Z. angew. Math. Phys. 49 (1998) 827–837
0044-2275/98/000827-12 $ 1.50+0.20/0

Zeitschrift für angewandte Mathematik und Physik ZAMP

Behavior of entropy across shock waves in dusty gases

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Abstract. The behavior of entropy across shock waves in dusty gases is calculated by using the Navier–Stokes equations for the gas phase and the particle phase. The resulting system of six nonlinear ordinary differential equations is reduced to a system of four autonomous nonlinear differential equations which are solved exactly. This solution is obtained formally by assuming that both the velocity and temperature of particles in the gasdynamic region are constant. A careful study of the equation that governs the entropy shows that the entropy profile has a maximum value within the shock region. It is also shown that the entropy is increasing continuously across the shock wave with increasing both the Mach number and the particle concentration.

Mathematics Subject Classification (1991). 34G20, 76D05, 76L05, 76T05.

1. Introduction

There are many engineering applications for flows of a medium that consists of a suspension of powdered materials or liquid droplets in a gas. Most of these flows involve changes of the gas velocity and temperature. Gas-particle interaction through viscous drag and heat transfer produces corresponding changes in the particles. These processes are relatively slow, so that for fast changes in the gas phase, considerable deviations from equilibrium may occur. Thus, one has to deal with typical relaxation processes.

The structure of shock waves in dusty gases has been investigated by many authors [1]–[8]. It has been assumed in these papers, that the transport coefficient of viscosity and heat conductivity were negligible, so that the gas-dynamic shock wave appeared as a discontinuity. This assumption is not valid for weak shock waves, because the thickness L of a shock wave in a pure gas becomes very large when the Mach number M_0 of the gas approaches unity [9].

The transport coefficients of viscosity and heat conductivity have been taken into consideration by Hamad [10] in order to determine theoretically the structure of fully dispersed waves in dusty gases. The influence of the above mentioned coefficients is discussed in previous papers Hamad [11]–[13].

But in the present paper the behavior of entropy across shock waves is discussed by simplifying the previous work which is difficult. In a previous paper [10], the structure of shock waves in dusty gases was discussed for any Prandtl number. Four nonlinear autonomous differential equations were obtained and two singularities appeared in the phase space arising from the boundary conditions ahead and behind the shock wave. This feature made it very difficult to deal with, and it was impossible to study the variation of entropy within the shock region. To overcome these difficulties, here in the present paper, it is assumed that the velocity and temperature of the existing particles are constant (in the gasdynamic region) and take the Prandtl number to be 3/4. This not only reduced the number of equations to two but also changed the existing singularities in such a way, that made it possible to discuss the entropy variation.

The equilibrium properties of a dusty gas can be reduced formally to the equilibrium properties of a simple gas by introducing certain parameters of the mixture, for example the Mach number of the mixture:

$$\bar{M} = M_0 \frac{1-\phi}{\sqrt{1-\mu}} \sqrt{\frac{1+(\mu/(1-\mu))(c/c_v)}{1+(\mu/(1-\mu))(c/c_p)}}, \tag{1}$$

where μ is the mass fraction of the particles, c the specific heat of the particle material, c_v and c_p the specific heats of the gas at constant volume and pressure respectively. The volume fraction of the particles ϕ will be neglected in the present paper. The Mach number of the gas M_0 is based on the gas velocity and on the speed of sound far upstream.

