4^{-1}T_{1b}^{\mathrm T}+H_{21b}L_{1b}Z_{14}^{-1}L_{1b}^{\mathrm T},$$

$$\tilde\Omega_1=\begin{bmatrix}\varphi_{11} & \varphi_{12} & \varphi_{13} & L_{11} & -T_{11} & L_{13} & -T_{13}\\ * & \varphi_{14} & 0 & L_{12} & -T_{12} & 0 & 0\\ * & * & \varphi_{15} & 0 & 0 & L_{14} & -T_{14}\\ * & * & * & -\mathrm e^{-\alpha h_1}Q_{11} & 0 & 0 & 0\\ * & * & * & * & -\mathrm e^{-\alpha h_2}Q_{12} & 0 & 0\\ * & * & * & * & * & -\mathrm e^{-\alpha h_3}Q_{13} & 0\\ * & * & * & * & * & * & -\mathrm e^{-\alpha h_4}Q_{14}\end{bmatrix},$$

$$\varsigma(t)=\begin{bmatrix}x(t)\\ x(t-d_1(t))\\ x(t-d_2(t))\\ x(t-h_1)\\ x(t-h_2)\\ x(t-h_3)\\ x(t-h_4)\end{bmatrix},$$

$$Y_{1a}^{\mathrm T}=\begin{bmatrix}Y_{11}^{\mathrm T} & Y_{12}^{\mathrm T} & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

$$
\begin{aligned}
T_{1a}^{\mathrm{T}} &= \begin{bmatrix} T_{11}^{\mathrm{T}} & T_{12}^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
L_{1a}^{\mathrm{T}} &= \begin{bmatrix} L_{11}^{\mathrm{T}} & L_{12}^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
Y_{1b}^{\mathrm{T}} &= \begin{bmatrix} Y_{13}^{\mathrm{T}} & Y_{14}^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
T_{1b}^{\mathrm{T}} &= \begin{bmatrix} T_{13}^{\mathrm{T}} & T_{14}^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
L_{1b}^{\mathrm{T}} &= \begin{bmatrix} L_{13}^{\mathrm{T}} & L_{14}^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\bar{A} &= [\, A \quad E \quad B \quad 0 \quad 0 \quad 0 \quad 0 \,].
\end{aligned}
$$

Thus, it follows from Schur complements [2] that $\Theta_1 < 0$ is equivalent to (8). Then, it is obvious that

$$\dot{V}_1(t) + \alpha V_1(t) \le 0. \tag{24}$$

Furthermore, by integration, the inequality (9) is obtained, which completes the proof. □

Remark 2 It is clear to see that Lemma 1 implies exponential stability of the normal time delay systems. According to (7), it is true that

$$a_1 \|x(t)\|^2 \le V_1(t), \qquad V_1(t_0) \le b_1 \|x(t_0)\|_\theta^2, \tag{25}$$

where

$$
\begin{aligned}
a_1 &= \lambda_{\min}(P_1), \\
b_1 &= \lambda_{\max}(P_1) + \left(1 - \mathrm{e}^{-\alpha h_1}\right)\lambda_{\max}(Q_{11})/\alpha \\
&\quad + \left(1 - \mathrm{e}^{-\alpha h_2}\right)\left(\lambda_{\max}(Q_{12}) + \lambda_{\max}(Q_{15})\right)/\alpha \\
&\quad + \left(\mathrm{e}^{-\alpha h_2} + \alpha h_2 - 1\right)\lambda_{\max}(Z_{11})/\alpha^2 \\
&\quad + \left(\mathrm{e}^{-\alpha h_2} + \alpha h_2 - \mathrm{e}^{-\alpha h_1} - \alpha h_1\right)\lambda_{\max}(Z_{12})/\alpha^2 \\
&\quad + \left(1 - \mathrm{e}^{-\alpha h_3}\right)\lambda_{\max}(Q_{13})/\alpha \\
&\quad + \left(1 - \mathrm{e}^{-\alpha h_4}\right)\left(\lambda_{\max}(Q_{14}) + \lambda_{\max}(Q_{16})\right)/\alpha \\
&\quad + \left(\mathrm{e}^{-\alpha h_4} + \alpha h_4 - 1\right)\lambda_{\max}(Z_{13})/\alpha^2 \\
&\quad + \left(\mathrm{e}^{-\alpha h_4} + \alpha h_4 - \mathrm{e}^{-\alpha h_3} - \alpha h_3\right)\lambda_{\max}(Z_{14})/\alpha^2.
\end{aligned}
$$

Combining (9) and (25) yields

$$\|x(t)\| \le \sqrt{\frac{b_1}{a_1}}\mathrm{e}^{-\alpha(t-t_0)/2}\|x(t_0)\|_\theta. \tag{26}$$

Then, system (4) is exponentially stable. Moreover, if $\alpha = 0$, which means asymptotical stability, the results obtained by Lemma 1 are found in [6]. Thus, Lemma 1 is also applicable to normal interval time-varying delay systems.

