Z. Angew. Math. Phys. (2024) 75:185

https://doi.org/10.1007/s00033-024-02325-5

Zeitschrift für angewandte Mathematik und Physik ZAMP

On a quasilinear two-species chemotaxis system with general kinetic functions and interspecific competition

Yifeng Huili

Abstract. In this paper, we study the following two-species chemotaxis system with generalized volume-filling effect and general kinetic functions

$$\begin{cases} u_t = \nabla \cdot (D_1(u)\nabla u) - \nabla \cdot (\chi_1(u)\nabla w) + f_1(u) - \mu_1 a_1 uv, & (x,t) \in \Omega \times (0,\infty), \\ v_t = \nabla \cdot (D_2(v)\nabla v) - \nabla \cdot (\chi_2(v)\nabla w) + f_2(v) - \mu_2 a_2 uv, & (x,t) \in \Omega \times (0,\infty), \\ \tau w_t = \Delta w - w + g_1(u) + g_2(v), & (x,t) \in \Omega \times (0,\infty), \end{cases}$$

under homogeneous Neumann boundary conditions in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$, where a_1, a_2, μ_1, μ_2 are positive constants. When the functions D_i, S_i, f_i, g_i $(i = 1, 2)$ belong to C^2 fulfilling some suitable hypotheses, we study the global existence and boundedness of classical solutions for the above system and find that under the case of $\tau = 1$ or $\tau = 0$, either the higher-order nonlinear diffusion or strong logistic damping can prevent blow-up of classical solutions for the problem. In addition, when the functions are replaced to Lotka–Volterra competitive kinetic functional response term and linear signal generations, by constructing some appropriate Lyapunov functionals, we show that the solution convergences to the constant steady state in $L^\infty(\Omega)$ in the case of $a_1, a_2 \in (0, 1)$ or $a_1 \geq 1 > a_2 > 0$ under some more concise conditions than [2], which improved the existing conditions to some extent.

Mathematics Subject Classification. 35K55, 35B35, 35B40, 92C17, 35A01.

Keywords. Global existence, Asymptotic behavior, Two-species chemotaxis model, Lotka–Volterra dynamics.

1. Introduction and main results

Chemotaxis is one of the most important components in the process of reproduction and migration, which describes the partial movement of biological species or cells to the gradient of chemotactic substances. The classic chemotactic model was proposed by Keller and Segel [10] as the following

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u - S(u)\nabla v) + f(u), & x \in \Omega, \quad t > 0, \\ v_t = \Delta v - v + g(u), & x \in \Omega, \quad t > 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$. In the system (1.1), $u(x,t)$ and $v(x,t)$ represent the density of the population and the concentration of the chemical substance at space x and time t, respectively. $D(u)$ and $S(u)$ are the density-dependent diffusion function and the density-dependent sensitivity function, respectively. The function $f(u)$ is the logistic source and $g(u)$ is the production or consumption of chemical substances. When $f(u) = g(u) = 0$, the asymptotic property of $\frac{S(u)}{D(u)} \simeq u^{\frac{2}{n}}$ is the critical condition for blow-up and global boundedness (see [27,40]). Further studies even have provided further information in which the respective blow-up phenomenon either can occur within finite time (see [4–6]), or only arises in the sense of an infinite-time grow-up (see [4,6,41]). Moreover, some rigorous results of (1.1) are shown that the sub-logistic source $f(u)$ can prevent the blow-up of solutions (see [39,42]). In addition, more

Published online: 17 September 2024

fruitful results have been obtained for the classical chemotaxis model (1.1) and its variants forms, we refer [3,8,11,22,28,37] for further reading.

One of the well-known variants form of (1.1) is the two-species and one-stimuli type, that is, two species respond to the same chemical signaling substance produced by themselves. To describe the movement of two species, the following chemotaxis system

$$\begin{cases} u_t = \nabla\cdot(D_1(u)\nabla u - S_1(u)\nabla w) + f_1(u,v), & x\in\Omega, \quad t>0,\\ v_t = \nabla\cdot(D_2(v)\nabla v - S_2(v)\nabla w) + f_2(u,v), & x\in\Omega, \quad t>0,\\ \tau w_t = \Delta w - w + g(u,v), & x\in\Omega, \quad t>0, \end{cases} \tag{1.2}$$

was proposed by Tello and Winkler in [29], where $\tau\in\{0,1\}$. In the system (1.2), $u(x,t)$ and $v(x,t)$ represent the density of different species respectively, and $w(x,t)$ denotes the concentrations of chemical substances. The functions $f_i (i=1,2)$ contain the logistic source and interaction between species, and $g(u,v)$ is the production or consumption of chemical substances. So far, most of the conclusions are focused on the Lotka–Volterra case that $f_1(u,v)=\mu_1 u(1-u-a_1 v)$, $f_2(u,v)=\mu_2 v(1-v-a_2 u)$ and $g(u,v)=u+v$. Specifically, when the linear case $D_i(s)=1$ and $S_i(s)=s$ for $i=1,2$, in the case of $\tau=0$, Tello and Winkler obtained the global existence and asymptotic behavior of solutions when $a_1,a_2\in[0,1)$ (see [29]). When $a_1>1>a_2\geq 0$, Stinner et al. proved that the semi-trivial steady state is asymptotically stable (see [25]). In case $\tau=1$, Bai and Winkler derived the global existence of classical solutions when $n\leq 2$ and asymptotical behavior when the damping terms are suitably strong (i.e., μ_1 and μ_2 are large enough) in [2]; the bounded and asymptotic results are optimized in [16]. Moreover, in the quasilinear case, when $f_i=0$, it has been proved in [31] that if $D_i(s)$ and $S_i(s)$ satisfy $K_{0i}(s+1)^{l_i-1}\leq D_i(s)\leq K_{1i}(s+1)^{L_i-1}$ and $\frac{S_i(s)}{D_i(s)}\leq K_i(s+1)^{\alpha_i}, (i=1,2)$, then the solutions are globally bounded under the conditions that $0<\alpha_i<\frac{2}{n}$; and the finite-time blow-up of solution was also obtained. When the functions $D_i, S_i, f_i (i=1,2)$ and g satisfy some conditions, Pan and Wang proved that this system possesses a global bounded smooth solution under some specific conditions with or without the logistic functions $f_i(s)$ and further obtain the asymptotic stability for the solutions of system (1.2) (see [21]). More related interesting results can be found in [17,18,20,46]. Furthermore, for results of the system with two-species and two signaling substances, we can refer to [32,43,44].

Recently, inspired by the work [42], Li found several explicit conditions involving the kinetic functions f,g, the parameters χ,λ and the initial mass $\|u_0\|_{L^1(\Omega)}$ to ensure the global-in-time existence and uniform boundedness for a chemotaxis model with indirect signal production and general kinetic function (see [12]). Subsequently, Shan and Zheng applied this idea to a chemotactic model describing the immune system, and obtained some global boundedness results (see [24,47]).

Motivated by the above works, this paper is concerned with the following two-species chemotaxis system with generalized volume-filling effect and general kinetic functions

$$\begin{cases} u_t = \nabla\cdot(D_1(u)\nabla u) - \nabla\cdot(\chi_1(u)\nabla w) + f_1(u) - \mu_1 a_1 uv, & (x,t)\in\Omega\times(0,\infty),\\ v_t = \nabla\cdot(D_2(v)\nabla v) - \nabla\cdot(\chi_2(v)\nabla w) + f_2(v) - \mu_2 a_2 uv, & (x,t)\in\Omega\times(0,\infty),\\ \tau w_t = \Delta w - w + g_1(u) + g_2(v), & (x,t)\in\Omega\times(0,\infty),\\ \dfrac{\partial u}{\partial\nu} = \dfrac{\partial v}{\partial\nu} = \dfrac{\partial w}{\partial\nu} = 0, & (x,t)\in\partial\Omega\times(0,\infty),\\ u(x,0)=u_0(x), v(x,0)=v_0(x), \tau w(x,0)=w_0(x), & x\in\Omega, \end{cases} \tag{1.3}$$

where $\tau\in\{0,1\}$, $\Omega\subset\mathbb{R}^n$ $(n\geq 1)$ is a smoothly bounded domain and u, v, w have the same meanings as in (1.2). The parameters $\mu_i, a_i>0 (i=1,2)$ and nonnegative initial data satisfies

$$u_0\in C^0(\bar\Omega), v_0\in C^0(\bar\Omega), \tau w_0\in C^1(\bar\Omega). \tag{1.4}$$

Firstly, we consider the global boundedness of solutions for (1.3) under the following hypotheses with $i = 1, 2$,

(S_1) $D_i(s) \geq (s+1)^{-\alpha_i}$ with $\alpha_i \in \mathbb{R}$;

(S_2) $\chi_i(s) \leq C_{\chi_i} s(s+1)^{\beta_i - 1}$ with $C_{\chi_i} \geq 0, \beta_i \in \mathbb{R}$;

(S_3) $f_i(0) \geq 0, \chi_i(0) = g_i(0) = 0$ and $g_i(s), \chi_i(s) > 0$ for all $s > 0$;

(S_4) $g_i(s) \leq s^{\gamma_i}$ with $\gamma_i > 0$;

(S_5) D_i, χ_i, f_i, g_i belong to C^2.

Based on $(S_1) - (S_5)$, our main results of global existence and boundedness are stated as follows.

Theorem 1.1. *Let $\tau = 1$, $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary. Assume $(S_1) - (S_5)$ hold. For $i = 1, 2$, if one of the following conditions hold:*

(i) $f_i \equiv 0$ *and* $\alpha_i < \min\left\{\frac{2}{n}, 1 + \frac{2}{n} - \gamma_i - \beta_i\right\}$;

(ii) $\exists \iota_i^\star = \iota_i^\star(\beta_1, \beta_2, \gamma_1, \gamma_2, C_{\chi_1}, C_{\chi_2}) > 0$ *such that* $\lim_{s\to\infty} \inf \left\{-\frac{f_i(s)}{s^{\max\{1, \beta_i + \gamma_i\}}}\right\} =: \iota_i \in (\iota_i^\star, \infty]$,

then system (1.3) possesses a classical solution (u, v, w), which is uniformly bounded in time.

Remark 1.1. Our results extend the results in [7,9,13,26,47] from one species to two species. Specifically, when system (1.3) meets linear diffusion and chemosensitivity (i.e., $\alpha_i = 0$ and $\beta_i = 1$ for $i = 1, 2$), the condition (i) of Theorem 1.1 indicates that suitable sublinear signal production ($\gamma_i < \frac{2}{n}$ with $n \geq 2$ and $i = 1, 2$) can ensure the uniform boundedness, which is similar to the results for simpler (single-species) Keller-Segel system (see Theorem 1.1 in [13]). When system (1.3) meets linear diffusion and signal production (i.e., $\alpha_i = 0$ and $\gamma_i = 1$ for $i = 1, 2$), the condition (i) indicates that suitable sublinear chemoattractant ($\beta_i < \frac{2}{n}$ with $n \geq 2$ and $i = 1, 2$) can ensure the uniform boundedness (see Theorem 4.1 in [9]). Furthermore, when $f_i(s) = S_i - \mu_i s^{k_i}$ with $S_i, \mu_i, k_i > 0$ for $i = 1, 2$, the condition (ii) of Theorem 1.1 indicates that if $\beta_i + \gamma_i < k_i$ or $\beta_i + \gamma_i = k_i$ with μ_i large enough for $i = 1, 2$, then problem (1.3) possesses a unique global classical solution that is bounded in $\Omega \times (0, \infty)$ (see Theorem 1.2 in [26] or Theorem 1.1 in [7]). Moreover, when system (1.3) meets linear chemosensitivity and signal production, the condition (ii) implies that the superquadratic degradation mechanisms or quadratic degradation mechanisms with sufficiently large damping coefficients of species can ensure the uniform boundedness of the solution, while the subquadratic case is still a problem that needs to be developed.