For a simple gas a shock wave can occur only if one has for the Mach number of the gas $M_0 > 1$. The Mach number of the mixture $\bar{M}$ becomes unity for:

$$\bar{M} = M_{min} = \sqrt{1-\mu} \sqrt{\frac{1+(\mu/(1-\mu))(c/c_v)}{1+(\mu/(1-\mu))(c/c_p)}}. \tag{2}$$

For Mach numbers in the range from $\bar{M} = M_{\min}$ to $\bar{M} = 1$ the Rankine–Hugoniot conditions predict a new equilibrium state which can be realized by fully dispersed waves. The changes of the thermodynamic state of the gas are caused by four relaxation processes: by the molecular processes of momentum and energy transfer (viscosity and heat conductivity) and by the macroscopic processes of friction and heat transfer between gas and suspended particles. The characteristic time for the molecular processes is the mean time between molecular collisions:

$$\tau = l/\bar{c}, \tag{3}$$

where l is the mean free path of the gas and $\bar{c} = (8\kappa T/\pi m)^{1/2}$ is a mean molecular velocity. The characteristic time for the relaxation of the macroscopic velocity is:

$$\tau_v = \frac{m_P}{3\pi\sigma_P\eta} \tag{4}$$

and the characterisitc time for the relaxation of the temperature of the gas is:

$$\tau_T = \frac{m_P c_p}{2\pi\sigma_P\lambda}, \tag{5}$$

where m_P is the mass of a particle, η the viscosity, λ the heat conductivity, σ_P the diameter of the spherical particles, ρ_P the mass density of the particle material, ρ_D the mass density of the gas, κ the ratio of the specific heats. It has been assumed that the viscosity and the heat conductivity of the gas may be described by the gaskinetic model of rigid spheres. Equations (4) and (5) show that $\tau_v \sim \tau_T$. In order to obtain an approximate solution for the structure of shock waves in dusty gases a model is used by considering that the relaxation times τ_v and τ_T are much longer than the relaxation time τ, i.e. the shock for the dust particles takes a much longer time than for the gas. This means that, we can assume that both the velocity and temperature of particles in the gasdynamic region are constant.

2. Basic equations

The structure of shock waves in dusty gases is described by the conservation equations for one-dimensional steady flow. These six equations have to be solved for the boundary conditions. $u_P = u_D = u_0$, $T_P = T_D = T_0$, for $x \to -\infty$ and $u_P = u_D = u_1$, $T_P = T_D = T_1$ for $x \to +\infty$, where the subscripts "P" and "D" refer to the particles and the gas respectively. The quantity u represents the velocity and the quantity T the temperature. Denoting the density (i.e. the mass of the gas or the particles per unit volume of the system) by ζ, the integrated continuity equations for gas and particles become

$$\zeta_D u_D = m_1 = m \tag{6}$$

and

$$\zeta_P u_P = m_2 = \beta m, \tag{7}$$

where the integration constant m and the abbreviation $\beta = \mu/(1-\mu)$ have been introduced.

The integration force F_{Px} and the heat Q_P exchanged between particles and gas are eliminated by adding the momentum and energy equations. The resulting two equations can be integrated once immediately, by introducing the integrated continuity equations (6) and (7), one obtains:

$$\zeta_D u_D^2 + \zeta_P u_P^2 + p_{xx} = P, \tag{8}$$

$$(e_D + u_D^2/2) + \beta(e_P + u_P^2/2) + p_{xx}u_D/m + q_x/m = E, \tag{9}$$

where P and E are integration constants. Here e is the internal energy, q_x the heat flux in the gas, and p_{xx} one component of the stress tensor. For further treatment, explicit expressions for the equations of state, the stress tensor and the heat flux are introduced by:

$$e_D = c_\nu T_D, \qquad e_P = cT_P, \tag{10}$$

$$p_{xx} = p - \frac{4}{3}\eta\frac{du_D}{dx}, \qquad q_x = -\lambda\frac{dT_D}{dx}, \tag{11}$$

$$p_p = \rho_M R_M T \text{ or } p = \zeta_D RT_D \quad (\varphi \ll 1). \tag{12}$$

Using these expressions equations (8) and (9) can be put in the form:

$$\frac{4\eta}{3m}\frac{du_D}{dx} = u_D + \beta u_P + \frac{RT_D}{u_D} - \frac{P}{m}, \tag{13}$$

$$\frac{\lambda}{m}\frac{dT_D}{dx} = (c_v T_D + u_D^2/2) + \beta(cT_P + u_P^2/2) + u_D(P/m - u_D - \beta u_P) - E. \tag{14}$$