Since $B_2 = 0$, subsystem 2 of system (5) can be rewritten as

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Ex\big(t - d_1(t)\big), \\ &x(\theta) = \phi(\theta), \quad \theta \in [-h_2, 0] \end{aligned} \tag{27}$$

and the Lyapunov functional candidate is chosen as

$$\begin{aligned} V_2(t) = {} & x^{\mathrm{T}}(t)P_2x(t) + \sum_{i=1}^{4}\int_{t-h_i}^{t} x^{\mathrm{T}}(s)\mathrm{e}^{\beta(t-s)}Q_{2i}x(s)\,\mathrm{d}s \\ & + \int_{-h_2}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)\mathrm{e}^{\beta(t-s)}Z_{21}\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta \\ & + \int_{-h_2}^{-h_1}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)\mathrm{e}^{\beta(t-s)}Z_{22}\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta \\ & + \int_{-h_4}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)\mathrm{e}^{\alpha(s-t)}Z_{23}\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta \\ & + \int_{-h_4}^{-h_3}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)\mathrm{e}^{\alpha(s-t)}Z_{24}\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta \\ & + \int_{t-d_1(t)}^{t} x^{\mathrm{T}}(s)\mathrm{e}^{\beta(t-s)}Q_{25}x(s)\,\mathrm{d}s \\ & + \int_{t-d_2(t)}^{t} x^{\mathrm{T}}(s)\mathrm{e}^{\alpha(s-t)}Q_{26}x(s)\,\mathrm{d}s, \end{aligned} \tag{28}$$

where $P_2 > 0$, $Z_{2k} > 0$ and $Q_{2j} > 0$, $j = 1, 2, \ldots, 6$, $k = 1, 2, 3, 4$ are symmetric matrices.

Lemma 2 *For* (27) *under* (2) *and given* $\beta > 0$, *if there exist symmetric matrices* $P_2 > 0$, $Z_{2k} > 0$ *and* $Q_{2j} > 0$, $j = 1, 2, \ldots, 6$ *and matrices* Y_{2k}, T_{2k}, L_{2k}, $k = 1, 2, 3, 4$ *such that*

$$\begin{bmatrix} \Omega_2 & R_2 \\ * & -S_2 \end{bmatrix} < 0, \tag{29}$$

then one can have

$$V_2(t) \le \mathrm{e}^{\beta(t-t_0)}V_2(t_0), \tag{30}$$

where

$$\Omega_2 = \begin{bmatrix} \varphi_{21} & \varphi_{22} & \varphi_{23} \\ * & \varphi_{24} & 0 \\ * & * & \varphi_{25} \end{bmatrix},$$

$$R_2=\begin{bmatrix} L_{21} & -T_{21} & L_{23} & -T_{23} & Y_{21} & T_{21} & L_{21} & Y_{23} & T_{23} & L_{23} & A^{\mathrm{T}}U_2 \\ L_{22} & -T_{22} & 0 & 0 & Y_{22} & T_{22} & L_{22} & 0 & 0 & 0 & E^{\mathrm{T}}U_2 \\ 0 & 0 & L_{24} & -T_{24} & 0 & 0 & 0 & Y_{24} & T_{24} & L_{24} & B^{\mathrm{T}}U_2 \end{bmatrix},$$

$$S_2=\operatorname{diag}\{\mathrm{e}^{-\alpha h_1}Q_{21},\mathrm{e}^{-\alpha h_2}Q_{22},\mathrm{e}^{-\alpha h_3}Q_{23},\mathrm{e}^{-\alpha h_4}Q_{24},H_{20a}^{-1}Z_{21},$$
$$H_{21a}^{-1}Z_5,H_{21a}^{-1}Z_{22},H_{20b}^{-1}Z_{23},H_{21b}^{-1}Z_6,H_{21b}^{-1}Z_{24},U_2\},$$

$$H_{20a}=\frac{1-\mathrm{e}^{-\beta h_2}}{\beta},\qquad H_{21a}=\frac{\mathrm{e}^{-\beta h_1}-\mathrm{e}^{-\beta h_2}}{\beta},$$

$$H_{20b}=\frac{1-\mathrm{e}^{-\beta h_4}}{\beta},\qquad H_{21b}=\frac{\mathrm{e}^{-\beta h_3}-\mathrm{e}^{-\beta h_4}}{\beta},$$

