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Output Feedback H_∞ Control for 2-D State-Delayed Systems

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Abstract In this paper, we consider a class of two-dimensional (2-D) local state-space (LSS) Fornasini–Marchesini (FM) second models with delays in the states, and we study delay-independent and delay-dependent H_∞ control problems via output feedback. First, based on the definition of H_∞ disturbance attenuation γ for 2-D state-delayed systems, we propose a delay-dependent bounded real lemma. Specifically, a new Lyapunov functional candidate is introduced and free-weighting matrices are added to the difference Lyapunov functional for 2-D systems possessing two directions. Then delay-independent and delay-dependent output feedback H_∞ controllers are developed that ensure that the closed-loop system is asymptotically stable and has H_∞ performance γ in terms of linear matrix inequality (LMI) feasibility. Furthermore, the minimum H_∞ norm bound γ is obtained by solving linear objective optimization problems. Numerical examples demonstrate the effectiveness and advantages of the LMI approach to H_∞ control problems for 2-D state-delayed systems.

Keywords 2-D state-delayed systems · H_∞ disturbance attenuation · Output feedback controller · Delay-independent · Delay-dependent · LMI

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1 Introduction

Along with the increasing development of modern industry and civil economy, one must deal with an increasing amount of multivariable systems and multidimensional signals, most of which are expressed as two-dimensional (2-D) discrete-system models [11]. These applications provide a rich engineering physical background for 2-D system theory. When controlling a real plant, it is also desirable to design a control system which is not only stable, but also guarantees an adequate level of performance, such as H_∞ control [3], guaranteed cost control [7] and H_∞ mode reduction [19], for 2-D discrete systems.

The existence of delays is frequently a source of instability, and much work has been done in this area [1, 13, 17], especially for networked control systems (NCSs) [18]. Current efforts to achieve robust stability for one-dimensional (1-D) time-delay systems mainly focus on the delay-dependent criteria [9, 10, 21], which include information on the size of the delay. In general, the delay-dependent stability conditions are less conservative than the delay-independent ones, especially when the size of the delay is small. Recently, Xu and Lam [21] and He et al. [9] devised a new method that uses free-weighting matrices to express the relationships among terms in the Leibniz–Newton formula. This method overcomes the conservatism that results from the descriptor model transformation approach [4]. Because the states are not fully measurable, it is important to solve the stabilization and H_∞ control problems via output feedback. The 1-D output feedback H_∞ control problem was extensively developed in [22] and the references cited therein.

The need for 2-D stability and stabilization problems was motivated by the practical relevance of 2-D discrete linear systems with delays [5]. Most results for the 2-D problems focused on systems without delays, though for specific stability, control and filtering problems of 2-D state-delayed systems were considered in [14–16] and [2, 20], respectively. It is only for convenience and for the avoidance of analytical, structural and computational complexities that there are no delay-independent or delay-dependent H_∞ output feedback control results for 2-D systems.

The delay-independent and delay-dependent H_∞ control problem for 2-D state-delayed systems in this paper is to design dynamic output feedback controllers ensuring the H_∞ disturbance attenuation γ. First, a delay-dependent bounded real lemma is presented, which can be transformed into a delay-independent one. To avoid the complex result for two different directions changing at the same time in 2-D systems, a new Lyapunov functional candidate is introduced and free-weighting matrices are added to the difference Lyapunov function. Then, 2-D H_∞ controllers are designed through the solvability of linear matrix inequalities (LMIs). Furthermore, optimization problems consisting of LMIs are proposed to solve the minimum H_∞ disturbance attenuation γ. Numerical examples demonstrate the effectiveness of our results. The delay-dependent H_∞ controller design guarantees a smaller H_∞ performance γ than the delay-independent one and is still effective when the delay-independent one is infeasible.

2 Bounded Real Lemma

Consider the well-known 2-D Fornasini–Marchesini (FM) local state-space (LSS) second model with state delays in each of the two independent directions of information propagation

$$x(i+1,j+1) = A_1x(i+1,j) + A_2x(i,j+1) + A_{1d}x(i+1,j-d_1) + A_{2d}x(i-d_2,j+1) + B_1\omega(i+1,j) + B_2\omega(i,j+1), \quad (1)$$

$$z(i,j) = Cx(i,j) + D\omega(i,j) \quad (2)$$

where $x(i,j) \in R^n$ is the state input, $\omega(i,j) \in R^m$ is the bounded noise disturbance which belongs to l_2, $z(i,j) \in R^p$ is the control output and $i,j \in Z^+$. A_k, A_{kd}, B_k $(k=1,2)$, C and D are constant matrices with appropriate dimensions. Here, d_1 and d_2 are constant positive scalars representing delays along the vertical direction and horizontal direction, respectively.

The boundary conditions are assumed as

$$\begin{aligned} &\{x(i,j) = \varphi_{ij}\}, \quad \forall i \geq 0;\ j = -d_1, -d_1+1, \ldots, 0, \\ &\{x(i,j) = \psi_{ij}\}, \quad \forall j \geq 0;\ i = -d_2, -d_2+1, \ldots, 0,\ \varphi_{00} = \psi_{00}. \end{aligned} \quad (3)$$

The H_∞ performance measure for 2-D system (1)–(2) with zero boundary conditions ($\varphi_{i0} = \psi_{oj} = 0$) is defined as follows.

Definition 1 (Paszke et al. [15]) 2-D state-delayed system (1)–(2) with zero boundary conditions is said to have H_∞ disturbance attenuation γ if it is asymptotically stable and has H_∞ performance γ, i.e. $\|z\|_2 < \gamma\|\omega\|_2$, where $z = [z_1^{\mathrm{T}}\ z_2^{\mathrm{T}}]^{\mathrm{T}}$, $\omega = [\omega^{\mathrm{T}}(i+1,j)\ \omega^{\mathrm{T}}(i,j+1)]^{\mathrm{T}}$, $z_1 = z(i+1,j)$, $z_2 = z(i,j+2)$ with the l_2-norms defined by

$$\|z\|_2^2 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\|z_1\|_2^2 + \|z_2\|_2^2\right),$$

$$\|\omega\|_2^2 = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\left\|\omega(i+1,j)\right\|_2^2 + \left\|\omega(i,j+1)\right\|_2^2\right).$$

2.1 Delay-Independent Condition

Lemma 1 (Paszke et al. [15]) *2-D state-delayed system* (1)–(2) *with zero boundary conditions has H_∞ disturbance attenuation $\gamma > 0$ if there exist matrices $P > 0$, $Q > 0$ and $Q_k > 0$ $(k=1,2)$ such that the following LMI holds*:

$$\begin{bmatrix} A^{\mathrm{T}}PA - W_1 + C_d^{\mathrm{T}}C_d & A^{\mathrm{T}}PA_d & C_d^{\mathrm{T}}D_d + A^{\mathrm{T}}PB \\ A_d^{\mathrm{T}}PA & A_d^{\mathrm{T}}PA_d - W_2 & A_d^{\mathrm{T}}PB \\ B^{\mathrm{T}}PA + D_d^{\mathrm{T}}C_d & B^{\mathrm{T}}PA_d & D_d^{\mathrm{T}}D_d - \gamma^2 I + B^{\mathrm{T}}PB \end{bmatrix} < 0 \quad (4)$$

where $A = [A_1\ \ A_2]$, $A_d = [A_{1d}\ \ A_{2d}]$, $B = [B_1\ \ B_2]$, $C_d = \mathrm{diag}\{C, C\}$, $D_d = \mathrm{diag}\{D, D\}$, $W_1 = \mathrm{diag}\{P - Q - Q_1 - Q_2, Q\}$, $W_2 = \mathrm{diag}\{Q_1, Q_2\}$.

Remark 1 Similar to the case of 1-D time-delay systems, Lemma 1 is said to be the bounded real lemma for 2-D state-delayed systems.

2.2 Delay-Dependent Condition

Theorem 1 *2-D state-delayed system* (1)–(2) *with the boundary conditions* (3) *has delay-dependent* H_∞ *disturbance attenuation* γ *for any delay* d_k *satisfying* $0 \le d_k \le d_k^*$ $(k=1,2)$ *and* $d^* = \max\{d_1^*, d_2^*\}$ *if there exist matrices* $P>0$, $Q>0$, $R_k>0$, $S_k>0$, Y_{kl}, W_{kl}, M_{kl} $(k,l=1,2)$, $X_{l_1l_2}>0$, X_{l_12} $(l_1=1,4,6;\ l_2=1,3)$ *and* $X_{l_3l_4}$ $(l_3=2,3,5;\ l_4=1,2,3,4)$ *such that the following LMIs hold*:

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & \Phi_{15} \\ \Phi_{12}^{\mathrm{T}} & -\gamma^2 I + d^* X_6 & \Phi_{23} & L_{1d}^{\mathrm{T}} & \Phi_{25} \\ \Phi_{13}^{\mathrm{T}} & \Phi_{23}^{\mathrm{T}} & -P & 0 & 0 \\ \Phi_{14}^{\mathrm{T}} & L_{1d} & 0 & -I & 0 \\ \Phi_{15}^{\mathrm{T}} & \Phi_{25}^{\mathrm{T}} & 0 & 0 & -d^* S \end{bmatrix} < 0, \tag{5}$$

$$\Psi = \begin{bmatrix} X_1 & X_2 & X_3 & Y \\ X_2^{\mathrm{T}} & X_4 & X_5 & W \\ X_3^{\mathrm{T}} & X_5^{\mathrm{T}} & X_6 & M \\ Y^{\mathrm{T}} & W^{\mathrm{T}} & M^{\mathrm{T}} & S \end{bmatrix} \ge 0 \tag{6}$$

where

$$\Phi_{11} = \begin{bmatrix} \overline{Y}_1 & Y_{12} + Y_{21}^{\mathrm{T}} & -Y_{11} + W_{11}^{\mathrm{T}} & -Y_{12} + W_{21}^{\mathrm{T}} \\ Y_{21} + Y_{12}^{\mathrm{T}} & \overline{Y}_2 & -Y_{21} + W_{12}^{\mathrm{T}} & -Y_{22} + W_{22}^{\mathrm{T}} \\ -Y_{11}^{\mathrm{T}} + W_{11} & -Y_{21}^{\mathrm{T}} + W_{12} & \overline{W}_1 & -W_{12} - W_{21}^{\mathrm{T}} \\ -Y_{12}^{\mathrm{T}} + W_{21} & -Y_{22}^{\mathrm{T}} + W_{22} & -W_{21} - W_{12}^{\mathrm{T}} & \overline{W}_2 \end{bmatrix}$$

$$+ d^* \begin{bmatrix} X_1 & X_2 \\ X_2^t & X_4 \end{bmatrix},$$

$$\overline{Y}_1 = Y_{11} + Y_{11}^{\mathrm{T}} - Q + R_1, \qquad \overline{Y}_2 = Y_{22} + Y_{22}^{\mathrm{T}} - P + Q + R_2,$$

$$\overline{W}_1 = -W_{11} - W_{11}^{\mathrm{T}} - R_1, \qquad \overline{W}_2 = -W_{22} - W_{22}^{\mathrm{T}} - R_2,$$

$$\Phi_{12} = \begin{bmatrix} M_{11}^{\mathrm{T}} & M_{21}^{\mathrm{T}} \\ M_{12}^{\mathrm{T}} & M_{22}^{\mathrm{T}} \\ -M_{11}^{\mathrm{T}} & -M_{21}^{\mathrm{T}} \\ -M_{12}^{\mathrm{T}} & -M_{22}^{\mathrm{T}} \end{bmatrix} + d^* \begin{bmatrix} X_3 \\ X_5 \end{bmatrix},$$

$$\Phi_{13}=\begin{bmatrix}A_1^{\mathrm{T}}P\\ A_2^{\mathrm{T}}P\\ A_{1d}^{\mathrm{T}}P\\ A_{2d}^{\mathrm{T}}P\end{bmatrix},\qquad \Phi_{15}=\begin{bmatrix}\overline{S}_1 & d^*A_1^{\mathrm{T}}S_2\\ d^*A_2^{\mathrm{T}}S_1 & \overline{S}_2\\ d^*A_{1d}^{\mathrm{T}}S_1 & d^*A_{1d}^{\mathrm{T}}S_2\\ d^*A_{2d}^{\mathrm{T}}S_1 & d^*A_{2d}^{\mathrm{T}}S_2\end{bmatrix},$$

$$\Phi_{14}=\begin{bmatrix}L_d^{\mathrm{T}}\\ 0_{2n\times 2p}\end{bmatrix},\qquad \Phi_{23}=\begin{bmatrix}B_1^{\mathrm{T}}P\\ B_2^{\mathrm{T}}P\end{bmatrix},\qquad \Phi_{25}=\begin{bmatrix}d^*B_1^{\mathrm{T}}S_1 & d^*B_1^{\mathrm{T}}S_2\\ d^*B_2^{\mathrm{T}}S_1 & d^*B_2^{\mathrm{T}}S_2\end{bmatrix},$$

$$\overline{S}_1=d^*(A_1-I)^{\mathrm{T}}S_1,\qquad \overline{S}_2=d^*(A_2-I)^{\mathrm{T}}S_2,\qquad S=\mathrm{diag}\{S_1,S_2\},$$

$$X_{l_1}=\begin{bmatrix}X_{l_1 1} & X_{l_1 2}\\ X_{l_1 2}^{\mathrm{T}} & X_{l_1 3}\end{bmatrix},\qquad X_{l_3}=\begin{bmatrix}X_{l_3 1} & X_{l_3 2}\\ X_{l_3 3} & X_{l_3 4}\end{bmatrix},\qquad Y=\begin{bmatrix}Y_{11} & Y_{12}\\ Y_{21} & Y_{22}\end{bmatrix},$$

$$W=\begin{bmatrix}W_{11} & W_{12}\\ W_{21} & W_{22}\end{bmatrix},\qquad M=\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}.$$

Proof First, it will be shown that LMIs (5)–(6) ensure the asymptotic stability of system (1) with $\omega(i,j)=0$.

Since LMIs (5)–(6) imply that

$$\Phi_1=\begin{bmatrix}\Phi_{11} & \Phi_{13} & \Phi_{15}\\ \Phi_{13}^{\mathrm{T}} & -P & 0\\ \Phi_{15}^{\mathrm{T}} & 0 & -d^*S\end{bmatrix}<0, \tag{7}$$

$$\Psi_1=\begin{bmatrix}X_1 & X_2 & Y\\ X_2^{\mathrm{T}} & X_4 & W\\ Y^{\mathrm{T}} & W^{\mathrm{T}} & S\end{bmatrix}\ge 0 \tag{8}$$

hold, we only need to verify that the system (1) ($\omega(i,j)=0$) is asymptotically stable if LMIs (7)–(8) hold.

Denote

$$x_{\xi,\eta}=x(i+\xi,j+\eta),$$

$$
\begin{aligned}
V_{11}(i,j) &= x_{1,1}^{\mathrm{T}} P x_{1,1} + \sum_{l=-d_1}^{-1} x_{1,l+1}^{\mathrm{T}} R_1 x_{1,l+1} + \sum_{l=-d_2}^{-1} x_{l+1,1}^{\mathrm{T}} R_2 x_{l+1,1} \\
&\quad + \sum_{\theta=-d_1+1}^{0} \sum_{l=-1+\theta}^{-1} \overline{y}_{1,l+1}^{\mathrm{T}} S_1 \overline{y}_{1,l+1} + \sum_{\theta=-d_2+1}^{0} \sum_{l=-1+\theta}^{-1} \overline{y}_{l+1,1}^{\mathrm{T}} S_2 \overline{y}_{l+1,1}, \\
V_{d1}(i,j) &= x_{1,0}^{\mathrm{T}} Q x_{1,0} + \sum_{l=-d_1}^{-1} x_{1,l}^{\mathrm{T}} R_1 x_{1,l} + \sum_{\theta=-d_1+1}^{0} \sum_{l=-1+\theta}^{-1} \overline{y}_{1,l}^{\mathrm{T}} S_1 \overline{y}_{1,l}, \\
V_{d2}(i,j) &= x_{0,1}^{\mathrm{T}} (P-Q) x_{0,1} + \sum_{l=-d_2}^{-1} x_{l,1}^{\mathrm{T}} R_2 x_{l,1} + \sum_{\theta=-d_2+1}^{0} \sum_{l=-1+\theta}^{-1} \overline{y}_{l,1}^{\mathrm{T}} S_2 \overline{y}_{l,1}
\end{aligned}
\tag{9}
$$

where $\overline{y}_{1,l} = x_{1,l+1} - x_{1,l}$, $\overline{y}_{l,1} = x_{l+1,1} - x_{l,1}$ and $P > 0$, $Q > 0$, $R_k > 0$ and $S_k > 0$ $(k = 1, 2)$ are to be determined.

Due to

$$
x_{1,-d_1} = x_{1,0} - \sum_{l=-d_1}^{-1} \overline{y}_{1,l}, \qquad x_{-d_2,1} = x_{0,1} - \sum_{l=-d_2}^{-1} \overline{y}_{l,1}, \tag{10}
$$

connecting the two equations in (10) yields

$$
\alpha = 2\big(x^{\mathrm{T}} Y + x_d^{\mathrm{T}} W\big)\left(x - x_d - \sum_{l=-d}^{-1} \Delta x\right) = 0 \tag{11}
$$

for any matrices Y and W, where $x = [x_{1,0}^{\mathrm{T}}\ x_{0,1}^{\mathrm{T}}]^{\mathrm{T}}$, $x_d = [x_{1,-d_1}^{\mathrm{T}}\ x_{-d_2,1}^{\mathrm{T}}]^{\mathrm{T}}$, $\Delta x = [\overline{y}_{1,l}^{\mathrm{T}}\ \overline{y}_{l,1}^{\mathrm{T}}]^{\mathrm{T}}$, $d = \min\{d_1, d_2\}$.

On the other hand, for any semi-positive definite matrix $X = \begin{bmatrix} X_1 & X_2 \\ X_2^{\mathrm{T}} & X_4 \end{bmatrix} \geq 0$, the following equation holds:

$$
\beta = d\xi^{\mathrm{T}} X \xi - \sum_{l=-d}^{-1} \xi^{\mathrm{T}} X \xi = 0 \tag{12}
$$

where $\xi = [x^{\mathrm{T}}, x_d^{\mathrm{T}}]^{\mathrm{T}}$.