The integration constants P and E can be expressed in terms of the variable of state ahead of the shock wave as:

$$\frac{P}{m} = (1+\beta)u_0 + \frac{RT_0}{u_0} = (1 + \beta + \frac{1}{\kappa M_0^2})u_0 \tag{15}$$

and

$$E = \left[\frac{1 + \beta c/c_p}{(\kappa - 1)M_0^2} + (1+\beta)/2\right] u_0^2. \tag{16}$$

Two equations have been lost by adding the momentum and energy equations. Following Marble [6] these equations are replaced by the relaxation equations:

$$\tau_\nu \frac{du_P}{dx} = -\frac{u_P - u_D}{u_P}, \tag{17}$$

$$\tau_T \frac{dT_P}{dx} = -\frac{T_P - T_D}{u_P}. \tag{18}$$

The four equations (13), (14), (17), (18) are put into the dimensionless form:

$$\bar{\eta}\frac{d\omega_D}{dx} = \omega_D + \frac{\theta_D}{\omega_D} + \beta\omega_P - 1, \tag{19}$$

$$\bar{\lambda}\frac{d\theta_D}{dx} = \theta_D - \delta[(1-\omega_D)^2 + \alpha] + \beta\delta(\omega_P^2 - 2\omega_P\omega_D) + \beta c\theta_P/c_v, \tag{20}$$

$$\bar{\tau}_\nu \frac{d\omega_P}{dx} = -\frac{\omega_P - \omega_D}{\omega_P}, \tag{21}$$

$$\bar{\tau}_T \frac{d\theta_P}{dx} = -\frac{\theta_P - \theta_D}{\omega_P}, \tag{22}$$

where

$$\omega_D = \frac{mu_D}{P}, \qquad \theta_D = \frac{m^2 RT_D}{P^2}, \tag{23}$$

$$\omega_P = \frac{mu_P}{P}, \qquad \theta_P = \frac{m^2 R T_P}{P^2}, \tag{24}$$

$$\alpha + 1 = \frac{2Em^2}{P^2}, \qquad \delta = \frac{R}{2c_v} = \frac{\kappa - 1}{2}, \tag{25}$$

$$\bar{\eta} = 4\eta/3m, \qquad \bar{\lambda} = \lambda/(c_v m), \tag{26}$$

$$\bar{\tau}_v = \tau_\nu P/m, \qquad \bar{\tau}_T = \tau_T P/m. \tag{27}$$

For the special case of a simple gas, i.e. for $\beta = 0$, these equations are of course identical with Gilbarg and Paolucci's [14] equations for the shock wave in a simple gas. The above system of four differential equations (19)–(22) is difficult to solve numerically because of the nonlinarities and the two singularities of the direction field in phase space. These singularities correspond to the equilibrium conditions ahead of and behind the shock wave. In a previous paper, this system was solved analytically in phase space by expanding the variables of state in power series [11].

In order to simplify this solution a model is obtained by setting $\omega_P = \text{const} = \omega_0$ and $\theta_P = \text{const} = \theta_0$, then the equations (19) and (20) become:

$$\bar{\eta}\omega_D \frac{d\omega_D}{dx} = \omega_D^2 + \theta_D + (\beta\omega_0 - 1)\omega_D, \tag{28}$$

$$\bar{\lambda}\frac{d\theta_D}{dx} = \theta_D - \delta\omega_D^2 + 2\delta(1 - \beta\omega_0)\omega_D - F. \tag{29}$$

Here the following abbreviations have been used:

$$\kappa = c_p/c_v, \qquad \delta = (\kappa - 1)/2, \tag{30}$$

$$F = \delta(1 + \alpha) - \beta\delta\omega_0^2 - \beta\frac{c}{c_v}\theta_0. \tag{31}$$

