Z. angew. Math. Phys. 51 (2000) 573–590
0044-2275/00/040573-18 $ 1.50+0.20/0

Zeitschrift für angewandte Mathematik und Physik ZAMP

Concentrated force solutions for an inhomogeneous thick elastic plate

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Dedicated to the memory of Eric Reissner

Abstract. The technique developed in refs. [1-5] is applied to the problem of a concentrated line force acting in the interior of an infinite plate. The plate is of arbitrary thickness, is isotropic, but is inhomogeneous in that the elastic moduli are any specified functions, not necessarily continuous, of the through-thickness coordinate. The mechanical properties of the plate are not necessarily symmetric about the mid-surface. The solution is based on the classical solution for a concentrated force in a thin elastic plate. This classical solution is extended to give exact closed form solutions for the displacement and stress in the thick inhomogeneous plate. For a plate that is not symmetric an in-plane force gives rise to bending as well as stretching deformations. Higher order force singularities are also considered, as is the problem of a concentrated force on the boundary of a semi-infinite symmetric plate.

Mathematics Subject Classification (2000). 73B27, 73C02, 73K10.

Keywords. Elastic plates, inhomogeneous material, concentrated forces.

1. Introduction

In a series of earlier papers, Rogers and Spencer [1], Rogers [2], Spencer [3,4], Mian and Spencer [5] and Spencer and Selvadurai [6] have developed and applied a procedure for deriving exact solutions of the equations of linear elasticity for materials that are isotropic but inhomogeneous in a specified direction. Such inhomogeneous materials occur frequently both naturally and in man-made structures. For example, many geomaterials are layered or have properties that vary with depth; laminated and sandwich plates and shells are extensively used in aerospace and automotive structures, and currently there is growing interest in functionally graded materials which are deliberately constructed to have mechanical and thermal properties that have continuous spatial variation.

The origins of the method reside in classical solutions by Michell [7] for plane stress of moderately thick elastic plates, and a reformulation of Michell's equations by Kaprielian, Rogers and Spencer [8]. The principal result obtained in [1-5] is that any solution of the classical thin plate or classical laminate theory equations (which describe a two-dimensional theory) can be applied by straightforward substitutions, to generate an exact solution of the three-dimensional linear elasticity equations for a material with arbitrary inhomogeneity in a specified direction, which is here taken to be the direction normal to the surface of a thick flat plate. Thus, if this direction is taken to be the z direction of a system of rectangular Cartesian coordinates $Oxyz$, the Lamé elastic moduli (or Young's modulus and Poisson's ratio) can be arbitrary specified functions of z, subject only to the usual strain-energy positive-definiteness requirements. The dependence need not be continuous, so the important special case of a laminated or sandwich material, in which the moduli are piecewise constant, is included.

A defect of the method is that, although it constructs exact solutions of the field equations, the solutions are usually not sufficiently general to allow satisfaction of the standard point-by point boundary conditions at the edge of a plate. However they normally admit specification of the usual combinations of stress resultants and moments, or of average or mid-plane displacements, at a boundary edge. Hence the solutions should strictly be regarded as interior solutions in a plate, and for completeness need to be supplemented by edge boundary layer solutions. This problem does not arise in the case of an infinite plate, with no boundaries. There are some interesting solutions in two-dimensional elasticity theory which describe the effects of concentrated forces, force pairs, and higher order singularities, in plates. These solutions are fundamental, because solutions due to distributed forces can be constructed by appropriate superpositions of solutions due to concentrated forces. The main purpose of this paper is to derive the corresponding exact solution for a concentrated force in an inhomogeneous plate.

The general theory is summarized in Section 2. In Section 3 we state the classical solution for a concentrated force in a homogeneous thin plate. This solution is applied in Section 4 to develop the solution for displacement and stress due to a concentrated force in an inhomogeneous thick plate. Unless the plate is symmetric about the mid-plane with respect to its mechanical properties, an in-plane force gives rise to bending as well as stretching deformation modes. Some higher order singularities, such as a centre of compression and a centre of rotation, are considered in Section 5. In Section 6 we consider the three-dimensional solution generated by the classical solution for a concentrated force on the surface of a semi-infinite plate. In this solution a concentrated force is applied at the origin, but only resultant tractions can be specified to be zero at the remainder of the surface.

2. General theory

We employ a system of rectangular Cartesian coordinates (x, y, z) and cylindrical polar coordinates (r, θ, z) such that

$$x = r\cos\theta, \quad y = r\sin\theta.$$

For the most part the cylindrical polar system will be used. In this system, displacement components are denoted by u, v and w, and the components of the infinitesimal strain tensor $\mathbf{e}$ and the stress tensor σ as

$$\sigma = \begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{r\theta} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{rz} & \sigma_{\theta z} & \sigma_{zz} \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} e_{rr} & e_{r\theta} & e_{rz} \\ e_{r\theta} & e_{\theta\theta} & e_{\theta z} \\ e_{rz} & e_{\theta z} & e_{zz} \end{bmatrix}. \tag{2.1}$$

Then

$$\begin{aligned} e_{rr} &= \frac{\partial u}{\partial r}, \quad e_{\theta\theta} = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad e_{zz} = \frac{\partial w}{\partial z}, \\ 2e_{r\theta} &= \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}, \quad 2e_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}, \quad 2e_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}. \end{aligned} \tag{2.2}$$

For an isotropic elastic solid, the stress-strain relations can be expressed in the form

$$\begin{bmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ \sigma_{r\theta} & \sigma_{\theta\theta} & \sigma_{\theta z} \\ \sigma_{rz} & \sigma_{\theta z} & \sigma_{zz} \end{bmatrix} = \lambda(e_{rr} + e_{\theta\theta} + e_{zz}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 2\mu \begin{bmatrix} e_{rr} & e_{r\theta} & e_{rz} \\ e_{r\theta} & e_{\theta\theta} & e_{\theta z} \\ e_{rz} & e_{\theta z} & e_{zz} \end{bmatrix}, \tag{2.3}$$

where λ and μ are the Lamé elastic moduli. The equations of equilibrium are

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} &= 0, \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} &= 0, \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} &= 0. \end{aligned} \tag{2.4}$$

We consider a plate or slab of linearly elastic material, which is not necessarily thin, bounded by the planes $z = \pm h$. The material is isotropic, but may be inhomogeneous in the z direction, so that, in general, λ and μ are specified functions of z. The dependence of λ and μ on z need not be continuous, so that the important special case of a laminated or layered material, in which λ and μ are piecewise constant functions of z, is included. The case in which the dependence is continuous corresponds to a functionally graded material.

The underlying idea is that exact solutions of the three-dimensional elasticity equations for the inhomogeneous plate are generated by solutions of the classical two-dimensional thin elastic plate or classical laminate theory equations. The development of this theory is given in [5] and earlier papers. The following is a summary of required results that are described in detail in [5].

