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The random attractor of the stochastic Lorenz system

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Abstract. We study the existence of attractors for dynamical systems under the influence of random parameters. Such a random attractor is a measurable multi-function with compact images fulfilling particular invariance and attracting properties. In particular, this uniquely determined attractor attracts measurable multi-functions. Under certain conditions we estimate the Hausdorff dimension of the attractor. These results will be applied to the Lorenz system under the influence of a random external parameter perturbation.

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1. Introduction

An attractor is a set in the *phase space* (distinct from the complete phase space) which attracts the states of a dynamical system, and it is invariant with respect to the operators of the associated *semigroup.* Such an attractor was found by E. Lorenz [18] for a dynamical system generated by a three dimensional autonomous differential equation describing a meteorological problem. One can prove that the dimension of the attractor of this dynamical system is less than three, see Temam [22] page 293 ff.

This paper investigates the question of existence of attractors of dynamical systems under the influence of a random parameter. For instance, such a random dynamical system can be generated by a stochastic differential equation. In contrast to the autonomous differential equations, stochastic differential equations are not autonomous. Basic statements about random dynamical systems one can find in Arnold [2] or Arnold and Crauel [3]. That means, stochastic differential equations do not generate semigroups (with respect to the trajectories), so we need a generalization of a semigroup. This generalization is called a cocycle.

In the present paper we formulate conditions that ensure the existence of random attractors. These attractors depend on the random parameter, and the invariance property w.r.t. the operators of the cocycle is only true if we adjust the random parameter appropriately in time. The parameterized sets of the random attractor attract bounded sets of initial states depending on a random parameter. The attraction is to be understood with respect to a particular *pull back* procedure or with respect to convergence in probability.

We also consider the linearization of the random dynamical system which generates the evolution of n–dimensional volume elements having small edges. For particular dimensions these volumes are shrinking. The estimate of these dimensions is based on Birkhoff's ergodic theorem. From this shrinking condition we obtain an estimate for the *Hausdorff–dimension* for the random attractor.

We apply these general results to the Lorenz system augmented by a multiplicative noise term. For existence and uniqueness of the solution of this equation we refer to Keller [17]. In particular, we prove the existence of an attractor for this equation. We can also show that the Hausdorff–dimension of these sets is less than three. We also mention that we can apply these results to other finite or infinite dimensional stochastic differential equations.

The finiteness of the dimension of the state space of the Lorenz system is no restriction for the general theory. It is not hard to apply these results to random dynamical systems with infinite state space generated by random partial differential equations.

Attractors for differential equations with very particular random coefficients were introduced in Brzeźniak, Capiński and Flandoli [7]. Ito equations can not be handled with this particular definition of attractors. Another kind of attractor for Ito differential equations that attracts *deterministic* bounded sets was introduced in Crauel and Flandoli [10], [11] and in Schmalfuß [19]. However, in these articles difficulties occur in proving the uniqueness of the random attractor. To overcome these difficulties we require the attraction of random multi-functions in the definition of an attractor. These random sets fulfill particular subexponential growth conditions. In difference to Crauel and Flandoli [11] and Schmalfuß [19] these attractors attract larger classes of sets than deterministic bounded sets. Using this fact one can prove that an attractor with this stronger attraction properties is unique. On the other hand, one can also show that the examples treated in Crauel and Flandoli [11] also have an attractor in this new sense. We also mention some interesting applications in Arnold and Schmalfuß [4] and Schmalfuß [20].

Crauel and Flandoli [10] also give estimates of the Hausdorff dimension which can only be applied under very restrictive boundedness conditions. Roughly speaking they need the derivative in direction of the phase space to be uniformly bounded with respect to the random parameter. This condition is violated in the stochastic Lorenz system. Here we give also conditions for the estimate of the Hausdorff dimension based on a growth condition for stationary processes.

This article is organized as follows. In Section 2 we introduce the basic definitions for general random dynamical systems and prove the existence of random attractors. Section 3 is devoted to the study of the Hausdorff dimension of general random dynamical systems. In the fourth section we define the particular random dynamical system generated by the stochastic Lorenz equation and in Section 5 we apply the general results to the stochastic Lorenz equation.

2. Co-cycles and their attractors

Here we are going to extend the definition of a global attractor of a semigroup, cf. Babin and Vishik [6], [5] Chapter 2, Hale [15] Chapter 2, 3 and Temam [22] Chapter I. Stochastic differential equations are non-autonomous objects, so they do not generate semigroups on the state space. Hence we can not use directly the definition of attractors of the above references. On the other hand, Sell [21] proved the existence of attractors of skew-product semigroups generated by non-autonomous differential equations but assumed the set of non-autonomous parts (in our case the set of the paths of the noise) to be compact. But this compactness assumption for the parameter set is too restrictive for white noise. However, Crauel and Flandoli [11] and Schmalfuß [19] give a definition with applications to Ito equations. We will now introduce a strengthening of this definition.

First, we give some basic assumptions that allow us to treat attractors for stochastic differential equations. We are going to develop a concept for random attractors based on a generalization of semigroups called *co-cycles*.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A group of operators $\theta = \{\theta_t\}_{t\in\mathbb{R}}$ defined on the non-empty set Ω of elementary events such that

$$\theta_t : \Omega \to \Omega \qquad \text{for all } t \in \mathbb{R}$$

and satisfying

$$\theta_t \circ \theta_\tau = \theta_{t+\tau} \qquad \text{for all } t,\, \tau \in \mathbb{R}, \quad \theta_0 = \mathrm{id}$$

is called a *flow*. By this definition the inverse operator of θ_t is given by θ_{-t}. We assume the $\mathcal{B}_{\mathbb{R}} \otimes \mathcal{F}, \mathcal{F}$ measurability of

$$\mathbb{R} \times \Omega \ni (t, \omega) \mapsto \theta_t\omega \in \Omega,$$

where $\mathcal{B}_X$ is the Borel-σ-algebra of a metric space X. In addition, we assume that the operators θ_t, $t \in \mathbb{R}$, preserve $\mathbb{P}$ that is $\mathbb{P}(\theta_t A) = \mathbb{P}(A)$.

We now introduce an Ω–parameterized family of operators. These operators fulfill a particular composition property w.r.t. the flow θ. Let (H, d) be a complete separable metric space. A family of maps $\phi(t, \omega, \cdot) : H \to H$, $\omega \in \Omega$, $t \in \mathbb{R}^+$, is called *cocycle* w.r.t. θ if we have for all $\omega \in \Omega$ the following properties

$$\begin{aligned} &\phi(0, \omega, \cdot) = \mathrm{id}, \\ &\phi(t + \tau, \omega, \cdot) = \phi(t, \theta_\tau\omega, \phi(\tau, \omega, \cdot)) \qquad \text{for any } t,\, \tau \in \mathbb{R}^+. \end{aligned} \tag{1}$$

In addition, we assume the $\mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{F} \otimes \mathcal{B}_H, \mathcal{B}_H$-measurability of ϕ. The five-tuple $(\Omega, \mathcal{F}, \mathbb{P}, \theta, \phi)$ is called *random dynamical system*.

A *multi- function*

$$\Omega \ni \omega \mapsto D(\omega) \neq \emptyset$$

where the images are closed sets in H is said to be measurable if for any $x \in H$ the map $\omega \mapsto \inf_{y\in D(\omega)} d(x,y)$ is $\mathcal{F}, \mathcal{B}_{\mathbb{R}^+}$-measurable. If the multi-function D has compact values then this condition is equivalent to the $\mathcal{F}, \mathcal{B}_\mathcal{C}$ measurability of $D(\omega)$ where $(\mathcal{C}, d_\mathcal{C})$ is the metric space of compact subsets in H and $d_\mathcal{C}$ is the Hausdorff metric, cf. Castaing and Valadier [8] III.1, III.2, III.9. In particular, if $\omega \mapsto D(\omega) \neq \emptyset$ is defined to be a measurable multi-function with closed images in a separable complete metric space then there exists a countable set of measurable maps x_i, $i \in \mathbb{N}$, called selectors such that

$$\overline{\bigcup_{i\in\mathbb{N}} x_i(\omega)} = D(\omega), \tag{2}$$

for any $\omega \in \Omega$, see Castaing and Valadier [8] Theorem III.9. Let $\mathcal{D}$ be a system of measurable multi-functions having the following properties.

- The set $D(\omega)$ is non-empty and closed for any $\omega \in \Omega$.
- Suppose D', D are two measurable set functions with non-empty and closed images such that $D \in \mathcal{D}$ and

$$D'(\omega) \subset D(\omega), \quad \omega \in \Omega.$$

  Then $D' \in \mathcal{D}$.

Definition 2.1. *A measurable multi-function $A \in \mathcal{D}$ with compact values is called random $\mathcal{D}$-attractor of the random dynamical system* $(\Omega, \mathcal{F}, \mathbb{P}, \theta, \phi)$ *if for any* $\omega \in \Omega$

$$\phi(t, \omega, A(\omega)) = A(\theta_t\omega) \qquad \text{for any } t \geq 0, \tag{3}$$

$$\operatorname{dist}(\overline{\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega))}, A(\omega)) \to 0 \quad \text{if } t \to \infty \text{ for any } D \in \mathcal{D}. \tag{4}$$

where $\operatorname{dist}(A, B) = \sup_{a\in A} \inf_{b\in B} d(a, b)$.

Remark 2.2. (i) Assume Ω only contains the element ω_0. Then the random $\mathcal{D}$-attractor fulfills the properties of the definition of a semigroup attractor.