In addition, we get rid of the constraint of $\gamma_1 = \gamma_2 \leq 1$ relative to [21], which relies on the Sobolev regularity estimate (Lemma 3.1) and condition $\alpha_i < \frac{2}{n}$ for $i = 1, 2$.

Theorem 1.2. *Let $\tau = 0$, $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary, $\gamma = \max\{\gamma_1, \gamma_2\}$. Assume $(S_1) - (S_5)$ hold. For $i = 1, 2$, if one of the following conditions hold:*

(i) $f_i \equiv 0$ *and* $\alpha_i < 1 + \frac{2}{n} - \gamma - \beta_i$;

(ii) $\exists \tilde{\iota_i} = \tilde{\iota_i}(\beta_1, \beta_2, \gamma_1, \gamma_2, C_{\chi_1}, C_{\chi_2}) > 0$ *such that* $\lim_{s\to\infty} \inf \left\{-\frac{f_i(s)}{s^{\beta_i + \gamma}}\right\} =: \iota_i \in (\tilde{\iota_i}, \infty]$,

then system (1.3) possesses a classical solution (u, v, w), which is uniformly bounded in time.

Remark 1.2. Compared with Theorem 1.1, the conditions of Theorem 1.2 have been relatively simplified, thanks to the elliptic properties of the third equation in (1.3). Moreover, Our results extend the results from one species to two species. Specifically, when system (1.3) meets linear diffusion and chemosensitivity, the condition (i) of Theorem 1.2 indicates that suitable sublinear signal production ($\gamma < \frac{2}{n}$ with $n \geq 2$ and $i = 1, 2$) can ensure the uniform boundedness, which is similar to the results for single-species Keller-Segel system (see Proposition 1.3 in [36] and Theorem 1.1 in [33]). When $f_i(s) = S_i - \mu_i s^{k_i}$ with $S_i, \mu_i, k_i > 0$ for $i = 1, 2$, the condition (ii) of Theorem 1.2 indicates that when $\beta_i + \gamma_i < k_i$ or $\beta_i + \gamma_i = k_i$ with μ_i large enough for $i = 1, 2$, problem (1.3) possesses a uniformly bounded solution (see [35,45]). In addition, our results highlight the independence of β_i and α_i for $i = 1, 2$ respectively, compared with [23].

After the globally bounded solutions obtained, we next consider the large time behavior of solutions to the system (1.3). First, we taking into account the effects of Lotka–Volterra competitive kinetic functional response term and linear signal generations, i.e., f_i, g_i $(i = 1, 2)$ satisfy

$$(S_6)\ f_i(s) = \mu_i s(1 - s) \text{ and } g_i(s) = s \text{ for all } s > 0.$$

Then, it is easy to obtain that the system (1.3) has four possible constant steady states $(u_\star, v_\star, w_\star)$:

$$\begin{cases} P_1 \text{ or } P_2 \text{ or } P_3 \text{ or } P_*, & \text{if } a_1, a_2 < 1 \text{ or } a_1, a_2 > 1, \\ P_1 \text{ or } P_2 \text{ or } P_3, & \text{other cases,} \end{cases}$$

where P_1 is the extinction state, P_2 and P_3 are two semi trivial steady states, and P_* is the coexistence steady states with

$$\begin{cases} P_1 := (0, 0, 0), \quad P_2 := (1, 0, 1), \quad P_3 := (0, 1, 1) \\ P_\star = (u_\star, v_\star, w_\star) := \left(\frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2}, \frac{2-a_1-a_2}{1-a_1a_2}\right). \end{cases}$$

In view of the existing works [2,14–16], the weakly competitive case $(a_1, a_2 \in [0, 1))$ and the strongly-weakly competitive case $(a_1 \geq 1 > a_2 \geq 0)$ have been concerned. Moreover, the case of strong competition $(a_1, a_2 \geq 1)$ have been partly given in [20]. Along with the previous work, we further consider weakly competitive and the strongly-weakly competitive cases, and give a more concise result to ensure the asymptotic stability of the solutions in the case of $\tau = 1$ or $\tau = 0$.

Moreover, it follows from Theorems 1.1 and 1.2 that we can find positive constants k_1 and k_2 satisfying

$$k_1 = \max_{0 \leq u \leq \|u\|_{L^\infty(\Omega)}} (u + 1)^{2\beta_1+\alpha_1-2}, k_2 = \max_{0 \leq v \leq \|v\|_{L^\infty(\Omega)}} (v + 1)^{2\beta_2+\alpha_2-2}. \tag{1.5}$$

Then, we can get the following results about large time behavior of solutions for (1.3).

Theorem 1.3. (Strongly-weakly competitive case) *Let $\tau = \{0, 1\}$ and the assumption (S_6) hold. Assume that $a_1 \geq 1 > a_2 > 0$. If*

$$\mu_2 > \frac{C_{\chi_2}^2}{8} k_2, \tag{1.6}$$

then the unique global bounded solution (u, v, w) of (1.3) obtained by Theorem 1.1 or 1.2 satisfies

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t) - 1\|_{L^\infty(\Omega)} + \|w(\cdot, t) - 1\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty.$$

Theorem 1.4. (Weakly competitive case) *Let $\tau = \{0, 1\}$ and the assumption (S_6) hold. Assume that $0 < a_1, a_2 < 1$. If*

$$\mu_2 > \frac{C_{\chi_1}^2 u_\star \mu_2 (2 - a_2)}{8a_1\mu_1} k_1 + \frac{C_{\chi_2}^2 v_\star}{8} k_2, \tag{1.7}$$

then the unique global bounded solution (u, v, w) of (1.3) obtained by Theorem 1.1 or 1.2 satisfies

$$\|u(\cdot, t) - u_\star\|_{L^\infty(\Omega)} + \|v(\cdot, t) - v_\star\|_{L^\infty(\Omega)} + \|w(\cdot, t) - w_\star\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty.$$

Remark 1.3. Our results in Theorems 1.3 and 1.4 are more concise than the existing results in [2,14–16], and the condition of Theorem 1.3 is free from the influence of parameters a_1 and a_2. Specifically, when $a_1 \geq 1 > a_2 > 0$, the conditions in [2] are

$$\mu_2 > \frac{C_{\chi_2}^2 v_\star (a_1' + a_2 - 2a_1' a_2)}{16a_2 (1 - a_1' a_2)}. \tag{1.8}$$

for some $a_1' \in (1, a_1]$ such that $a_1' a_2 < 1$. It is not hard to get, $\frac{(a_1'+a_2-2a_1'a_2)}{2a_2(1-a_1'a_2)} > 1$ when $a_2 > \frac{1}{2}$ for all $a_1' > 1$. Therefore, Our results in Theorem 1.3 partly improved the existing work in [2]. For the condition (1.7) in Theorem 1.4, it can be rewritten to $\mu_2 > \frac{C_{\chi_2}^2 v_\star}{8d_\star} k_2$, where $d_\star := 1 - \frac{C_{\chi_1}^2 u_\star(2-a_2)}{8a_1\mu_1} k_1 > 0$ when $\mu_1 > \frac{C_{\chi_1}^2 u_\star(2-a_2)}{8a_1} k_1$, which implies that if μ_1 is sufficiently large, then the condition of μ_2 can be relaxed accordingly.

The rest of the article is organized as follows. In Sect. 2, we give some preliminary lemmas and the local existence of solution for system (1.3). In Sects. 3 and 4, we study the global existence and boundedness of solutions for system (1.3), and prove Theorems 1.1 and 1.2. In Sect. 5, we study the asymptotic behavior of global solutions for system (1.3), and prove Theorems 1.3 and 1.4. In the following content, we let $u(\cdot,t) = u(x,t)$ and shall use $K_i, C_i (i = 1, 2, \dots)$ to denote a generic positive constant which may vary in the context. Without confusion, the integration sign dx and $dxdt$ will be omitted.

2. Local existence and preliminaries

The following local existence for solutions of the system (1.3) can be obtained by adapting established techniques. The details can be found in [1,2,9,25,46] and can therefore be omitted here.

Lemma 2.1. *Let $\tau \in \{0,1\}$ and $\Omega \subset \mathbb{R}^n (n \ge 1)$ be a smoothly bounded domain. Assume that the functions D_i, χ_i, f_i and g_i $(i = 1, 2)$ satisfy (S_3) and (S_5), as well as the nonnegative initial data satisfies (1.4. Then there exist $T_{\max} \in (0, \infty]$ and uniquely determined nonnegative functions*

$$\begin{cases} u, v \in C^0\left(\bar\Omega \times [0, T_{\max})\right) \cap C^{2,1}\left(\bar\Omega \times (0, T_{\max})\right), \\ w \in C^0\left(\bar\Omega \times [0, T_{\max})\right) \cap C^{2,1}\left(\bar\Omega \times (0, T_{\max})\right) \cap L^\infty_{loc}\ \left([0, T_{\max}); W^{1,q}(\Omega)\right) \end{cases}$$

such that (u, v, w) solves problem (1.3)classically in $\Omega \times (0, T_{\max})$, where $q > n$ and $T_{\max}$ is the maximal existence time. Moreover, if $T_{\max} < \infty$, then

$$\sup\left\{||u(\cdot,t)||_{L^\infty(\Omega)} + ||v(\cdot,t)||_{L^\infty(\Omega)} + ||w(\cdot,t)||_{W^{1,\infty}(\Omega)}\right\} \to \infty, \quad as\, t \nearrow T_{\max}. \tag{2.1}$$

Lemma 2.2. (see [19]) *Let $\Omega \subset \mathbb{R}^n$, $n \ge 1$ be a bounded domain with smooth boundary, and let $p \geqslant 1$, $q \in (0, p)$. Then there exists a constant $C_{GN} > 0$ such that*

$$||u||_{L^p(\Omega)} \le C_{GN}(||\nabla u||^\delta_{L^2(\Omega)} ||u||^{1-\delta}_{L^q(\Omega)} + ||u||_{L^q(\Omega)}),$$

where $\delta = \frac{\frac{n}{q} - \frac{n}{p}}{1 - \frac{n}{2} + \frac{n}{q}} \in (0, 1)$.

Lemma 2.3. *Let the conditions in Lemma 2.1 hold and $f_i = 0 (i = 1, 2)$, then there exists $K_1 > 0$ such that the solution components u, v of (1.3) satisfy*

$$||u(\cdot,t)||_{L^1(\Omega)} + ||v(\cdot,t)||_{L^1(\Omega)} \le K_1 \quad \text{for all } t \in (0, T_{\max}). \tag{2.2}$$

Proof. Integrating the first two equations in (1.3) with respect to $x \in \Omega$, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u + a_1\mu_1 \int_\Omega uv = 0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v + a_2\mu_2\int_\Omega uv,$$

due to $f_i = 0 (i = 1, 2)$. Then, we have

$$\|u(\cdot,t)\|_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)} \text{ and } \|v(\cdot,t)\|_{L^1(\Omega)} \le \|v_0\|_{L^1(\Omega)} \tag{2.3}$$

for all $t \in (0, T_{\max})$. ∎

3. Global boundedness for $\tau = 1$

In this section, with the aid of damping term and diffusion respectively, we study the global boundedness of system (1.3) when $n \geq 1$ and $\tau = 1$. The ideas come from [11,21,47]. Firstly, we need the following estimate.