Let $\overline{u}(r,\theta), \overline{v}(r,\theta)$ and $\overline{w}(r,\theta)$ be displacements which may be interpreted as average or mid-surface displacements of a thin plate, and denote

$$\Delta = \frac{\partial \overline{u}}{\partial r} + \frac{\overline{u}}{r} + \frac{1}{r}\frac{\partial \overline{v}}{\partial \theta}, \quad \Omega = \frac{\partial \overline{v}}{\partial r} + \frac{\overline{v}}{r} - \frac{1}{r}\frac{\partial \overline{u}}{\partial \theta}. \tag{2.5}$$

Then the classical equations are

$$\begin{aligned}\nabla^4\overline{w} &= 0,\\ \kappa_1\frac{\partial\Delta}{\partial r} - \frac{1}{r}\frac{\partial\Omega}{\partial\theta} + \kappa_2\frac{\partial\nabla^2\overline{w}}{\partial r} &= 0,\\ \kappa_1\frac{1}{r}\frac{\partial\Delta}{\partial\theta} + \frac{\partial\Omega}{\partial r} + \kappa_2\frac{1}{r}\frac{\partial\nabla^2\overline{w}}{\partial\theta} &= 0,\end{aligned} \tag{2.6}$$

where $\kappa_1$and κ_2 are the constants

$$\kappa_1 = \frac{4\int_{-h}^{h}\{\mu(\lambda+\mu)/(\lambda+2\mu)\}dz}{2h\overline{\mu}}, \quad \kappa_2 = -\frac{4\int_{-h}^{h}\{\mu(\lambda+\mu)/(\lambda+2\mu)\}zdz}{2h\overline{\mu}}, \tag{2.7}$$

and $\overline{\mu}$ is the through-thickness average value of μ, and therefore

$$2h\overline{\mu} = \int_{-h}^{h}\mu dz. \tag{2.8}$$

It follows from (2.6) that Δ and Ω are harmonic functions, so that

$$\nabla^2\Delta = 0, \quad \nabla^2\Omega = 0, \tag{2.9}$$

and ∇^2 is the two-dimensional Laplacian in polar coordinates; thus

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

If the plate is symmetric, so that $\lambda(z) = \lambda(-z)$ and $\mu(z) = \mu(-z)$, then $\kappa_2 = 0$, and the stretching deformations, represented by $\overline{u}$ and $\overline{v}$, uncouple from the bending deformations represented by $\overline{w}$.

It is shown in [5] that if $\overline{u}, \overline{v}$ and $\overline{w}$ satisfy (2.6) then an exact solution of the three-dimensional elasticity equations (2.2), (2.3) and (2.4) is

$$\begin{aligned}u(r,\theta,z) &= \overline{u}(r,\theta) - z\frac{\partial\overline{w}(r,\theta)}{\partial r} + F(z)\frac{\partial\Delta(r,\theta)}{\partial r} + B(z)\frac{\partial\nabla^2\overline{w}(r,\theta)}{\partial r},\\ v(r,\theta,z) &= \overline{v}(r,\theta) - z\frac{1}{r}\frac{\partial\overline{w}(r,\theta)}{\partial\theta} + F(z)\frac{1}{r}\frac{\partial\Delta(r,\theta)}{\partial\theta} + B(z)\frac{1}{r}\frac{\partial\nabla^2\overline{w}(r,\theta)}{\partial\theta},\\ w(r,\theta,z) &= \overline{w}(r,\theta) + G(z)\Delta(r,\theta) + C(z)\nabla^2\overline{w}(r,\theta).\end{aligned} \tag{2.10}$$

The functions $B(z), C(z), G(z)$ and $F(z)$ are determined by the equations

$$\frac{dG}{dz} = -\frac{\lambda}{\lambda+2\mu}, \quad \frac{dC}{dz} = z\frac{\lambda}{\lambda+2\mu},$$

$$\frac{d}{dz}\left\{\mu\left(\frac{dF}{dz}+G\right)\right\}=\mu\kappa_1-\frac{4\mu(\lambda+\mu)}{\lambda+2\mu},$$
$$\frac{d}{dz}\left\{\mu\left(\frac{dB}{dz}+C\right)\right\}=\mu\kappa_2+z\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}, \tag{2.11}$$

and the boundary conditions

$$\frac{dF}{dz}+G=0, \qquad \frac{dB}{dz}+C=0, \qquad \text{at } \; z=\pm h. \tag{2.12}$$

In formulating (2.10)-(2.12) it has been assumed that the lateral surfaces $z=\pm h$ of the plate are free from tractions, so that

$$\sigma_{rz}=0, \quad \sigma_{\theta z}=0, \quad \sigma_{zz}=0, \qquad \text{at } \; z=\pm h. \tag{2.13}$$

The theory developed in [5] is rather more general than this, in that it allows less restrictive boundary conditions on the lateral surfaces. However (2.10)-(2.12) are sufficient for the purposes of this paper.

From (2.2) the strain associated with the displacement (2.10) is

$$\begin{aligned}
e_{rr}&=\frac{\partial\overline{u}}{\partial r}-z\frac{\partial^2\overline{w}}{\partial r^2}+F\frac{\partial^2\Delta}{\partial r^2}+B\frac{\partial^2\nabla^2\overline{w}}{\partial r^2},\\
e_{\theta\theta}&=\frac{\overline{u}}{r}+\frac{1}{r}\frac{\partial\overline{v}}{\partial\theta}-z\left(\frac{1}{r}\frac{\partial\overline{w}}{\partial r}+\frac{1}{r^2}\frac{\partial^2\overline{w}}{\partial\theta^2}\right)+F\left(\frac{1}{r}\frac{\partial\Delta}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Delta}{\partial\theta^2}\right)\\
&\quad+B\left(\frac{1}{r}\frac{\partial\nabla^2\overline{w}}{\partial r}+\frac{1}{r^2}\frac{\partial^2\nabla^2\overline{w}}{\partial\theta^2}\right),\\
e_{zz}&=\frac{dG}{dz}\Delta+\frac{dC}{dz}\nabla^2\overline{w},\\
2e_{r\theta}&=\frac{1}{r}\frac{\partial\overline{u}}{\partial\theta}+\frac{\partial\overline{v}}{\partial r}-\frac{\overline{v}}{r}-z\left(\frac{2}{r}\frac{\partial^2\overline{w}}{\partial r\partial\theta}-\frac{2}{r^2}\frac{\partial\overline{w}}{\partial\theta}\right)+F\left(\frac{2}{r}\frac{\partial^2\Delta}{\partial r\partial\theta}-\frac{2}{r^2}\frac{\partial\Delta}{\partial\theta}\right)\\
&\quad+B\left(\frac{2}{r}\frac{\partial^2\nabla^2\overline{w}}{\partial r\partial\theta}-\frac{2}{r^2}\frac{\partial\nabla^2\overline{w}}{\partial\theta}\right),\\
2e_{rz}&=\left(\frac{dF}{dz}+G\right)\frac{\partial\Delta}{\partial r}+\left(\frac{dB}{dz}+C\right)\frac{\partial\nabla^2\overline{w}}{\partial r},\\
2e_{\theta z}&=\left(\frac{dF}{dz}+G\right)\frac{1}{r}\frac{\partial\Delta}{\partial\theta}+\left(\frac{dB}{dz}+C\right)\frac{1}{r}\frac{\partial\nabla^2\overline{w}}{\partial\theta},
\end{aligned} \tag{2.14}$$

and hence, from (2.6) and (2.9)

$$e_{rr}+e_{\theta\theta}+e_{zz}=\left(1+\frac{dG}{dz}\right)\Delta-\left(z-\frac{dC}{dz}\right)\nabla^2\overline{w}. \tag{2.15}$$