(ii) The ω-wise attraction property of a set $D \in \mathcal{D}$ given by (4) is called *pull back* convergence. However, from this property one can derive the *forward* convergence to the attractor w.r.t. convergence in probability. By the invariance of $\mathbb{P}$ w.r.t. the operators of θ we have the convergence in probability

$$\lim_{t\to\infty} \mathbb{P}\{d(\overline{\phi(t, \omega, D(\omega))}, A(\theta_t\omega)) > \delta\} = 0$$

for any $\delta > 0$ which gives an interpretation for technical and physical applications. By (2) the term inside of the probability is contained in $\mathcal{F}$. That means the trajectories of the system starting in a the set $D(\omega)$ will be attracted by the moving compact sets $A(\theta_t\omega)$ which are uniquely determined. To see this replace ω by $\theta_t\omega$ in (4).

(iii) (3) is a generalization of the invariance property.

Using Definition 2.1 one can show in contrast to Crauel and Flandoli [11] and Schmalfuß [19] the uniqueness of a random $\mathcal{D}$-attractor.

Lemma 2.3. *The random $\mathcal{D}$-attractor introduced in Definition 2.1 is unique.*

Proof. Suppose we have two attractors $A_i \in \mathcal{D}$, $i = 1, 2$. It follows for any $\omega \in \Omega$:

$$\mathrm{dist}(A_1(\omega), A_2(\omega)) = \lim_{t\to\infty} \mathrm{dist}(\phi(t, \theta_{-t}\omega, A_1(\theta_{-t}\omega)), A_2(\omega)) = 0.$$

Therefore, $A_1(\omega) \subset A_2(\omega)$ for any $\omega \in \Omega$. Similarly, we can find the contrary inclusion, so the $\mathcal{D}$-attractor is unique. □

We take a simple example. Consider the linear Ito equation

$$du + au\,dt = f\,dt + d\omega, \quad a > 0,\ f \in \mathbb{R}, \quad u(0) = u_0 \in \mathbb{R} \tag{5}$$

where $\omega(t)$ is a one dimensional *two-sided* canonical Wiener process on $\mathbb{R}$. We can construct such a process if we glue together two independent one-sided Wiener processes having the same distribution, with one of them being defined on the negative of the other on the positive semi axis, cf. Arnold [2]. The paths of $\omega(t)$ are contained in the Fréchet space (cf. Dieudonné [13] 12.14.6) C_0 of continuous functions on $\mathbb{R}$ with values in $\mathbb{R}$ such that $\omega(0) = 0$. On C_0 we define the flow by the operators

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \qquad t \in \mathbb{R}. \tag{6}$$

Let $\mathbb{P}$ be the Wiener measure defined on $\mathcal{B}_{C_0}$. This measure is the distribution of the Wiener process. In particular, $\mathbb{P}$ is ergodic (and invariant) w.r.t. the flow θ. The solution of equation (5) is given by

$$\phi(t, \omega, u_0) = u_0 e^{-at} + \frac{f}{a}(1 - e^{-at}) + \int_0^t e^{-a(t-\tau)} d\omega(\tau). \tag{7}$$

Equation (5) has a unique stationary solution $t \mapsto u^s(t, \omega)$ where $u^s(t, \omega)$ is defined by the random variable

$$U(\omega) = \frac{f}{a} + \int_{-\infty}^0 e^{a\tau} d\omega(\tau),$$

such that $u^s(t, \omega) = U(\theta_t\omega)$. Without proof we mention that there exists a $\{\theta_t\}_{t\in\mathbb{R}}$ invariant set $\Omega \in \mathcal{B}_{C_0}$ of full $\mathbb{P}$-measure such that

$$\frac{\log^+ U(\theta_{-t}\omega)}{t} \to 0 \tag{8}$$

for $t \to \infty$ and that $U(\omega)$ is defined for $\omega \in \Omega$. For a similar property see Lemma 5.1 below. Let $\mathcal{F}$ be the trace-σ-algebra of $\mathcal{B}_{C_0}$ on Ω. We now can define a random dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta, \phi)$ where the flow is introduced in (6) and the cocycle is given by (7) with state space $H = \mathbb{R}$.

The family of multi-functions $\mathcal{D}$ is given by the set of measurable multi-functions $D(\omega)$ with closed images such that

$$r_D(\theta_{-t}\omega)e^{-ct} \to 0, \quad r_D(\omega) = \sup_{x \in D(\omega)} \|x\| < \infty \tag{9}$$

for $t \to \infty$ and any $c > 0$, $\omega \in \Omega$. The measurability of r_D follows from (2). It is easy to check that the random dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta, \phi)$ has the random $\mathcal{D}$-attractor A where $A(\omega) = \{U(\omega)\}$. By (8) we have $\{U(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$. We also mention that we obtain the same attractor as in Crauel and Flandoli [11].

A measurable multi-function B is called *positively invariant* if

$$\phi(t, \omega, B(\omega)) \subset B(\theta_t \omega) \tag{10}$$

for all $\omega \in \Omega$, $t \geq 0$. A measurable multi-function $B \in \mathcal{D}$ is called $\mathcal{D}$-*absorbing* if for any $D \in \mathcal{D}$, $\omega \in \Omega$ there exists a $t_D(\omega) \geq 0$ such that for any $t > t_D(\omega)$

$$\phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega). \tag{11}$$

We now formulate the main theorem of this section. For the proof we use similar arguments to find appropriate conclusions as in Crauel and Flandoli [11] and Schmalfuß [19]. In difference to these articles this theorem covers a larger class of sets that will be attracted by the random attractor.

Theorem 2.4. *We assume the continuity of the mappings $x \mapsto \phi(t, \omega, x)$ for any $t \geq 0$, $\omega \in \Omega$. In addition, we assume the existence of a measurable positively invariant and $\mathcal{D}$-absorbing multi-function B of compact sets. Then the random dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta, \phi)$ has a unique random $\mathcal{D}$-attractor given by*

$$A(\omega) = \bigcap_{t \in \mathbb{N}} \phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)). \tag{12}$$

Proof. The family $\{\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega))\}_{t \in \mathbb{N}}$ is a decreasing sequence of compact sets. Indeed, we have by the positive invariance of B for $t > s \geq 0$

$$\begin{aligned} \phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) &= \phi(s, \theta_{-s}\omega, \phi(t-s, \theta_{-t}\omega, B(\theta_{-t}\omega))) \\ &\subset \phi(s, \theta_{-s}\omega, B(\theta_{-s}\omega)). \end{aligned}$$

Hence $\{\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega))\}_{t \in \mathbb{N}}$ tends to the compact and non-empty set $A(\omega)$ w.r.t. the Hausdorff distance for any $\omega \in \Omega$. By (2) the multi-function with compact images

$$\omega \mapsto \phi(t, \omega, B(\omega))$$

is measurable for any $t \in \mathbb{R}^+$. Proposition III.4 in Castaing and Valadier [8] gives the measurability of $A(\omega)$.

To prove (4) we choose for any $\varepsilon > 0$, $\omega \in \Omega$ a $t_2 = t_2(\omega, \varepsilon)$ so that

$$\operatorname{dist}(\phi(t_2, \theta_{-t_2}\omega, B(\theta_{-t_2}\omega)), A(\omega)) < \varepsilon.$$

The attraction property (4) follows from the cocycle property:

$$\phi(t_1 + t_2, \theta_{-t_1-t_2}\omega, D(\theta_{-t_1-t_2}\omega)) = \phi(t_2, \theta_{-t_2}\omega, \phi(t_1, \theta_{-t_1-t_2}\omega, D(\theta_{-t_1-t_2}\omega)).$$

Indeed, the third argument of the right side is contained in $B(\theta_{-t_2}\omega)$ for $t_1 > t_D(\theta_{-t_2}\omega)$. This follows by the $\mathcal{D}$–absorbing property of B.

We now prove (3). First, it is easily seen that

$$\begin{aligned}\phi(t, \omega, A(\omega)) &\subset \bigcap_{\tau \in \mathbb{N}} \phi(t, \omega, \phi(\tau, \theta_{-\tau}\omega, B(\theta_{-\tau}\omega))) \\ &= \bigcap_{\tau \in \mathbb{N}} \phi(t + \tau, \theta_{-t-\tau}\theta_t\omega, B(\theta_{-\tau-t}\theta_t\omega))) = A(\theta_t\omega).\end{aligned} \tag{13}$$

Second, suppose $x \in A(\theta_t\omega)$. By (1), (10) we find for any sufficiently large $\sigma \in \mathbb{N}$ an $x_\sigma \in \phi(\sigma, \theta_{-\sigma}\omega, B(\theta_{-\sigma}\omega)) \subset B(\omega)$ such that $x = \phi(t, \omega, x_\sigma)$. A compactness argument gives the existence of a cluster point x_0 of (x_σ). This cluster point has to be in $A(\omega)$. By the continuity of $\phi(t, \omega, \cdot)$ we have $x = \phi(t, \omega, x_0)$. It follows $A(\theta_t\omega) \subset \phi(t, \omega, A(\omega))$ for $t \geq 0$. Formula (10) shows that A is a *measurable* multi-functions with compact images. In addition, by (12) $A(\omega) \subset B(\omega)$ for all $\omega \in \Omega$. Consequently, $A \in \mathcal{D}$. □

3. The Hausdorff dimension of random attractors

We want to find a bound for the Hausdorff dimension of $A(\omega)$. For the definition of the Hausdorff dimension we refer to Temam [22] page 278f. Let H be a complete separable metric space, and let $A \subset H$ be a compact set. We define

$$\mu_H(A, d, \varepsilon) = \inf \sum_{i \in I} r_i^d, \ |I| \text{ finite} .$$

The infimum is taken over all finite coverings of A by balls with radii $r_i \leq \varepsilon$. The infimum of the numbers d such that

$$\mu_H(A, d) = \sup_{\varepsilon > 0} \mu_H(A, d, \varepsilon) = \lim_{\varepsilon \downarrow 0} \mu_H(A, d, \varepsilon) = 0$$

is called the Hausdorff dimension of A. The map $\varepsilon \mapsto \mu_H(A, d, \varepsilon)$ is decreasing.