$$U_2=h_2Z_{21}+(h_2-h_1)Z_{22}+h_4Z_{23}+(h_4-h_3)Z_{24},$$

$$Z_5=Z_{21}+Z_{22},\qquad Z_6=Z_{23}+Z_{24},$$

$$\varphi_{21}=P_2A+A^{\mathrm{T}}P_2+\sum_{i=1}^{6}Q_{2i}+Y_{21}+Y_{21}^{\mathrm{T}}+Y_{23}+Y_{23}^{\mathrm{T}}-\beta P_2,$$

$$\varphi_{22}=P_2E-Y_{21}+Y_{22}^{\mathrm{T}}+T_{21}-L_{21},\qquad \varphi_{23}=P_2B-Y_{23}+Y_{24}^{\mathrm{T}}+T_{23}-L_{23},$$

$$\varphi_{24}=(d_1-1)\mathrm{e}^{\beta h_2}Q_{25}-Y_{22}-Y_{22}^{\mathrm{T}}+T_{22}+T_{22}^{\mathrm{T}}-L_{22}-L_{22}^{\mathrm{T}},$$

$$\varphi_{25}=(d_2-1)\mathrm{e}^{\beta h_4}Q_{26}-Y_{24}-Y_{24}^{\mathrm{T}}+T_{24}+T_{24}^{\mathrm{T}}-L_{24}-L_{24}^{\mathrm{T}}.$$

Proof The proof is similar to that of Lemma 1. Thus, it is omitted. □

Remark 3 Note that Lemmas 1 and 2 present the decay and growth estimations of $V_1(t)$ and $V_2(t)$, respectively. Their sufficient conditions are given in the form of LMIs.

Theorem 1 *Under* (2) *and* (3), *given* $\alpha>0$, $\beta>0$, $h_k>0$, $d_k>0$ *and* $\mu\ge 1$, *if there exist symmetric matrices* $P_p>0$, $Q_{pi}>0$, $Z_{pk}>0$, $p\in N$ *satisfying*

$$P_s\le\mu P_l,\quad Q_{si}\le\mu Q_{li},\quad Z_{sk}\le\mu Z_{lk},\quad \forall s,l\in N \tag{31}$$

and matrices Y_{pk},T_{pk},L_{pk}, $k=1,2,3,4$, $i=1,2,\dots,6$ *such that LMIs* (8) *and* (29) *hold, then system* (5) *is exponentially stable under such conditions, satisfying*

$$f_T\le\frac{\alpha-\alpha^*}{\alpha+\beta},\quad \alpha^*\in(0,\alpha) \tag{32}$$

and

$$f_N\le\frac{\alpha_0}{\ln\mu^2\mu_1},\quad \alpha_0\in(0,\alpha^*). \tag{33}$$

Then, the state decay of system (5) *can be estimated as*

$$\|x(t)\| \le \sqrt{\frac{b_1}{a_1}} \mathrm{e}^{-\lambda(t-t_0)} \|x(t_0)\|_\theta \tag{34}$$

where $\lambda = (\alpha^* - \alpha_0)/2$, $\mu_1 = \mathrm{e}^{(\alpha+\beta) max\{h_2, h_4\}}$.

Proof We choose the following piecewise Lyapunov functional candidate:

$$V_{\sigma(t)}(t) = \begin{cases} V_1(t_{2k}), & \text{if } t \in [t_{2k}, t_{2k+1}), \\ V_2(t_{2k+1}), & \text{if } t \in [t_{2k+1}, t_{2k+2}), \end{cases} \tag{35}$$

where $k = 0, 1, 2, \dots, \infty$. $V_1(t)$ and $V_2(t)$ are defined in (7) and (28), respectively.

From (31), it is obtained on the switching point t that

$$V_1(t) \le \mu V_2(t^-), \qquad V_2(t) \le \mu\mu_1 V_1(t^-), \tag{36}$$

where $t^- = \lim_{\tau \to t} \tau$. Noting (A3), based on Lemmas 1 and 2, by induction during $t \in [t_{2k+2}, \infty)$, we have

$$\begin{aligned} V_1(t) &\le \mathrm{e}^{-\alpha(t-t_{2k+2})} V_1(t_{2k+2}) \le \mathrm{e}^{-\alpha(t-t_{2k+2})} \mu V_2(t_{2k+2}^-) \\ &\le \mathrm{e}^{-\alpha(t-t_{2k+2})} \mu \mathrm{e}^{\beta(t_{2k+2}-t_{2k+1})} V_2(t_{2k+1}) \\ &\le \mathrm{e}^{-\alpha(t-t_{2k+2})} \mu^2 \mu_1 \mathrm{e}^{\beta(t_{2k+2}-t_{2k+1})} V_1(t_{2k+1}^-) \\ &\le \mathrm{e}^{-\alpha(t-t_{2k+2})} \mu^2 \mu_1 \mathrm{e}^{\beta(t_{2k+2}-t_{2k+1})} \mathrm{e}^{-\alpha(t_{2k+1}-t_{2k})} V_1(t_{2k+1}^-) \\ &\le \cdots \le \mu^{2k+2} \mu_1^{k+1} \mathrm{e}^{-\alpha(t-t_0-T_c(t))} \mathrm{e}^{\beta T_c(t)} V_1(t_0). \end{aligned} \tag{37}$$