It follows from (2.3) and (2.11) that the stress is

$$\begin{aligned}
\sigma_{rr} &= \frac{2\lambda\mu}{\lambda+2\mu}(\Delta - z\nabla^2\overline{w}) + 2\mu\left\{\frac{\partial\overline{u}}{\partial r} - z\frac{\partial^2\overline{w}}{\partial r^2} + F\frac{\partial^2\Delta}{\partial r^2} + B\frac{\partial^2\nabla^2\overline{w}}{\partial r^2}\right\},\\
\sigma_{\theta\theta} &= \frac{2\lambda\mu}{\lambda+2\mu}(\Delta - z\nabla^2\overline{w}) + 2\mu\left\{\frac{\overline{u}}{r} + \frac{1}{r}\frac{\partial\overline{v}}{\partial\theta} - z\left(\frac{1}{r}\frac{\partial\overline{w}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\overline{w}}{\partial\theta^2}\right)\right\}\\
&\quad + 2\mu\left\{F\left(\frac{1}{r}\frac{\partial\Delta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Delta}{\partial\theta^2}\right) + B\left(\frac{1}{r}\frac{\partial\nabla^2\overline{w}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\nabla^2\overline{w}}{\partial\theta^2}\right)\right\},\\
\sigma_{r\theta} &= \mu\left\{\frac{1}{r}\frac{\partial\overline{u}}{\partial\theta} + \frac{\partial\overline{v}}{\partial r} - \frac{\overline{v}}{r} - z\left(\frac{2}{r}\frac{\partial^2\overline{w}}{\partial r\partial\theta} - \frac{2}{r^2}\frac{\partial\overline{w}}{\partial\theta}\right)\right\}\\
&\quad + \mu\left\{F\left(\frac{2}{r}\frac{\partial^2\Delta}{\partial r\partial\theta} - \frac{2}{r^2}\frac{\partial\Delta}{\partial\theta}\right) + B\left(\frac{2}{r}\frac{\partial^2\nabla^2\overline{w}}{\partial r\partial\theta} - \frac{2}{r^2}\frac{\partial\nabla^2\overline{w}}{\partial\theta}\right)\right\},\\
\sigma_{rz} &= \mu\left\{\left(\frac{dF}{dz} + G\right)\frac{\partial\Delta}{\partial r} + \left(\frac{dB}{dz} + C\right)\frac{\partial\nabla^2\overline{w}}{\partial r}\right\},\\
\sigma_{\theta z} &= \mu\left\{\left(\frac{dF}{dz} + G\right)\frac{1}{r}\frac{\partial\Delta}{\partial\theta} + \left(\frac{dB}{dz} + C\right)\frac{1}{r}\frac{\partial\nabla^2\overline{w}}{\partial\theta}\right\},\\
\sigma_{zz} &= 0. \qquad (2.16)
\end{aligned}$$

Also, from (2.11)

$$\begin{aligned}
G(z) &= -\int_0^z \frac{\lambda(\zeta)}{\lambda(\zeta)+2\mu(\zeta)}d\zeta + G_0,\\
C(z) &= \int_0^z \frac{\zeta\lambda(\zeta)}{\lambda(\zeta)+2\mu(\zeta)}d\zeta + C_0, \qquad (2.17)
\end{aligned}$$

and without loss of generality we may take $G_0 = 0,\ C_0 = 0$, because terms such as $G_0\Delta(r,\theta)$ and $C_0\nabla^2\overline{w}(r,\theta)$ in (2.10) can be absorbed into $\overline{w}(r,\theta)$. Further, from (2.7), (2.11) and (2.12)

$$\begin{aligned}
\mu\left(\frac{dF}{dz} + G\right) &= \kappa_1\int_{-h}^z \mu(\zeta)d\zeta - 4\int_{-h}^z \frac{\mu(\zeta)\{\lambda(\zeta)+\mu(\zeta)\}}{\lambda(\zeta)+2\mu(\zeta)}d\zeta,\\
\mu\left(\frac{dB}{dz} + C\right) &= -\kappa_2\int_{-h}^z \mu(\zeta)d\zeta + 4\int_{-h}^z \frac{\zeta\mu(\zeta)\{\lambda(\zeta)+\mu(\zeta)\}}{\lambda(\zeta)+2\mu(\zeta)}d\zeta, \qquad (2.18)
\end{aligned}$$

and hence

$$\begin{aligned}
F(z) &= \int_0^z\int_{-h}^\zeta \frac{\mu(\zeta)}{\mu(\xi)}\left\{\kappa_1 - 4\frac{\{\lambda(\zeta)+\mu(\zeta)\}}{\lambda(\zeta)+2\mu(\zeta)}\right\}d\xi d\zeta + \int_0^z \frac{(z-\zeta)\lambda(\zeta)}{\lambda(\zeta)+2\mu(\zeta)}d\zeta + F_0,\\
B(z) &= \int_0^z\int_{-h}^\zeta \frac{\mu(\zeta)}{\mu(\xi)}\left\{\kappa_2 + 4\frac{\zeta\{\lambda(\zeta)+\mu(\zeta)\}}{\lambda(\zeta)+2\mu(\zeta)}\right\}d\xi d\zeta + \int_0^z \frac{\zeta(z-\zeta)\lambda(\zeta)}{\lambda(\zeta)+2\mu(\zeta)}d\zeta + B_0. \qquad (2.19)
\end{aligned}$$

and we may also, without loss of generality, take $F_0 = 0,\ B_0 = 0$,because terms involving these constants can be absorbed into $\overline{u}(r,\theta)$ and $\overline{v}(r,\theta)$ in (2.10).

If the plate is symmetric, so that λ and μ are even functions of z, then F and C are even functions of z, and G and B are odd functions of z.