Lemma 3.1. *Let $(\mathcal{C}, d_{\mathcal{C}})$ be the metric space of compact subsets of the complete separable space H where $d_{\mathcal{C}}$ denotes the usual Hausdorff metric, cf. Castaing and Valadier, Chapter II. Then the maps*

$$\mathcal{C} \times \mathbb{R}^+ \times \mathbb{R}^+ \setminus \{0\} \ni (C, d, \varepsilon) \to \mu_H(C, d, \varepsilon),$$
$$\mathcal{C} \times \mathbb{R}^+ \ni (C, d) \to \mu_H(C, d) = \lim_{\varepsilon \downarrow 0} \mu_H(C, d, \varepsilon) \in \mathbb{R}^+ \cup \{+\infty\}$$

are measurable for any $\varepsilon > 0$ w.r.t. the associated σ-algebras.

Proof. Let $\{x_i\}_{i\in\mathbb{N}}$ be a dense set in the separable space H and let $\bar{x} = (x_1, \cdots, x_n)$, $\bar{r} = (r_1, \cdots, r_n)$, $r_i \in \mathbb{Q} \setminus \{0\}$ be n-tuples where n is an arbitrary integer. We define

$$U_{\bar{x},\bar{r}} = \bigcup_{i=1}^{n} K(x_i, r_i)$$

where $K(x_i, r_i)$ is an open ball with center x_i and radius r_i. The set $\mathcal{U}_{\bar{x},\bar{r}}$ of all compact sets contained in $U_{\bar{x},\bar{r}}$ is open in $(\mathcal{C}, d_{\mathcal{C}})$, cf. Castaing and Valadier [8] page 42. The mapping

$$\mathcal{C} \times \mathbb{R}^+ \times \mathbb{R}^+ \setminus \{0\} \ni (C, d, \varepsilon) \mapsto \sum_{i=1}^{n} (\min(r_i, \varepsilon))^d, \qquad \bar{r} \text{ fixed}$$

is continuous. If we change this mapping by $+\infty$ for $C \notin \mathcal{U}_{\bar{x},\bar{r}}$ then this new mapping $\mu_H(C, d, \varepsilon, \bar{x}, \bar{r})$ is measurable. Thus,

$$(C, d, \varepsilon) \mapsto \mu_H(C, d, \varepsilon) = \inf\{\mu_H(C, d, \varepsilon, \bar{x}, \bar{r})\}$$

is measurable where we take the infimum over all finite tuples $\bar{x}$, $\bar{r}$ defined above. Hence

$$(C, d) \mapsto \mu_H(C, d) = \lim_{j\to\infty} \mu_H(C, d, 2^{-j})$$

is also measurable. □

Now let H be a separable Hilbert space. Let $\mathcal{E}$ be an ellipsoid in H with semi-axis $\alpha_1 \geq \alpha_2 \geq \cdots$ and define $V_n(\mathcal{E})$, $n \in \mathbb{N}$, as the product $\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n$. If d is not necessarily an integer, $d = n + s$, $s \in (0, 1]$, then we define $V_d(\mathcal{E})$ to be by interpolation

$$V_d(\mathcal{E}) := (V_n(\mathcal{E}))^{1-s} (V_{n+1}(\mathcal{E}))^s.$$

For a linear compact operator L we define $V_d(L)$ to be $V_d(LK_1)$ where LK_1 is the image of the unit ball $K_1$with respect to L. This set is an ellipsoid. We suppose that $\phi(t, \omega, x)$ has a compact Fréchet derivative $\phi'(t, \omega, x)$ for the x-variable for all $x \in H$.

For a chosen positive number $d = n + s$, $n \in \mathbb{N}$, $s \in (0,1]$ we take a $k \in (0,1)$ such that the following inequalities are satisfied:

$$(d+1)^{1/2} k^{1/d} < \frac{1}{4}, \qquad \beta_d k < \left(\frac{1}{4}\right)^{d+1}, \quad \beta_d := 2^n (n+1)^{d/2}. \tag{14}$$

For such a pair (d, k) we assume the existence of a random variable $t(\omega)$ with positive values fulfilling the inequality

$$\sup_{x \in A(\omega)} V_d(\phi'(t(\omega), \omega, x)) \leq k. \tag{15}$$

Let the following uniform differentiability condition be fulfilled:

$$\sup_{u,v \in A(\omega), \|u-v\| \leq \varepsilon} \frac{\|\phi(t,\omega,v) - \phi(t,\omega,u) - \phi'(t,\omega,u)(v-u)\|}{\|u-v\|} \leq \eta(t,\omega) \cdot \varepsilon \tag{16}$$

where $(t,\, \omega) \mapsto \eta(t,\omega) \geq 0$ is $\mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{F}$, $\mathcal{B}_{\mathbb{R}^+}$-measurable. Let $m(\omega)$ be a random variable satisfying the conditions

$$m^d(\omega) \geq k, \qquad \sup_{x \in A(\omega)} \|\phi'(t(\omega), \omega, x)\| \leq m(\omega) \tag{17}$$

for a fixed d, k fulfilling (14).

We define the term $Z(\omega)$ similar to Temam [22], (V.3.7), (V.3.8):

$$Z(\omega) = \left(\frac{m(\omega)^n}{k}\right)^{\frac{1}{s}} \eta(t(\omega), \omega). \tag{18}$$

The following theorem gives an estimate of the Hausdorff dimension of the random $\mathcal{D}$-attractor A. The additional assumptions will be needed to treat ω-dependent attractors in generalization to Temam [22]. In particular, we have to deal with the fact that $\mu_H(A(\theta_t\omega), d, \varepsilon)$ is time dependent and not uniformly bounded in t.

Theorem 3.2. *Let H be a separable Hilbert space. Suppose that $x \mapsto \phi(t,\omega,x)$ has the compact Fréchet derivative $\phi'(t,\omega,x)$ in x for $t \geq 0$ such that (16) is fulfilled. In addition, we suppose the existence of $d \geq 0$, $k \in (0,1)$ such that (14) is satisfied and a random variable $t(\omega) < \infty$ depending on $(d,\, k)$ such that (15) is fulfilled for $\omega \in \Omega$. We define the (measurable) mapping*

$$\omega \mapsto \tilde\theta\omega := \theta_{t(\omega)}\omega,$$

and suppose that $\tilde\theta$ preserves $\mathbb{P}$ and

$$\lim_{i\to\infty} \frac{\log^+ Z(\tilde\theta_i\omega)}{i} = 0 \quad a.s. \qquad \tilde\theta_i := \begin{cases} \mathrm{id} & i = 0 \\ \underbrace{\tilde\theta \circ \cdots \circ \tilde\theta}_{i} & i \in \mathbb{N}\,. \end{cases} \tag{19}$$

Then the Hausdorff dimension of $A(\omega)$ is less than or equal to d, a.s.

Proof. The following proof is inspired by Temam [22] Theorem 3.1. We cover $A(\omega)$ by open balls $K(x_i, r_i)$, $x_i \in A(\omega)$, $r_i \le \varepsilon$. Because of the compactness of $A(\omega)$ we can select a finite number of balls having these properties. We obtain

$$A(\theta_t\omega) = \phi(t, \omega, A(\omega)) = \bigcup_{i=1}^{m} \phi(t, \omega, K(x_i, r_i) \cap A(\omega)).$$

Because of (16) we have for any $i = 1, \dots, m$ and any $x \in K(x_i, r_i) \cap A(\omega)$

$$\|\phi(t, \omega, x) - \phi(t, \omega, x_i) - \phi'(t, \omega, x_i)(x - x_i)\| \le \eta(t, \omega)\|x - x_i\| r_i.$$

That means

$$\phi(t, \omega, K(x_i, r_i) \cap A(\omega)) \subset \phi(t, \omega, x_i) + \phi'(t, \omega, x_i)K(0, r_i) + K(0, \eta(t, \omega)r_i^2). \quad (20)$$

The sets $\phi'(t, \omega, x_i)K(0, r_i)$ are ellipsoids. We introduce the ω-dependent polynomials in the ε-variable R, R_1 given by

$$\begin{aligned} \varepsilon &\mapsto R(\omega, \varepsilon) := (n + 1)^{1/2} k^{1/d}(1 + Z(\omega)\varepsilon)\varepsilon, \\ \varepsilon &\mapsto R_1(\omega, \varepsilon) := \beta_d k(1 + Z(\omega)\varepsilon)^d. \end{aligned}$$