From (32), it holds that

$$-\alpha\big(t - t_0 - T_c(t_0, t)\big) + \beta T_c(t_0, t) \le -\alpha^*(t - t_0). \tag{38}$$

Notice Definition 2, $N_c(t_0, t) = k + 1$ for $t \in [t_{2k+2}, \infty)$; then from (33), it is obtained that

$$\mu^{2(k+1)} \mu_1^{k+1} = \mu^{2N_c(t_0,t)} \mu_1^{N_c(t_0,t)} \le \mathrm{e}^{\alpha_0(t-t_0)}. \tag{39}$$

Combining (37), (38) and (39) yields

$$V_1(t) \le \mathrm{e}^{-(\alpha^*-\alpha_0)(t-t_0)} V_1(t_0). \tag{40}$$

Combining (25) and (40) leads to (34). The proof is completed. □

Remark 4 Note that to obtain μ with respect to the failure frequency of the controller f_N, we first solve LMIs (8) and (29), then find μ according to (31). This method may lead to a larger μ. Another method is that we first select a known constant μ, and then solve (8), (29) and (31) simultaneously. If these LMIs have no solutions, a larger μ will be selected to yield a solution. On the contrary, if a solution is found, a smaller μ will be chosen. So, a proper μ could be obtained.

Remark 5 Note that Theorem 1 considers the case where the derivative of the interval time-varying delay is bounded by (2) and (3). As for the delay-derivative-free case, however,

$$0 \le h_1 \le d_1(t) \le h_2, \qquad 0 \le h_3 \le d_2(t) \le h_4. \tag{41}$$

Theorem 1 is still applicable, provided that the terms containing Q_{15}, Q_{16} in (8) and Q_{25}, Q_{26} in (29) are removed. On the other hand, Theorem 1 assumes that the delay in state and control input are different. Nevertheless, when they are the same, that is, when $h_1 = h_3$ and $h_2 = h_4$, similar results can be obtained if the terms concerning $d_2(t)$ are removed.

4 Numerical Examples

In this section, two examples are given to show the effectiveness of the proposed methods. Example 1 is for the case of (2) and (3) and Example 2 is for the case of (41).

Example 1 Consider system (1) under (2) and (3) with

$$A = \begin{bmatrix} 0.2 & 0 \\ 0 & 1.1 \end{bmatrix}, \qquad E = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \qquad F = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$h_1 = 0.1, \qquad h_2 = 0.5, \qquad d_1 = 2.0.$$

System (1) is assumed to be controlled through a network by a feedback controller $u(t) = Kx(t - d_2(t))$, where $K = [0.4\ \ -0.3]$ and $h_3 = 0.3$, $h_4 = 0.9$, $d_2 = 3.0$. Under initial state $\phi(\theta) = [10\ 5]^{\mathrm{T}}$, $\theta \in [-0.9, 0]$, the state trajectories of system (1) with and without the controller are shown in Figs. 1 and 2, respectively. Figure 1 shows that the designed controller can stabilize the resulting closed-loop system, which implies that Assumption (A2) is satisfied. Figure 2 depicts the state response of the system when $u(t) = 0$. It is easy to see that the open-loop system is unstable, that is, assumption (A1) is satisfied. Therefore, system (1) can be transformed to (5). Under this mode, applying Theorem 1 with $\alpha = 0.22$ and $\beta = 0.39$, the corresponding matrices are obtained as

$$P_1 = \begin{bmatrix} 4.0469 & -0.3935 \\ -0.3935 & 0.5530 \end{bmatrix}, \qquad Q_{11} = \begin{bmatrix} 0.5174 & -0.0017 \\ -0.0017 & 0.0012 \end{bmatrix},$$

$$Q_{12} = \begin{bmatrix} 0.5767 & -0.0018 \\ -0.0018 & 0.0009 \end{bmatrix}, \qquad Q_{13} = \begin{bmatrix} 0.4678 & 0.0001 \\ 0.0001 & 0.0007 \end{bmatrix},$$

$$Q_{14} = \begin{bmatrix} 0.4891 & 0.0001 \\ 0.0001 & 0.0006 \end{bmatrix}, \qquad Q_{15} = \begin{bmatrix} 0.2124 & -0.0001 \\ -0.0001 & 0.0001 \end{bmatrix},$$

$$Q_{16} = \begin{bmatrix} 0.1730 & -0.0004 \\ -0.0004 & 0.0000 \end{bmatrix}, \qquad Z_{11} = \begin{bmatrix} 2.9277 & 0.1562 \\ 0.1562 & 2.8221 \end{bmatrix},$$

$$Z_{12} = \begin{bmatrix} 0.7213 & -0.0227 \\ -0.0227 & 0.0371 \end{bmatrix}, \qquad Z_{13} = \begin{bmatrix} 2.0010 & -1.1354 \\ -1.1354 & 0.8734 \end{bmatrix},$$