3. Equilibrium

Far in front of the wave and far behind the wave all gradients of the variables of state become zero. Under this condition the equilibrium state can be calculated from equations (28) and (29):

$$\omega_{0,1} = \frac{\kappa}{\kappa + 1}(A \pm \varepsilon), \tag{32}$$

$$\theta_{0,1} = \frac{\kappa}{(\kappa + 1)^2}[A^2 - \kappa\varepsilon^2 \mp (\kappa - 1)A\varepsilon], \tag{33}$$

where

$$A = \frac{1 + \kappa M_0^2}{1 + \kappa(1 + \beta)M_0^2}, \tag{34}$$

$$\varepsilon = \frac{M_0^2 - 1}{1 + \kappa(1+\beta)M_0^2}, \tag{35}$$

the parameter ε is a measure for the strength of the change in the variables of state. One has:

$$M_0^2 = \frac{1}{\kappa(1+\beta)} \frac{1+\varepsilon}{\frac{1}{\kappa(1+\beta)} - \varepsilon}. \tag{36}$$

For very strong shock waves with $M_0 \to \infty$, one has:

$$\varepsilon \to \frac{1}{\kappa(1+\beta)}. \tag{37}$$

The limit $\varepsilon \to 0$ for very weak shock waves is obtained for the Mach number $M_0 = 1$.

4. The shock profile

Multiplying the differential equation (28) by 2, dividing differential equation (29) by δ, and dropping the index D, then adding, we get one differential equation in the following form:

$$\bar{\eta}\frac{d}{dx}(\omega^2 + \frac{\bar{\lambda}}{\delta\bar{\eta}}\theta) = \omega^2 + \frac{\kappa}{\delta}\theta - F/\delta. \tag{38}$$

Suppose that $\bar{\lambda}/\bar{\eta} = \kappa$ (see Appendix) and let $y = \omega^2 + \kappa\theta/\delta$, then equation (38) takes the form:

$$\bar{\eta}\frac{dy}{dx} = y - F/\delta. \tag{39}$$

By integration, it yields:

$$y = F/\delta \quad \text{then} \quad \theta = \frac{\delta}{\kappa}(\frac{F}{\delta} - \omega^2). \tag{40}$$

Eliminating the temperature θ from the differential equation (28) by using (40), one obtains:

$$\bar{\eta}\omega\frac{d\omega}{dx} = \frac{\delta+1}{\kappa}\omega^2 - (1 - \beta\omega_0)\omega + \frac{F}{\kappa}. \tag{41}$$

Equation (41) can be integrated and the solution $\omega(x)$ can take the following form:

$$(\omega_0 - \omega)^l/(\omega - \omega_1)^m = e^{nx}, \tag{42}$$

where

$$l = \omega_0/(\omega_0 - \omega_1) \quad m = \omega_1/(\omega_0 - \omega_1) \quad n = (1+\delta)/\kappa\bar{\eta}. \tag{43}$$

The solution (42) which represents a shock profile of the gas velocity must join the two singularities P_0 and P_1.

Setting $\beta = 0$ in equation (41) and equations (43), one obtains from (42) Becker's solution for the shock wave in a simple gas [9].

5. Entropy structure in a shock wave

The second law of thermodynamic yields:

$$dS_M = c_{v,M}\frac{dT}{T} - \frac{p}{T}\frac{d\rho_M}{\rho_M^2}, \tag{44}$$

where S_M is the entropy of the mixture, ρ_M is the mixture density and $c_{v,M}$ is the specific heat at constant volume of a gas-particle mixture, therefore it is related to that of the gas by:

$$c_{v,M} = (1-\mu)c_v + \mu c \tag{45}$$

where c_v is the specific heat of the gas at constant volume, c is the specific heat of the particles, and μ is the mass-fraction of the particles. Using equation (12), we get:

$$\frac{p}{T} = \rho_M R_M, \tag{46}$$

where R_M is the effective gas constant of the mixture, Rudinger [3] which is given by:

$$R_M = (1-\mu)R. \tag{47}$$

Substituting from (45), (46) and (47) in (44), one obtains:

$$dS_M = (1-\mu)\left[(c_p + \beta c)\frac{dT}{T} - R\frac{dp}{p}\right]. \tag{48}$$