Replacing t by $t(\omega)$ in (20) the sets

$$\phi'(t(\omega), \omega, x_i)K(0, r_i) + K(0, \eta(t(\omega), \omega)r_i^2)$$

are embedded in ellipsoids $\mathcal{E}'_i$ so that $V_d(\mathcal{E}'_i) \le \beta_d^{-1} R_1(\omega, r_i) r_i^d$, cf. Temam [22] Lemma V.3.2 and page 283. To apply this lemma we have to use (17) such that the length of the principal axis of $\phi'(t(\omega), \omega, x_i)K(0, r_i)$ is bounded by $m(\omega)r_i$. In addition, we use

$$V_d(\phi'(t(\omega), \omega, x_i)K(0, r_i)) \le r_i^d k \quad (21)$$

which follows from the definition of $t(\omega)$. The definition of R_1, (21) and Temam [22] (V.3.6) induce the relation

$$\mu_H(\mathcal{E}'_i, d, (n + 1)^{1/2}(\beta_d^{-1} R_1(\omega, \varepsilon))^{1/d} r_i) \le \beta_d V_d(\mathcal{E}'_i) \le R_1(\omega, \varepsilon) r_i^d. \quad (22)$$

Moreover, using the method of Temam [22], page 283–284, (22) and the decreasing monotonicity of the mapping $\varepsilon \mapsto \mu_H(A, d, \varepsilon)$ gives

$$\mu_H(A(\tilde\theta\omega), d, R(\omega, \varepsilon)) \le R_1(\omega, \varepsilon)\mu_H(A(\omega), d, \varepsilon). \quad (23)$$

This estimate holds for any $\varepsilon > 0$. We set $R^1(\omega,\varepsilon) = R(\omega,\varepsilon)$, $R_1^1(\omega,\varepsilon) = R_1(\omega,\varepsilon)$. If we define recursively for $i \geq 2$

$$\begin{aligned} R^i(\omega,\varepsilon) &= R(\tilde\theta_{i-1}\omega, R^{i-1}(\omega,\varepsilon)),\\ R_1^i(\omega,\varepsilon) &= R_1(\tilde\theta_{i-1}\omega, R^{i-1}(\omega,\varepsilon)) \cdot R_1^{i-1}(\omega,\varepsilon), \end{aligned}$$

then we have by iteration of (23)

$$\mu_H(A(\tilde\theta_i\omega), d, R^i(\omega,\varepsilon)) \leq R_1^i(\omega,\varepsilon)\mu_H(A(\omega),d,\varepsilon) \qquad \text{for } i \in \mathbb{N}. \tag{24}$$

For instance for $i = 2$ we replace ω by $\tilde\theta\omega$, ε by $R(\omega,\varepsilon)$ and estimate the μ_H–term on the right hand side of (23) obtaining (24).

By (19) we find a random variable $\varepsilon_0(\omega) > 0$ with

$$Z(\tilde\theta_i\omega)\frac{1}{2^i}\varepsilon \leq 1 \tag{25}$$

for any $0 < \varepsilon < \varepsilon_0(\omega)$, $i \in \mathbb{Z}^+$. The structure of R, R_1 and (25) yields

$$R^i(\omega,\varepsilon) \leq 2^{-i}\varepsilon \quad \text{and} \quad R_1^i(\omega,\varepsilon) \leq 2^{-di} \tag{26}$$

for any $i \in \mathbb{N}$, $\varepsilon < \varepsilon_0(\omega)$. Indeed, (25) gives $R(\omega,\varepsilon) \leq \varepsilon/2$ for $i = 1$. On the other hand $\varepsilon \mapsto R(\omega,\varepsilon)$ is an increasing function such that

$$\begin{aligned} R^2(\omega,\varepsilon) &= R(\tilde\theta_1\omega,\cdot) \circ R(\omega,\varepsilon) \leq \frac{1}{4}(1 + Z(\tilde\theta_1\omega)\frac{1}{2}\varepsilon) \cdot \frac{1}{2}\varepsilon \leq \frac{1}{2^2}\varepsilon,\\ R^i(\omega,\varepsilon) &= \frac{1}{4}(1 + Z(\tilde\theta_{i-1}\omega)\frac{1}{2^{i-1}}\varepsilon)\frac{1}{2^{i-1}}\varepsilon \leq \frac{1}{2^i}\varepsilon \end{aligned}$$

for $i = 3, 4, \cdots$. Similarly, we estimate R_1^i. For $i \in \mathbb{N}$ by the $\tilde\theta_i$-invariance of $\mathbb{P}$ we have by (24) and (26)

$$\begin{aligned} &\mathbb{P}\left\{\mu_H\left(A(\omega),d,\varepsilon\right) > \delta\right\} = \lim_{i\to\infty}\mathbb{P}\left\{\mu_H\left(A(\tilde\theta_i\omega),d,\varepsilon\right) > \delta\right\}\\ &\quad\leq \limsup_{i\to\infty}\mathbb{P}\left\{\mu_H\left(A(\tilde\theta_i\omega),d,R^i(\omega,\varepsilon)\right) > \delta, \varepsilon < \varepsilon_0(\omega)\right\} + \mathbb{P}\left\{\varepsilon \geq \varepsilon_0(\omega)\right\}\\ &\quad\leq \limsup_{i\to\infty}\mathbb{P}\left\{R_1^i(\omega,\varepsilon)\mu_H\left(A(\omega),d,\varepsilon\right) > \delta, \varepsilon < \varepsilon_0(\omega)\right\} + \mathbb{P}\left\{\varepsilon \geq \varepsilon_0(\omega)\right\}\\ &\quad\leq \mathbb{P}\left\{\varepsilon \geq \varepsilon_0(\omega)\right\} \end{aligned}$$

for any δ, $\varepsilon > 0$. Indeed, for ω, ε with $\varepsilon < \varepsilon_0(\omega)$ the term $R_1^i(\omega,\varepsilon)\mu_H\left(A(\omega),d,\varepsilon\right)$ tends to zero for $i \to \infty$. By Lemma 3.1 the expressions in the last formula are well defined. Here we have used the fact that a measurable multi-function with compact images is measurable iff the random variable $A(\omega)$ is measurable w.r.t.

the Borel-σ-algebra of $(\mathcal{C}, d_{\mathcal{C}})$. Hence by the monotonicity of μ_H in ε for any $\delta > 0$ we obtain

$$
\begin{aligned}
0 &= \lim_{j\to\infty} \mathbb{P}\left\{\mu_H\left(A(\omega), d, 2^{-j}\right) > \delta\right\} = \mathbb{P}\left(\bigcup_{j\in\mathbb{N}} \left\{\mu_H\left(A(\omega), d, 2^{-j}\right) > \delta\right\}\right) \\
&\geq \mathbb{P}\{\sup_{j\in\mathbb{N}} \mu_H(A(\omega), d, 2^{-j}) > \delta\} = \mathbb{P}\{\sup_{\varepsilon>0} \mu_H(A(\omega), d, \varepsilon) > \delta\} \\
&= \mathbb{P}\{\mu_H(A(\omega), d) > \delta\}.
\end{aligned}
$$

□.

Remark 3.3. If $\mathbb{P}$ is ergodic w.r.t. the flow θ then the Hausdorff dimension of $A(\omega)$ is constant, a.s. Indeed, by the invariance of $A(\omega)$ and the proof of Temam [22] Proposition VI 3.1 we have $\mu_H(A(\theta_1\omega), d) \leq \mu_H(A(\omega), d)$. By the ergodicity of $\mathbb{P}$ the term $\mu_H(A(\omega), d)$ has to be equal to a constant a.s. depending on d.

4. The stochastic Lorenz system

In this section we give the basic properties of the stochastic Lorenz system which will be needed for the following section. For the deterministic background of this equation see Temam [22] page 33.

We shall write this equation containing a linear multiplicative noise part as

$$
d\phi_x(t) + (A\phi_x(t) + F(\phi_x(t)))dt = f dt + q\phi_x(t) \circ d\omega,\ \phi_x(0) = x \in \mathbb{R}^3 \qquad (27)
$$

where ω is defined to be a two-sided canonical Wiener process with trajectories in C_0, q is a real parameter, and the differential $\circ d\omega$ is to be understood in the sense of Stratonovich, cf. [1] Section 10.2.

The linear operator A is given by the matrix

$$
A = \begin{pmatrix} \sigma & -\sigma & 0 \\ \sigma & 1 & 0 \\ 0 & 0 & b \end{pmatrix}
$$

where b, $\sigma > 0$. We note here this operator is positive definite with

$$
(A\phi, \phi) \geq l\|\phi\|^2, \qquad l = \min(1, b, \sigma). \qquad (28)
$$

Here $\|\cdot\|$ is the Euclidean norm and $(\cdot,\cdot)$ the inner product in $\mathbb{R}^3$. The non-linear function F is given by $F(\phi) = \tilde{F}(\phi, \phi)$ where

$$
\tilde{F}(\phi_1, \phi_2) = (0, x_1 z_2, -x_1 y_2), \quad \phi_i = (x_i, y_i, z_i) \in \mathbb{R}^3,\ i = 1, 2.
$$

$\tilde{F}$ has the properties

$$\begin{aligned}
&(\tilde{F}(\phi_1,\phi_2),\phi_2)=0,\\
&\|\tilde{F}(\phi_1,\phi_2)\|\le\|\phi_1\|\|\phi_2\|,\\
&(\tilde{F}(\phi_1,\phi_2),\phi_3)=-(\tilde{F}(\phi_1,\phi_3),\phi_2),\quad \phi_1,\ \phi_2,\ \phi_3\in\mathbb{R}^3.
\end{aligned}\tag{29}$$

The external force is given by

$$f=(0,0,-b(r+\sigma))\in\mathbb{R}^3,\quad r>0.$$

Let $\mathbb{P}$ be the Wiener measure defined on $\mathcal{B}_{C_0}$ which gives the distribution of the Wiener process. This measure is ergodic (and invariant) w.r.t. the flow θ defined by (6). For standard properties of a Wiener process we refer to Arnold [1] Chapter 1 and Gihman and Skorohod [14] §1.