Now, for system (1) ($\omega(i,j) = 0$), define $\Delta V(i,j)$ as

$$
\Delta V(i,j) = V_{11}(i,j) - V_{d1}(i,j) - V_{d2}(i,j) + \alpha + \beta \leq \xi^{\mathrm{T}} \Theta \xi - \sum_{l=-d}^{-1} \zeta^{\mathrm{T}} \Psi_1 \zeta \tag{13}
$$

where $\zeta = [x^{\mathrm{T}}\ x_d^{\mathrm{T}}\ \Delta x^{\mathrm{T}}]^{\mathrm{T}}$ and $\Theta = \begin{bmatrix} \Theta_1 & \Theta_2 \\ \Theta_2^{\mathrm{T}} & \Theta_3 \end{bmatrix}$ with

$$
\Theta_1 = A^{\mathrm{T}} P A + Y + Y^{\mathrm{T}} - \overline{Q} + R + dX_1 + \sum_{k=1}^{2} d_k \overline{A}_k^{\mathrm{T}} S_k \overline{A}_k, \quad \overline{Q} = \mathrm{diag}\{Q, P-Q\},
$$

$$\Theta_2 = A^{\mathrm{T}} P A_d - Y + W^{\mathrm{T}} + d X_2 + \sum_{k=1}^{2} d_k \overline{A}_k^{\mathrm{T}} S_k A_d,$$

$$\overline{A}_1 = [A_1 - I \quad A_2], \ \overline{A}_2 = [A_1 \quad A_2 - I],$$

$$\Theta_3 = A_d^{\mathrm{T}} P A_d - W - W^{\mathrm{T}} - R + d X_4 + \sum_{k=1}^{2} d_k A_d^{\mathrm{T}} S_k A_d, \qquad R = \mathrm{diag}\{R_1, R_2\}.$$

Since $\Psi_1 \geq 0$ is ensured by LMI (8), if $\Theta < 0$, it is obtained that $\Delta V(i, j) < 0$ for any $\xi \neq 0$.

Applying Schur's complement and LMI (7), for all d_k satisfying $0 < d_k \leq d_k^*$ and $d^* = \max\{d_1^*, d_2^*\}$, we have

$$\begin{aligned} &\begin{bmatrix} A^{\mathrm{T}} P A + Y + Y^{\mathrm{T}} - \overline{Q} + R & A^{\mathrm{T}} P A_d - Y + W^{\mathrm{T}} \\ A_d^{\mathrm{T}} P A - Y^{\mathrm{T}} + W & A_d^{\mathrm{T}} P A_d - W - W^{\mathrm{T}} - R \end{bmatrix} + dX \\ &\quad + d_1 \begin{bmatrix} \overline{A}_1^{\mathrm{T}} S_1 \\ A_d^{\mathrm{T}} S_1 \end{bmatrix} S_1^{-1} [\, S_1 \overline{A}_1 \quad S_1 A_d \,] + d_2 \begin{bmatrix} \overline{A}_2^{\mathrm{T}} S_2 \\ A_d^{\mathrm{T}} S_2 \end{bmatrix} S_2^{-1} [\, S_2 \overline{A}_2 \quad S_2 A_d \,] \\ &\quad < 0 \end{aligned} \tag{14}$$

which shows $\Theta < 0$. So $\Delta V(i, j) < 0$ for any $\xi \neq 0$, if LMIs (7)–(8) hold.

Noting this, using equation (13) and then following a similar line as in the proof of Theorem 3 in [20], we have that system (1) with $\omega(i, j) = 0$ is asymptotically stable if LMIs (5) and (6) are feasible.

Next, we shall prove $\|z\|_2 < \gamma \|\omega\|_2$ under zero initial conditions for any nonzero $\omega(i, j) \in l_2\{[0, \infty), [0, \infty)\}$. To this end, we introduce

$$J = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[z^T z - \gamma^2 \omega^T \omega \right]. \tag{15}$$

In view of the stability of the system and the zero initial condition, we have that, for any nonzero $\omega(i, j) \in l_2\{[0, \infty), [0, \infty)\}$,

$$J \leq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left[z^T z - \gamma^2 \omega^T \omega + \Delta V(i, j) \right]. \tag{16}$$

Similar to (11) and (12), the following two equations:

$$\alpha_1 = 2\left(x^{\mathrm{T}} Y + x_d^{\mathrm{T}} W + \omega^{\mathrm{T}} M\right) \left(x - x_d - \sum_{l=-d}^{-1} \Delta x \right) = 0 \tag{17}$$

and

$$\beta_1 = d \xi_1^{\mathrm{T}} X_1 \xi_1 - \sum_{l=-d}^{-1} \xi_1^{\mathrm{T}} X_1 \xi_1 = 0, \tag{18}$$

hold for any matrices Y, W, M and semi-positive definite matrix

$$X_1 = \begin{bmatrix} X_1 & X_2 & X_3 \\ X_2^{\mathrm{T}} & X_4 & X_5 \\ X_3^{\mathrm{T}} & X_5^{\mathrm{T}} & X_6 \end{bmatrix} \ge 0,$$

where $\xi_1 = [x^{\mathrm{T}},\ x_d^{\mathrm{T}},\ \omega^{\mathrm{T}}]^{\mathrm{T}}$.

In the same way, we can compute $\Delta V(i,j)$ as

$$\Delta V(i,j) = V_{11}(i,j) - V_{d1}(i,j) - V_{d2}(i,j) + \alpha_1 + \beta_1$$

and then it follows that

$$J \le \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left[\xi_1^{\mathrm{T}}\overline{\Theta}\xi_1 - \sum_{l=-d}^{-1}\zeta_1^{\mathrm{T}}\Psi\zeta_1\right]$$

where $\zeta_1 = [x^{\mathrm{T}}\ x_d^{\mathrm{T}}\ \omega^{\mathrm{T}}\ \Delta x^{\mathrm{T}}]^{\mathrm{T}}$ and

$$\overline{\Theta} = \begin{bmatrix} \Theta_1 + L_d^{\mathrm{T}}L_d & \Theta_2 & \Theta_4 \\ \Theta_2^{\mathrm{T}} & \Theta_3 & \Theta_5 \\ \Theta_4^{\mathrm{T}} & \Theta_5^{\mathrm{T}} & \Theta_6 \end{bmatrix},$$

$$\Theta_4 = A^{\mathrm{T}}PB + L_d^{\mathrm{T}}L_{1d} + M^{\mathrm{T}} + dX_3 + \sum_{k=1}^{2} d_k\overline{A}_k^{\mathrm{T}}S_kB$$

$$\Theta_5 = A_d^{\mathrm{T}}PB - M^{\mathrm{T}} + dX_5 + \sum_{k=1}^{2} d_kA_d^{\mathrm{T}}S_kB,$$

$$\Theta_6 = B^{\mathrm{T}}PB + L_{1d}^{\mathrm{T}}L_{1d} - \gamma^2 I + dX_6 + \sum_{k=1}^{2} d_kB^{\mathrm{T}}S_kB.$$

Since $\Psi \ge 0$ is ensured by LMI (6), if $\overline{\Theta} < 0$, it is obtained that $H(x,\omega,i,j) < 0$ for any $\xi_1 \ne 0$.

In terms of inequalities (5) and (14), we have

$$\begin{bmatrix} A^{\mathrm{T}}PA + Y + Y^{\mathrm{T}} - \overline{Q} + R + L_d^{\mathrm{T}}L_d & A^{\mathrm{T}}PA_d - Y + W^{\mathrm{T}} & A^{\mathrm{T}}PB + L_d^{\mathrm{T}}L_{1d} + M^{\mathrm{T}} \\ A_d^{\mathrm{T}}PA - Y^{\mathrm{T}} + W & A_d^{\mathrm{T}}PA_d - W - W^{\mathrm{T}} - R & A_d^{\mathrm{T}}PB - M^{\mathrm{T}} \\ B^{\mathrm{T}}PA + L_{1d}^{\mathrm{T}}L_d + M & B^{\mathrm{T}}PA_d - M & B^{\mathrm{T}}PB + L_{1d}^{\mathrm{T}}L_{1d} - \gamma^2 I \end{bmatrix}$$
$$+ \sum_{k=1}^{2} d_k \begin{bmatrix} \overline{A}_k^{\mathrm{T}}S_k \\ A_d^{\mathrm{T}}S_k \\ B^{\mathrm{T}}S_k \end{bmatrix} S_k^{-1}\begin{bmatrix} S_k\overline{A}_k & S_kA_d & S_kB \end{bmatrix} + dX < 0$$

which, by Schur's complement, is equivalent to $\overline{\Theta} < 0$ for all d_k satisfying $0 < d_k \leq d_k^*$ and $d^* = \max\{d_1^*, d_2^*\}$. So LMI (5) implies $\overline{\Theta} < 0$, i.e. $\|z\|_2 < \gamma\|\omega\|_2$ is guaranteed by LMIs (5) and (6).

Summarizing the above two points demonstrates that 2-D system (1)–(2) has H_∞ disturbance attenuation γ if LMIs (5) and (6) are true. This completes the proof. □

Remark 2 Because a 2-D system has two directions: horizontal direction i and vertical direction j, $V_{11}(i,j)$, $V_{d1}(i,j)$ and $V_{d2}(i,j)$ involve two new variables $\overline{y}_{1,l}$ and $\overline{y}_{l,1}$ expressed by the changes of system states $x_{1,l}$ and $x_{l,1}$. So, the computation of $\Delta V(i,j)$ in (13) involves state delays d_1 and d_2, and the delay-dependent H_∞ disturbance attenuation γ can be guaranteed through the proof of Theorem 1. This treatment avoids the complicated $\Delta V(i,j)$ brought by i and j changing simultaneously.