By integration, we get

$$\frac{S_M}{c_p} = \ln\frac{T}{T_0} - \frac{\kappa-1}{\kappa(1+\beta)}\ln\frac{p}{p_0}, \tag{49}$$

where $c = c_p$ and $\beta = \mu/(1-\mu)$.

By using equations (23) and (24), we can put equation (49) in a dimensionless form as follows:

$$\bar{S}_M = \ln\frac{\theta}{\theta_0} - \frac{\kappa-1}{\kappa(1+\beta)}\ln(\frac{\theta\omega_0}{\theta_0\omega}) = \frac{1+\kappa\beta}{\kappa(1+\beta)}\ln\frac{\theta}{\theta_0} - \frac{\kappa-1}{\kappa(1+\beta)}\ln(\frac{\omega_0}{\omega}). \tag{50}$$

Eliminating the temperature θ between equation (40) and equation (50), we get:

$$\bar{S}_M = \frac{1}{\kappa(1+\beta)}\left[(1+\kappa\beta)\ln\left\{\frac{\delta M_0^2}{\omega_0^2}\left(\frac{F}{\delta} - \omega^2\right)\right\} - (\kappa-1)\ln\frac{\omega_0}{\omega}\right], \tag{51}$$

where $\bar{S}_M = S_M/c_p$, $\kappa\theta_0 = \omega_0^2/M_0^2$ (see Appendix).

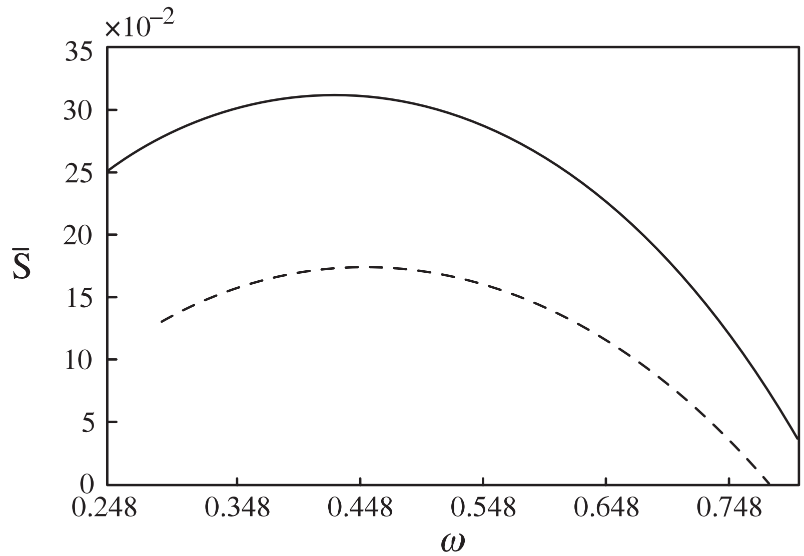

Figure 1.
Entropy $\bar{S}$ as a function of dimensionless velocity ω for $\beta = 0.1$ and $--$ $M_0 = 2$ — $M_0 = 2.5$.