3. Concentrated force in a homogeneous thin plate

There is a well-known solution for the stress and displacement in an infinite homogeneous thin elastic plate subject to a concentrated force acting at the origin in the plane of the plate in, say, the x-direction. The stress solution is given, for example, by Love [9] and Timoshenko and Goodier [10]. In the present context it is more convenient to start with the displacement . A little consideration shows that, in the classical two-dimensional theory, the required displacement is, to within a rigid body displacement, of the form

$$\begin{aligned}\overline{u}(r,\theta) &= (\alpha \ln r + \beta)\cos\theta,\\ \overline{v}(r,\theta) &= (-\alpha \ln r + \beta)\sin\theta,\end{aligned}\tag{3.1}$$

where α and β are constants to be determined. Correspondingly, from (2.5)

$$\Delta = (\alpha + 2\beta)\frac{\cos\theta}{r},\quad \Omega = (-\alpha + 2\beta)\frac{\sin\theta}{r},\tag{3.2}$$

and it follows that (2.6), with $\overline{w} = 0$, are satisfied provided that

$$\alpha(\kappa_1 - 1) + 2\beta(\kappa_1 + 1) = 0,\tag{3.3}$$

or since for homogeneous material $\kappa_1 = 4(\lambda+\mu)/(\lambda+2\mu)$, that

$$\alpha(3\lambda + 2\mu) + 2\beta(5\lambda + 6\mu) = 0.\tag{3.4}$$

It then follows from (2.3) that in the case of plane stress, with $\sigma_{zz} = 0$, the in-plane stress components are given as

$$\begin{aligned}\sigma_{rr} &= \frac{2\mu}{\lambda+2\mu}\left\{2\alpha(\lambda+\mu) + 2\beta\lambda\right\}\frac{\cos\theta}{r},\\ \sigma_{\theta\theta} &= \frac{2\mu}{\lambda+2\mu}\left\{\alpha\lambda + 4\beta(\lambda+\mu\right\}\frac{\cos\theta}{r},\\ \sigma_{r\theta} &= 2\mu(\alpha+2\beta)\frac{\sin\theta}{r}.\end{aligned}\tag{3.5}$$

After eliminating α by the use of (3.4), these become

$$\sigma_{rr} = -\frac{4\mu\beta(7\lambda+6\mu)}{3\lambda+2\mu}\frac{\cos\theta}{r},$$

$$\sigma_{\theta\theta} = \frac{4\mu\beta(\lambda+2\mu)}{3\lambda+2\mu}\frac{\cos\theta}{r},$$
$$\sigma_{r\theta} = \frac{4\mu\beta(\lambda+2\mu)}{3\lambda+2\mu}\frac{\sin\theta}{r}. \tag{3.6}$$

The resultant force exerted by a circle of radius r with centre at the origin acts in the x direction, and its magnitude P is given by

$$P = -2h\int_{-\pi}^{\pi}(\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta)rd\theta \tag{3.7}$$

from which, with (3.6), there follows

$$P = \frac{64\pi\mu\beta(\lambda+\mu)h}{3\lambda+2\mu} \tag{3.8}$$

and therefore, from (3.4) and (3.8)

$$\alpha = -\frac{P(5\lambda+6\mu)}{32\pi\mu(\lambda+\mu)h}, \qquad \beta = \frac{P(3\lambda+2\mu)}{64\pi\mu(\lambda+\mu)h}, \tag{3.9}$$

so that

$$\sigma_{rr} = -\frac{P(7\lambda+6\mu)}{16\pi(\lambda+\mu)}\frac{\cos\theta}{hr},$$
$$\sigma_{\theta\theta} = \frac{P(\lambda+2\mu)}{16\pi(\lambda+\mu)}\frac{\cos\theta}{hr},$$
$$\sigma_{r\theta} = \frac{P(\lambda+2\mu)}{16\pi(\lambda+\mu)}\frac{\sin\theta}{hr}. \tag{3.10}$$

It is also straightforward to show that the resultant force in the y direction on the semi-circle of radius r with $0 \leq \theta \leq \pi$ is zero, and so the solution corresponds to a simple line force at the origin of magnitude P.

4. Concentrated force in an inhomogeneous thick plate

The two-dimensional solution developed in Section 3 will now be used to generate an exact three-dimensional solution for a concentrated force in an inhomogeneous plate of arbitrary thickness. To allow the possibility that the plate is not symmetric (in which case there is a coupling between stretching and bending deformation modes) it is necessary to generalize the two-dimensional solution slightly, so that in place of (3.1) we take the required solution of (2.6) to be of the form

$$\overline{u}(r,\theta) = (\alpha\ln r + \beta)\cos\theta,$$
$$\overline{v}(r,\theta) = (-\alpha\ln r + \beta)\sin\theta,$$
$$\overline{w}(r,\theta) = \gamma r\ln r\ \cos\theta \tag{4.1}$$

and it follows that

$$\Delta = (\alpha + 2\beta)\frac{\cos\theta}{r}, \quad \Omega = (-\alpha + 2\beta)\frac{\sin\theta}{r}, \quad \nabla^2\overline{w} = 2\gamma\frac{\cos\theta}{r}, \tag{4.2}$$

and that (2.6) are satisfied provided that

$$\alpha(\kappa_1 - 1) + 2\beta(\kappa_1 + 1) + 2\gamma\kappa_2 = 0. \tag{4.3}$$

Then, from (2.10), the corresponding three-dimensional displacement in the thick plate is

$$\begin{aligned}
u(r,\theta,z) &= \left\{\alpha \ln r + \beta - \gamma z(1+\ln r) - F\frac{\alpha+2\beta}{r^2} - B\frac{2\gamma}{r^2}\right\}\cos\theta, \\
v(r,\theta,z) &= \left\{-\alpha \ln r + \beta + \gamma z \ln r - F\frac{\alpha+2\beta}{r^2} - B\frac{2\gamma}{r^2}\right\}\sin\theta, \\
w(r,\theta,z) &= \left\{\gamma r \ln r + G\frac{\alpha+2\beta}{r} + C\frac{2\gamma}{r}\right\}\cos\theta,
\end{aligned} \tag{4.4}$$

and the associated strain components are

$$\begin{aligned}
e_{rr} &= \left\{\frac{\alpha}{r} - \frac{\gamma z}{r} + F\frac{2(\alpha+2\beta)}{r^3} + B\frac{4\gamma}{r^3}\right\}\cos\theta, \\
e_{\theta\theta} &= \left\{\frac{2\beta}{r} - \frac{\gamma z}{r} - F\frac{2(\alpha+2\beta)}{r^3} - B\frac{4\gamma}{r^3}\right\}\cos\theta, \\
e_{zz} &= \left\{\frac{dG}{dz}\frac{(\alpha+2\beta)}{r} + \frac{dC}{dz}\frac{2\gamma}{r}\right\}\cos\theta, \\
2e_{r\theta} &= \left\{-\frac{(\alpha+2\beta)}{r} + \frac{2\gamma z}{r} + F\frac{4(\alpha+2\beta)}{r^3} + B\frac{8\gamma}{r^3}\right\}\sin\theta, \\
2e_{rz} &= \left\{-\left(\frac{dF}{dz} + G\right)\frac{(\alpha+2\beta)}{r^2} - \left(\frac{dB}{dz} + C\right)\frac{2\gamma}{r^2}\right\}\cos\theta, \\
2e_{\theta rz} &= \left\{-\left(\frac{dF}{dz} + G\right)\frac{(\alpha+2\beta)}{r^2} - \left(\frac{dB}{dz} + C\right)\frac{2\gamma}{r^2}\right\}\sin\theta.
\end{aligned} \tag{4.5}$$