Remark 3 On the other hand, in terms of the introduction of variables $y_{1,l}$ and $y_{l,1}$, Equation (11) holds. Similar to the Leibniz–Newton formula for 1-D time-delay systems in [10], (11) and (17) are employed to obtain the delay-dependent bounded real lemma of 2-D system (1). The free-weighting matrices Y, W and M are used to express the relationship among the terms x, x_d, and $\sum_{l=-d}^{-1} \Delta x$ and they can easily be determined by solving LMIs (5) and (6). This method avoids the conservatism that results from any system transformation.

Remark 4 Theorem 1 provides a sufficient H_∞ performance criterion for 2-D state-delayed system (1)–(2). Now, we will show that although this condition is dependent on the size of delays, by a certain choice of matrices, it also implies an extension of the previous delay-independent one. Choosing the following matrices:

$$X = \frac{\varepsilon I_{(4n+2m)\times(4n+2m)}}{d^*}, \qquad Y = W = 0_{2n\times 2n},$$
$$M = 0_{2m\times 2n}, \qquad S = \frac{\varepsilon I_{2n\times 2n}}{d^*}$$

for some sufficiently small positive scalar ε, (5) and (6) imply the well-known delay-independent sufficient condition of H_∞ performance; see, for instance, Theorem 5 in [15] and Theorem 2 in [2]. This implies that Theorem 1 is powerful in the sense that it provides sufficient conditions for both the delay-dependent and the delay-independent cases. In other words, for 2-D systems where the delay-independent bounded real lemma can find feasible solutions, the delay-dependent result presented here is also feasible for $d^* \to \infty$.

3 H_∞ Control via Dynamic Output Feedback

Based on the analysis of H_∞ disturbance attenuation in the previous section, in this section, we shall study the delay-independent and delay-dependent H_∞ control problems for 2-D state-delayed systems via dynamic output feedback.

Now, we consider a 2-D state-delayed system with control input and state delays as

$$\begin{aligned} x(i+1,j+1) = {} & A_1 x(i+1,j) + A_2 x(i,j+1) + A_{1d} x(i+1,j-d_1) \\ & + A_{2d} x(i-d_2,j+1) + B_{11} u(i+1,j) + B_{12} u(i,j+1) \\ & + B_{21}\omega(i+1,j) + B_{22}\omega(i,j+1), \end{aligned} \tag{19}$$

$$y(i,j) = C_1 x(i,j) + C_2 u(i,j), \tag{20}$$

$$z(i,j) = D_1 x(i,j) + D_2 u(i,j) + D_3\omega(i,j) \tag{21}$$

where $u(i,j) \in R^p$ is the control input and $y(i,j) \in R^q$ is the measurable output, respectively. B_{kl}, C_k, D_k $(k,l = 1,2)$ and D_3 are constant matrices with appropriate dimensions. The boundary conditions are also of the form (3). Without loss of generality, we assume $C_2 = 0$.

Introduce the following dynamic output feedback controller:

$$\begin{aligned} \widehat{x}(i+1,j+1) = {} & A_{c1}\widehat{x}(i+1,j) + A_{c2}\widehat{x}(i,j+1) + A_{c1d}\widehat{x}(i+1,j-d_1) \\ & + A_{c2d}\widehat{x}(i-d_2,j+1) + B_{c1} y(i+1,j) \\ & + B_{c2} y(i,j+1), \end{aligned} \tag{22}$$

$$u(i,j) = C_c\widehat{x}(i,j) + D_c y(i,j) \tag{23}$$

where $\widehat{x}(i,j) \in R^{n_c}$. Then the closed-loop system obtained by substituting controller (22)–(23) into system (19)–(21) is represented as

$$\begin{aligned} \overline{x}(i+1,j+1) = {} & \overline{A}_1\overline{x}(i+1,j) + \overline{A}_2\overline{x}(i,j+1) + \overline{A}_{1d}\overline{x}(i+1,j-d_1) \\ & + \overline{A}_{2d}\overline{x}(i-d_2,j+1) + \overline{B}_1\omega(i+1,j) + \overline{B}_2\omega(i,j+1), \end{aligned} \tag{24}$$

$$z(i,j) = \overline{D}\overline{x}(i,j) + D_3\omega(i,j) \tag{25}$$

where $\overline{x}(i,j) = [x^{\mathrm{T}}(i,j)\ \widehat{x}^{\mathrm{T}}(i,j)]^{\mathrm{T}}$ and

$$\begin{aligned} & \overline{A}_k = \begin{bmatrix} A_k + B_{1k}D_cC_1 & B_{1k}C_c \\ B_{ck}C_1 & A_{ck} \end{bmatrix}, \qquad \overline{A}_{kd} = \begin{bmatrix} A_{kd} & 0 \\ 0 & A_{ckd} \end{bmatrix}, \\ & \overline{B}_k = [B_{2k}^{\mathrm{T}}\ 0]^{\mathrm{T}} \quad (k = 1,2), \qquad \overline{D} = [D_1 + D_2D_cC_1\ D_2C_c]. \end{aligned} \tag{26}$$

Accordingly, the boundary conditions are assumed as

$$\begin{aligned} & \overline{x}(i,j) = \{\varphi_{i,j}^{\mathrm{T}}, 0\}^{\mathrm{T}}, \quad \forall i \ge 0,\ j = -d_1, -d_1+1, \ldots, 0; \\ & \overline{x}(i,j) = \{\psi_{i,j}^{\mathrm{T}}, 0\}^{\mathrm{T}}, \quad \forall j \ge 0,\ i = -d_2, -d_2+1, \ldots, 0,\ \varphi_{0,0} = \psi_{0,0}. \end{aligned} \tag{27}$$

Theorem 2 and Theorem 3 in the following sections realize delay-independent and delay-dependent H_∞ control for system (19)–(21) via controller (22)–(23), respectively, which makes the closed-loop system (24)–(25) have H_∞ disturbance attenuation γ.

3.1 Delay-Independent H_∞ Controller

Theorem 2 *Consider* 2-*D state-delayed system* (19)–(21) *with boundary conditions* (3). *Given a positive scalar* $\gamma > 0$, *the delay-independent* H_∞ *control problem is solvable via the dynamic output feedback controller* (22)–(23) *if there exist matrices* $X > 0$, $Y > 0$, $J_Q > 0$, $J_{Q_k} > 0$ $(k = 1, 2)$, D_c, $\overline{Z}_k$, $\widehat{Z}_k$, $\widetilde{Z}_k$ $(k = 1, 2)$ *and* Z *such that the following LMI holds*:

$$\begin{bmatrix} -J_P + J_Q + \sum_{k=1}^{2} J_{Q_k} & 0 & J_{A_1}^{\mathrm{T}} & 0 & 0 & J_D^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & -J_Q & J_{A_2}^{\mathrm{T}} & 0 & 0 & 0 & J_D^{\mathrm{T}} & 0 & 0 \\ J_{A_1} & J_{A_2} & -J_P & J_{A_{1d}} & J_{A_{2d}} & 0 & 0 & J_{B_1} & J_{B_2} \\ 0 & 0 & J_{A_{1d}}^{\mathrm{T}} & -J_{Q_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & J_{A_{2d}}^{\mathrm{T}} & 0 & -J_{Q_2} & 0 & 0 & 0 & 0 \\ J_D & 0 & 0 & 0 & 0 & -I & 0 & D_3 & 0 \\ 0 & J_D & 0 & 0 & 0 & 0 & -I & 0 & D_3 \\ 0 & 0 & J_{B_1}^{\mathrm{T}} & 0 & 0 & D_3^{\mathrm{T}} & 0 & -\gamma^2 I & 0 \\ 0 & 0 & J_{B_2}^{\mathrm{T}} & 0 & 0 & 0 & D_3^{\mathrm{T}} & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{28}$$

where

$$J_P = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}, \qquad J_{A_k} = \begin{bmatrix} XA_k + \overline{Z}_k C_1 & \widehat{Z}_k \\ A_k + B_{1k} D_c C_1 & A_k Y + B_{1k} Z \end{bmatrix},$$
$$J_{A_{kd}} = \begin{bmatrix} XA_{kd} & \widetilde{Z}_k \\ A_{kd} & A_{kd} Y \end{bmatrix},$$
$$J_{B_k} = [\, B_{2k}^{\mathrm{T}} X \quad B_{2k}^{\mathrm{T}} \,]^{\mathrm{T}} \quad (k = 1, 2), \qquad J_D = [\, D_1 + D_2 D_c C_1 \quad D_1 Y + D_2 Z \,].$$

Furthermore, the system matrices A_{ck}, A_{ckd}, B_{ck} $(k = 1, 2)$ *and* C_c *of output feedback controller* (22)–(23) *can be solved as*

$$\begin{aligned} A_{ck} &= (\widehat{P}_{12})^{-1}\big(\widehat{Z}_k - XA_kY - \overline{Z}_k C_1 Y - XB_{1k} C_c P_{12}^{\mathrm{T}}\big)\big(P_{12}^{\mathrm{T}}\big)^{-1}, \\ B_{ck} &= (\widehat{P}_{12})^{-1}(\overline{Z}_k - XB_{1k} D_c), \\ A_{ckd} &= (\widehat{P}_{12})^{-1}(\widetilde{Z}_k - XA_{kd}Y)\big(P_{12}^{\mathrm{T}}\big)^{-1}, \\ C_c &= (Z - D_c C_1 Y)\big(P_{12}^{\mathrm{T}}\big)^{-1} \quad (k = 1, 2). \end{aligned} \tag{29}$$