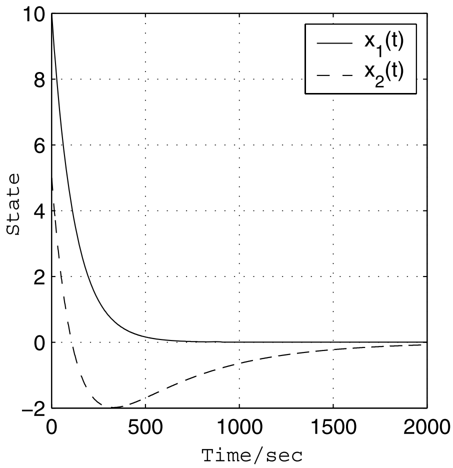

Fig. 1 State response of system (1) with the controller (Example 1)

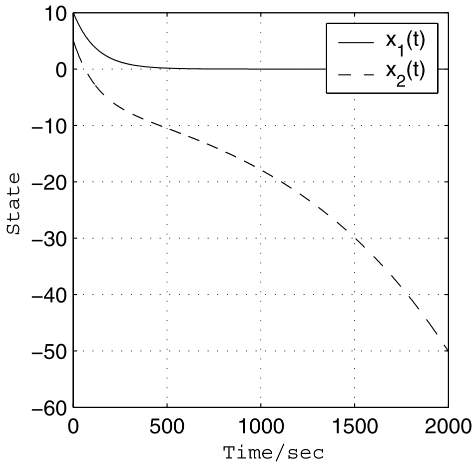

Fig. 2 State response of system (1) without the controller (Example 1)

$$Z_{14}=\begin{bmatrix}0.6272 & -0.0108\\ -0.0108 & 0.0246\end{bmatrix},\qquad P_2=\begin{bmatrix}2.5627 & -0.1994\\ -0.1994 & 0.4401\end{bmatrix},$$

$$Q_{21}=\begin{bmatrix}0.6266 & -0.0020\\ -0.0020 & 0.0024\end{bmatrix},\qquad Q_{22}=\begin{bmatrix}0.6846 & -0.0021\\ -0.0021 & 0.0019\end{bmatrix},$$

$$Q_{23}=\begin{bmatrix}0.5770 & 0.0001\\ 0.0001 & 0.0014\end{bmatrix},\qquad Q_{24}=\begin{bmatrix}0.5904 & 0.0002\\ 0.0002 & 0.0011\end{bmatrix},$$

$$Q_{25}=\begin{bmatrix}0.2626 & -0.0002\\ -0.0002 & 0.0001\end{bmatrix},\qquad Q_{26}=\begin{bmatrix}0.2059 & -0.0004\\ -0.0004 & 0.0001\end{bmatrix},$$

$$Z_{21}=\begin{bmatrix}1.9893 & -0.0269\\ -0.0269 & 1.2602\end{bmatrix},\qquad Z_{22}=\begin{bmatrix}0.7829 & -0.0381\\ -0.0381 & 0.0732\end{bmatrix},$$

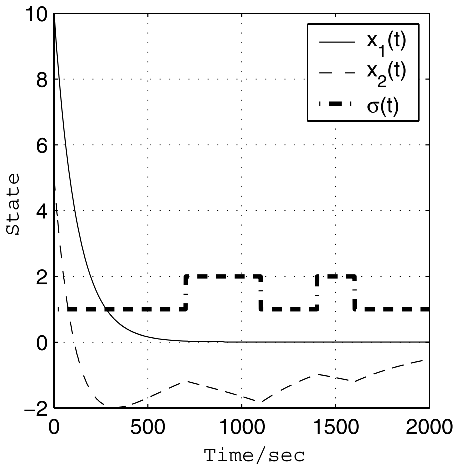

Fig. 3 State response of system (1) with controller failures (Example 1)

$$Z_{23}=\begin{bmatrix}1.1185 & -0.5023\\ -0.5023 & 0.5062\end{bmatrix},\qquad Z_{24}=\begin{bmatrix}0.6985 & -0.0204\\ -0.0204 & 0.0491\end{bmatrix}.$$