Furthermore, from (2.16), the stress is given as

$$\begin{aligned}
\sigma_{rr} &= \left\{\frac{2\mu}{\lambda+2\mu}\frac{\{2\alpha(\lambda+\mu) + 2\beta\lambda - \gamma(3\lambda+2\mu)z\}}{r} + \frac{4\mu\{F(\alpha+2\beta) + 2B\gamma\}}{r^3}\right\}\cos\theta, \\
\sigma_{\theta\theta} &= \left\{\frac{2\mu}{\lambda+2\mu}\frac{\{2\alpha\lambda + 2\beta(\lambda+\mu) - \gamma(3\lambda+2\mu)z\}}{r} - \frac{4\mu\{F(\alpha+2\beta) + 2B\gamma\}}{r^3}\right\}\cos\theta, \\
\sigma_{r\theta} &= \mu\left\{\frac{\{-(\alpha+2\beta) + 2\gamma z\}}{r} + \frac{4\mu\{F(\alpha+2\beta) + 2B\gamma\}}{r^3}\right\}\sin\theta,
\end{aligned}$$

$$\sigma_{rz} = \mu\left\{-\left(\frac{dF}{dz}+G\right)\frac{(\alpha+2\beta)}{r^2} - \left(\frac{dB}{dz}+C\right)\frac{2\gamma}{r^2}\right\}\cos\theta,$$
$$\sigma_{\theta z} = \mu\left\{-\left(\frac{dF}{dz}+G\right)\frac{(\alpha+2\beta)}{r^2} - \left(\frac{dB}{dz}+C\right)\frac{2\gamma}{r^2}\right\}\sin\theta,$$
$$\sigma_{zz} = 0. \tag{4.6}$$

In general, the stress components σ_{rz} and $\sigma_{\theta z}$ are non-zero, and give rise to non-zero stress resultants $\int_{-h}^{h}\sigma_{rz}\,dz$ and $\int_{-h}^{h}\sigma_{\theta z}\,dz$. However, these stress resultants are zero, and so only in-plane resultant forces are present, if γ is chosen such that

$$(\alpha+2\beta)\int_{-h}^{h}\mu\left(\frac{dF}{dz}+G\right)dz + 2\gamma\int_{-h}^{h}\mu\left(\frac{dB}{dz}+C\right)dz = 0. \tag{4.7}$$

From (2.7), (2.8) and (2.18)

$$\begin{aligned}\int_{-h}^{h}\mu\left(\frac{dF}{dz}+G\right)dz &= \kappa_1\int_{-h}^{h}\int_{-h}^{z}\mu(\zeta)d\zeta dz - 4\int_{-h}^{h}\int_{-h}^{z}\frac{\mu(\zeta)\{\lambda(\zeta)+\mu(\zeta)\}}{\lambda(\zeta)+2\mu(\zeta)}d\zeta dz\\
&= \kappa_1\int_{-h}^{h}\int_{\zeta}^{h}\mu(\zeta)dzd\zeta - 4\int_{-h}^{h}\int_{\zeta}^{h}\frac{\mu(\zeta)\{\lambda(\zeta)+\mu(\zeta)\}}{\lambda(\zeta)+2\mu(\zeta)}dzd\zeta\\
&= \kappa_1\int_{-h}^{h}(h-\zeta)\mu(\zeta)d\zeta - 4\int_{-h}^{h}(h-\zeta)\frac{\mu(\zeta)\{\lambda(\zeta)+\mu(\zeta)\}}{\lambda(\zeta)+2\mu(\zeta)}d\zeta\\
&= \kappa_1(2h^2\overline{\mu}-2h\widehat{\mu}) - 2h^2\overline{\mu}\kappa_1 - 2h\overline{\mu}\kappa_2\\
&= -2h(\kappa_1\widehat{\mu}+\kappa_2\overline{\mu}),\end{aligned} \tag{4.8}$$

where

$$2h\widehat{\mu} = \int_{-h}^{h} z\mu(z)dz. \tag{4.9}$$

Similarly

$$\int_{-h}^{h}\mu\left(\frac{dB}{dz}+C\right)dz = -2h(\kappa_2\widehat{\mu}+\kappa_3\overline{\mu}), \tag{4.10}$$

where

$$\kappa_3 = \frac{4\int_{-h}^{h}\{\mu(\lambda+\mu)/(\lambda+2\mu)\}z^2dz}{2h\overline{\mu}}. \tag{4.11}$$

Therefore (4.7) can be written as

$$(\alpha+2\beta)(\kappa_1\widehat{\mu}+\kappa_2\overline{\mu}) + 2\gamma(\kappa_2\widehat{\mu}+\kappa_3\overline{\mu}) = 0 \tag{4.12}$$

If the plate is symmetric, then $\kappa_2 = 0$ and $\widehat{\mu} = 0$, and therefore in this case $\gamma = 0$.

The resultant force exerted by a cylinder of radius r with its axis coincident with the $z-$axis on the material outside the cylinder acts in the $x-$direction and has magnitude P, where

$$
\begin{aligned}
P &= -\int_{-h}^{h}\int_{-\pi}^{\pi}(\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta) r d\theta dz \\
&= -\int_{-h}^{h}\int_{-\pi}^{\pi}\Big\{\frac{2\mu\{2\alpha(\lambda+\mu)+2\beta\lambda-\gamma(3\lambda+2\mu)z\}}{(\lambda+2\mu)r}\cos^2\theta \\
&- \frac{\mu\{-(\alpha+2\beta)+2\gamma z\}}{r}\sin^2\theta\Big\} r d\theta dz \\
&= -\pi\int_{-h}^{h}\left\{\frac{\mu}{\lambda+2\mu}\{\alpha(5\lambda+6\mu)+2\beta(3\lambda+2\mu)-8\gamma(\lambda+\mu)z\}\right\}dz.
\end{aligned}
\tag{4.13}
$$

However, from (2.7) and (2.8)

$$
\int_{-h}^{h}\frac{\mu(5\lambda+6\mu)}{\lambda+2\mu}dz = 2h\overline{\mu}(\kappa_1+1), \quad \int_{-h}^{h}\frac{\mu(3\lambda+2\mu)}{\lambda+2\mu}dz = 2h\overline{\mu}(\kappa_1-1), \tag{4.14}
$$

and therefore, from (2.7), (4.13) and (4.14)

$$
P = -2\pi h\overline{\mu}\{\alpha(\kappa_1+1)+2\beta(\kappa_1-1)+2\gamma\kappa_2\}. \tag{4.15}
$$

Hence, from ((4.3)

$$
P = -4\pi h\overline{\mu}(\alpha-2\beta), \tag{4.16}
$$

which is the same as the result (3.8) for a homogeneous thin plate subject to the deformation ((3.1), with α and β given by (3.9). Since P is independent of r, (4.16) remains valid formally as the radius of the cylinder tends to zero, and the solution can be interpreted as describing the displacement and stress in the plate due to a line force at $r = 0$ acting in the x direction. Of course, as in standard plane stress or plane strain theory for a homogeneous material, the displacement and stress are singular as $r \to 0$, thus contradicting the underlying small strain assumption of linear elasticity theory. Therefore, as in the theory for homogeneous materials, the solution should be regarded as an asymptotic solution as $r \to \infty$. It is of interest that P does not depend on γ, and so even for a non-symmetric plate P is not affected by the bending displacement mode that is included in (4.1). Another surprising feature is that although in the inhomogeneous plate the stress includes terms of order r^{-3} as well as terms of order r^{-1}, these higher order singular terms do not contribute to the resultant force on a cylinder of radius r.