Lemma 3.1. (see Lemma 2.3 in [34]) *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a smoothly bounded domain, and let $0 \leq t_0 < T_{\max} \leq \infty$ and $p \in (n, +\infty)$. Assume that $z_0 \in W^{2,p}(\Omega)$ with $\partial_\nu z_0 = 0$ on $\partial\Omega$ and $h_1, h_2 \in L^p([0, T_{\max}); L^p(\Omega))$. Then the problem*

$$\begin{cases} z_t = \Delta z - z + h_1 + h_2, & (x,t) \in \Omega \times (0, T_{\max}), \\ \dfrac{\partial z}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0, T_{\max}), \\ z(\cdot, 0) = z_0, & x \in \Omega, \end{cases}$$

exists a unique solution $z \in W^{1,p}([0, T_{\max}); L^p(\Omega)) \bigcap L^p\left([0, T_{\max}); W^{2,p}(\Omega)\right)$ and there exists $C_S(p) > 0$ such that

$$\int_{t_0}^{t} e^{p\tau} \int_\Omega |\Delta z(\cdot,\tau)|^p \mathrm{d}\tau \leq C_S(p) \int_{t_0}^{t} e^{p\tau} \int_\Omega (|h_1(\cdot,s)|^p + |h_2(\cdot,s)|^p)\, \mathrm{d}\tau + C_S(p) e^{pt_0} ||\Delta z(\cdot, t_0)||^p_{L^p(\Omega)}$$

for any $t \in (t_0, T_{\max})$.

Next, we establish a priori estimates about u, v, which are of great help to get the main result.

Lemma 3.2. *Let the conditions in Lemma 2.1 hold and $\tau = 1$. For any $p_i > p_i^\star := \max\{1, \alpha_i, 1 - \beta_i\}$ and $\epsilon_i > 0$ with $i = 1, 2$, the solution components u, v of (1.3) satisfy*

$$\begin{aligned} &\frac{1}{p_1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^{p_1} + \frac{4(p_1-1)}{(p_1-\alpha_1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{p_1-\alpha_1}{2}}\right|^2 \\ &\quad \leq \epsilon_1 \int_\Omega (u+1)^{p_1+\beta_1+\gamma_1-1} + M_1(\epsilon_1)\int_\Omega |\Delta w|^{\tilde{p_1}} + \int_\Omega (u+1)^{p_1-1} f_1(u), \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &\frac{1}{p_2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (v+1)^{p_2} + \frac{4(p_2-1)}{(p_2-\alpha_2)^2}\int_\Omega \left|\nabla (v+1)^{\frac{p_2-\alpha_2}{2}}\right|^2 \\ &\quad \leq \epsilon_2 \int_\Omega (v+1)^{p_2+\beta_2+\gamma_2-1} + M_2(\epsilon_2)\int_\Omega |\Delta w|^{\tilde{p_2}} + \int_\Omega (v+1)^{p_2-1} f_2(v), \end{aligned} \tag{3.2}$$

where $\tilde{p_i} = \frac{p_i+\beta_i+\gamma_i-1}{\gamma_i}$ and

$$M_i(\epsilon_i) = \frac{\gamma_i}{p_i+\beta_i+\gamma_i-1}\left(\epsilon_1 \cdot \frac{p_i+\beta_i+\gamma_i-1}{p_i+\beta_i-1}\right)^{\frac{1-p_i-\beta_i}{\gamma_i}} \left(\frac{C_{\chi_i}(p_i-1)}{p_i+\beta_i-1}\right)^{\frac{p_i+\beta_i+\gamma_i-1}{\gamma_i}}, \tag{3.3}$$

for $i = 1, 2$.

Proof. Testing the first equation in (1.3) by $(u+1)^{p_1-1}$ for $p_1>1$, we have

$$\begin{aligned}&\frac{1}{p_1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^{p_1}\\&=-(p_1-1)\int_\Omega (u+1)^{p_1-2}D_1(u)|\nabla u|^2+(p_1-1)\int_\Omega (u+1)^{p_1-2}\chi_1(u)\nabla u\cdot\nabla w\\&\quad+\int_\Omega (u+1)^{p_1-1}f_1(u).\end{aligned}\tag{3.4}$$

Due to (S_1), (S_2), $p_1>\alpha_1$ and $p_1>1-\beta_1$, we get

$$\begin{aligned}-(p_1-1)\int_\Omega (u+1)^{p_1-2}D_1(u)|\nabla u|^2&\le -(p_1-1)\int_\Omega (u+1)^{p-\alpha_1-2}|\nabla u|^2\\&\le \frac{4\,(p_1-1)}{(p_1-\alpha_1)^2}\int_\Omega \left|\nabla\,(u+1)^{\frac{p_1-\alpha_1}{2}}\right|^2,\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}(p_1-1)\int_\Omega (u+1)^{p_1-2}\chi_1(u)\nabla u\cdot\nabla w&=-(p_1-1)\int_\Omega\left(\int_0^u\Gamma(s)ds\right)\Delta w\\&\le (p_1-1)\int_\Omega\left|\int_0^u\Gamma(s)ds\right||\Delta w|\\&\le \frac{C_{\chi_1}(p_1-1)}{p_1+\beta_1-1}\int_\Omega (u+1)^{p_1+\beta_1-1}|\Delta w|,\end{aligned}\tag{3.6}$$

where

$$\Gamma(s)=(s+1)^{p_1-2}\chi_1(s)\le C_{\chi_1}(s+1)^{p_1+\beta_1-2}.$$

For the right term of (3.6), since $\gamma_1>0$ and $p_1>1-\beta_1$, using the fact $\frac{\gamma_1}{p_1+\beta_1+\gamma_1-1}\in(0,1)$ and Young's inequality that for any $\epsilon_1>0$, we obtain

$$\frac{C_{\chi_1}(p_1-1)}{p_1+\beta_1-1}\int_\Omega (u+1)^{p_1+\beta_1-1}|\Delta w|\le \epsilon_1\int_\Omega (u+1)^{p_1+\beta_1+\gamma_1-1}+M_1(\epsilon_1)\int_\Omega |\Delta w|^{\tilde{p_1}}\tag{3.7}$$

where $\tilde{p_1}=\frac{p_1+\beta_1+\gamma_1-1}{\gamma_1}$ and $M_1(\epsilon_1)$ is defined in (3.3). Then, collecting (3.4), (3.5) and (3.7) that we prove (3.1). Similarly, (3.2) can be obtained by using the same framework. □

Then, based on the above lemmas, we establish the uniform boundedness of $\|u\|_{L^{p_1}(\Omega)}$, $\|v\|_{L^{p_2}(\Omega)}$ under some appropriate parameter conditions.

Lemma 3.3. *Let (u,v,w) be a solution ensured in Lemma 2.1 and the assumptions in Theorem 1.1 hold. Then there exists a positive constant K_2 such that for all $p_1,p_2>1$*

$$\|u(\cdot,t)\|_{L^{p_1}(\Omega)}+\|v(\cdot,t)\|_{L^{p_2}(\Omega)}\le K_2\quad \textit{for all}\, t\in(0,T_{\max}).\tag{3.8}$$

Proof. Applying Lemma 3.1 to the third equation in (1.3) and combining (S_4), for $i=1,2$, there exists $C_S(p_i)>0$ such that

$$\begin{aligned} &\int_{t_0}^{t} e^{\tilde{p_i}\tau}\int_{\Omega}|\Delta w(\cdot,\tau)|^{\tilde{p_i}}\mathrm{d}\tau \\ &\le C_S(p_i)\int_{t_0}^{t} e^{\tilde{p_i}\tau}\int_{\Omega}(u^{\gamma_1}+v^{\gamma_2})^{\tilde{p_i}}\,\mathrm{d}\tau + C_S(p_i)e^{\tilde{p_i}t_0}\left\|\Delta w(\cdot,t_0)\right\|^{\tilde{p_i}}_{L^{\tilde{p_i}}(\Omega)} \\ &\le C_S(p_i)2^{\tilde{p_i}}\int_{t_0}^{t} e^{\tilde{p_i}\tau}\int_{\Omega}\left\{(u(\cdot,\tau)+1)^{\tilde{p_i}\gamma_1}+(v(\cdot,\tau)+1)^{\tilde{p_i}\gamma_2}\right\}\mathrm{d}\tau \\ &\quad + C_S(p_i)e^{\tilde{p_i}t_0}\left\|\Delta w(\cdot,t_0)\right\|^{\tilde{p_i}}_{L^{\tilde{p_i}}(\Omega)}. \end{aligned} \tag{3.9}$$

Let $\tilde{p_1}=\tilde{p_2}$. Next, we divide our proof into the following two parts.

Part 1 We shall deal with the hypothesis (i) in Theorem 1.1 by using the aid of the diffusion when $f_i=0(i=1,2)$.

For $i=1,2$, $\psi_i\in W^{1,2}(\Omega)\cap L^{\frac{2}{p_i-\alpha_i}}(\Omega)$ and the same p_i in Lemma 3.2, combining Lemmas 2.2, 2.3 and Young's inequality, there exist positive constants C_1^i, C_2^i, C_3^i such that,

$$\begin{aligned} \frac{\tilde{p_i}}{p_i}\int_{\Omega}(\psi_i+1)^{p_i} &= \frac{\tilde{p_i}}{p_i}\left\|(\psi_i+1)^{\frac{p_i-\alpha_i}{2}}\right\|^{\frac{2p_i}{p_i-\alpha_i}}_{L^{\frac{2p_i}{p_i-\alpha_i}}(\Omega)} \\ &\le C_1^i\left\|\nabla(\psi_i+1)^{\frac{p_i-\alpha_i}{2}}\right\|^{\frac{2p_i}{p_i-\alpha_i}\delta_1^i}_{L^2(\Omega)}\left\|(\psi_i+1)^{\frac{p_i-\alpha_i}{2}}\right\|^{\frac{2p_i}{p_i-\alpha_i}\left(1-\delta_1^i\right)}_{L^{\frac{2}{p_i-\alpha_i}}(\Omega)} \\ &\quad + C_1^i\left\|(\psi_i+1)^{\frac{p_i-\alpha_i}{2}}\right\|^{\frac{2p_i}{p_i-\alpha_i}}_{L^{\frac{2}{p_i-\alpha_i}}(\Omega)} \\ &\le C_2^i\left\|\nabla(\psi_i+1)^{\frac{p_i-\alpha_i}{2}}\right\|^{\frac{2p_i}{p_i-\alpha_i}\delta_1^i}_{L^2(\Omega)} + C_2^i \\ &\le \frac{2(p_i-1)}{(p_i-\alpha_i)^2}\int_{\Omega}\left|\nabla(\psi_i+1)^{\frac{p_i-\alpha_i}{2}}\right|^2 + C_3^i, \end{aligned} \tag{3.10}$$

where $\delta_1^i := \frac{\frac{n(p_i-\alpha_i)}{2}\left(1-\frac{1}{p_i}\right)}{1-\frac{n}{2}+\frac{n(p_i-\alpha_i)}{2}}\in(0,1)$ and $\frac{2p_i}{p_i-\alpha_i}\delta_1^i\in(0,2)$ due to $p_i>1$ and $\alpha_i<\frac{2}{n}$ with $n\ge 1$.