Furthermore, it is obtained from (31) that $\mu = 2.7$. From (32), selecting $\alpha^* = 0.02$ yields $f_T \le 1/3$, which means that the controller failure time is not more than $1/3$ of the total time. Given $\alpha_0 = 0.01$, it holds from (33) that $f_N \le 0.0039$, which implies that the controller failure frequency is permitted to be 0.0039. Thus, if the two conditions are satisfied, then system (1) is exponentially stable with the state delay

$$\|x(t)\| \le 3.6754\mathrm{e}^{-0.01t}\|x(0)\|_\theta.$$

Choose the following switching signal:

$$\sigma(t)=\begin{cases}1, & t\in\Pi=[0,700]\cup[1000,1300]\cup[1600,2000],\\ 2, & t\in[0,2000]/\Pi,\end{cases}$$

which satisfies the restriction conditions. In this switching signal, the total time and the number of controller failures are 600 seconds and 2 times, respectively. Figure 3 gives the state trajectories of system (1) under $\sigma(t)$ in 2000 seconds. Comparing the state responses in Figs. 1 and 3, it can be observed that the states of system (1) are not seriously affected for a less unavailable rate and failure frequency of the controller.

Example 2 Consider system (1) under (41) with

$$A=\begin{bmatrix}0.5 & 0.1\\ 0 & 0.8\end{bmatrix},\qquad E=\begin{bmatrix}-0.3 & 0\\ -0.2 & -1.1\end{bmatrix},\qquad F=\begin{bmatrix}1\\ 0\end{bmatrix},$$
$$h_1=0.1,\qquad h_2=0.8.$$

A state feedback controller $u(t) = Kx(t - d_2(t))$ has been designed, where $K = [-0.4\ 0.2]$ and $h_3 = 0.3$, $h_4 = 1.0$. The controller stabilizes system (1) over an unreliable path. Under initial state $\phi(\theta) = [12\ 9]^{\mathrm{T}}$, $\theta \in [-1.0, 0]$, the state trajectories of

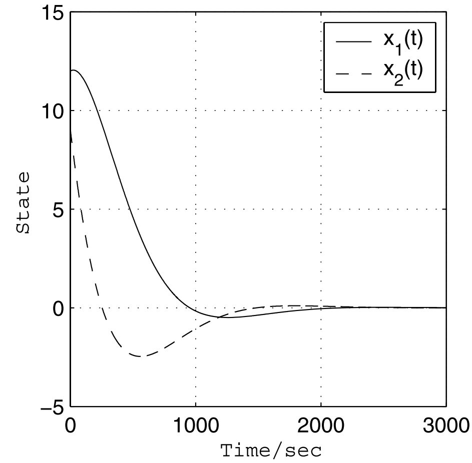

Fig. 4 State response of system (1) with the controller (Example 2)

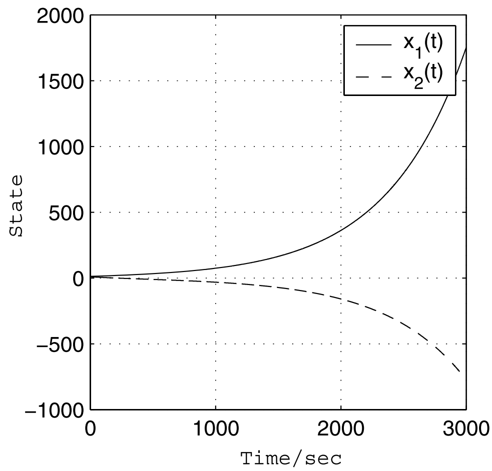

Fig. 5 State response of system (1) without the controller (Example 2)

system (1) with and without the controller are shown in Figs. 4 and 5, respectively. Figure 4 shows that the designed controller can stabilize the system. This implies that the resulting closed-loop system is stable. Figure 5 depicts the state response of the system when $u(t)=0$. It can be clearly seen that the open-loop system is unstable. Then Assumptions (A2) and (A1) are satisfied. Therefore, system (1) can be formulated as (5). By applying the methods mentioned in Remark 5 with $\alpha=0.15$, $\beta=0.41$ and $\mu=4.35$ to solve this problem, the obtained results are

$$P_1=\begin{bmatrix}0.1054 & 0.0204\\ 0.0204 & 0.1755\end{bmatrix},\qquad Q_{11}=\begin{bmatrix}0.0021 & 0.0002\\ 0.0002 & 0.0028\end{bmatrix},$$

$$Q_{12}=\begin{bmatrix}0.0024 & 0.0002\\ 0.0002 & 0.0032\end{bmatrix},\qquad Q_{13}=\begin{bmatrix}0.0018 & 0.0001\\ 0.0001 & 0.0035\end{bmatrix},$$