We now show the existence of a map $(t,\omega,x)\in\mathbb{R}^+\times C_0\times\mathbb{R}^3\mapsto\phi(t,\omega,x)\in\mathbb{R}^3$ that satisfies the cocycle property. The mapping $t\mapsto\phi(t,\omega,x)$ is the solution process of (27) with initial condition x for $\omega\in C_0$. We then obtain a random dynamical system $(C_0,\,\mathcal{B}_{C_0},\,\mathbb{P},\,\theta,\,\phi)$ with state space $H=\mathbb{R}^3$. More generally, we consider a system of stochastic differential equations given by (27) and its linearization. Let C_0^1 be the set of differentiable paths contained in C_0.

Theorem 4.1. *For any $x,\ h\in\mathbb{R}^3$ there exists a unique solution of the Stratonovich equations*

$$\begin{aligned}
&d\phi_x(t)+(A\phi_x(t)+F(\phi_x(t)))dt=f\,dt+q\phi_x(t)\circ d\omega(t),\quad \phi_x(0)=x\\
&d\psi_{x,h}(t)+(A\psi_{x,h}(t)+\tilde{F}(\phi_x(t),\psi_{x,h}(t))dt+\tilde{F}(\psi_{x,h}(t),\phi_x(t)))dt=\\
&\qquad=q\psi_{x,h}(t)\circ d\omega(t),\qquad \psi_{x,h}(0)=h.
\end{aligned}\tag{30}$$

The mapping $(t,\,\omega,\,x,\,h)\mapsto\Phi(t,\omega,x,h)=(\phi_x(t,\omega),\psi_{x,h}(t,\omega))$ defines a cocycle which depends continuously on (t,ω,x,h).

Proof. We only consider the first equation of (30). Standard methods (cf. Dieudonné, [12], 10.7.1) and particular a priori estimates for the solution give that the ordinary differential equation

$$\frac{d\bar{\phi}_x}{dt}+A\bar{\phi}_x+e^{q\omega(t)}F(\bar{\phi}_x)=fe^{-q\omega(t)},\quad \bar{\phi}_x(0,\omega)=x$$

has a unique solution for any $\omega\in C_0$ depending on (t,ω,x) continuously. Indeed, the equation has a similar structure as the deterministic Lorenz system which has a unique global solution. Ito's formula (see Gihman and Skorohod [14] §3 gives that the process $\phi_x(t,\omega):=e^{q\omega(t)}\bar{\phi}_x(t,\omega)$ for fixed x solves the first equation of (27). By this definition the map $(t,\omega,x)\mapsto\phi_x(t,\omega)$ is continuous on $\mathbb{R}^+\times C_0\times\mathbb{R}^3$.

The cocycle property follows immediately by $(\theta_t\omega(\cdot))' = \omega'(\cdot + t)$ for $\omega \in C_0^1$. The set C_0^1 is dense in the space C_0. Hence we can extend the cocycle property on $\mathbb{R}^+ \times C_0 \times \mathbb{R}^3$. □

We denote the first component of $\Phi(t, \omega, x, h)$ by $\phi(t, \omega, x)$.

Lemma 4.2. *For any $t \geq 0$, $x \in \mathbb{R}^3$, $\omega \in C_0$, the linear mapping $h \mapsto \psi_{x,h}(t, \omega) : \mathbb{R}^3 \to \mathbb{R}^3$ is the Fréchet derivative of $\phi(t, \omega, \cdot)$ in x which will be denoted by $\phi'(t, \omega, x)$. Moreover, we have the following estimate for any $h \in H$*

$$\begin{aligned}&\|\phi(t,\omega,x+h) - \phi(t,\omega,x) - \phi'(t,\omega,x)h\| \\ &\leq \|h\|^2 \frac{1}{(8l^2)^{\frac{1}{2}}} \exp\left(2\int_0^t \|\phi(\tau,\omega,x)\| d\tau\right) \exp\left(\sup_{\tau\in[0,t]} q\omega(\tau) + q\omega(t)\right).\end{aligned} \tag{31}$$

The right hand side is continuous (and $\mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{B}_{C_0} \otimes \mathcal{B}_{\mathbb{R}^3}$-measurable) for any $h \in H$.

Proof. Based on (29) we have

$$(F(\phi_1) - F(\phi_2), \phi_1 - \phi_2) = (\tilde{F}(\phi_1 - \phi_2, \phi_1), \phi_1 - \phi_2) \leq \|\phi_1 - \phi_2\|^2 \|\phi_1\|. \tag{32}$$

For $\delta(t, \omega) := \phi(t, \omega, x + h) - \phi(t, \omega, x)$, by the chain rule for $\omega \in C_0^1$ with $d/dx\|x\|^4 = 4\|x\|^2(x, \cdot)$ we obtain

$$\begin{aligned}&\frac{d}{dt}\|\delta(t,\omega)\|^4 + 4\|\delta(t,\omega)\|^2 (A\delta(t,\omega), \delta(t,\omega)) \\ &\quad + 4\|\delta(t,\omega)\|^2 (F(\phi(t,\omega,x+h)) - F(\phi(t,\omega,x)), \delta(t,\omega)) = 4\|\delta(t,\omega)\|^4 q\omega'(t).\end{aligned}$$

Formulas (28), (32) and the last equation yield

$$\frac{d}{dt}\|\delta(t,\omega)\|^4 + 4l\|\delta(t,\omega)\|^4 \leq 4\|\delta(t,\omega)\|^4 \|\phi(t,\omega,x)\| + 4q\omega'(t)\|\delta(t,\omega)\|^4,$$

hence

$$\|\delta(t,\omega)\|^4 \leq \|h\|^4 \exp\left(-4l\,t + 4\int_0^t \|\phi(\tau,\omega,x)\| d\tau + 4q\omega(t)\right) \tag{33}$$

We set $\Delta(t, \omega) := \phi(t, \omega, x + h) - \phi(t, \omega, x) - \psi_{x,h}(t, \omega)$. Δ fulfills the equation

$$\begin{aligned}&\frac{d}{dt}\Delta(t,\omega) + A\Delta(t,\omega) + \tilde{F}(\delta(t,\omega), \delta(t,\omega)) + \\ &\qquad + \tilde{F}(\phi(t,\omega,x), \Delta(t,\omega)) + \tilde{F}(\Delta(t,\omega), \phi(t,\omega,x)) = q\Delta(t,\omega)\omega'(t).\end{aligned}$$

To find this formula we use Constantin, Foias and Temam [9] Formula (4.11) for the non-random coefficients. We also use the fact that $\phi(t,\omega,x+h)-\phi(t,\omega,x)=\delta(t,\omega)$ solves the second equation in (30). Similarly to the first part of this proof we obtain

$$
\begin{aligned}
\frac{d}{dt}\|\Delta(t,\omega)\|^2 &\leq -2l\|\Delta(t,\omega)\|^2+2(\|\delta(t,\omega)\|^2\|\Delta(t,\omega)\|+\\
&\quad+\|\phi(t,\omega,x)\|\|\Delta(t,\omega)\|^2)+2\|\Delta(t,\omega)\|^2 q\omega'(t)\\
&\leq \frac{1}{2l}\|\delta(t,\omega)\|^4+4\|\phi(t,\omega,x)\|\|\Delta(t)\|^2+2\|\Delta(t)\|^2 q\omega'(t),\\
\|\Delta(t,\omega)\|^2 &\leq \frac{1}{2l}\int_0^t \|\delta(s,\omega)\|^4 e^{4\int_s^t \|\phi(\tau,\omega,x)\|^2 d\tau+2q\omega(t)-2q\omega(s)}ds.
\end{aligned}
$$

by the variation of constants formula. Inserting for $\|\delta(s,\omega)\|^4$ the right hand side of (33) it follows from the last formula that

$$
\|\Delta(t)\|\leq \|h\|^2\frac{1}{(8l^2)^{\frac{1}{2}}}\exp\left(2\int_0^t\|\phi(\tau,\omega,x)\|d\tau\right)\exp\left(\sup_{s\in[0,t]}q\omega(s)+q\omega(t)\right).
$$

By the extension argument the last relation remains true for any $\omega\in C_0$. □

5. The random attractor of the stochastic Lorenz system

We start with a lemma that will be used frequently in this section. By the law of large numbers we have a θ-invariant set Ω_1 of full Wiener measure $\mathbb{P}$ such that the functions contained in this set have a sublinear growth:

$$
\lim_{t\to\pm\infty}\frac{\omega(t)}{t}=0 \tag{34}
$$

for $\omega\in\Omega_1$, see Arnold [1] Chapter 3.1.