Moreover, let

$$\begin{aligned} M_3^1 &= \epsilon_1 + M_1(\epsilon_1)C_S(p_1)2^{\tilde{p_1}} + M_2(\epsilon_2)C_S(p_2)2^{\tilde{p_2}}, \\ M_3^2 &= \epsilon_2 + M_1(\epsilon_1)C_S(p_1)2^{\tilde{p_1}} + M_2(\epsilon_2)C_S(p_2)2^{\tilde{p_2}}, \end{aligned} \tag{3.11}$$

and $p_i > \max\left\{2-\beta_i-\gamma_i, \alpha_i+\frac{n-2}{n}, \frac{n\alpha_i}{2}+\frac{n-2}{2}(\beta_i+\gamma_i-1)\right\}$ with $i=1,2$, combining Lemmas 2.2, 2.3 and Young's inequality again, for $i=1,2$, there exist positive constants C_4^i, C_5^i, C_6^i such that

$$\begin{aligned} M_3^i \int_\Omega (\psi_i+1)^{p_i+\beta_i+\gamma_i-1} &= M_3^i \left\|(\psi_i+1)^{\frac{p_i-\alpha_i}{2}}\right\|_{L^{\frac{2(p_i+\beta_i+\gamma_i-1)}{p_i-\alpha_i}}(\Omega)}^{\frac{2(p_i+\beta_i+\gamma_i-1)}{p_i-\alpha_i}} \\ &\le C_4^i \left\|\nabla(\psi_i+1)^{\frac{p_i-\alpha_i}{2}}\right\|_{L^2(\Omega)}^{\frac{2(p_i+\beta_i+\gamma_i-1)}{p_i-\alpha_i}\delta_2^i} \left\|(\psi_i+1)^{\frac{p_i-\alpha_i}{2}}\right\|_{L^{\frac{2}{p_i-\alpha_i}}(\Omega)}^{\frac{2(p_i+\beta_i+\gamma_i-1)}{p_i-\alpha_i}\left(1-\delta_2^i\right)} \\ &+ C_4^i \left\|(\psi_i+1)^{\frac{p_i-\alpha_i}{2}}\right\|_{L^{\frac{2}{p_i-\alpha_i}}(\Omega)}^{\frac{2(p_i+\beta_i+\gamma_i-1)}{p_i-\alpha_i}} \\ &\le C_5^i \left\|\nabla(\psi_i+1)^{\frac{p_i-\alpha_i}{2}}\right\|_{L^2(\Omega)}^{\frac{2(p_i+\beta_i+\gamma_i-1)}{p_i-\alpha_i}\delta_2^i} + C_5^i \\ &\le \frac{2(p_i-1)}{(p_i-\alpha_i)^2}\int_\Omega \left|\nabla(\psi_i+1)^{\frac{p_i-\alpha_i}{2}}\right|^2 + C_6^i, \end{aligned} \tag{3.12}$$

where $\delta_2^i := \frac{\frac{n(p_i-\alpha_i)}{2}\left(1-\frac{1}{p_i+\beta_i+\gamma_i-1}\right)}{1-\frac{n}{2}+\frac{n(p_i-\alpha_i)}{2}} \in (0,1)$ and $\frac{2(p_i+\beta_i+\gamma_i-1)}{p_i-\alpha_i}\delta_2^i \in (0,2)$ due to $\alpha_i+\beta_i+\gamma_i < 1+\frac{2}{n}$ with $n \ge 1$.

Together (3.10), (3.12) with (3.1) and (3.2), we have

$$\begin{aligned} \frac{1}{p_1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^{p_1} + \frac{\tilde{p_1}}{p_1}\int_\Omega (u+1)^{p_1} + M_3^1\int_\Omega (u+1)^{p_1+\beta_1+\gamma_1-1} \\ \le \epsilon_1 \int_\Omega (u+1)^{p_1+\beta_1+\gamma_1-1} + M_1(\epsilon_1)\int_\Omega |\Delta w|^{\tilde{p_1}}, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \frac{1}{p_2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (v+1)^{p_2} + \frac{\tilde{p_2}}{p_2}\int_\Omega (v+1)^{p_2} + M_3^2\int_\Omega (v+1)^{p_2+\beta_2+\gamma_2-1} \\ \le \epsilon_2 \int_\Omega (v+1)^{p_2+\beta_2+\gamma_2-1} + M_2(\epsilon_2)\int_\Omega |\Delta w|^{\tilde{p_2}}, \end{aligned} \tag{3.14}$$

due to $f_i = 0 (i=1,2)$. Integrating (3.13) and (3.14) from t_0 to t yields

$$\begin{aligned} \frac{1}{p_1}\int_\Omega (u+1)^{p_1}(\cdot,t) + \left(M_3^1-\epsilon_1\right)e^{-\tilde{p_1}t}\int_{t_0}^t e^{\tilde{p_1}\tau}\int_\Omega (u(\cdot,\tau)+1)^{p_1+\beta_1+\gamma_1-1} \\ \le M_1(\epsilon_1)e^{-\tilde{p_1}t}\int_{t_0}^t e^{\tilde{p_1}\tau}\int_\Omega |\Delta w(\cdot,\tau)|^{\tilde{p_1}} + \frac{1}{p_1}e^{\tilde{p_1}(t_0-t)}\int_\Omega (u(\cdot,t_0)+1)^{p_1} \end{aligned} \tag{3.15}$$

and

$$\frac{1}{p_2}\int_\Omega (v+1)^{p_2}(\cdot,t) + \left(M_3^2-\epsilon_2\right)e^{-\tilde{p_2}t}\int_{t_0}^t e^{\tilde{p_2}\tau}\int_\Omega (v(\cdot,\tau)+1)^{p_2+\beta_2+\gamma_2-1} \\ \le M_2(\epsilon_1)e^{-\tilde{p_2}t}\int_{t_0}^t e^{\tilde{p_2}\tau}\int_\Omega |\Delta w(\cdot,\tau)|^{\tilde{p_2}} + \frac{1}{p_2}e^{\tilde{p_2}(t_0-t)}\int_\Omega (v(\cdot,t_0)+1)^{p_2}. \tag{3.16}$$

Together (3.15) and (3.16) with (3.9), we obtain that for all $t\in(0,T_{\max})$,

$$\frac{1}{p_1}\int_\Omega (u+1)^{p_1}(\cdot,t) + \frac{1}{p_2}\int_\Omega (v+1)^{p_2}(\cdot,t) \le C_7 \tag{3.17}$$

for some constant $C_7>0$ due to $\tilde{p_1}=\tilde{p_2}$.

Part 2 We deal with the hypothesis (ii) in Theorem 1.1 by using the effect of damping terms. Firstly, it follows from $M_1(\epsilon_1)$ and $M_2(\epsilon_2)$ in (3.3) and simple calculations that $M_3^1(\epsilon_1)$ and $M_3^2(\epsilon_2)$ defined in (3.11) attain the minimum value

$$\min_{\epsilon_1>0} M_3^1(\epsilon_1) = \frac{2C_{\chi_1}(p_1-1)C_S(p_1)^{\frac{1}{\tilde{p_1}}}}{p_1+\beta_1-1} + \frac{2\gamma_2 C_{\chi_2}(p_2-1)C_S(p_2)^{\frac{1}{\tilde{p_2}}}}{(p_2+\beta_2+\gamma_2-1)(p_2+\beta_2-1)} := M_4^1 \tag{3.18}$$

and

$$\min_{\epsilon_2>0} M_3^2(\epsilon_2) = \frac{2C_{\chi_2}(p_2-1)C_S(p_2)^{\frac{1}{\tilde{p_2}}}}{p_2+\beta_2-1} + \frac{2\gamma_1 C_{\chi_1}(p_1-1)C_S(p_1)^{\frac{1}{\tilde{p_1}}}}{(p_1+\beta_1+\gamma_1-1)(p_1+\beta_1-1)} := M_4^2 \tag{3.19}$$

respectively, when

$$\epsilon_i = \frac{2C_{\chi_i}(p_i-1)C_S(p_i)^{\frac{1}{\tilde{p_i}}}}{p_i+\beta_i+\gamma_i-1} \tag{3.20}$$

for $i=1,2$.

Adding $\frac{\tilde{p_1}}{p_1}\int_\Omega (u+1)^{p_1}$ and $\frac{\tilde{p_2}}{p_2}\int_\Omega (v+1)^{p_2}$ to both sides of (3.1) and (3.2), respectively, we have

$$\frac{1}{p_1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^{p_1} + \frac{\tilde{p_1}}{p_1}\int_\Omega (u+1)^{p_1} \\ \le \frac{\tilde{p_1}}{p_1}\int_\Omega (u+1)^{p_1} + \epsilon_1\int_\Omega (u+1)^{p_1+\beta_1+\gamma_1-1} + M_1(\epsilon_1)\int_\Omega |\Delta w|^{\tilde{p_1}} \\ + \int_\Omega (u+1)^{p_1-1}f_1(u), \tag{3.21}$$

and

$$\frac{1}{p_2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (v+1)^{p_2} + \frac{\tilde{p_2}}{p_2}\int_\Omega (v+1)^{p_2} \\ \le \frac{\tilde{p_2}}{p_2}\int_\Omega (v+1)^{p_2} + \epsilon_2\int_\Omega (v+1)^{p_2+\beta_2+\gamma_2-1} + M_2(\epsilon_2)\int_\Omega |\Delta w|^{\tilde{p_2}} \\ + \int_\Omega (v+1)^{p_2-1}f_2(v). \tag{3.22}$$

Integrating (3.21), (3.22) from t_0 to t and using (3.9) yields

$$\begin{aligned}
&\frac{1}{p_1}\int_\Omega (u+1)^{p_1}(\cdot,t)+\frac{1}{p_2}\int_\Omega (v+1)^{p_2}(\cdot,t)\\
&\quad\le \frac{\tilde{p_1}}{p_1}e^{-\tilde{p_1}t}\int_{t_0}^{t} e^{\tilde{p_1}\tau}\int_\Omega (u(\cdot,\tau)+1)^{p_1}+M_3^1(\epsilon_1)e^{-\tilde{p_1}t}\int_{t_0}^{t} e^{\tilde{p_1}\tau}\int_\Omega (u(\cdot,\tau)+1)^{p_1+\beta_1+\gamma_1-1}\\
&\qquad+e^{-\tilde{p_1}t}\int_{t_0}^{t} e^{\tilde{p_1}\tau}\int_\Omega (u+1)^{p_1-1}f_1(u)(\cdot,\tau)+\frac{1}{p_1}e^{\tilde{p_1}(t_0-t)}\int_\Omega (u(\cdot,t_0)+1)^{p_1}\\
&\qquad+\frac{\tilde{p_2}}{p_2}e^{-\tilde{p_2}t}\int_{t_0}^{t} e^{\tilde{p_2}\tau}\int_\Omega (v(\cdot,\tau)+1)^{p_2}+M_3^2(\epsilon_2)e^{-\tilde{p_2}t}\int_{t_0}^{t} e^{\tilde{p_2}\tau}\int_\Omega (v(\cdot,\tau)+1)^{p_2+\beta_2+\gamma_2-1}\\
&\qquad+e^{-\tilde{p_2}t}\int_{t_0}^{t} e^{\tilde{p_2}\tau}\int_\Omega (v+1)^{p_2-1}f_2(v)(\cdot,\tau)+\frac{1}{p_2}e^{\tilde{p_2}(t_0-t)}\int_\Omega (v(\cdot,t_0)+1)^{p_2}
\end{aligned}\tag{3.23}$$