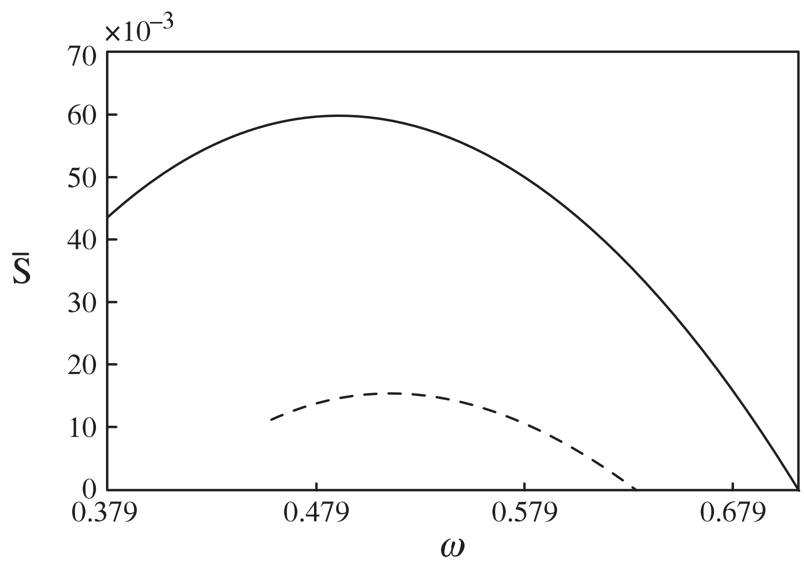

Figure 2.
Entropy $\bar{S}$ as a function of dimensionless velocity ω for $\beta = 0.1$ and $--$ $M_0 = 1.2$ — $M_0 = 1.5$.

6. Conclusion

The variation of the entropy $\bar{S}_M$ with the dimensionless velocity ω within the shock region as given by equation (51) is shown in figures 1–3. Figures 1 and 2 illustrate the variation of entropy $\bar{S}_M$ for $\beta = 0.1$ and the different values of the Mach number $M_0 = 1.2$, $M_0 = 1.5$, $M_0 = 2$ and $M_0 = 2.5$. It can be seen that the entropy $\bar{S}_M$ increases as the dimensionless velocity ω decreases within the shock, till it reaches a maximum value and then it decreases to its boundary value behind the shock wave. It is also clear that the entropy increases with M_0.

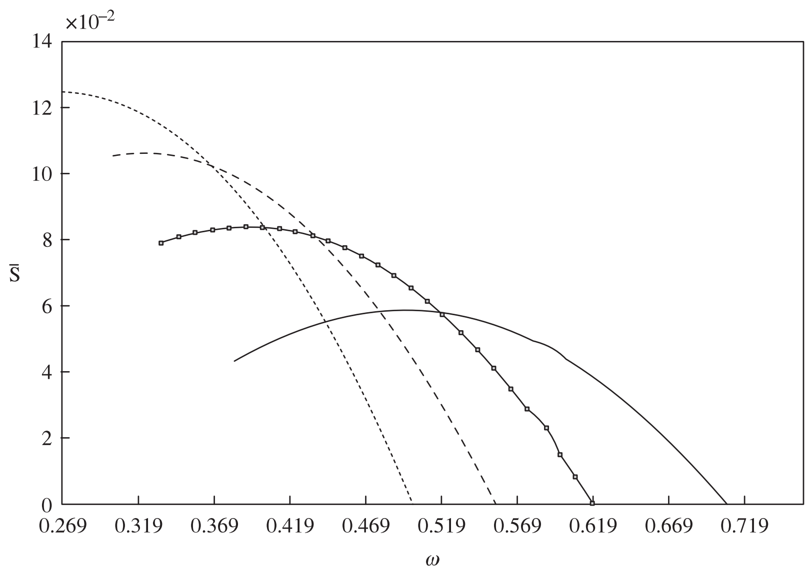

Figure 3.
Entropy $\bar{S}$ as a function of dimensionless velocity ω for $M_0 = 1.5$ and — $\beta = 0.1$, –◇–◇–◇– $\beta = 0.3$, – – – $\beta = 0.5$ and - - - - $\beta = 0.7$.