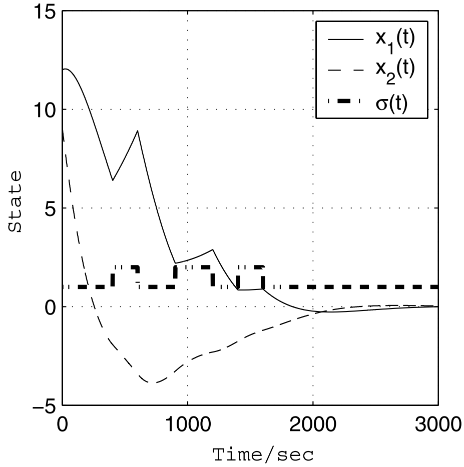

Fig. 6 State response of system (1) with controller failures (Example 2)

$$Q_{14} = \begin{bmatrix} 0.0020 & -0.0001 \\ -0.0001 & 0.0038 \end{bmatrix}, \qquad Z_{11} = \begin{bmatrix} 0.1176 & 0.0818 \\ 0.0818 & 0.5119 \end{bmatrix},$$

$$Z_{12} = \begin{bmatrix} 0.0130 & 0.0004 \\ 0.0004 & 0.0156 \end{bmatrix}, \qquad Z_{13} = \begin{bmatrix} 0.1268 & -0.0475 \\ -0.0475 & 0.0354 \end{bmatrix},$$

$$Z_{14} = \begin{bmatrix} 0.0110 & 0.0008 \\ 0.0008 & 0.0169 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 0.0637 & 0.0021 \\ 0.0021 & 0.0708 \end{bmatrix},$$

$$Q_{21} = \begin{bmatrix} 0.0067 & 0.0006 \\ 0.0006 & 0.0086 \end{bmatrix}, \qquad Q_{22} = \begin{bmatrix} 0.0074 & 0.0007 \\ 0.0007 & 0.0098 \end{bmatrix},$$

$$Q_{23} = \begin{bmatrix} 0.0057 & 0.0002 \\ 0.0002 & 0.0105 \end{bmatrix}, \qquad Q_{24} = \begin{bmatrix} 0.0064 & -0.0002 \\ -0.0002 & 0.0115 \end{bmatrix},$$

$$Z_{21} = \begin{bmatrix} 0.0703 & 0.0145 \\ 0.0145 & 0.1427 \end{bmatrix}, \qquad Z_{22} = \begin{bmatrix} 0.0328 & -0.0001 \\ -0.0001 & 0.0329 \end{bmatrix},$$

$$Z_{23} = \begin{bmatrix} 0.0687 & -0.0123 \\ -0.0123 & 0.0362 \end{bmatrix}, \qquad Z_{24} = \begin{bmatrix} 0.0289 & 0.0009 \\ 0.0009 & 0.0353 \end{bmatrix}.$$

Furthermore, according to (32), selecting $\alpha^* = 0.02$ leads to $f_T \le 0.23$. Given $\alpha_0 = 0.01$, it holds from (33) that $f_N \le 0.029$. Thus, system (1) is exponentially stable with

$$\|x(t)\| \le 1.7055\mathrm{e}^{-0.005t} \|x(0)\|_\theta,$$

if the conditions $f_T \le 0.23$ and $f_N \le 0.029$ are satisfied with the following switching signal:

$$\sigma(t) = \begin{cases} 1, & t \in \Pi = [0, 400] \cup [600, 900] \cup [1200, 1400] \cup [1600, 3000], \\ 2, & t \in [0, 3000]/\Pi, \end{cases}$$

which means that the total time and the number of controller failures are 700 seconds and 3 times, respectively. Figure 6 gives the state trajectories of system (1) under

$\sigma(t)$ in 3000 seconds. It can be seen from the comparison of the state responses in Figs. 4 and 6 that the state responses of system (1) do not degenerate seriously for a less unavailable rate and failure frequency of the controller. It follows from the preceding two examples that the systems with controller failure still remain stable under a certain failure frequency and an unavailable rate of the controller. We also find that Theorem 1 is applicable to both cases, where the derivative of the time delay is either known or unknown.

5 Conclusions

This paper has studied the problem of controller failure for systems with interval time-varying delay, which are controlled by a predesigned state feedback controller. Using the idea of switched control, this problem is first converted into a stability analysis problem for a class of switched delay systems composed of both stable and unstable subsystems. Then, two lemmas have been developed to reflect the change property of the Lyapunov functional for each subsystem. Finally, based on the piecewise Lyapunov functional method, a class of switching laws has been proposed to guarantee exponential stability of the switched delay system. We have also extended the results to the case where there is no restriction on the derivative of the interval time-varying delay, which allows a fast time-varying delay. Simulation results have been provided to illustrate the effectiveness of the proposed methods.