due to $\tilde{p_1}=\tilde{p_2}$. With the ϵ_i in (3.20) for $i=1,2$, we obtain

$$\begin{aligned}
&\frac{1}{p_1}\int_\Omega (u+1)^{p_1}(\cdot,t)+\frac{1}{p_2}\int_\Omega (v+1)^{p_2}(\cdot,t)\\
&\quad\le e^{-\tilde{p_1}t}\int_{t_0}^{t} e^{\tilde{p_1}\tau}\int_\Omega F_1(u)(\cdot,\tau)+e^{-\tilde{p_2}t}\int_{t_0}^{t} e^{\tilde{p_2}\tau}\int_\Omega F_2(v)(\cdot,\tau)+C_8
\end{aligned}\tag{3.24}$$

where $C_8=\frac{1}{p_1}e^{\tilde{p_1}(t_0-t)}\int_\Omega (u(\cdot,t_0)+1)^{p_1}+\frac{1}{p_2}e^{\tilde{p_2}(t_0-t)}\int_\Omega (v(\cdot,t_0)+1)^{p_2}$ and

$$F_i(s):=\frac{\tilde{p_i}}{p_i}(s+1)^{p_i}+M_4^i(s+1)^{p_i+\beta_i+\gamma_i-1}+(s+1)^{p_i-1}f_i(s)\tag{3.25}$$

for $i=1,2$. Let

$$\iota_1^\star=\begin{cases}\frac{\tilde{p_1}}{p_1}, & \text{when } \beta_1+\gamma_1<1;\\ \frac{\tilde{p_1}}{p_1}+M_4^1, & \text{when } \beta_1+\gamma_1=1;\\ M_4^1, & \text{when } \beta_1+\gamma_1>1,\end{cases}\tag{3.26}$$

and

$$\iota_2^\star=\begin{cases}\frac{\tilde{p_2}}{p_2}, & \text{when } \beta_2+\gamma_2<1;\\ \frac{\tilde{p_2}}{p_2}+M_4^2, & \text{when } \beta_2+\gamma_2=1;\\ M_4^2, & \text{when } \beta_2+\gamma_2>1,\end{cases}\tag{3.27}$$

then we can divide the three cases (a) $\beta_1+\gamma_1<1$, (b) $\beta_1+\gamma_1=1$ and (c) $\beta_1+\gamma_1>1$ to deal with F_1, and F_2 is treated in a similar way. Here we only give the proof of case (a).

If $\beta_1+\gamma_1<1$, it follows from $\lim_{s\to\infty}\inf\left\{-\frac{f_1(s)}{s}\right\}=:\iota_1\in(\iota_1^\star,\infty]$ for $\iota_1^\star=\frac{\tilde{p_1}}{p_1}$ that

$$\begin{aligned}
\lim_{s\to\infty}\inf\frac{1}{(s+1)^{p_1}}F_1(s)&=\frac{\tilde{p_1}}{p_1}+M_4^1\lim_{s\to\infty}\inf(s+1)^{\beta_1+\gamma_1-1}-\iota_1\\
&=\frac{\tilde{p_1}}{p_1}-\iota_1<0,
\end{aligned}\tag{3.28}$$

so that

$$\exists \bar{s}_1 > 0, \text{ s.t. } F_1(s) < 0 \text{ for all } s > \bar{s}_1.$$

Therefore, we further have

$$\int_\Omega F_1(u)(\cdot,\tau) = \int_{u\le \bar{s}_1} F_1(u)(\cdot,\tau) + \int_{u>\bar{s}_1} F_1(u)(\cdot,\tau) \le \sup_{0<s\le \bar{s}_1} F_1(s)|\Omega| < \infty. \tag{3.29}$$

In this similar way, we can also get

$$\int_\Omega F_1(u)(\cdot,\tau) < \infty \quad \text{when case } (b),(c) \tag{3.30}$$

and

$$\int_\Omega F_2(v)(\cdot,\tau) < \infty \quad \text{when case } (a),(b),(c) \tag{3.31}$$

under the condition $\lim_{s\to\infty} \inf \left\{ -\frac{f_i(s)}{s^{\max\{1,\beta_i+\gamma_i\}}} \right\} =: \iota_i \in (\iota_i^\star, \infty]$ by the definition of $\iota_i^\star$ in (3.26) and (3.27) with $i = 1, 2$. Combining (3.24) with (3.30) and (3.31), we obtain the existence of positive constant C_9 such that

$$\frac{1}{p_1}\int_\Omega (u+1)^{p_1}(\cdot,t) + \frac{1}{p_2}\int_\Omega (v+1)^{p_2}(\cdot,t) \le C_9 \tag{3.32}$$

for all $t \in (0, T_{\max})$. The proof of Lemma 3.3 is complete. □

By Lemma 3.3, we can complete the proof of Theorem 1.1.

Proof of Theorem 1.1. By using the well-known $L^p - L^q$ estimate in [38] and selecting the appropriate values for p_1, p_2 in Lemma 3.3, we can obtain the boundedness of $\|w(\cdot,t)\|_{W^{1,\infty}}$. Then, it follows form Moser iteration technique (Appendix A of [27]) that there exists a constant $K_3 > 0$ independent of t such that

$$\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty} + \|w(\cdot,t)\|_{W^{1,\infty}} \le K_3,$$

which together with the extension criterion in Lemma 2.1 proves Theorem 1.1.

4. Global boundedness for $\tau = 0$

In this section, we shall prove the global boundedness of solutions for (1.3) in any space dimension when $\tau = 0$. Firstly, we give the coupled estimate of $\int_\Omega u^{p_1}$ and $\int_\Omega v^{p_2}$ for some suitably large $p_i > 1 (i = 1, 2)$.

Lemma 4.1. *Let $\tau = 0$ and (u, v, w) be a solution ensured in Lemma 2.1. Then the solution (u, v, w) satisfies*

$$\begin{aligned} &\frac{1}{p_1}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega (u+1)^{p_1} + \frac{4(p_1-1)}{(p_1-\alpha_1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{p_1-\alpha_1}{2}}\right|^2 \\ &\le \frac{C_{\chi_1}(p_1-1)}{(p_1+\beta_1-1)}\int_\Omega (u+1)^{p_1+\beta_1+\gamma_1-1} + \frac{C_{\chi_1}(p_1-1)}{p_1+\beta_1+\gamma_2-1}\int_\Omega (u+1)^{p_1+\beta_1+\gamma_2-1} \\ &\quad + \frac{C_{\chi_1}\gamma_2(p_1-1)}{(p_1+\beta_1+\gamma_2-1)(p_1+\beta_1-1)}\int_\Omega (v+1)^{p_1+\beta_1+\gamma_2-1} - \int_\Omega f_1(u)(u+1)^{p_1-1} \end{aligned} \tag{4.1}$$

and

$$
\begin{aligned}
&\frac{1}{p_2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (v+1)^{p_2} + \frac{4(p_2-1)}{(p_2-\alpha_2)^2}\int_\Omega \left|\nabla (v+1)^{\frac{p_2-\alpha_2}{2}}\right|^2 \\
&\quad\le \frac{C_{\chi_2}(p_2-1)}{(p_2+\beta_2-1)}\int_\Omega (v+1)^{p_2+\beta_2+\gamma_2-1} + \frac{C_{\chi_2}(p_2-1)}{p_2+\beta_2+\gamma_1-1}\int_\Omega (v+1)^{p_2+\beta_2+\gamma_1-1} \\
&\qquad + \frac{C_{\chi_2}\gamma_1(p_2-1)}{(p_2+\beta_2+\gamma_1-1)(p_2+\beta_2-1)}\int_\Omega (u+1)^{p_2+\beta_2+\gamma_1-1} - \int_\Omega f_2(v)(v+1)^{p_2-1}
\end{aligned} \tag{4.2}
$$

for all $t \in (0, T_{\max})$*, where*

$$
p_i > \max\{1, 1-\beta_i, \alpha_i\} \quad \text{with } i = 1, 2. \tag{4.3}
$$

Proof. Testing the first equation in (1.3) by $(u+1)^{p_1-1}$, since (S_1) and $p_1 > \alpha_1$, we have

$$
\begin{aligned}
&\frac{1}{p_1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+1)^{p_1} \\
&= \int_\Omega (u+1)^{p_1-1}\nabla\cdot(D_1(u)\nabla u) - \int_\Omega (u+1)^{p_1-1}\nabla\cdot(\chi_1(u)\nabla w) - \int_\Omega f_1(u)(u+1)^{p_1-1} \\
&\le \frac{-4(p_1-1)}{(p_1-\alpha_1)^2}\int_\Omega \left|\nabla (u+1)^{\frac{p_1-\alpha_1}{2}}\right|^2 + (p_1-1)\int_\Omega (u+1)^{p_1-2}\chi_1(u)\nabla u\cdot\nabla w \\
&\quad - \int_\Omega f_1(u)(u+1)^{p_1-1}.
\end{aligned} \tag{4.4}
$$

Due to $(S_2), (S_4)$ and $p_1 > 1-\beta_1$, using Young's inequality, we obtain

$$
\begin{aligned}
&(p_1-1)\int_\Omega (u+1)^{p_1-2}\chi_1(u)\nabla u\cdot\nabla w \\
&\quad = (p_1-1)\int_\Omega \nabla\varpi(u)\cdot\nabla w \\
&\quad = -(p_1-1)\int_\Omega \varpi(u)\cdot\Delta w \\
&\quad \le (p_1-1)\int_\Omega \varpi(u)(u^{\gamma_1}+v^{\gamma_2}-w) \\
&\quad \le \frac{C_{\chi_1}(p_1-1)}{(p_1+\beta_1-1)}\int_\Omega (u+1)^{p_1+\beta_1-1}\left(u^{\gamma_1}+v^{\gamma_2}\right) \\
&\quad \le \frac{C_{\chi_1}(p_1-1)}{(p_1+\beta_1-1)}\left\{\int_\Omega (u+1)^{p_1+\beta_1+\gamma_1-1} + \frac{p_1+\beta_1-1}{p_1+\beta_1+\gamma_2-1}\int_\Omega (u+1)^{p_1+\beta_1+\gamma_2-1}\right. \\
&\qquad \left. + \frac{\gamma_2}{p_1+\beta_1+\gamma_2-1}\int_\Omega (v+1)^{p_1+\beta_1+\gamma_2-1}\right\}
\end{aligned} \tag{4.5}
$$

where

$$\varpi(u)=\int_0^u (s+1)^{p_1-2}\chi_1(s)ds\le\int_0^u (s+1)^{p_1+\beta_1-2}ds.$$

Then, collecting (4.4) and (4.5) that we prove (4.1). Similarly, (4.2) can be obtained by using the same framework. □

Then, based on the above lemmas, we establish the uniform boundedness of $\|u\|_{L^{p_1}(\Omega)}$, $\|v\|_{L^{p_2}(\Omega)}$ under some appropriate parameter conditions.

Lemma 4.2. *Let (u,v,w) be a solution ensured in Lemma 2.1 and the assumptions in Theorem 1.2 hold. Then, for all $p_1,p_2>1$, there exists a positive constant K_4 such that*

$$\|u(\cdot,t)\|_{L^{p_1}(\Omega)}+\|v(\cdot,t)\|_{L^{p_2}(\Omega)}\le K_4 \quad \textit{for all}\, t\in(0,T_{\max}). \tag{4.6}$$

Proof. Let $p_1+\beta_1+\gamma_2=p_2+\beta_2+\gamma_1$ in Lemma 4.1. Similar to Lemma 3.3, we divide our proof into the following two parts.