To complete the solution the coefficients α, β and γ need to be expressed in terms of P and the elastic moduli. From (4.3), (4.12) and (4.15) it follows straightforwardly that

$$
\alpha = -\frac{P}{8\pi h\overline{\mu}}\left\{1+\frac{\kappa_3\overline{\mu}+\kappa_2\widehat{\mu}}{\overline{\mu}(\kappa_1\kappa_3-\kappa_2^2)}\right\},
$$

$$\beta = \frac{P}{16\pi h\overline{\mu}}\left\{1 - \frac{\kappa_3\overline{\mu} + \kappa_2\widehat{\mu}}{\overline{\mu}(\kappa_1\kappa_3 - \kappa_2^2)}\right\},$$
$$\gamma = \frac{P(\kappa_2\overline{\mu} + \kappa_1\widehat{\mu})}{8\pi h\overline{\mu}^2((\kappa_1\kappa_3 - \kappa_2^2)}. \tag{4.17}$$

If the plate is symmetric, then $\kappa_2 = 0$ and $\widehat{\mu} = 0$, and (4.17) reduce to

$$\alpha = -\frac{P}{8\pi h\overline{\mu}}\left\{1 + \frac{1}{\kappa_1}\right\}, \quad \beta = \frac{P}{16\pi h\overline{\mu}}\left\{1 - \frac{1}{\kappa_1}\right\}, \quad \gamma = 0. \tag{4.18}$$

If the plate is homogeneous, so that λ and μ are constants, then it is easily shown that (4.18) reduce further to (3.9).

The stress and displacement due to distributed in-plane forces can be constructed by superposition of solutions for concentrated forces.

5. Higher order singularities

Stress and displacement singularities of higher order can be derived by differentiation, just as for homogeneous materials, and as described by, for example, Timoshenko and Goodier [10, Chap. 4]. For this purpose it is more convenient to express the displacement and stress in terms of Cartesian coordinates, so in this section vector and tensor quantities are referred to the coordinate system $Oxyz$. In this system the displacement (4.4) is

$$u_x = \alpha\ln r - \gamma z(\frac{1}{2} + \ln r) + \left\{\beta - \frac{1}{2}\gamma z - F\frac{(\alpha + 2\beta)}{r^2} - B\frac{2\gamma}{r^2}\right\}\frac{x^2 - y^2}{r^2},$$
$$u_y = 2\left\{\beta - \frac{1}{2}\gamma z - F\frac{(\alpha + 2\beta)}{r^2} - B\frac{2\gamma}{r^2}\right\}\frac{xy}{r^2},$$
$$u_z = w = \left\{\gamma\ln r + G\frac{\alpha + 2\beta}{r^2} + C\frac{2\gamma}{r^2}\right\}x, \tag{5.1}$$

and the stress (4.6) is

$$\sigma_{xx} = \frac{2\mu}{\lambda + 2\mu}\{2\alpha(\lambda + \mu) + 2\beta\lambda - \gamma(3\lambda + 2\mu)z\}\frac{x}{r^2} + 8\mu(\beta - \frac{1}{2}\gamma z)\frac{xy^2}{r^4}$$
$$+ 4\mu\{F(\alpha + 2\beta) + 2B\gamma\}\frac{x(x^2 - 3y^2)}{r^6},$$
$$\sigma_{yy} = \frac{2\lambda\mu}{\lambda + 2\mu}\{\alpha + 2\beta)\lambda - 2\gamma z\}\frac{x}{r^2} + 4\mu(\beta - \frac{1}{2}\gamma z)\frac{x(x^2 - y^2)}{r^4}$$
$$- 4\mu\{F(\alpha + 2\beta) + 2B\gamma\}\frac{x(x^2 - 3y^2)}{r^6},$$

$$\sigma_{xy} = 2\mu(\beta - \frac{1}{2}\gamma z)\frac{y(y^2 - 3x^2)}{r^4} + 4\mu\{F(\alpha + 2\beta) + 2B\gamma\}\frac{y(3y^2 - x^2)}{r^6},$$
$$\sigma_{xz} = \mu\left\{-\left(\frac{dF}{dz} + G\right)(\alpha + 2\beta) - \left(\frac{dB}{dz} + C\right)2\gamma\right\}\frac{x^2 - y^2}{r^4},$$
$$\sigma_{yz} = \mu\left\{-\left(\frac{dF}{dz} + G\right)(\alpha + 2\beta) - \left(\frac{dB}{dz} + C\right)2\gamma\right\}\frac{2xy}{r^4},$$
$$\sigma_{zz} = 0. \tag{5.2}$$

Consider for example a pair of equal opposite concentrated forces of magnitude P situated at $x = \pm\delta$, $y = 0$ and directed in the positive and negative x directions respectively. Denote the displacement (5.1) by $\mathbf{u}(x, y, z)$and the stress (4.6) by $\sigma(x, y, z)$. In the terminology of Love [9, Chap.9] this comprises a 'force pair without moment'. Then the displacement $\mathbf{u}_1$ and stress σ_1 due to the pair of forces are given by superposition as

$$\mathbf{u}_1 = \mathbf{u}(x + \delta,\ y,\ z) - \mathbf{u}(x - \delta,\ y,\ z), \quad \sigma_1 = \sigma(x + \delta,\ y,\ z) - \sigma(x - \delta,\ y,\ z). \tag{5.3}$$

Hence in the limit $\delta \to 0$,

$$\mathbf{u}_1 = 2\delta\frac{\partial \mathbf{u}}{\partial x}, \quad \sigma_1 = 2\delta\frac{\partial \sigma}{\partial x}, \tag{5.4}$$

and it is assumed that $P \to \infty$, $\delta \to 0$ in such a way that $Q = 2P\delta$ is finite. It follows from (5.1) that the components of $\mathbf{u}_1$ are

$$u_x^{(1)} = (\overline{\alpha} - \overline{\gamma}z)\frac{x}{r^2} + 4(\overline{\beta} - \frac{1}{2}\overline{\gamma}z)\frac{xy^2}{r^4} + 2\{F(\overline{\alpha} + 2\overline{\beta}) + 2B\overline{\gamma}\}\frac{x(x^2 - 3y^2)}{r^6},$$
$$u_y^{(1)} = -2(\overline{\beta} - \frac{1}{2}\overline{\gamma}z)\frac{y(x^2 - y^2)}{r^4} - 2\{F(\overline{\alpha} + 2\overline{\beta}) + 2B\overline{\gamma}\}\frac{y(x^2 - 3y^2)}{r^6},$$
$$u_z^{(1)} = \overline{\gamma}\left\{\ln r + \frac{x^2}{r^2}\right\} - \{G(\overline{\alpha} + 2\overline{\beta}) + 2B\overline{\gamma}\}\frac{(x^2 - y^2)}{r^4}, \tag{5.5}$$

where

$$\overline{\alpha} = -\frac{Q}{8\pi h\overline{\mu}}\left\{1 + \frac{\kappa_3\overline{\mu} + \kappa_2\widehat{\mu}}{\overline{\mu}(\kappa_1\kappa_3 - \kappa_2^2)}\right\},$$
$$\overline{\beta} = \frac{Q}{16\pi h\overline{\mu}}\left\{1 - \frac{\kappa_3\overline{\mu} + \kappa_2\widehat{\mu}}{\overline{\mu}(\kappa_1\kappa_3 - \kappa_2^2)}\right\},$$
$$\overline{\gamma} = \frac{Q(\kappa_2\overline{\mu} + \kappa_1\widehat{\mu})}{8\pi h\overline{\mu}^2((\kappa_1\kappa_3 - \kappa_2^2)}. \tag{5.6}$$

The stress can be obtained by differentiation in a similar way.