Lemma 5.1. (i) *Let θ be the flow defined in (6), l be a positive constant, $q\in\mathbb{R}$. Then the mapping*

$$
\omega\mapsto\rho_{l,q}(\omega)=\begin{cases}\int_{-\infty}^0\exp(l\tau-q\omega(\tau))d\tau & \text{for } \omega\in\Omega_1\\ 0 & \text{for } \omega\notin\Omega_1\end{cases} \tag{35.}
$$

is a $\mathcal{B}_{C_0},\mathcal{B}_{\mathbb{R}^+}$ measurable random variable where Ω_1 is defined by (34)
(ii) *For any $\omega\in\Omega_1$, c, $l>0$ and $q\in\mathbb{R}$ we have*

$$
\lim_{t\to-\infty}e^{ct}\int_{-\infty}^0\exp(l\tau-q\theta_t\omega(\tau))d\tau=0.
$$

(iii) *Suppose $q^2/2 < l$. Then there exists a θ-invariant set Ω_2 of full measure (depending on l, q) so that for any $\omega \in \Omega_2$*

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t \rho_{l,q}(\theta_s\omega)ds = \frac{1}{l-q^2/2}.$$

Proof. (i) The mappings

$$\omega \mapsto \rho^n_{l,q}(\omega) = \begin{cases} \int_{-n}^0 \exp(l\tau - q\omega(\tau))d\tau & \omega\in\Omega_1 \\ 0 & \omega\notin\Omega_1 \end{cases}$$

are finite and measurable and $\rho_{l,q}(\omega)$ is the pointwise limit of these maps for $n \to \infty$. A path $\omega(t) \in \Omega_1$ has a growth less than linear for $t \to \pm\infty$, so the integral in the definition of $\rho_{l,q}$ is finite.
(ii) We have the following estimate:

$$\begin{aligned} &\sup_{t,\tau\in(-\infty,0]} \frac{c}{2}t + \frac{l}{2}\tau - q\theta_t\omega(\tau) \\ &\quad\le \sup_{t,\tau\in(-\infty,0]} \frac{c}{2}t + \frac{l}{2}\tau + |q\omega(t+\tau)| + |q\omega(t)| \\ &\quad\le 2\sup_{s\in(-\infty,0]}\left(\frac{\min(c,l)}{4}s + |q\omega(s)|\right) =: k_{l,c,q}(\omega). \end{aligned}$$

The sublinear growth of $\omega\in\Omega_1$ implies that $k_{l,c,q}$ is finite. Thus

$$e^{ct}\int_{-\infty}^0 \exp(l\tau - q\theta_t\omega(\tau))d\tau \le e^{\frac{c}{2}t}\frac{2}{l}e^{k_{l,c,q}(\omega)}, \quad t\le 0.$$

(iii) The simple Ito equation

$$dy + ly\,dt = 1\,dt + \frac{q^2}{2}y\,dt + y\,d\omega, \qquad y(0)=y_0. \tag{36}$$

has the solution

$$y(t,\omega) = y_0e^{-lt+q\omega(t)} + \int_0^t e^{-l(t-s)+q\omega(t)-q\omega(s)}ds, \tag{37}$$

see Gihman and Skorohod [14] §5. Replacing y_0 by $\rho_{l,q}(\omega)$ shows that the process $t\mapsto \rho_{l,q}(\theta_t\omega)$ solves (37). Thus this process is the stationary solution of (36).

Indeed, the distribution of $\rho(\theta_t\omega)$ is independent of t. Taking the expectation over (36) it follows straightforwardly that

$$\mathbb{E}\rho_{l,q} = \frac{1}{l - q^2/2}.$$

The Birkhoff ergodic theorem gives the existence of the θ-invariant set Ω_2 with $\mathbb{P}(\Omega_2) = 1$. □

We now show that the stochastic Lorenz system has a random attractor with Hausdorff dimension less than 3. We set $H = \mathbb{R}^3$. Let $\Omega \in \mathcal{B}_{C_0}$ be a set of full $\mathbb{P}$ measure. The trace of $\mathcal{B}_{C_0}$ on Ω will be denoted by $\mathcal{F}$. The restriction of $\mathbb{P}$ on $\mathcal{F}$ is also denoted by $\mathbb{P}$.

Let $\mathcal{D}$ be the set of measurable multi-functions $D = D(\omega) \subset \mathbb{R}^3$ with closed and non-empty images such that (9) is fulfilled. It is easily seen that the family of sets $\mathcal{D}$ has the properties stated in Section 2.

Theorem 5.2. *Let $\Omega = \Omega_1$ where Ω_1 was defined at the beginning of this section. Then the random dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta, \phi)$ generated by (6), (27) has a unique random $\mathcal{D}$-attractor for b, r, $\sigma > 0$ and $q \in \mathbb{R}$.*

Proof. We first show that the random dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta, \phi)$ generated by (6), (27) has for any $\delta > 0$ the $\mathcal{D}$-absorbing and positively invariant multi-function

$$B(\omega) := B^\delta(\omega) = \left\{x \in \mathbb{R}^3 : \|x\| \le (1+\delta)b(r+\sigma)\rho_{l,q}(\omega)\right\} \tag{38}$$

where $\rho_{l,q}$ is defined in (35) for q introduced in (28) and l introduced in (28).)[1] Indeed, because of (28) and (29) $\|\phi(t,\omega,x)\|^2$ fulfills the inequality

$$\frac{d}{dt}\|\phi(t,\omega,x)\|^2 + 2l\|\phi(t,\omega,x)\|^2 \le 2b(r+\sigma)\|\phi(t,\omega,x)\| + 2q\|\phi(t,\omega,x)\|^2\omega'(t)$$

for all $\omega \in C_0^1$. Therefore

$$\|\phi(t,\omega,x)\| \le \|x\|\exp(-lt + q\omega(t)) + b(r+\sigma)\int_0^t \exp(l(\tau - t) + q\omega(t) - q\omega(\tau))d\tau$$

for any $\omega \in C_0$ by the continuity of the right and the left side in ω. Replacing $\|x\|$ by $(1+\delta)b(r+\sigma)\rho_{l,q}(\omega)$ yields immediately

$$\sup_{x\in B(\omega)} \|\phi(t,\omega,x)\| \le (1+\delta)b(r+\sigma)\rho_{l,q}(\theta_t\omega)$$

[1] We take the restriction of $\rho_{l,q}$ on Ω

which proves (10). For any $D \in \mathcal{D}$ we have

$$\sup_{x\in D(\theta_{-t}\omega)} \|\phi(t,\theta_{-t}\omega,x)\| \leq r_D(\theta_{-t}\omega)\exp(-lt-q\omega(t)) + b(r+\sigma)\int_{-t}^{0}\exp(l\tau-q\omega(\tau))d\tau.$$

For $\omega \in \Omega$ the first term of the right side tends to zero. The second term tends to $b(r+\sigma)\rho_{l,q}(\omega)$. This proves (11). The multi-function B is measurable because the random radius of this ball is measurable. Lemma 5.1 (ii) ensures $B \in \mathcal{D}$. Thus Theorem 2.3 and the continuity of $\phi(t,\omega,\cdot)$ give the assertions of the theorem. □

Remark 5.3. We note that the main idea to prove existence of $\mathcal{D}$-attractors for stochastic differential equations is to have an absorbing set $B \in \mathcal{D}$. This absorbing set is generated by a stationary solution of an affine stochastic differential equation which is *exponentially stable*. This exponential stability ensures the absorption of sets $D \in \mathcal{D}$ (having a subexponential growth). In the same way we can construct absorbing sets for the examples treated in Crauel and Flandoli [11]. So one can hope that these examples also have attractors in the sense of Definition 2.1.

We are now going to show that $A(\omega)$ has Hausdorff dimension less than 3. We now assume that Ω is given by $\Omega_1 \cap \Omega_2$, where Ω_2 is defined in Lemma 5.1. To estimate $V_n(\phi'(t,\omega,x))$ one has to calculate

$$\sup\{|\psi_{x,h_1}(t,\omega)\wedge\cdots\wedge\psi_{x,h_n}(t,\omega)|,\ \|h_i\|\leq 1,\ i=1,\cdots,n\},$$

where $c_1\wedge\cdots\wedge c_n,\ c_i \in H$, denotes the exterior product of $c_1,\cdots,c_n$, see Temam [22] Section V.1. The term $|c_1\wedge\cdots\wedge c_n|^2$ can be calculated by the Gramian determinant $\det((c_i,c_j)_{i,j=1,\cdots,n})$ and

$$(c_1\wedge\cdots\wedge c_n, c_1'\wedge\cdots\wedge c_n') := \det((c_i,c_j')_{i,j=1,\cdots,n}), \quad c_i,\ c_i' \in H.$$

The Differentiation of the square of a determinant $V_{x,h_1,\cdots,h_n}(t,\omega)^2 := |\psi_{x,h_1}(t,\omega)\wedge\cdots\wedge\psi_{x,h_n}(t,\omega)|^2$ yields

$$\begin{aligned}
&dV_{x,h_1,\cdots,h_n}(t,\omega)^2\\
&=2\sum_{i=1}^{n}(\psi_{x,h_1}\wedge\cdots\wedge\psi_{x,h_{i-1}}\wedge d\psi_{x,h_i}\wedge\psi_{x,h_{i+1}}\cdots\wedge\psi_{x,h_n},\psi_{x,h_1}\wedge\cdots\wedge\psi_{x,h_n})\\
&=-2V_{x,h_1,\cdots,h_n}(t,\omega)^2\mathrm{Tr}_n(AQ_n\cdot+\tilde F(\phi(\tau,\omega,x),Q_n\cdot)+\tilde F(Q_n\cdot,\phi(\tau,\omega,x))dt\\
&+2qV_{x,h_1,\cdots,h_n}(t,\omega)^2\,\mathrm{Tr}_n\mathrm{id}\circ d\omega
\end{aligned}$$

where the last equality follows from Temam [22] Lemma V.1.2. and page 276. Here $Q_n = Q_n(t,\omega)$ is the orthonormal projector on the span of the vectors