Part 1 We shall deal with the hypothesis (i) in Theorem 1.2 by using the aid of the diffusion when $f_i=0(i=1,2)$. Combining (4.1)–(4.2) and apply Young's inequality, there exist some constant $C_{10},C_{11}>0$ such that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{p_1}\int_\Omega (u+1)^{p_1}+\frac{1}{p_2}\int_\Omega (v+1)^{p_2}\right\}\\&\quad+\frac{4(p_1-1)}{(p_1-\alpha_1)^2}\int_\Omega\left|\nabla(u+1)^{\frac{p_1-\alpha_1}{2}}\right|^2+\frac{4(p_2-1)}{(p_2-\alpha_2)^2}\int_\Omega\left|\nabla(v+1)^{\frac{p_2-\alpha_2}{2}}\right|^2\\&\le C_{10}\int_\Omega (u+1)^{p_1+\beta_1+\gamma-1}+C_{11}\int_\Omega (v+1)^{p_2+\beta_2+\gamma-1}\end{aligned} \tag{4.7}$$

due to $\gamma=\max\{\gamma_1,\gamma_2\}$, $p_1+\beta_1+\gamma_2=p_2+\beta_2+\gamma_1$ and $f_i=0(i=1,2)$. Since $\alpha_i<1+\frac{2}{n}-\gamma-\beta_i$ for $i=1,2$, using the similar techniques as in (3.12) we obtain

$$\begin{aligned}&(C_{10}+1)\int_\Omega (u+1)^{p_1+\beta_1+\gamma_1-1}\\&\quad\le\frac{4\,(p_1-1)}{(p_1-\alpha_1)^2}\int_\Omega\left|\nabla\,(u+1)^{\frac{p_1-\alpha_1}{2}}\right|^2+\tilde{C}_{10}\end{aligned}$$

and

$$\begin{aligned}&(C_{11}+1)\int_\Omega (v+1)^{p_2+\beta_2+\gamma_2-1}\\&\quad\le\frac{4\,(p_2-1)}{(p_2-\alpha_2)^2}\int_\Omega\left|\nabla\,(v+1)^{\frac{p_2-\alpha_2}{2}}\right|^2+\tilde{C}_{11},\end{aligned}$$

where $\tilde{C}_{10}, \tilde{C}_{11}$ are some positive constants. Together them with (4.7) that we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{p_1}\int_\Omega (u+1)^{p_1}+\frac{1}{p_2}\int_\Omega (v+1)^{p_2}\right\}\\&\quad+\int_\Omega (u+1)^{p_1+\beta_1+\gamma-1}+\int_\Omega (v+1)^{p_2+\beta_2+\gamma-1}\\&\leq C_{12},\end{aligned}\tag{4.8}$$

where C_{12} is a positive constant. This along with the ODE comparison argument, we obtain the existence of positive constant C_{13} such that

$$\frac{1}{p_1}\int_\Omega (u+1)^{p_1}(\cdot,t)+\frac{1}{p_2}\int_\Omega (v+1)^{p_2}(\cdot,t)\leq C_{13}\tag{4.9}$$

for all $t\in(0,T_{\max})$.

Part 2 We deal with the hypothesis (ii) in Theorem 1.2 by using the effect of damping terms.

Together (4.1) with (4.2) and adding $\int_\Omega (u+1)^{\frac{p_1}{2}}+\int_\Omega (v+1)^{\frac{p_2}{2}}$ to the both sides, since $p_1,p_2>1$ and $p_1+\beta_1+\gamma_2=p_2+\beta_2+\gamma_1$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{p_1}\int_\Omega (u+1)^{p_1}+\frac{1}{p_2}\int_\Omega (v+1)^{p_2}\right\}+\int_\Omega (u+1)^{\frac{p_1}{2}}+\int_\Omega (v+1)^{\frac{p_2}{2}}\\&\leq\int_\Omega \varrho_1(u)+\int_\Omega \varrho_2(v),\end{aligned}\tag{4.10}$$

where

$$\begin{aligned}\varrho_1(u)=&\left(\frac{C_{\chi_1}(p_1-1)}{p_1+\beta_1+\gamma_2-1}+\frac{C_{\chi_2}\gamma_1(p_2-1)}{(p_2+\beta_2+\gamma_1-1)(p_2+\beta_2-1)}\right)(u+1)^{p_1+\beta_1+\gamma_2-1}\\&+\frac{C_{\chi_1}(p_1-1)}{(p_1+\beta_1-1)}(u+1)^{p_1+\beta_1+\gamma_1-1}+(u+1)^{\frac{p_1}{2}}-f_1(u)(u+1)^{p_1-1}\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}\varrho_2(v)=&\left(\frac{C_{\chi_2}(p_2-1)}{p_2+\beta_2+\gamma_1-1}+\frac{C_{\chi_1}\gamma_2(p_1-1)}{(p_1+\beta_1+\gamma_2-1)(p_1+\beta_1-1)}\right)(v+1)^{p_2+\beta_2+\gamma_1-1}\\&+\frac{C_{\chi_2}(p_2-1)}{(p_2+\beta_2-1)}(v+1)^{p_2+\beta_2+\gamma_2-1}+(v+1)^{\frac{p_2}{2}}-f_2(v)(v+1)^{p_2-1}.\end{aligned}\tag{4.12}$$

Let $p_i>2(\beta_i+\gamma-2)$ $(i=1,2)$ and

$$\tilde{\iota_1}=\begin{cases}\frac{C_{\chi_1}(p_1-1)}{p_1+\beta_1+\gamma_2-1}+\frac{C_{\chi_2}\gamma_1(p_2-1)}{(p_2+\beta_2+\gamma_1-1)(p_2+\beta_2-1)}, & \text{when } \gamma_1<\gamma_2;\\ \frac{C_{\chi_1}(p_1-1)}{p_1+\beta_1+\gamma_2-1}+\frac{C_{\chi_2}\gamma_1(p_2-1)}{(p_2+\beta_2+\gamma_1-1)(p_2+\beta_2-1)}+\frac{C_{\chi_1}(p_1-1)}{(p_1+\beta_1-1)}, & \text{when } \gamma_1=\gamma_2;\\ \frac{C_{\chi_1}(p_1-1)}{(p_1+\beta_1-1)}, & \text{when } \gamma_1>\gamma_2,\end{cases}\tag{4.13}$$

and

$$\tilde{\iota_2}=\begin{cases}\frac{C_{\chi_2}(p_2-1)}{p_2+\beta_2+\gamma_1-1}+\frac{C_{\chi_1}\gamma_2(p_1-1)}{(p_1+\beta_1+\gamma_2-1)(p_1+\beta_1-1)}, & \text{when } \gamma_1>\gamma_2;\\ \frac{C_{\chi_2}(p_2-1)}{p_2+\beta_2+\gamma_1-1}+\frac{C_{\chi_1}\gamma_2(p_1-1)}{(p_1+\beta_1+\gamma_2-1)(p_1+\beta_1-1)}+\frac{C_{\chi_2}(p_2-1)}{(p_2+\beta_2-1)}, & \text{when } \gamma_1=\gamma_2;\\ \frac{C_{\chi_2}(p_2-1)}{(p_2+\beta_2-1)}, & \text{when } \gamma_1<\gamma_2,\end{cases}\tag{4.14}$$

then we can divide the three cases (a) $\gamma_1 < \gamma_2$, (b) $\gamma_1 = \gamma_2$ and (c) $\gamma_1 > \gamma_2$ to deal with ϱ_1, and ϱ_2 is treated in a similar way. Here we only give the proof of case (a).

If $\gamma_1 < \gamma_2$, then $\gamma = \gamma_2$, it follows from $\lim_{s\to\infty}\inf\left\{-\frac{f_1(s)}{s^{\beta_1+\gamma}}\right\} =: \iota_1 \in (\tilde{\iota_1}, \infty]$ for $\tilde{\iota_1} = \frac{C_{\chi_1}(p_1-1)}{p_1+\beta_1+\gamma_2-1} + \frac{C_{\chi_2}\gamma_1(p_2-1)}{(p_2+\beta_2+\gamma_1-1)(p_2+\beta_2-1)}$ that

$$\begin{aligned}\lim_{s\to\infty}\inf\frac{1}{(s+1)^{p_1+\beta_1+\gamma-1}}\varrho_1(s) &= \frac{C_{\chi_1}(p_1-1)}{p_1+\beta_1+\gamma_2-1} + \frac{C_{\chi_2}\gamma_1(p_2-1)}{(p_2+\beta_2+\gamma_1-1)(p_2+\beta_2-1)} \\ &\quad + \frac{C_{\chi_1}(p_1-1)}{(p_1+\beta_1-1)}\lim_{s\to\infty}\inf(s+1)^{\gamma_1-\gamma_2} - \iota_1 \\ &< 0,\end{aligned} \tag{4.15}$$

so that

$$\exists \bar{s}_2 > 0, \text{ s.t. } \varrho_1(s) < 0 \text{ for all } s > \bar{s}_2.$$

Therefore, we further have

$$\int_\Omega \varrho_1(u)(\cdot,\tau) = \int_{u\le\bar{s}_2} \varrho_1(u)(\cdot,\tau) + \int_{u>\bar{s}_2} \varrho_1(u)(\cdot,\tau) \le \sup_{0<s\le\bar{s}_2}\varrho_1(s)|\Omega| < \infty. \tag{4.16}$$

In this similar way, we can also get

$$\int_\Omega \varrho_1(u)(\cdot,\tau) < \infty \quad \text{when case } (b),(c) \tag{4.17}$$

and

$$\int_\Omega \varrho_2(v)(\cdot,\tau) < \infty \quad \text{when case } (a),(b),(c) \tag{4.18}$$

under the condition $\lim_{s\to\infty}\inf\left\{-\frac{f_i(s)}{s^{\beta_i+\gamma}}\right\} =: \iota_i \in (\tilde{\iota_i}, \infty]$ by the definition of $\tilde{\iota_i}$ in (4.13) and (4.14) with $i = 1, 2$. Combining this with (4.10), we obtain the existence of positive constant C_{14} such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{p_1}\int_\Omega (u+1)^{p_1} + \frac{1}{p_2}\int_\Omega (v+1)^{p_2}\right\} + \int_\Omega (u+1)^{\frac{p_1}{2}} + \int_\Omega (v+1)^{\frac{p_2}{2}} \le C_{14} \tag{4.19}$$

for all $t \in (0, T_{\max})$. This along with the Lemma 5.1 in Chapter III of [30], we obtain the existence of positive constant C_{15} such that

$$\frac{1}{p_1}\int_\Omega (u+1)^{p_1}(\cdot,t) + \frac{1}{p_2}\int_\Omega (v+1)^{p_2}(\cdot,t) \le C_{15} \tag{4.20}$$

for all $t \in (0, T_{\max})$. The proof of Lemma 4.2 is complete. □

Remark 4.1. When $\beta_i \ge 0$ with $i = 1, 2$, for any $p_i > 0$, it is interesting that (4.13) and (4.14) can be simplified to

$$\tilde{\iota_1} = \begin{cases} C_{\chi_1}, & \text{when } \gamma_1 < \gamma_2; \\ 2C_{\chi_1}, & \text{when } \gamma_1 = \gamma_2; \\ C_{\chi_1}, & \text{when } \gamma_1 > \gamma_2, \end{cases} \quad \text{and} \quad \tilde{\iota_2} = \begin{cases} C_{\chi_2}, & \text{when } \gamma_1 > \gamma_2; \\ 2C_{\chi_2}, & \text{when } \gamma_1 = \gamma_2; \\ C_{\chi_2}, & \text{when } \gamma_1 < \gamma_2, \end{cases} \tag{4.21}$$

due to $p_1 + \beta_1 + \gamma_2 = p_2 + \beta_2 + \gamma_1$, and the condition (ii) in Theorem 1.2 can be changed into $\lim_{s\to\infty} \inf \left\{ -\frac{f_i(s)}{s^{\beta_i+\gamma}} \right\} =: \iota_i \in [\tilde{\iota_i}, \infty]$. In the other hand, when $\beta_i < 0$, since the continuity and

$$\lim_{p_1,p_2\to\infty} \left\{ \frac{C_{\chi_1}(p_1-1)}{p_1+\beta_1+\gamma_2-1} + \frac{C_{\chi_2}\gamma_1(p_2-1)}{(p_2+\beta_2+\gamma_1-1)(p_2+\beta_2-1)} \right\} = C_{\chi_1},$$

$$\lim_{p_1\to\infty} \left\{ \frac{C_{\chi_1}(p_1-1)}{(p_1+\beta_1-1)} \right\} = C_{\chi_1},$$

$$\lim_{p_1,p_2\to\infty} \left\{ \frac{C_{\chi_2}(p_2-1)}{p_2+\beta_2+\gamma_1-1} + \frac{C_{\chi_1}\gamma_2(p_1-1)}{(p_1+\beta_1+\gamma_2-1)(p_1+\beta_1-1)} \right\} = C_{\chi_2},$$

and

$$\lim_{p_2\to\infty} \left\{ \frac{C_{\chi_2}(p_2-1)}{(p_2+\beta_2-1)} \right\} = C_{\chi_2},$$

(4.13) and (4.14) can also be simplified to (4.21) such that the condition (ii) in Theorem 1.2 is reasonable for sufficiently large p_1, p_2.