Proof By applying Lemma 1 to the closed-loop system (24)–(25), a new LMI holds by substituting $\overline{A}_k$, $\overline{A}_{kd}$, $\overline{B}_k$ $(k = 1, 2)$, $\overline{D}$ and D_3 for A_k, A_{kd}, B_k $(k = 1, 2)$, C and D, respectively. Pre- and post-multiplying the left-hand side of the new LMI by $\mathrm{diag}\{P^{-1}, P^{-1}, P^{-1}, P^{-1}, P^{-1}, I, I, I, I\}$ and setting $\widetilde{P} = P^{-1}$, $\widetilde{Q} = P^{-1}QP^{-1}$, $\widetilde{Q}_k = P^{-1}Q_kP^{-1}$ $(k = 1, 2)$, the delay-independent H_∞ control problem is solvable if the following matrix inequality holds:

$$\begin{bmatrix} -\widetilde{P}+\widetilde{Q}+\widetilde{Q}_1+\widetilde{Q}_2 & 0 & \widetilde{P}\overline{A}_1^{\mathrm{T}} & 0 & 0 & \widetilde{P}\overline{D}^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & -\widetilde{Q} & \widetilde{P}\overline{A}_2^{\mathrm{T}} & 0 & 0 & 0 & \widetilde{P}\overline{D}^{\mathrm{T}} & 0 & 0 \\ \overline{A}_1\widetilde{P} & \overline{A}_2\widetilde{P} & -\widetilde{P} & \overline{A}_{1d}\widetilde{P} & \overline{A}_{2d}\widetilde{P} & 0 & 0 & \overline{B}_1 & \overline{B}_2 \\ 0 & 0 & \widetilde{P}\overline{A}_{1d}^{\mathrm{T}} & -\widetilde{Q}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \widetilde{P}\overline{A}_{2d}^{\mathrm{T}} & 0 & -\widetilde{Q}_2 & 0 & 0 & 0 & 0 \\ \overline{D}\widetilde{P} & 0 & 0 & 0 & 0 & -I & 0 & D_3 & 0 \\ 0 & \overline{D}\widetilde{P} & 0 & 0 & 0 & 0 & -I & 0 & D_3 \\ 0 & 0 & \overline{B}_1^{\mathrm{T}} & 0 & 0 & D_3^{\mathrm{T}} & 0 & -\gamma^2 I & 0 \\ 0 & 0 & \overline{B}_2^{\mathrm{T}} & 0 & 0 & 0 & D_3^{\mathrm{T}} & 0 & -\gamma^2 I \end{bmatrix} < 0. \tag{30}$$

Next, partition $\widetilde{P}$ and $\widetilde{P}^{-1}$ as follows:

$$\widetilde{P} = \begin{bmatrix} Y & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22} \end{bmatrix}, \qquad \widetilde{P}^{-1} = \begin{bmatrix} X & \widehat{P}_{12} \\ \widehat{P}_{12}^{\mathrm{T}} & \widehat{P}_{22} \end{bmatrix}$$

where $Y, X, P_{12}, \widehat{P}_{12} \in R^{n\times n}$ and $P_{12}\widehat{P}_{12}^{\mathrm{T}} = I - YX$.

Set

$$J = \begin{bmatrix} X & I \\ \widehat{P}_{12}^{\mathrm{T}} & 0 \end{bmatrix}, \qquad \widetilde{J} = \begin{bmatrix} I & Y \\ 0 & P_{12}^{\mathrm{T}} \end{bmatrix}.$$

Then it follows that $\widetilde{P}J = \widetilde{J}$, $J^{\mathrm{T}}\widetilde{P}J = \left[\begin{smallmatrix} X & I \\ I & Y \end{smallmatrix}\right] > 0$ and

$$\begin{aligned} J^{\mathrm{T}}\overline{A}_k\widetilde{P}J &= \begin{bmatrix} XA_k + \overline{Z}_k C_1 & \widehat{Z}_k \\ A_k + B_{1k}D_cC_1 & A_kY + B_{1k}Z \end{bmatrix}, \\ J^{\mathrm{T}}\overline{A}_{kd}\widetilde{P}J &= \begin{bmatrix} XA_{kd} & \widetilde{Z}_k \\ A_{kd} & A_{kd}Y \end{bmatrix}, \\ J^{\mathrm{T}}\overline{B}_k &= [\, B_{2k}^{\mathrm{T}}X \quad B_{2k}^{\mathrm{T}} \,]^{\mathrm{T}} \quad (k=1,2), \\ \overline{D}\widetilde{P}J &= [\, D_1 + D_2D_cC_1 \quad D_1Y + D_2Z \,] \end{aligned}$$

where

$$\begin{aligned} &Z = D_cC_1Y + C_cP_{12}^{\mathrm{T}}, \qquad \overline{Z}_k = XB_{1k}D_c + \widehat{P}_{12}B_{ck}, \\ &\widetilde{Z}_k = XA_{kd}Y + \widehat{P}_{12}A_{ckd}P_{12}^{\mathrm{T}}, \\ &\widehat{Z}_k = XA_kY + \overline{Z}_kC_1Y + XB_{1k}C_cP_{12}^{\mathrm{T}} + \widehat{P}_{12}A_{ck}P_{12}^{\mathrm{T}} \quad (k=1,2). \end{aligned} \tag{31}$$

Therefore, let $J_Q = J^{\mathrm{T}}\widetilde{Q}J$, $J_{Q_k} = J^{\mathrm{T}}\widetilde{Q}_kJ$ $(k=1,2)$ and take a congruence transformation to (30) by diag$\{J, J, J, J, J, I, I, I, I\}$; then we obtain (28). So, (28) is a sufficient condition for the delay-independent H_∞ control problem of system (19)–(21).

If LMI (28) has feasible results, the invertible matrices $\widehat{P}_{12}$ and P_{12} can be computed in terms of the nonsingularity of $P_{12}\widehat{P}_{12}^{\mathrm{T}} = I - YX$. Finally, the control system matrices are solved by (29). This completes the proof. □

Remark 5 In Theorem 2, the delay-dependent dynamic output feedback controller (22)–(23) is constructed, and the main results are in the form of an LMI. Via the proposed method, a delay-independent controller can also be designed, but the induced results are not a strict LMI. To solve the undemanding LMI, we must add some limitations on the matrices in the LMI, which would bring more conservatism.

In Theorem 2, γ is regarded as given. However, (28) is still an LMI when γ is also a variable. Thus, it is possible to formulate the following optimization problem to find a delay-independent controller with the smallest H_∞ norm bound γ.

Problem 1

$$\text{minimize } \delta$$
$$\text{subject to } X > 0, \quad Y > 0, \quad J_Q > 0, \quad J_{Q_1} > 0, \quad J_{Q_2} > 0 \quad \text{and LMI} \quad (28)$$

applying MINCX in MATLAB Toolbox, where $\gamma = \sqrt{\delta}$.If the optimization problem is feasible, the optimal controller in the form of (22)–(23) can also be obtained using (29).

3.2 Delay-Dependent H_∞ Controller

The design of the delay-dependent H_∞ controller is obtained from the following theorem.

Theorem 3 *We are given scalars t, t_1 and t_2. 2-D state-delayed system* (19)–(21) *with the boundary conditions* (3) *has a delay-dependent H_∞ disturbance attenuation γ under the action of the controller* (22)–(23) *for any delay d_k satisfying $0 \le d_k \le d_k^*$ $(k = 1, 2)$ and $d^* = \max\{d_1^*, d_2^*\}$ if there exist matrices $X > 0$, $Y > 0$, $\widetilde{Q} > 0$, $\widetilde{R}_k > 0$, D_c, Z, $\overline{Z}_k$, $\widetilde{Z}_k$, $\widehat{Z}_k$, $\widetilde{Y}_{kl}$, $\widetilde{W}_{kl}$, $\widetilde{M}_{kl}$ $(k, l = 1, 2)$, $\widetilde{X}_{l_1,l_2} > 0$, $\widetilde{X}_{l_1,2}$ $(l_1 = 1, 4, 6;\ l_2 = 1, 3)$ and $\widetilde{X}_{l_3,l_4}$ $(l_3 = 2, 3, 5;\ l_4 = 1, 2, 3, 4)$ such that the following LMIs hold*:

$$\widetilde{\Phi} = \begin{bmatrix} \widetilde{\Gamma}_1 & \widetilde{\Gamma}_2 \\ \widetilde{\Gamma}_2^{\mathrm{T}} & \widetilde{\Gamma}_3 \end{bmatrix} < 0, \tag{32}$$

$$\widetilde{\Psi} = \begin{bmatrix} \widetilde{X}_1 & \widetilde{X}_2 & \widetilde{X}_3 & \widetilde{Y} \\ \widetilde{X}_2^{\mathrm{T}} & \widetilde{X}_4 & \widetilde{X}_5 & \widetilde{W} \\ \widetilde{X}_3^{\mathrm{T}} & \widetilde{X}_5^{\mathrm{T}} & \widetilde{X}_6 & \widetilde{M} \\ \widetilde{Y}^{\mathrm{T}} & \widetilde{W}^{\mathrm{T}} & \widetilde{M}^{\mathrm{T}} & \widetilde{S} \end{bmatrix} \ge 0 \tag{33}$$

where

$$\widetilde{\Gamma}_1 = \begin{bmatrix} \widetilde{\Gamma}_{11} & \Gamma_{12} \\ \Gamma_{12}^{\mathrm{T}} & \Gamma_{22} \end{bmatrix},$$