By superposing two such force pairs, with lines of action along the x and y axes respectively. there results the displacement corresponding to a centre of dilatation, namely

$$u_x = \left\{ \{(\overline{\alpha} - \overline{\gamma} z) + 2(\overline{\beta} - \frac{1}{2}\overline{\gamma} z)\} - 4\{F(\overline{\alpha} + 2\overline{\beta}) + 2B\overline{\gamma}\}\frac{1}{r^2} \right\} \frac{x}{r^2},$$
$$u_y = \left\{ \{(\overline{\alpha} - \overline{\gamma} z) + 2(\overline{\beta} - \frac{1}{2}\overline{\gamma} z)\} - 4\{F(\overline{\alpha} + 2\overline{\beta}) + 2B\overline{\gamma}\}\frac{1}{r^2} \right\} \frac{y}{r^2},$$
$$u_z = \overline{\gamma}\{\ln r + 1\}. \tag{5.7}$$

This result can also be obtained by considering solutions of (2.6) in which $\overline{u}, \overline{v}$, and $\overline{w}$ depend only on r.

In a similar way, a 'force pair with moment' is formed by two concentrated forces, of equal magnitude P, with the first acting in the positive x-direction at $(0,\delta)$,and the second in the negative x-direction at $(0, -\delta)$. Hence the pair of forces exerts a moment of magnitude $2\delta P = Q$ about the z -axis. In this case the displacement $\mathbf{u}_2$ and the stress σ_2 are given by

$$\mathbf{u}_2 = 2\delta\frac{\partial \mathbf{u}}{\partial y}, \quad \sigma_2 = 2\delta\frac{\partial \sigma}{\partial y}. \tag{5.8}$$

This gives the displacement

$$u_x^{(2)} = (\overline{\alpha} - \overline{\gamma} z)\frac{y}{r^2} - 4(\overline{\beta} - \frac{1}{2}\overline{\gamma} z)\frac{x^2 y}{r^4} + 2\{F(\overline{\alpha} + 2\overline{\beta}) + 2B\overline{\gamma}\}\frac{y(3x^2 - y^2)}{r^6},$$
$$u_y^{(2)} = -2(\overline{\beta} - \frac{1}{2}\overline{\gamma} z)\frac{x(x^2 - y^2)}{r^4} - 2\{F(\overline{\alpha} + 2\overline{\beta}) + 2B\overline{\gamma}\}\frac{x(x^2 - 3y^2)}{r^6},$$
$$u_z^{(2)} = \left\{ \overline{\gamma} - 2\{G(\overline{\alpha} + 2\overline{\beta}) + 2C\overline{\gamma}\}\frac{1}{r^2} \right\} \frac{xy}{r^2}. \tag{5.9}$$

By superposing two such force pairs oriented along the x and y axes there follows the solution for a centre of rotation about the z -axis, which is

$$u_x = -(\overline{\alpha} + 2\overline{\beta})\frac{y}{r^2},$$
$$u_y = (\overline{\alpha} + 2\overline{\beta})\frac{x}{r^2},$$
$$u_z = 0. \tag{5.10}$$

This solution also may be derived by considering solutions of (2.6) in which $\overline{u}, \overline{v}$, and $\overline{w}$ depend only on r.

Higher order singularities may be described by higher order derivatives of $\mathbf{u}$ and σ with respect to x and y. For example, the displacement and stress that are generated by

$$\frac{1}{4\delta^2}\frac{\partial^2 \mathbf{u}}{\partial x \partial y} \quad \text{and} \quad \frac{1}{4\delta^2}\frac{\partial^2 \sigma}{\partial x \partial y}$$

correspond to a system of four concentrated forces of equal magnitudes P, of which two act in the positive x-direction at (δ, δ) and $(-\delta, -\delta)$, and two in the negative x-direction at $(\delta, -\delta)$ and $(-\delta, \delta)$, in the limit $P \to \infty$, $\delta \to 0$, with $P\delta^2$ remaining finite.

6. Concentrated force on a half-plane

The displacement (3.1) is not the only one that gives rise to point force solutions in a thin plate. For simplicity, in this section only symmetric plates are considered, and attention is restricted to stretching deformation modes, so that $\overline{w} = 0$. Then a relevant solution of (2.5) and (2.6) is

$$\begin{aligned} \overline{u}(r,\theta) &= (\alpha \ln r + \beta)\cos\theta + \varepsilon\theta\sin\theta, \\ \overline{v}(r,\theta) &= (-\alpha \ln r + \beta)\sin\theta + \varepsilon\theta\cos\theta, \end{aligned} \tag{6.1}$$

and then

$$\Delta = (\alpha + 2\beta + \varepsilon)\frac{\cos\theta}{r}, \quad \Omega = -(\alpha - 2\beta + \varepsilon)\frac{\sin\theta}{r} \tag{6.2}$$

and (2.6) are satisfied provided that

$$(\kappa_1 - 1)(\alpha + \varepsilon) + (\kappa_1 + 1)2\beta = 0 \tag{6.3}$$

When $\varepsilon \neq 0$, the displacement (6.1) is not single-valued, and so cannot represent the displacement in a whole uncut plane. However it can be applied in a half-plane or wedge, and with appropriate choice of the coefficients, yields the solution for the application of a point force to a half-space.

For an inhomogeneous (but symmetric) plate, (6.1) gives rise, through (2.10), to the three-dimensional displacement

$$\begin{aligned} u(r,\theta,z) &= (\alpha \ln r + \beta)\cos\theta + \varepsilon\theta\sin\theta - F(z)(\alpha + 2\beta + \varepsilon)\frac{\cos\theta}{r^2}, \\ v(r,\theta,z) &= (-\alpha \ln r + \beta)\sin\theta + \varepsilon\theta\cos\theta - F(z)(\alpha + 2\beta + \varepsilon)\frac{\sin\theta}{r^2}, \\ w(r,\theta,z) &= G(z)(\alpha + 2\beta + \varepsilon)\frac{\cos\theta}{r}, \end{aligned} \tag{6.4}$$

and the stress

$$\begin{aligned} \sigma_{rr} &= \left\{ \left(\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\alpha + \frac{2\lambda\mu}{\lambda+2\mu}(2\beta+\varepsilon) \right)\frac{1}{r} + 4F\mu(\alpha+2\beta+\varepsilon)\frac{1}{r^3} \right\}\cos\theta, \\ \sigma_{\theta\theta} &= \left\{ \left(\frac{2\lambda\mu}{\lambda+2\mu}\alpha + \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}(2\beta+\varepsilon) \right)\frac{1}{r} - 4F\mu(\alpha+2\beta+\varepsilon)\frac{1}{r^3} \right\}\cos\theta, \end{aligned}$$

$$\sigma_{r\theta} = \mu\left\{-(\alpha+2\beta-\varepsilon)\frac{1}{r} + 4F\mu(\alpha+2\beta+\varepsilon)\frac{1}{r^3}\right\}\sin\theta,$$
$$\sigma_{rz} = -\mu\left(\frac{dF}{dz}+G\right)\frac{(\alpha+2\beta+\varepsilon)}{r^2}\cos\theta,$$
$$\sigma_{\theta z} = -\mu\left(\frac{dF}{dz}+G\right)\frac{(\alpha+2\beta+\varepsilon)}{r^2}\sin\theta,$$
$$\sigma_{zz} = 0. \tag{6.5}$$