By Lemma 4.2, we can complete the proof of Theorem 1.2.

Proof of Theorem 1.2. applying the elliptic estimate to the third equation in (1.3) and combining Sobolev embedding theorem, we obtain

$$\|w(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C_{16}\|w(\cdot,t)\|_{W^{2,p}(\Omega)} \le C_{17}\|u(\cdot,t)+v(\cdot,t)\|_{L^p(\Omega)} \le C_{18} \tag{4.22}$$

for some positive constants C_{16}, C_{17}, C_{18}. Then, using Moser iteration technique (Appendix A of [27]), there exists a constant $K_5 > 0$ independent of t such that

$$\|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty} + \|w(\cdot,t)\|_{W^{1,\infty}} \le K_5,$$

which together with the extension criterion in Lemma 2.1 proves Theorem 1.2.

5. Asymptotic behavior

In this section, we show the asymptotic behavior of solutions by constructing the proper Lyapunov functionals under some assumptions to prove Theorems 1.3 and 1.4, separately. To achieve our goals and apart from constructing the energy functionals, we first give the following key lemma

Lemma 5.1. (see [2]) *Let (u, v, w) be a nonnegative global bounded classical solution of (1.3) with initial data (u_0, v_0, w_0) satisfying (1.4). Then there exist $\theta \in (0,1)$ and $K_6 > 0$ such that*

$$\|u\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar{\Omega}\times[t,t+1])} + \|v\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar{\Omega}\times[t,t+1])} + \|w\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar{\Omega}\times[t,t+1])} \le K_6 \quad \text{for all } t \ge 1.$$

In addition, Let $f(t) : (1,\infty) \to \mathbb{R}$ be a nonnegative and uniformly continuous function that satisfies $\int_1^\infty f(t)dt < \infty$. Then, $f(t) \to 0$ as $t \to \infty$.

5.1. Proof of Theorem 1.3

Lemma 5.2. *Let $\tau = 1$ and (u, v, w) is global bounded solution of (1.3). Suppose that $\rho_1, \rho_2 > 0$ are arbitrary, then the function*

$$E_1(t) := \int_\Omega u + \rho_1 \int_\Omega (v - 1 - \ln v) + \frac{\rho_2}{2} \int_\Omega (w-1)^2$$

satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_1(t)&\\ \le &-(\mu_1-\rho_1\mu_2)\int_\Omega u^2-(\mu_1a_1+\rho_1\mu_2a_2-2\rho_1\mu_2)\int_\Omega uv-(2\rho_1\mu_2-\mu_1-\rho_1\mu_2a_2)\int_\Omega u\\ &-\left(2\rho_1\mu_2-\frac{k_2\rho_1C_{\chi_2}^2}{4}\right)\int_\Omega|\nabla w|^2-\rho_1\mu_2\int_\Omega(u+v-w)^2-\rho_1\mu_2\int_\Omega(w-1)^2\end{aligned}\tag{5.1}$$

for all $t>0$ when $\rho_2=2\rho_1\mu_2$, where k_2 is defined in (1.5).

Proof. Let $A(t), B(t)$ and $D(t)$ be defined as

$$A_1(t):=\int_\Omega u,\quad B_1(t):=\int_\Omega(v-1-\ln v)\quad\text{ and }\quad G_1(t):=\frac{1}{2}\int_\Omega(w-1)^2.$$

From the first equation in (1.3) and (S_6), by a straightforward calculation that

$$\frac{\mathrm{d}}{\mathrm{d}t}A_1(t)=-\mu_1\int_\Omega u^2+\mu_1\int_\Omega u\,(1-a_1v)\tag{5.2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}G_1(t)=-\int_\Omega|\nabla w|^2+\int_\Omega(w-1)\,(u+v-w)\,.\tag{5.3}$$

Together with (S_1), (S_2) and (S_6), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}B_1(t)&=-\int_\Omega D_2(v)\frac{|\nabla v|^2}{v^2}+\int_\Omega\frac{S_2(v)}{v^2}\nabla v\cdot\nabla w+\mu_2\int_\Omega(v-1)\,(1-v-a_2u)\\ &\le\frac{C_{\chi_2}^2}{4}k_2\int_\Omega|\nabla w|^2+\mu_2\int_\Omega(v-1)\,(1-v-a_2u)\end{aligned}\tag{5.4}$$

by using Young's inequality and the definition of k_2 in (1.5). For arbitrary $\rho_1,\rho_2>0$, a direct linear combination (5.2)$+\rho_2\times$ (5.3)$+\rho_1\times$ (5.4), the consequence thus obtained then reads

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_1(t)\le&-\left(\mu_1-\frac{\rho_2}{2}\right)\int_\Omega u^2-(\mu_1a_1+\rho_1\mu_2a_2-\rho_2)\int_\Omega uv-\left(\rho_2-\frac{k_2\rho_1C_{\chi_2}^2}{4}\right)\int_\Omega|\nabla w|^2\\ &-\int_\Omega\left(\frac{\rho_2}{2}u^2+\rho_1\mu_2v^2+\frac{\rho_2}{2}w^2-\rho_2uw-\rho_2vw+\rho_2uv\right)-\int_\Omega(\frac{\rho_2}{2}w^2-\rho_2w+p_1\mu_2)\\ &-(2\rho_1\mu_2-\rho_2)\int_\Omega v-(\rho_2-\mu_1-\rho_1\mu_2a_2)\int_\Omega u.\end{aligned}$$

Then, let $\rho_2=2\rho_1\mu_2$, it immediately complete the proof of this lemma. □

Next we will use the energy functional constructed in Lemma 5.2 to obtain the large time behavior of solutions to (1.3) when $a_1\ge1>a_2>0$.

Lemma 5.3. *Suppose that the conditions in Theorem 1.3 hold and $\tau = 1$, then there exist some $\rho_1, \sigma_1 > 0$ such that the functions*

$$E_1(t) := \int_\Omega u + \rho_1 \int_\Omega (v - 1 - \ln v) + \rho_1\mu_2 \int_\Omega (w-1)^2$$

and

$$F_1(t) := \int_\Omega u^2 + \int_\Omega (v + u - w)^2 + \int_\Omega (w-1)^2$$

fulfill

$$E_1'(t) \le -\sigma_1 F_1(t) \quad \textit{for all } t > 0. \tag{5.5}$$

Proof. Since $a_1 \ge 1$ and $a_2 < 1$, we have

$$\frac{\mu_1}{\mu_2(2-a_2)} \le \frac{\mu_1 a_1}{\mu_2(2-a_2)} \text{ and } \frac{\mu_1}{\mu_2(2-a_2)} < \frac{\mu_1}{\mu_2}, \tag{5.6}$$

which ensures the existence of ρ_1 satisfying

$$\rho_1 < \frac{\mu_1}{\mu_2} \text{ and } \rho_1 \in \left[\frac{\mu_1}{\mu_2(2-a_2)}, \frac{\mu_1 a_1}{\mu_2(2-a_2)}\right] \tag{5.7}$$

such that $\mu_1 - \rho_1\mu_2 > 0$ and $\min\{\mu_1 a_1 + \rho_1\mu_2 a_2 - 2\rho_1\mu_2, 2\rho_1\mu_2 - \mu_1 - \rho_1\mu_2 a_2\} \ge 0$. Therefore, by choosing suitable ρ_1 and combining (1.6), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1(t) \le -(\mu_1 - \rho_1\mu_2)\int_\Omega u^2 - \rho_1\mu_2 \int_\Omega (u+v-w)^2 - \rho_1\mu_2 \int_\Omega (w-1)^2. \tag{5.8}$$

□

Lemma 5.4. *Suppose that the conditions in Theorem 1.3 hold and $\tau = 0$, $\tau = 1$, then there exist some $\rho_1, \sigma_2 > 0$ such that the function*

$$E_2(t) := \int_\Omega u + \rho_1 \int_\Omega (v - 1 - \ln v)$$

fulfills

$$E_2'(t) \le -\sigma_2 F_1(t) \quad \textit{for all } t > 0, \tag{5.9}$$

where $F_1(t)$ is same in Lemma 5.3.

Proof. Testing the third equation in (1.3) by $w - 1$, we deduce that

$$0 = -\int_\Omega |\nabla w|^2 + \int_\Omega (w-1)(u+v-w), \tag{5.10}$$

then by a direct linear combination (5.2)$+2\rho_1\mu_2\times$(5.10) $+\rho_1\times$ (5.4) and choosing the ρ_1 satisfying (5.6), we can complete the proof of Lemma 5.4 through the same steps as in Lemma 5.3. □

Proof of Theorem 1.3. Because of $s - 1 - \ln s \ge 0$ for all $s > 0$, $E_1(t), E_2(t)$ are nonnegative. Integrating (5.5) or (5.9) over $(0, t)$ and letting $t \to \infty$, we obtain

$$\int_0^\infty \int_\Omega u^2 + \int_0^\infty \int_\Omega (v+u-w)^2 + \int_0^\infty \int_\Omega (w-1)^2 \le K_7$$

with some $K_7 > 0$. Therefore, combining this with Lemma 5.1 then yields

$$\|u(\cdot,t)\|_{L^2(\Omega)} + \|v(\cdot,t) + u(\cdot,t) - w(\cdot,t)\|_{L^2(\Omega)} + \|w(\cdot,t) - 1\|_{L^2(\Omega)} \to 0 \quad \text{as } t \to \infty,$$

this implies

$$\|u(\cdot,t)\|_{L^2(\Omega)}+\|v(\cdot,t)-1\|_{L^2(\Omega)}+\|w(\cdot,t)-1\|_{L^2(\Omega)}\to 0 \quad \text{as } t\to\infty.$$

Invoking the Gagliardo-Nirenberg inequality to find $K_8>0$ fulfilling

$$\|\varphi\|_{L^\infty(\Omega)}\le K_8\|\varphi\|_{W^{1,\infty}(\Omega)}^{n/(n+2)}\|\varphi\|_{L^2(\Omega)}^{2/(n+2)} \quad \text{for all } \varphi\in W^{1,\infty}(\Omega).$$

Applying the above Gagliardo-Nirenberg inequality to u,v,w for $t>0$, and using Lemma 5.1, we conclude that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)-1\|_{L^\infty(\Omega)}+\|w(\cdot,t)-1\|_{L^\infty(\Omega)}\to 0 \quad \text{as } t\to\infty.$$

Hence the proof of Theorem 1.3 is completed.