$$\widetilde{\Gamma}_{11} = \begin{bmatrix} \overrightarrow{Y}_{11} & \widetilde{Y}_{12} + \widetilde{Y}_{21}^{\mathrm{T}} & -\widetilde{Y}_{11} + \widetilde{W}_{11}^{\mathrm{T}} & -\widetilde{Y}_{12} + \widetilde{W}_{21}^{\mathrm{T}} \\ \widetilde{Y}_{21} + \widetilde{Y}_{12}^{\mathrm{T}} & \overrightarrow{Y}_{22} & -\widetilde{Y}_{21} + \widetilde{W}_{12}^{\mathrm{T}} & -\widetilde{Y}_{22} + \widetilde{W}_{22}^{\mathrm{T}} \\ -\widetilde{Y}_{11}^{\mathrm{T}} + \widetilde{W}_{11} & -\widetilde{Y}_{21}^{\mathrm{T}} + \widetilde{W}_{12} & \overrightarrow{W}_{11} & -\widetilde{W}_{12} - \widetilde{W}_{21}^{\mathrm{T}} \\ -\widetilde{Y}_{12}^{\mathrm{T}} + \widetilde{W}_{21} & -\widetilde{Y}_{22}^{\mathrm{T}} + \widetilde{W}_{22} & -\widetilde{W}_{21} - \widetilde{W}_{12}^{\mathrm{T}} & \overrightarrow{W}_{22} \end{bmatrix} + d^* \begin{bmatrix} \widetilde{X}_1 & \widetilde{X}_2 \\ \widetilde{X}_2^{\mathrm{T}} & \widetilde{X}_4 \end{bmatrix},$$

$$\overrightarrow{Y}_{11} = \widetilde{Y}_{11} + \widetilde{Y}_{11}^{\mathrm{T}} - \widetilde{Q} + \widetilde{R}_1, \qquad \overrightarrow{Y}_{22} = \widetilde{Y}_{22} + \widetilde{Y}_{22}^{\mathrm{T}} - tJ_P + \widetilde{Q} + \widetilde{R}_2,$$

$$\overrightarrow{W}_{11} = -\widetilde{W}_{11} - \widetilde{W}_{11}^{\mathrm{T}} - \widetilde{R}_1, \qquad \overrightarrow{W}_{22} = -\widetilde{W}_{22} - \widetilde{W}_{22}^{\mathrm{T}} - \widetilde{R}_2,$$

$$\Gamma_{12} = \begin{bmatrix} \widetilde{M}_{11}^{\mathrm{T}} & \widetilde{M}_{21}^{\mathrm{T}} \\ \widetilde{M}_{12}^{\mathrm{T}} & \widetilde{M}_{22}^{\mathrm{T}} \\ -\widetilde{M}_{11}^{\mathrm{T}} & -\widetilde{M}_{21}^{\mathrm{T}} \\ -\widetilde{M}_{12}^{\mathrm{T}} & -\widetilde{M}_{22}^{\mathrm{T}} \end{bmatrix} + d^* \begin{bmatrix} \widetilde{X}_3 \\ \widetilde{X}_5 \end{bmatrix},$$

$$\Gamma_{22} = \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} + d^* \widetilde{X}_6,$$

$$\widetilde{\Gamma}_2 = \begin{bmatrix} tJ_{A_1}^{\mathrm{T}} & J_C^{\mathrm{T}} & 0 & d^* t_1 (J_{A_1}^{\mathrm{T}} - J_P) & d^* t_2 J_{A_1}^{\mathrm{T}} \\ tJ_{A_2}^{\mathrm{T}} & 0 & J_C^{\mathrm{T}} & d^* t_1 J_{A_2}^{\mathrm{T}} & d^* t_2 (J_{A_2}^{\mathrm{T}} - J_P) \\ tJ_{A_{1d}}^{\mathrm{T}} & 0 & 0 & d^* t_1 J_{A_{1d}}^{\mathrm{T}} & d^* t_2 J_{A_{1d}}^{\mathrm{T}} \\ tJ_{A_{2d}}^{\mathrm{T}} & 0 & 0 & d^* t_1 J_{A_{2d}}^{\mathrm{T}} & d^* t_2 J_{A_{2d}}^{\mathrm{T}} \\ tJ_{B_1}^{\mathrm{T}} & D_3^{\mathrm{T}} & 0 & d^* t_1 J_{B_1}^{\mathrm{T}} & d^* t_2 J_{B_1}^{\mathrm{T}} \\ tJ_{B_2}^{\mathrm{T}} & 0 & D_3^{\mathrm{T}} & d^* t_1 J_{B_2}^{\mathrm{T}} & d^* t_2 J_{B_2}^{\mathrm{T}} \end{bmatrix},$$

$$J_P = \begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0,$$

$$J_{A_k} = \begin{bmatrix} XA_k + \overline{Z}_k C_1 & \widehat{Z}_k \\ A_k + B_{1k} D_c C_1 & A_k Y + B_{1k} Z \end{bmatrix}, \qquad J_{A_{kd}} = \begin{bmatrix} XA_{kd} & \widetilde{Z}_k \\ A_{kd} & A_{kd} Y \end{bmatrix},$$

$$J_{B_k} = \begin{bmatrix} B_{2k}^{\mathrm{T}} X & B_{2k}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \quad (k = 1, 2), \qquad J_C = [\, D_1 + D_2 D_c C_1 \quad D_1 Y + D_2 Z \,],$$

$$\widetilde{\Gamma}_3 = \mathrm{diag}\{-tJ_P, -I, -I, -d^* t_1 J_P, -d^* t_2 J_P\}, \qquad \widetilde{S} = \mathrm{diag}\{t_1 J_P, t_2 J_P\},$$

$$\widetilde{X}_{l_1} = \begin{bmatrix} \widetilde{X}_{l_1 1} & \widetilde{X}_{l_1 2} \\ \widetilde{X}_{l_1 2}^{\mathrm{T}} & \widetilde{X}_{l_1 3} \end{bmatrix}, \qquad X_{l_3} = \begin{bmatrix} \widetilde{X}_{l_3 1} & \widetilde{X}_{l_3 2} \\ \widetilde{X}_{l_3 3} & \widetilde{X}_{l_3 4} \end{bmatrix},$$

$$\widetilde{Y} = \begin{bmatrix} \widetilde{Y}_{11} & \widetilde{Y}_{12} \\ \widetilde{Y}_{21} & \widetilde{Y}_{22} \end{bmatrix}, \qquad \widetilde{W} = \begin{bmatrix} \widetilde{W}_{11} & \widetilde{W}_{12} \\ \widetilde{W}_{21} & \widetilde{W}_{22} \end{bmatrix}, \qquad \widetilde{M} = \begin{bmatrix} \widetilde{M}_{11} & \widetilde{M}_{12} \\ \widetilde{M}_{21} & \widetilde{M}_{22} \end{bmatrix}.$$

Furthermore, the delay-dependent control system matrices A_{ck}, A_{ckd}, B_{ck} $(k=1,2)$, C_c *and* D_c *can be derived from equation* (29), *i.e., the delay-dependent* H_∞ *control problem for system* (19)–(21) *is solved.*

Proof Applying Theorem 1, the closed-loop system (24)–(25) has H_∞ disturbance attenuation γ if LMIs (5)–(6) hold by substituting $\overline{A}_k$, $\overline{A}_{kd}$, $\overline{B}_k$ $(k=1,2)$, $\overline{D}$ and D_3 for A_k, A_{kd}, B_k $(k=1,2)$, C and D, respectively. Setting $P=t\overline{P}$, $S_1=t_1\overline{P}$, $S_2=t_2\overline{P}$, $\widetilde{P}=\overline{P}^{-1}$, and pre-multiplying and post-multiplying the left-hand side of (5) by diag$\{\widetilde{P},\widetilde{P},\widetilde{P},\widetilde{P},I,I,\widetilde{P},I,I,\widetilde{P},\widetilde{P}\}$ and (6) by diag$\{\widetilde{P},\widetilde{P},\widetilde{P},\widetilde{P},I,I,\widetilde{P},\widetilde{P}\}$ we obtain the matrix inequalities (8') and (9'), respectively, which are omitted here.

Furthermore, partition $\widetilde{P}$, $\widetilde{P}^{-1}$ and let J, $\widetilde{J}$ be as in the proof of Theorem 2; then taking a congruence transformation for (8') and (9') by diag$\{J,J,J,J,I,I,J,I,I,J,J\}$ and diag$\{J,J,J,J,I,I,J,J\}$, respectively, yields LMIs (32) and (33). Moreover, if LMIs (32)–(33) are feasible, similar to Theorem 2, we can compute the delay-dependent control system matrices A_{ck}, A_{ckd}, B_{ck} $(k=1,2)$, C_c and D_c using equation (29). This completes the proof. □

Remark 6 The results of Theorem 3 apply the tuning parameters t, t_1 and t_2. The question arises of how to find the optimal combination of these parameters. The optimal values can be found by the approach proposed in [4]. Moreover, in the proof of Theorem 3, if we do not introduce the parameters t, t_1 and t_2, the LMIs obtained by substituting $\overline{A}_k$, $\overline{A}_{kd}$, $\overline{B}_k$ $(k=1,2)$, $\overline{D}$ and D_3 for A_k, A_{kd}, B_k $(k=1,2)$, C and D in LMIs (5)–(6) are bilinear and must be solved under some limitations, which will bring a more conservative result. So, the choices of parameters t, t_1 and t_2 have reduced the conservatism of the result.