We consider a semi-infinite plate occupying the region $x \geq 0$. It is not possible to satisfy pointwise boundary conditions on the surface $x = 0$, but it is possible to impose conditions on stress resultants. Hence we introduce the resultants

$$(N_{rr}, N_{\theta\theta}, N_{r\theta}) = \int_{-h}^{h}(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta})dz \tag{6.6}$$

It follows from (6.4), with (2.7) and (2.8), that

$$N_{rr} = 2h\overline{\mu}\left\{\{\kappa_1\alpha + (\kappa_1-2)(2\beta+\varepsilon)\}\frac{1}{r} + 4\overline{F}(\alpha+2\beta+\varepsilon)\frac{1}{r^3}\right\}\cos\theta,$$
$$N_{\theta\theta} = 2h\overline{\mu}\left\{\{(\kappa_1-2)\alpha + \kappa_1(2\beta+\varepsilon)\}\frac{1}{r} - 4\overline{F}(\alpha+2\beta+\varepsilon)\frac{1}{r^3}\right\}\cos\theta,$$
$$N_{r\theta} = 2h\overline{\mu}\left\{-\{\alpha+2\beta-\varepsilon\}\frac{1}{r} + 4\overline{F}(\alpha+2\beta+\varepsilon)\frac{1}{r^3}\right\}\sin\theta, \tag{6.7}$$

where

$$\overline{F} = \frac{1}{2h\overline{\mu}}\int_{-h}^{h}F(z)\mu(z)dz, \tag{6.8}$$

and therefore, from (2.19)

$$\overline{F} = \frac{1}{2h\overline{\mu}}\left[\int_{-h}^{h}\int_{0}^{z}\int_{-h}^{\zeta}\frac{\mu(z)\mu(\zeta)}{\mu(\xi)}\left\{\kappa_1 - 4\frac{\{\lambda(\zeta)+\mu(\zeta)\}}{\lambda(\zeta)+2\mu(\zeta)}\right\}d\xi d\zeta dz + \frac{1}{2}\int_{-h}^{h}\frac{(h-z)^2\lambda(z)}{\lambda(z)-2\mu(z)}dz\right] \tag{6.9}$$

In a homogeneous material, in plane stress or plane strain, there exists a 'radial stress' solution with $\sigma_{r\theta} = 0$ and $\sigma_{\theta\theta} = 0$. Such a solution does not exist for an inhomogeneous material, but there is a solution with $N_{r\theta} = 0$ and $N_{\theta\theta} = 0$. For this it is required firstly that

$$(\kappa_1-2)\alpha + \kappa_1(2\beta+\varepsilon) = 0,$$
$$\alpha + 2\beta - \varepsilon = 0 \tag{6.10}$$

It is interesting that these are compatible with (6.3). Hence (6.3) and (6.10) are satisfied if

$$\alpha = \kappa_1 \varepsilon, \quad 2\beta = -(\kappa_1 - 1)\varepsilon \tag{6.11}$$

and with this choice

$$\begin{aligned} N_{rr} &= 8h\overline{\mu}\varepsilon \left\{ (\kappa_1 - 1)\frac{1}{r} + 2\overline{F}\frac{1}{r^3} \right\} \cos\theta, \\ N_{\theta\theta} &= -16h\varepsilon\overline{\mu}\overline{F}\frac{1}{r^3}\cos\theta, \\ N_{r\theta} &= 16h\varepsilon\overline{\mu}\overline{F}\frac{1}{r^3}\sin\theta. \end{aligned} \tag{6.12}$$

To eliminate the remainder of $N_{\theta\theta}$ and $N_{r\theta}$ it is necessary to superpose an additional solution of (2.6). The required solution is

$$\overline{u}(r,\theta) = \frac{2\varepsilon\overline{F}\cos\theta}{r^2}, \quad \overline{v}(r,\theta) = \frac{2\varepsilon\overline{F}\sin\theta}{r^2}, \tag{6.13}$$

which give $\Delta = 0$ and $\Omega = 0$, so that (2.6) are trivially satisfied. Therefore, corresponding to (6.13),

$$\overline{u}(r,\theta,z) = \frac{2\varepsilon\overline{F}\cos\theta}{r^2}, \quad \overline{v}(r,\theta,z) = \frac{2\varepsilon\overline{F}\sin\theta}{r^2},$$

the associated non-zero stress components are

$$\sigma_{rr} = -\frac{8\mu\varepsilon\overline{F}\cos\theta}{r^3}, \quad \sigma_{\theta\theta} = \frac{8\mu\varepsilon\overline{F}\cos\theta}{r^3}, \quad \sigma_{r\theta} = -\frac{8\mu\varepsilon\overline{F}\cos\theta}{r^3},$$

and the stress resultants are

$$N_{rr} = -\frac{16h\overline{\mu}\varepsilon\overline{F}\cos\theta}{r^3}, \quad N_{\theta\theta} = \frac{16h\overline{\mu}\varepsilon\overline{F}\cos\theta}{r^3}, \quad N_{r\theta} = -\frac{16h\overline{\mu}\varepsilon\overline{F}\cos\theta}{r^3}. \tag{6.14}$$

Then superimposing (6.12) and (6.14) gives the radial solution for the stress resultants

$$N_{rr} = 8h\overline{\mu}\varepsilon(\kappa_1 - 1)\frac{1}{r}\cos\theta, \quad N_{\theta\theta} = 0, \quad N_{r\theta} = 0.$$

The corresponding resultant concentrated force acting in the x direction on the surface of the semi-infinite plate is

$$P = \int_{-\pi/2}^{\pi/2} N_{rr}\cos\theta\, d\theta = 4\pi h\overline{\mu}\varepsilon(\kappa_1 - 1) \tag{6.15}$$

which determines ε if P is specified. The case of a tangential concentrated force on the surface of a semi-infinite plate can be solved in a similar manner, and hence, by superposition, the solution for any concentrated load can be obtained. From this, again by superposition, there follows the solution for any distributed surface force. However, in these solutions it is only possible to specify resultant forces on the surface. To satisfy pointwise boundary conditions additional boundary layer solutions must be applied.

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