$\{\psi_{x,h_i}(t,\omega), i = 1, \cdots, n\}$ and $\mathrm{Tr}_n = \mathrm{Tr}_n(t,\omega)$ is the trace with respect to this subspace. We mention that the derivative of $F(u)$ is given by $\tilde{F}(u,\cdot) + \tilde{F}(\cdot,u)$. Thus it follows that $V^2_{x,h_1,\cdots,h_n}(t,\omega) = |\psi_{x,h_1}(t,\omega) \wedge \cdots \wedge \psi_{x,h_n}(t,\omega)|^2$ fulfills the linear equation

$$\begin{aligned}
&V^2_{x,h_1,\cdots,h_n}(t,\omega) \\
&+ \int_0^t 2V_{x,h_1,\cdots,h_n}(\tau,\omega)^2 \mathrm{Tr}_n(AQ_n \cdot + \tilde{F}(\phi(\tau,\omega,x),Q_n\cdot) + \tilde{F}(Q_n\cdot,\phi(\tau,\omega,x)))d\tau \\
&= 2q\int_0^t V_{x,h_1,\cdots,h_n}(\tau,\omega)^2 \mathrm{Tr}_n \,\mathrm{id} \circ d\omega + |h_1 \wedge \cdots \wedge h_n|^2,
\end{aligned}$$

$\|h_i\| \le 1$, $n = 2$ or 3. To calculate Tr_2, Tr_3 we explain the method in Temam [22], page 294–296. We have

$$A + \tilde{F}(\phi,\cdot) + \tilde{F}(\cdot,\phi) = A_1 + A_2 + B(\phi)$$

where

$$A_1 = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & -\sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad B(\phi) = \begin{pmatrix} 0 & 0 & 0 \\ \phi_3 & 0 & \phi_1 \\ -\phi_2 & -\phi_1 & 0 \end{pmatrix},$$

$\phi = (\phi_1,\phi_2,\phi_3)$. Hence we have straightforwardly

$$-\mathrm{Tr}_3(AQ_3 \cdot + \tilde{F}(\phi(\tau,\omega,x),Q_3\cdot) + \tilde{F}(Q_3\cdot,\phi(\tau,\omega,x))) = -(\sigma + b + 1) =: C_1.$$

For an arbitrary orthonormal system $e^i = (e^i_1, e^i_2, e^i_3)$, $i = 1, 2, 3$ in $\mathbb{R}^3$ we have

$$\begin{aligned}
&\sum_{i=1}^2 ((A_1 + A_2)e^i, e^i) \ge \sigma + 1 + b - \max(1,b,\sigma) \\
&\left|\sum_{i=1}^2 (B(\phi), e^i, e^i)\right| \le \left|\sum_{i=1}^2 (\phi_3 e^i_1 e^i_2 - \phi_2 e^i_1 e^i_3)\right| = |-\phi_3 e^3_1 e^3_2 + e^3_1 e^3_3 \phi_2| \\
&\le |e^3_1| \left((e^3_2)^2 + (e^3_3)^2\right)^{\frac{1}{2}} (\phi_2^2 + \phi_3^2)^{\frac{1}{2}} \le \frac{1}{2}\|\phi\|^2,
\end{aligned}$$

hence

$$\begin{aligned}
&-\mathrm{Tr}_2(AQ_2 \cdot + \tilde{F}(\phi(\tau,\omega,x),Q_2\cdot) + \tilde{F}(Q_2\cdot,\phi(\tau,\omega,x))) \\
&\qquad \le -(\sigma + 1 + b) + \max(1,b,\sigma) + \frac{1}{2}\|\phi(\tau,\omega,x)\| =: C_2(\tau,\omega,x).
\end{aligned}$$

To find $\sup_{x\in A(\omega)} V_d(\phi'(t,\omega,x))$ for $d=n+s$, $s\in(0,1)$ one has to calculate the following exponential interpolation

$$\begin{aligned}&\sup_{x\in A(\omega)} V_d(\phi'(t,\omega,x)) = \\ &\quad \sup_{x\in A(\omega)} [(\sup_{\|h_i\|\le 1} V_{x,h_1,h_2}(t,\omega))^{1-s}(\sup_{\|h_i\|\le 1} V_{x,h_1,h_2,h_3}(t,\omega))^{s}]\end{aligned} \tag{39}$$

If the exponent of the logarithm of this expression (depending on s) fulfills particular conditions then this expression tends to zero for $t\to\infty$. In particular, if there exists an $\varepsilon>0$ such that for large t

$$\frac{1}{t}\left(\sup_{x\in A(\omega)}\int_0^t (sC_1+(1-s)C_2(x,\tau,\omega))d\tau+(2+s)q\omega(t)\right)\le -\varepsilon<0 \tag{40}$$

then for any $k\in(0,1)$ there exists a $t(\omega)$ such that $\sup_{A(\omega)} V_d(\phi'(t(\omega),\omega,x))\le k$. The big brackets in (40) give an upper bound for the logarithm of (39).

By (12) the set $A(\omega)$ is contained in $B^\delta(\omega)$ for $\delta=0$ where $B^\delta(\omega)$ is defined in (38). On account of this property and Lemma 5.1 (iii) it follows for any $\mu>0$ and $\omega\in\Omega$ the existence of a $t_0(\omega,\mu)\ge 0$ such that

$$\sup_{x\in A(\omega)}\int_0^t \|\phi(\tau,\omega,x)\|d\tau \le b(r+\sigma)\int_0^t \rho_{l,q}(\theta_\tau\omega)d\tau \le \left(\frac{b(r+\sigma)}{l-\frac{q^2}{2}}+\mu\right)t$$

for any $t>t_0(\omega,\mu)$. The relation (40) is fulfilled if $s=s(\varepsilon,\sigma,b,r,q)\in(0,1)$ is given by

$$\begin{aligned}sC_1+(1-s)&\left(C_3+\frac{1}{2}\frac{b(r+\sigma)}{l-q^2/2}\right)=-2\varepsilon<0,:0<\varepsilon\ll 1,\\ &C_3=-(b+\sigma+1)+\max(1,b,\sigma)\end{aligned} \tag{41}$$

which is the stochastic analogue of Temam [22] VI.1.23. We have used the fact that for $\omega\in\Omega_1$ the growth rate of $\omega(t)$ is less than linear for $t\to\infty$ so that $|\omega(t)/t|\le\varepsilon$ for large t. Let $t(\omega)$ be the *first time* such that

$$\begin{aligned}\log k &= (sC_1+(1-s)C_3)t\\ &\quad+\frac{(1-s)}{2}b(r+\sigma)\int_0^t \rho_{l,q}(\theta_\tau\omega)d\tau+(2+s)q\omega(t)\end{aligned} \tag{42}$$

for $k\in(0,1)$, $s=s(\varepsilon,b,r,\sigma,q)\in(0,1)$, $0<\varepsilon\ll 1$ fixed. Based on (40), (41) we find that (15) is fulfilled. The above considerations and Lemma 5.1 (iii) ensure the finiteness of the random variable $t(\omega)$ for *any* $\omega\in\Omega$ which depends on ω, b, r, σ, q

and on d, k. In particular, the sum of the first two terms on the right side of (42) has a negative linear growth on average, and $|\omega(\cdot)|$ has an average growth less than linear. Summarizing, we have

Lemma 5.4. *Suppose $q^2/2 < l$ where $l = \min(1, b, \sigma)$. Then for any $0 < \varepsilon \ll 1$, $k \in (0,1)$, $\omega \in \Omega := \Omega_1 \cap \Omega_2$ there exists a finite random variable $t(\omega)$ depending on ε, k, b, r, σ, q such that (15) is fulfilled for $d = 2 + s(\varepsilon, \sigma, b, r, q)$ where $s(\varepsilon, \sigma, b, r, q) \in (0,1)$ is defined in (41).*

Using this lemma we obtain the second main theorem.

Theorem 5.5. *Suppose $q^2/2 < l$ where $l = \min(1, b, \sigma)$. Then the Hausdorff dimension of $A(\omega)$, $\omega \in \Omega = \Omega_1 \cap \Omega_2$ is a constant a.s. which is bounded by $2 + s_0$ where $s_0 = s(\varepsilon, \sigma, b, r, q)|_{\varepsilon=0} \in (0,1)$ is defined in (41).*

Proof. We are going to apply Theorem 3.2. So we have only to prove (19) and the invariance of $\mathbb{P}$ w.r.t. $\tilde{\theta}$. We first prove (19). To do this we need a random variable $m(\omega)$ such that (17) is satisfied. Multiplying the second equation in (30) with $\psi_{x,h}(t,\omega)$ yields

$$\begin{aligned}&\|\psi_{x,h}(t,\omega)\|^2 + 2l\int_0^t \|\psi_{x,h}\|^2 d\tau \\ &\quad\le \|h\|^2 + 2\int_0^t \|\phi(\tau,\omega,x)\|\|\psi_{x,h}\|^2 d\tau + 2q\int_0^t \|\psi_{x,h}\|^2 \circ d\omega.\end{aligned}$$

Thus

$$\|\phi'(t,\omega,x)\|^2_{\mathcal{L}(\mathbb{R}^3)} \le \exp\left(2\int_0^t \|\phi(\tau,\omega,x)\|d\tau + 2q\omega(t)\right).$$

This formula and $A(\omega) \subset B^\delta(\omega)$ for $\delta = 0$ gives for

$$m(\omega) = \exp\left(b(r+\sigma)\int_0^{t(\omega)} \rho_{l,q}(\theta_\tau\omega)d\tau + q\omega(t(\omega))\right) \tag{43}$$

the second estimate in (17). On the other hand the *equation* (42) gives $k \le m(\omega)^d$.