Remark 7 Theorem 3 provides an approach to solve the delay-dependent H_∞ control problem for 2-D state-delayed system (19)–(21). For some sufficiently small positive scalar ε, if we choose the following matrices:

$$\begin{aligned} &t=1, \qquad t_1=t_2=\varepsilon, \qquad \widetilde{Y}=\widetilde{W}=0_{2n\times 2n},\\ &\widetilde{M}=0_{2m\times 2n}, \qquad \widetilde{X}=\frac{\varepsilon I_{(4n+2m)\times(4n+2m)}}{d^*} \end{aligned}$$

then (32)–(33) imply the sufficient condition of the delay-independent H_∞ control problem given in Theorem 2.

Similar to Problem 1, the scalar γ in Theorem 3 also can be included as a variable to obtain the minimum H_∞ noise attenuation level bound, yielding the following convex optimization problem.

Problem 2

$$\begin{aligned} &\text{minimize } \delta\\ &\text{subject to } X>0, \quad \widetilde{Y}>0, \quad \widetilde{Q}>0, \quad \widetilde{R}_k>0, \quad \widetilde{X}_{l_1,l_2}>0 \quad \text{and LMIs (32)–(33)} \end{aligned}$$

applying MINCX in MATLAB Toolbox for given state delays d_1 and d_2, *where* $\gamma = \sqrt{\delta}$. Furthermore, the corresponding controller is given by (29). In terms of the same choices of scalars and free-weighting matrices in Remark 7, Problem 2 is equivalent to Problem 1.

Remark 8 The preceding research is focused on 2-D discrete systems with constant state delays. As time delays are commonly time varying and some new results on the stability of time-varying systems have been proposed in [6], based on these results, we would further study 2-D time-varying state-delayed systems.

4 Numerical Examples

Example 1 First, we demonstrate the design of 2-D delay-independent and delay-dependent H_∞ output feedback control for a stationary random field in image processing using the LMI approach proposed in Problem 1 and Problem 2, respectively. Note that the delay-dependent approach provides a smaller H_∞ performance level than the delay-independent one.

It is known that the stationary random field can be modeled as the following 2-D system [11]:

$$\eta(i+1, j+1) = a_1\eta(i+1, j) + a_2\eta(i, j+1) - a_1a_2\eta(i, j) + \omega_1(i, j) \tag{34}$$

where $\eta(i, j)$ is the state of the random field at spacial coordinate (i, j), $a_1^2 < 1$ and $a_2^2 < 1$ as a_1 and a_2 are, respectively, the horizontal and vertical correlations of the random field.

Now, we consider the influence of time delays on system (34) and introduce two terms $\eta(i+1, j-d_1)$ and $\eta(i-d_2, j+1)$ in (34), resulting in

$$\begin{aligned}\eta(i+1, j+1) &= a_1\eta(i+1, j) + a_2\eta(i, j+1) + a_3\eta(i+1, j-d_1)\\ &\quad + a_4\eta(i-d_2, j+1) - a_1a_2\eta(i, j) + \omega(i, j)\end{aligned} \tag{35}$$

where $a_3^2 < 1$ and $a_4^2 < 1$ as a_3 and a_4 are also, respectively, the horizontal and vertical correlations of the random field, and $\omega(i, j)$ is the measurement noise.

Denote $x^{\mathrm{T}}(i, j) = [\,\eta^{\mathrm{T}}(i, j+1) - a_2\eta^{\mathrm{T}}(i, j)\;\; \eta^{\mathrm{T}}(i, j)\,]$, and assume that the measurement output is given by $y(i, j) = [3\ 1]x(i, j)$ and the signal to be estimated is $z(i, j) = 0.5\eta(i, j) + 0.4u(i, j) + 0.7\omega(i, j)$.

It is easy to see that the 2-D system (35) can be converted to the 2-D FM LSS model (19)–(21) with

$$\begin{aligned}
&A_1 = \begin{bmatrix} 0 & 0 \\ 1 & a_1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix},\\
&A_{1d} = \begin{bmatrix} a_3 & a_1a_3 \\ 0 & 0 \end{bmatrix}, \qquad A_{2d} = \begin{bmatrix} a_4 & a_1a_4 \\ 0 & 0 \end{bmatrix},\\
&B_{11} = [0.1 \quad 1]^{\mathrm{T}}, \qquad B_{12} = [0 \quad 0.3]^{\mathrm{T}}, \qquad B_{21} = 0,\\
&B_{22} = [1 \quad 0]^{\mathrm{T}}.
\end{aligned} \tag{36}$$

Let $a_1 = 0.2$, $a_2 = 0.3$, $a_3 = 0.15$, $a_4 = 0.03$. Then we use the optimization Problem 1 and obtain the minimum H_∞ norm bound $\gamma_{\mathrm{opt}} = 1.2189$ and the system matrices of H_∞ controller (22)–(23) as

$$\begin{aligned} A_{c1} &= \begin{bmatrix} -0.6025 & -37658.44 \\ -0.0000 & -0.000015 \end{bmatrix}, \qquad A_{c2} = \begin{bmatrix} -0.1408 & -11295.33 \\ 0.0000 & -0.0000 \end{bmatrix}, \\ B_{c1} &= \begin{bmatrix} 30567518.94 \\ -0.0106452 \end{bmatrix}, \qquad B_{c2} = \begin{bmatrix} -11650152.59 \\ 0.0346255 \end{bmatrix}, \\ A_{c1d} &= \begin{bmatrix} -0.0000 & -6.3628 \\ 0.0000 & 0.2639 \end{bmatrix}, \qquad A_{c2d} = \begin{bmatrix} 0.0000 & 1.7407 \\ 0.0000 & 0.0923 \end{bmatrix} \\ C_c &= [\,-0.000000002253 \quad -0.0001808\,], \qquad D_c = -0.1865. \end{aligned} \tag{37}$$

Figures 1 and 2 show the optimal state response and frequency response of the closed-loop system obtained by substituting (37) into (36), respectively. The 2-D system (36) can be stabilized by the above optimal controller (37) and the maximum frequency response is 1.0731, which is below 1.2189.

For a reasonable comparison, resolving Problem 2 when assuming $d_1 = 1$, $d_2 = 2$ and $t_1 = 0.01$, $t_2 = 0.101$ and $t_3 = 0.11$, we can obtain the minimum H_∞ norm bound

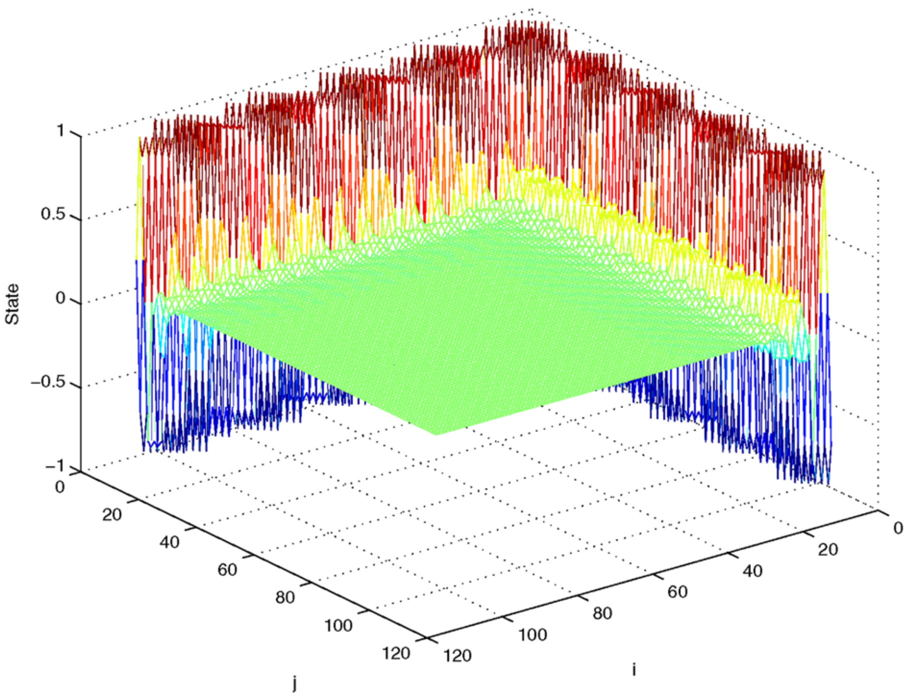

Fig. 1 The delay-independent optimal state response of closed-loop system by substituting (37) into (36)

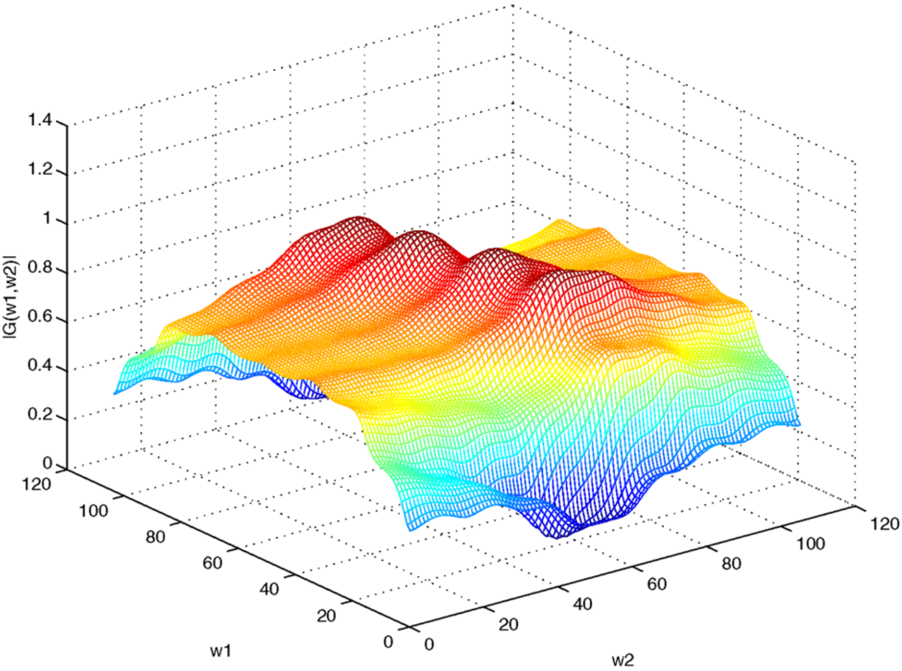

Fig. 2 The delay-independent optimal frequency response of closed-loop system by substituting (37) into (36)

