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# A note on the Burgers–Rott vortex with a free surface

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**Summary.** Rott's solution of the 'bathtub vortex' problem, which neglects the depression of the free surface, is extended by allowing for a mildly sloping free surface of a shallow, rotating fluid.

**Keywords.** Vortex, free surface.

## 1. Introduction

We consider here the depression of the free surface of a shallow, rotating fluid, following Rott [1], who applied Burgers's [2] description of the viscous core of a line vortex to the 'bathtub vortex' problem on the assumption of a level upper surface. The depressed-free-surface problem has since been solved by Lundgren [3], who allowed for any depression up to the limit of a swallowed vortex, but his formulation is rather elaborate and requires numerical integration of the resulting differential equations. It therefore seems worthwhile to consider an analytical extension of Rott's solution on the assumption of a mild depression (up to about half the outer depth).

## 2. Formulation

We posit a swirling flow of radial velocity $u(r)$ and azimuthal velocity $v(r)$ bounded below by the horizontal surface $z = 0$ and above by the free surface $z = h(r)$ in cylindrical polar coordinates $r$ and $z$. We replace Rott's modified stagnation-point flow, in which $u/r$ is constant, by a shallow-water flow in which the pressure is given by the hydrostatic approximation

$$p(r, z) = \rho g[h(r) - z] \tag{1}$$

and the vertical velocity $w_0(r)$ is prescribed at $z = 0$. Continuity requires the radial outflow across a cylinder of radius $r$ and depth $h$ $(0 < z < h)$ to be equal to the inflow across the base of that cylinder at $z = 0$:

$$2\pi rhu(r) = 2\pi \int_0^r w_0(r) r dr \equiv -Q(r), \tag{2}$$

where (by definition) $Q > 0$ for positive drainage. The radial and azimuthal components of the Navier–Stokes equations yield

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = g\frac{\partial h}{\partial r} = \frac{v^2}{r}, \quad u\frac{\partial v}{\partial r} + \frac{uv}{r} = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right), \tag{3a,b}$$

($\rho$ and $\nu$ are the density and kinematic viscosity of the fluid), which correspond to Rott's equations (7) and (5), respectively. But, whereas the radial viscous stress vanishes identically for $u = -ar$ in Rott's formulation, it is frankly neglected (as proves to be consistent with the present development) in (3a).

We now assume that $w_0 = -U$ is constant, so that $Q = \pi U r^2$ and (2) reduces to

$$u = -\frac{1}{2}Uh^{-1}r. \tag{4}$$

Substituting (4) into (3b) and introducing

$$x \equiv \frac{1}{4}(U/\nu H)r^2, \quad \mathfrak{h}(x) \equiv h(r)/H, \quad \mathcal{C}(x) \equiv rv(r)/C_\infty, \tag{5a-c}$$

and

$$\varepsilon \equiv \frac{1}{8}C_\infty^2 U/gH^2\nu, \tag{6}$$

where $H$ is the outer depth and $2\pi C_\infty$ is the outer circulation (Rott's $\Gamma_\infty$), we obtain

$$\mathfrak{h}'(x) \equiv d\mathfrak{h}/dx = \varepsilon x^{-2}\mathcal{C}^2, \qquad \mathfrak{h}\mathcal{C}'' + \mathcal{C}' = 0, \tag{7a,b}$$

which are equivalent to Lundgren's (2.3a,b). The boundary conditions are

$$\mathcal{C} = 0 \quad (x = 0); \quad \mathcal{C} \sim 1, \quad \mathfrak{h} \sim 1 \quad (x \uparrow \infty). \tag{8a-c}$$

## 3. Solution for $\varepsilon \ll 1$

The solution of (7) and (8) for sufficiently small $\varepsilon$ may be obtained by expanding $\mathfrak{h}$ and $\mathcal{C}$ in powers of $\varepsilon$, starting from the solution [1, 2] for a level upper boundary:

$$\mathfrak{h} \to 1, \quad \mathcal{C} \to 1 - e^{-x} \quad (\varepsilon \downarrow 0). \tag{9a,b}$$

The next approximations are

$$\mathfrak{h} = 1 + \varepsilon\mathfrak{h}_1(x) + \mathcal{O}(\varepsilon^2), \quad \mathcal{C} = 1 - e^{-x} + \varepsilon\mathcal{C}_1(x) + \mathcal{O}(\varepsilon^2), \tag{10a,b}$$

where

$$-\mathfrak{h}_1 = \int_x^\infty y^{-2}(1-e^{-y})^2 dy = x^{-1}(1-e^{-x})^2 + 2E_1(x) - 2E_1(2x), \tag{11}$$

$$E_1(x) = \int_x^\infty y^{-1}e^{-y}\,dy \tag{12}$$

is an exponential integral, and

$$\mathcal{C}_1 = e^{-x}\left[\int_0^\infty e^{-y}\mathfrak{h}_1(y)dy - \int_0^x \mathfrak{h}_1(y)dy\right] - \int_x^\infty e^{-y}\mathfrak{h}_1(y)dy. \tag{13}$$

Letting $x = 0$ in (10a) and (11), we obtain the central depression

$$\mathfrak{h}_0 = 1 - \varepsilon \ln 4 + \mathcal{O}(\varepsilon^2), \tag{14}$$

which agrees with Lundgren's [3] numerical results (within the accuracy with which his Fig. 2 can be read) for $\varepsilon(\equiv K/8) < 1/4$. Lundgren finds that $\mathfrak{h}_0$ is a monotonically decreasing function of $\varepsilon$ that vanishes (the drain vortex is swallowed) for $\varepsilon = 0.336$; however the present expansion of the solution in powers of $\varepsilon$ presumably fails in this limit in consequence of the singularity of (7b) at $\mathfrak{h} = 0$. In any event, higher-order terms in this expansion are complicated and, in view of the availability of Lundgren's solution, do not appear to be worth pursuing.

We remark that the neglect of the exponentially decaying terms in (10) yields the asymptotic approximations

$$\mathfrak{h} \sim 1 - \varepsilon x^{-1} = 1 - \frac{1}{2}(C_\infty^2/gH)r^{-2}, \quad v \sim C_\infty r^{-1}, \tag{15a,b}$$

which correspond to the outer flow for a Rankine vortex.

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