If we take the supremum over $A(\omega)$ in (31) then we can set

$$\eta(t,\omega) := (8l^2)^{-\frac{1}{2}} \exp\left(2b(r+\sigma)\int_0^t \rho_{l,q}(\theta_\tau\omega)d\tau\right) \exp\left(\sup_{\tau\in[0,t]} q\omega(\tau) + q\omega(t)\right)$$

such that (16) is fulfilled. Formula (18) and (43) yield for appropriate constants c_1, c_2, $c_3 > 0$ independent of ω, but depending on σ, b, r, q and d, k such that

$$\log Z(\omega) \le c_1 \int_0^{t(\omega)} \rho_{l,q}(\theta_r\omega)dr + c_2 \sup_{[0,t(\omega)]} |q\omega(\tau)| + c_3. \tag{44}$$

Let us define recursively

$$t_0(\omega)=0,\ t_i(\omega)=t(\theta_{t_{i-1}(\omega)}\omega)+t_{i-1}(\omega),\qquad i\in\mathbb{N},\tag{45}$$

which gives $\tilde\theta_i\omega=\theta_{t_i(\omega)}\omega$. We add the equations (42) for ω, $\omega=\theta_{t_1(\omega)}\omega,\cdots,\omega=\theta_{t_{i-1}(\omega)}\omega$ and divide this sum by $t_i(\omega)$. We find a constant $c_{d,k,\varepsilon}\in(0,\infty)$ depending on $d\ge 0$, $k\in(0,1)$, $\varepsilon>0$ (ε is defined in (41)) such that for $i\to\infty$ the sequence $t_i(\omega)\to\infty$, and moreover

$$\frac{t_i(\omega)}{i}\to c_{d,k,\varepsilon}\text{ for }\omega\in\Omega.\tag{46}$$

Hence Lemma 5.1(iii) and (46) gives

$$\begin{aligned}0&=\lim_{i\to\infty}\left(\frac{1}{t_i(\omega)}\int_0^{t_i(\omega)}\rho_{l,q}(\theta_r\omega)dr-\frac{1}{t_{i-1}(\omega)}\int_0^{t_{i-1}(\omega)}\rho_{l,q}(\theta_r\omega)dr\right)\\&=\lim_{i\to\infty}\frac{1}{t_i(\omega)}\int_0^{t(\theta_{t_{i-1}(\omega)}\omega)}\rho_{l,q}(\theta_r\theta_{t_{i-1}(\omega)}\omega)dr\\&=\lim_{i\to\infty}\frac{1}{i}\int_0^{t(\theta_{t_{i-1}(\omega)}\omega)}\rho_{l,q}(\theta_r\theta_{t_{i-1}(\omega)}\omega)dr=0\end{aligned}\tag{47}$$

for $\omega\in\Omega$. Indeed, (46) gives $t_i(\omega)/t_{i-1}(\omega)\to 1$ for $i\to\infty$.

By (46) for any $\omega\in\Omega$ we have an $i_0(\omega)\in\mathbb{N}$ such that for $i\ge i_0(\omega)$

$$\frac{1}{i}\sup_{\tau\in[0,t(\theta_{t_{i-1}(\omega)}\omega)]}|\theta_{t_{i-1}(\omega)}\omega(\tau)|\le\frac{2}{i}\sup_{\tau\in[0,t_i(\omega)]}|\omega(\tau)|\le\frac{2}{i}\sup_{[0,2c_{k,\varepsilon}i]}|\omega(\tau)|.\tag{48}$$

By the law of large numbers applied to the Wiener process ($\omega\in\Omega_1$) we have for any $n\in\mathbb{N}$ a $c(n,\omega)\in[0,\infty)$ such that for $t\ge 0$

$$\sup_{\tau\in[0,t]}|\omega(\tau)|\le c(n,\omega)+\frac{1}{n}|t|.$$

It follows by (44), (47) and (48) that

$$0\le\limsup_{i\to\infty}\frac{\log^+Z(\tilde\theta_i\omega)}{i}\le\frac{1}{n}\quad a.s.$$

which is true for any $n\in\mathbb{N}$. Hence (19) then follows.

Second, we prove that $\tilde\theta\omega$ is a Wiener process with distribution $\mathbb{P}$. We have to show that the random variable $t(\omega)$ is a stopping time, cf. Gihman and Skorohod

[14] §4. That means we have to prove the $\mathcal{F}_t$-measurability of the events $\{t(\omega) \le t\}$, $t \ge 0$, where $\mathcal{F}_t$ is the Ω-trace of $\mathcal{B}^t_{C_0}$:

$$\mathcal{B}^t_{C_0} := \chi_t^{-1}(\mathcal{B}_{C_0}), \qquad \chi_t(\omega) = \begin{cases} \omega(s) & s \le t \\ \omega(t) & s > t \end{cases},$$

cf. Ikeda and Watanabe [16] Chapter 4.1. Let $X(t,\omega)$ be the process defined by the right side of (42). To show $\{t(\omega) \le t\} \in \mathcal{F}_t$ for a $t \ge 0$ we have to check $\{\inf_{\tau\in[0,t]} X(\tau,\omega) \le \log k\} \in \mathcal{F}_t$, $\log k < 0$ since $\tau \mapsto X(\tau,\omega)$ is continuous and $X(0,\omega) = 0$ for *any* $\omega \in \Omega$. It follows from Lemma 5.1(i) that $\rho_{l,q}(\omega)$ is measurable w.r.t. $\mathcal{F}_0$, and $\omega(\tau)$ is measurable w.r.t. $\mathcal{F}_\tau$. On the other hand we have by the θ_τ-invariance of Ω and the $\mathcal{B}^\tau_{C_0}$, $\mathcal{B}^0_{C_0}$-measurability of θ_τ the $\mathcal{F}_\tau$, $\mathcal{F}_0$-measurability of θ_τ, which proves that the random variables $X(\tau,\omega)$ are $\mathcal{F}_\tau$-measurable. Hence $\tilde{\theta}\omega$ is a Wiener Process with distribution $\mathbb{P}$.

We have proved that the Hausdorff dimension of $A(\omega)$ is less than or equal to $2 + s(\varepsilon, b, r, \sigma, q)$ for any $\varepsilon > 0$. By the definition of of the Hausdorff dimension it is also bounded by $2 + s_0$. □

Remark 5.6. For the parameters of the Lorenz system $r = 28$, $b = 8/3$, $\sigma = 10$, $q = 0.1$ a bound of the Hausdorff dimension of $A(\omega)$ is given by $\approx 2.78 < 3$. A bound for the attractor of the deterministic Lorenz system is given by $\approx 2.54...$, see Temam [22] Theorem VI.1.1.

We approximate the Lorenz system with the above parameters by an Euler scheme with step size 0.001. In particular, we calculate an approximation for $\phi(t, \theta_{-t}\omega, x)$ for a sufficiently large number of time steps and a large number of $x \in \mathbb{R}^3$ for a *fixed* ω. We obtain an approximation of $A(\omega)$. Due to the negative shifted ω argument we have to take an approximated trajectory of the *negative* semi axis. We obtain a picture similar to the picture of the attractor of the deterministic Lorenz system. The second picture is the same set as the first picture viewed from another direction. The set of the pixels seems to be thin. Of course, it is an approximation of a set of Hausdorff dimension less than 3.

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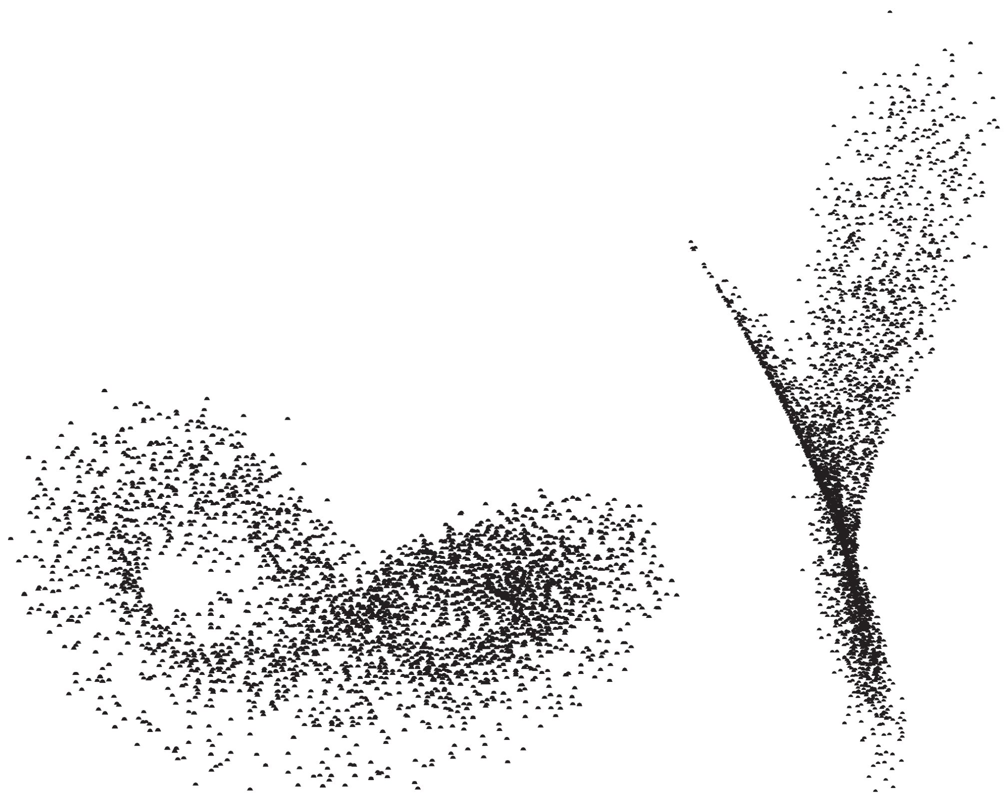

Figure 1. The random attractor of the stochastic Lorenz system

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