$\gamma_{\text{opt}} = 0.7000019$ and controller matrices as

$$
\begin{aligned}
&A_{c1} = \begin{bmatrix} -0.5664 & 4.6682 \times 10^{12} \\ 0.0000 & 5799.2 \end{bmatrix}, \qquad A_{c2} = \begin{bmatrix} -0.0698 & 1.7436 \times 10^{12} \\ 0.0000 & 8823.4 \end{bmatrix}, \\
&A_{c1d} = \begin{bmatrix} -0.0000 & 13488 \\ 0.0000 & 0.1855 \end{bmatrix}, \qquad A_{c2d} = \begin{bmatrix} 0.0000 & 5392.1 \\ 0.0000 & 0.0348 \end{bmatrix}, \\
&B_{c1} = \begin{bmatrix} 618606 \\ -0.003732 \end{bmatrix}, \qquad B_{c2} = [\,-1121904 \quad -0.005677\,]^{\mathrm{T}}, \\
&C_c = [\,-0.0000 \quad 364202.3068\,], \qquad D_c = -0.2343.
\end{aligned}
\tag{38}
$$

Figure 3 gives the maximum singular values plot of the transfer function of the closed-loop system obtained by substituting (38) into (36) over $0 \le \omega_1 \le 2\pi$, $0 \le \omega_2 \le 2\pi$. In the figure, the griddings denote the obtained H_∞ disturbance attenuations, and the maximum value is 0.699988, which is below 0.7000019.

Next it is shown that a smaller H_∞ norm bound γ can be obtained by using the delay-dependent approach than by using the delay-independent approach proposed in this paper.

Example 2 In this example, we will show that the delay-dependent approach is still feasible when the delay-independent one is invalid.

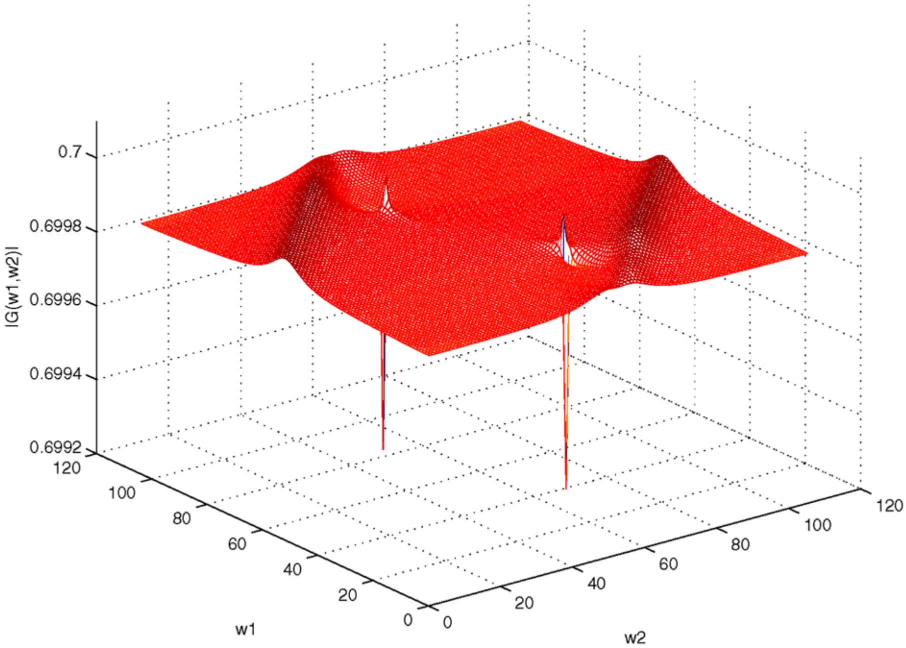

Fig. 3 The delay-dependent optimal frequency response of closed-loop system by substituting (38) into (36)

Now, consider system (19)–(21) with

$$A_{1d} = \begin{bmatrix} 1.1453 & 0.1489 \\ 0.0824 & 0.0536 \end{bmatrix}, \qquad A_{2d} = \begin{bmatrix} 0.0880 & 0.1367 \\ 0.1867 & 0.0425 \end{bmatrix} \tag{39}$$

and other system matrices given in (36). Note that for this special example, the delay-independent result proposed in this paper fails to find a feasible solution. By solving Problem 2 in this paper, however, we can obtain the minimum guaranteed H_∞ cost $\gamma_{\text{opt}} = 0.70020365$ when assuming $d_1 = 1$, $d_2 = 2$, with the associated controller matrices as

$$\begin{aligned} A_{c1} &= \begin{bmatrix} 5.0803 & 22.2296 \\ 0.4609 & 0.9303 \end{bmatrix}, \qquad A_{c2} = \begin{bmatrix} 3.0194 & 11.7306 \\ 1.7278 & 5.9413 \end{bmatrix}, \\ A_{c1d} &= \begin{bmatrix} -0.0240 & -0.0972 \\ 0.0429 & 0.1745 \end{bmatrix}, \qquad A_{c2d} = \begin{bmatrix} -0.0051 & -0.0225 \\ 0.0085 & 0.0349 \end{bmatrix}, \\ B_{c1} &= [\,-0.0031 \quad -0.0002\,]^{\mathrm{T}}, \qquad B_{c2} = [\,-0.0019 \quad -0.0009\,]^{\mathrm{T}}, \\ C_c &= [\,-640.1876 \quad 398.0370\,], \qquad D_c = -0.1434. \end{aligned} \tag{40}$$

Figure 4 gives the maximum singular value plots of the closed-loop system obtained by substituting (40) into (39), and the maximum H_∞ norm bound $\gamma_{\max} = 0.70006642$, which is below 0.70020365. So, it is obvious that the guaranteed H_∞ disturbance attenuation level is still effective.

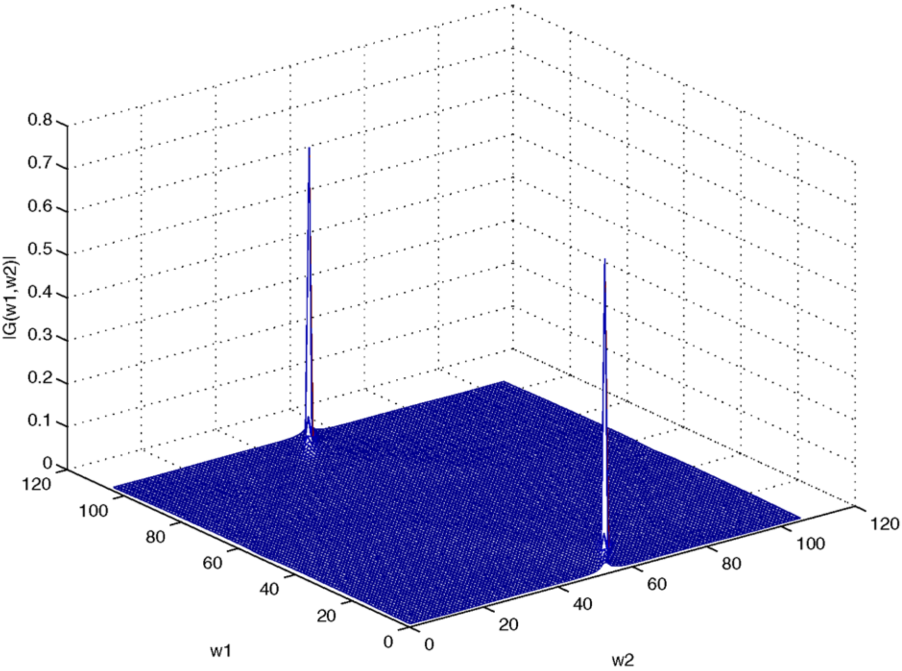

Fig. 4 The delay-dependent optimal frequency response of closed-loop system by substituting (40) into (39)

5 Conclusions

This paper studies H_∞ control problems of 2-D state-delayed systems described by the FM LSS model. First, a delay-dependent bounded real lemma has been derived to ensure the asymptotic stability and H_∞ performance γ. By a certain choice of free-weighting matrices, it also implies an extension of the existing delay-independent result. Then, the method of dynamic output feedback control is considered to solve H_∞ control problems. The sufficient conditions are given in terms of LMIs, so it is easy to verify the solvability of output feedback control problems applying MATLAB Toolbox. Optimization problems are also proposed to compute the minimum upper bound of H_∞ disturbance attenuation γ. Finally, numerical examples are given to show the effectiveness of our results and the superiority of the delay-dependent approach to the delay-independent condition. The augmented Lyapunov functional methods in [12] and [8] are effective in reducing the conservatism of stability criteria for time-delay systems, so they would be the main method applied in our future work.

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