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References

1. M. Basin, E. Sanchez, R. Martinez-Zuniga, Optimal linear filtering for systems with multiple state and observation delays. Int. J. Innov. Comput. Inf. Control **3**, 1309–1320 (2007)
2. S. Boyd, E.L. Ghaoui, E. Feron, V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory* (SIAM, Philadelphia, 1994)
3. R. DeCarlo, M.S. Branicky, S. Pettersson, B. Lennartson, Perspectives and results on the stability and stabilizability of hybrid systems. Proc. IEEE **88**, 1069–1082 (2000)
4. K. Gu, V.L. Kharitonov, J. Chen, *Stability of Time-Delay Systems* (Springer, Berlin, 2003)
5. M. Hassan, K. Boukas, Multilevel technique for large scale LQR with time-delays and systems constraints. Int. J. Innov. Comput. Inf. Control **3**, 419–434 (2007)
6. Y. He, Q.G. Wang, C. Lin, M. Wu, Delay-range-dependent stability for systems with time-varying delay. Automatica **43**, 371–376 (2007)
7. X.F. Jiang, Q.L. Han, Delay-dependent robust stability for uncertain linear systems with interval time-varying delay. Automatica **42**, 1059–1065 (2006)
8. D. Liberzon, A.S. Morse, Basic problems in stability and design of switched systems. IEEE Control Syst. Mag. **19**, 59–70 (1999)
9. D. Liberzon, *Switching in Systems and Control* (Birkhäuser, Boston, 2003)
10. J.P. Richard, Time-delay systems: An overview of some recent advances and open problems. Automatica **39**, 1667–1694 (2003)
11. P. Shi, E.K. Boukas, R. Agarwal, S. Shue, Robust H_∞ state feedback control of discrete time-delay linear systems with norm bounded uncertainty. Int. J. Syst. Sci. **31**, 409–415 (2000)
12. P. Shi, Y. Xia, G. Liu, D. Rees, On designing of sliding mode control for stochastic jump systems. IEEE Trans. Automat. Contr. **51**, 97–103 (2006)

13. Z. Sun, S.S. Ge, Analysis and synthesis of switched linear control systems. Automatica **41**, 181–195 (2005)
14. Z. Sun, S.S. Ge, *Switched Linear Systems: Control and Design* (Springer, New York, 2005)
15. X.M. Sun, J. Zhao, D.J. Hill, Stability and L_2-gain analysis for switched delay systems: A delay-dependent method. Automatica **42**, 1769–1774 (2006)
16. D. Wang, W. Wang, P. Shi, Exponential H_∞ filtering for switched linear systems with interval time-varying delay. Int. J. Robust Nonlinear Control (2008, in press). DOI: 10.1002/rnc.1334
17. D. Wang, W. Wang, P. Shi, Delay-dependent exponential stability for switched delay systems. Optim. Control Appl. Methods (2008, in press). DOI: 10.1002/oca.856
18. R. Wang, J. Zhao, Exponential stability analysis for discrete-time switched linear systems with time-delay. Int. J. Innov. Comput. Inf. Control **3**, 1557–1564 (2007)
19. M. Wu, Y. He, J.H. She, G.P. Liu, Delay-dependent criteria for robust stability of time-varying delay systems. Automatica **40**, 1435–1439 (2004)
20. D. Yue, Q.L. Han, J. Lam, Network-based robust H_∞ control of systems with uncertainty. Automatica **41**, 999–1007 (2005)
21. G.S. Zhai, S. Takai, K. Yasuda, Controller failure time analysis for linear time-invariant systems. Trans. Soc. Instrum. Control Eng. **36**, 1050–1052 (2000)
22. G.S. Zhai, S. Takai, K. Yasuda, Stability analysis of linear time-invariant systems with controller failures, in *Proceedings of the European Control Conference*, Porto, Portugal, pp. 138–142 (2001)
23. G.S. Zhai, X. Chen, S. Takai, K. Yasuda, Controller failure time analysis for H_∞ control systems, in *Proceedings of the 40th IEEE Conference on Decision and Control*, Orlando, Florida, USA, pp. 1029–1030 (2001)
24. G.S. Zhai, H. Lin, Controller failure time analysis for symmetric H_∞ control. Int. J. Control **77**, 598–605 (2004)
25. W. Zhang, M.S. Branicky, S.M. Phillips, Stability of networked control systems. IEEE Control Syst. Mag. **21**, 84–99 (2001)
26. L. Zhang, P. Shi, C. Wang, H. Gao, Robust H_∞ filtering for switched linear discrete-time systems with polytopic uncertainties. Int. J. Adapt. Control Signal Process. **20**, 291–304 (2006)