Figure 3 illustrates the variation of $\bar{S}_M$ as a function of the dimensionless velocity of ω for $M_0 = 1.5$ and the different values of $\beta = 0.1$, $\beta = 0.3$, $\beta = 0.5$ and $\beta = 0.7$. Here again the entropy $\bar{S}_M$ increases to a maximum value and then decreases to its boundary value behind the shock wave as ω decreases within the shock region. But for $\beta = 0.7$ we can observe that the entropy reaches its maximum value behind the shock wave, i.e. when the velocity reaches a minimum value ω_1. The entropy also increases with increasing β.

Figure 4 illustrates the behavior of entropy $\bar{S}$ as a function of dimensionless velocity ω for $M_0 = 1.5$ and $\beta = 0$, i.e. in the case of pure gas with both viscosity and heat conductivity.

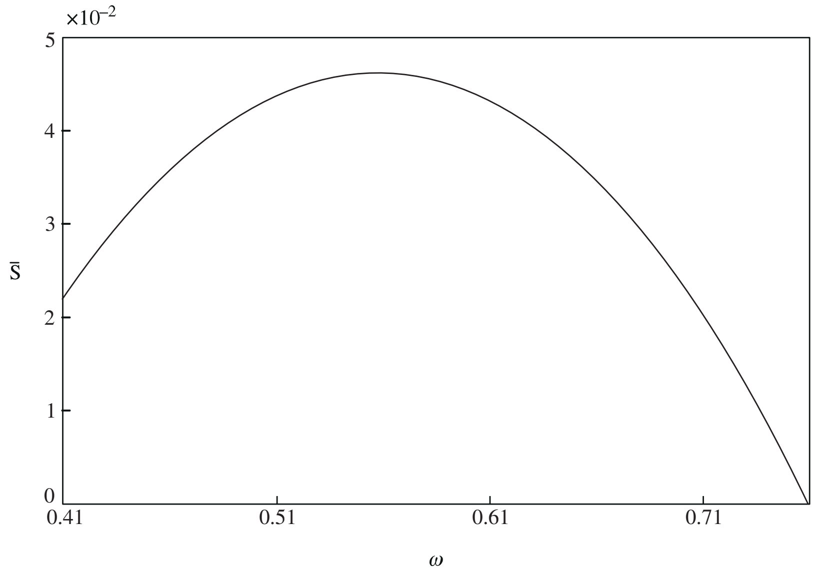

Figure 4.
Entropy $\bar{S}$ as a function of dimensionless velocity ω for $\beta = 0$ and $M_0 = 1.5$.

Appendix

1. From equations (23), one obtains:

$$\begin{aligned}
\omega_0 &= \frac{mu_0}{P} \text{ and } \theta_0 = \frac{m^2RT_0}{P^2},\\
\theta_0 &= \frac{m^2u_0}{P^2} \cdot \frac{RT_0}{u_0^2} = \omega_0^2 \frac{kRT_0}{ku_0^2},\\
&= \omega_0 \cdot \frac{1}{kM_0^2} \text{ where } M_0^2 = \frac{u_0^2}{a_0^2},\ a_0^2 = kRT_0,\\
k\theta_0 &= \frac{\omega_0}{M_0^2}.
\end{aligned}$$

2. For the assumption $\frac{\bar{\lambda}}{\bar{\eta}} = \kappa$, using equation (26), one has:

$$\kappa = \frac{\bar{\lambda}}{\bar{\eta}} = \frac{\lambda}{c_v m} \cdot \frac{3m}{4\eta} = \frac{3}{4} \cdot \frac{\lambda}{c_v \eta} = \frac{3}{4}\kappa \frac{\lambda}{c_p \eta} = \frac{3}{4} \cdot \frac{\kappa}{\mathrm{Pr}}$$

then $\mathrm{Pr} = \frac{3}{4}$, where Pr indicates the Prandtl number.

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(Received: January 13, 1997; revised: October 28, 1997)