5.2. Proof of Theorem 1.4

Lemma 5.5. *Let $\tau=1$ and (u,v,w) is global bounded solution of (1.3). Suppose that $\varsigma_1,\varsigma_2>0$ are arbitrary, then the function*

$$E_3(t):=\int_\Omega(u-u_\star-u_\star\ln\frac{u}{u_\star})+\varsigma_1\int_\Omega(v-v_\star-v_\star\ln\frac{v}{v_\star})+\frac{\varsigma_2}{2}\int_\Omega(w-w_\star)^2$$

satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_3(t)\le&-(\mu_1-\varsigma_1\mu_2)\int_\Omega(u-u_\star)^2-(\mu_1a_1+\varsigma_1\mu_2a_2-2\varsigma_1\mu_2)\int_\Omega(u-u_\star)(v-v_\star)\\&-\left(2\varsigma_1\mu_2-\frac{k_1u_\star C_{\chi_1}^2}{4}-\varsigma_1\frac{k_2v_\star C_{\chi_2}^2}{4}\right)\int_\Omega|\nabla w|^2-\varsigma_1\mu_2\int_\Omega(u+v-w)^2\\&-\varsigma_1\mu_2\int_\Omega(w-w_\star)^2\end{aligned}\tag{5.11}$$

for all $t>0$ when $\varsigma_2=2\varsigma_1\mu_2$, where k_1,k_2 are defined in (1.5).

Proof. Let $A_2(t),B_2(t)$ and $G_2(t)$ be defined as

$$A_2(t):=\int_\Omega(u-u_\star-u_\star\ln\frac{u}{u_\star}),\quad B_2(t):=\int_\Omega(v-v_\star-v_\star\ln\frac{v}{v_\star})\quad\text{and}\quad G_2(t):=\frac12\int_\Omega(w-w_\star)^2.$$

Together with (S_1), (S_2) and (S_6), by using Young's inequality and the definitions of k_1,k_2 in (1.5) that we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A_2(t)&=-u_\star\int_\Omega D_1(u)\frac{|\nabla u|^2}{u^2}+\int_\Omega\frac{u_\star S_1(u)}{u^2}\nabla u\cdot\nabla w+\mu_1\int_\Omega(u-u_\star)(1-u-a_1v)\\&\le\frac{C_{\chi_1}^2}{4}k_1u_\star\int_\Omega|\nabla w|^2-\mu_1\int_\Omega(u-u_\star)^2-a_1\mu_1\int_\Omega(u-u_\star)(v-v_\star)\end{aligned}\tag{5.12}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}B_2(t)\le\frac{C_{\chi_2}^2}{4}k_2v_\star\int_\Omega|\nabla w|^2-\mu_2\int_\Omega(v-v_\star)^2-a_2\mu_2\int_\Omega(v-v_\star)(u-u_\star)\tag{5.13}$$

as well as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}G_2(t) &= -\int_\Omega |\nabla w|^2 + \int_\Omega (w-w_\star)(u+v-w)\\ &= -\int_\Omega |\nabla w|^2 - \int_\Omega (w-w_\star)^2 + \int_\Omega (w-w_\star)(u-u_\star) + \int_\Omega (w-w_\star)(v-v_\star).\end{aligned} \tag{5.14}$$

For some $\varsigma_1, \varsigma_2 > 0$, a direct linear combination (5.12)+$\varsigma_1\times$ (5.13)+$\varsigma_2\times$(5.14) and let $\varsigma_2 = 2\varsigma_1\mu_2$, it immediately obtain (5.11) due to $u_\star + v_\star - w_\star = 0$. □

Next we will use the energy functional constructed in Lemma 5.5 to obtain the large time behavior of solutions to (1.3) for $a_1, a_2 < 1$.

Lemma 5.6. *Suppose that the conditions in Theorem 1.4 hold and $\tau = 1$, let $\varsigma_1 := \frac{a_1\mu_1}{\mu_2(2-a_2)}$, then there exists $\sigma_3 > 0$ such that the functions*

$$E_3(t) := \int_\Omega (u - u_\star - u_\star \ln \frac{u}{u_\star}) + \varsigma_1 \int_\Omega (v - v_\star - v_\star \ln \frac{v}{v_\star}) + \varsigma_1\mu_2 \int_\Omega (w-w_\star)^2$$

and

$$F_2(t) := \int_\Omega (u-u_\star)^2 + \int_\Omega (v+u-w)^2 + \int_\Omega (w-w_\star)^2$$

fulfill

$$E_3'(t) \le -\sigma_3 F_2(t) \quad \text{for all } t>0. \tag{5.15}$$

Proof. Since $a_1, a_2 < 1$, we have

$$\varsigma_1 = \frac{a_1\mu_1}{\mu_2(2-a_2)} < \frac{\mu_1}{\mu_2}, \tag{5.16}$$

which ensures that $\mu_1 - \varsigma_1\mu_2 > 0$ and $\mu_1 a_1 + \varsigma_1\mu_2 a_2 - 2\varsigma_1\mu_2 = 0$. Therefore, by choosing this ς_1 and combining (1.7), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_3(t) \le -(\mu_1 - \varsigma_1\mu_2)\int_\Omega (u-u_\star)^2 - \varsigma_1\mu_2 \int_\Omega (u+v-w)^2 - \varsigma_1\mu_2\int_\Omega (w-w_\star)^2. \tag{5.17}$$

□

Lemma 5.7. *Suppose that the condition in Theorem 1.4 hold and $\tau = 0$, let $\varsigma_1 := \frac{a_1\mu_1}{\mu_2(2-a_2)}$, then there exists $\sigma_4 > 0$ such that the function*

$$E_4(t) := \int_\Omega (u - u_\star - u_\star \ln \frac{u}{u_\star}) + \varsigma_1 \int_\Omega (v - v_\star - v_\star \ln \frac{v}{v_\star})$$

fulfills

$$E_4'(t) \le -\sigma_4 F_2(t) \quad \text{for all } t>0, \tag{5.18}$$

where $F_2(t)$ is same in Lemma 5.6.

Proof. Testing the third equation in (1.3) by $w - w_\star$, we deduce that

$$0 = -\int_\Omega |\nabla w|^2 + \int_\Omega (w-w_\star)(u+v-w), \tag{5.19}$$

then by a direct linear combination (5.12)+$\varsigma_1\times$ (5.13) +$2\varsigma_1\mu_2\times$(5.19), we can complete the proof of Lemma 5.7 through the same steps as in Lemma 5.6. □

We are now in the position to prove our main result.

Proof of Theorem 1.4. Firstly, we show the nonnegativity of $E_3(t), E_4(t)$. Let $y(s) := s - u_* \ln s$ for $s > 0$. By applying Taylor's formula, there exists $\sigma \in (0,1)$ such that

$$\begin{aligned} y(u) - y(u_*) &= y'(u_*) \cdot (u - u_*) + \frac{1}{2} y''[\sigma u + (1-\sigma)u_*] \cdot (u - u_*)^2 \\ &= \frac{u_*}{2[\sigma u + (1-\sigma)u_*]^2} (u - u_*)^2 \geq 0 \end{aligned}$$

for $x \in \Omega$ and $t > 0$, which implies that $A_2(t) = \int_\Omega (y(u) - y(u_*)) \geq 0$. Similarly, we can obtain $B_2(t) \geq 0$ for all $t \geq 0$. Thus, $E_3(t)$ and $E_4(t)$ are nonnegative. Integrating (5.15) or (5.18) over $(0,t)$ and letting $t \to \infty$, we obtain

$$\int_0^\infty \int_\Omega (u - u_\star)^2 + \int_0^\infty \int_\Omega (v + u - w)^2 + \int_0^\infty \int_\Omega (w - w_\star)^2 \leq K_8$$

with some $K_8 > 0$. combining this with Lemma 5.1 then yields

$$\|u(\cdot,t) - u_\star\|_{L^2(\Omega)} + \|v(\cdot,t) + u(\cdot,t) - w(\cdot,t)\|_{L^2(\Omega)} + \|w(\cdot,t) - w_\star\|_{L^2(\Omega)} \to 0 \quad \text{as } t \to \infty,$$

this implies

$$\|u(\cdot,t) - u_\star\|_{L^2(\Omega)} + \|v(\cdot,t) - v_\star\|_{L^2(\Omega)} + \|w(\cdot,t) - w_\star\|_{L^2(\Omega)} \to 0 \quad \text{as } t \to \infty.$$

Applying the above Gagliardo-Nirenberg inequality to u, v, w for $t > 0$, and using Lemma 5.1, we conclude that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t) - 1\|_{L^\infty(\Omega)} + \|w(\cdot,t) - 1\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty.$$

Hence the proof of Theorem 1.4 is completed.

Acknowledgements

The authors would like to deeply thank the editor and anonymous reviewers for their insightful and constructive comments.

Author contributions YH contributed to writing, editing and revising

Funding Not applicable.

Data availability Not applicable.

Declarations

Conflict of interest The author declares that there is no conflict of interest.

Ethical approval Not applicable.

References

[1] Amann, H.: Dynamic theory of quasilinear parabolic equations. II. Reaction–diffusion systems. Differ. Integral Equ. **3**, 13–75 (1990)

[2] Bai, X., Winkler, M.: Equilibration in a fully parabolic two-species chemotaxis system with competitive kinetics. Indiana Univ. Math. J. **65**, 553–583 (2016)

[3] Bellomo, N., Bellouquid, A., Tao, Y., Winkler, M.: Towards a mathematical theory of Keller–Segel models of pattern formation in biological tissues. Math. Models Methods Appl. Sci. **25**, 1663–1763 (2015)

[4] Cieślak, T., Stinner, C.: Finite-time blowup and global-in-time unbounded solutions to a parabolic-parabolic quasilinear Keller–Segel system in higher dimensions. J. Differ. Equ. **252**, 5832–5851 (2012)

[5] Cieślak, T., Stinner, C.: Finite-time blowup in a supercritical quasilinear parabolic-parabolic Keller–Segel system in dimension 2. Acta Appl. Math. **129**, 135–146 (2014)

[6] Cieślak, T., Stinner, C.: New critical exponents in a fully parabolic quasilinear Keller–Segel system and applications to volume filling models. J. Differ. Equ. **258**, 2080–2113 (2015)

[7] Ding, M., Wang, W., Zhou, S., Zheng, S.: Asymptotic stability in a fully parabolic quasilinear chemotaxis model with general logistic source and signal production. J. Differ. Equ. **268**(11), 6729–6777 (2020)

[8] Frassu, S., Viglialoro, G.: Boundedness for a fully parabolic Keller–Segel model with sublinear segregation and superlinear aggregation. Acta Appl. Math. **171**, 1–20 (2021)

[9] Horstmann, D., Winkler, M.: Boundedness vs. blow-up in a chemotaxis system. J. Differ. Equ. **215**, 52–107 (2005)

[10] Keller, E.F., Segel, L.A.: Initiation of slime mold aggregation viewed as an instability. J. Theor. Biol. **26**, 399–415 (1970)

[11] Hu, R., Zheng, P.: On a quasilinear fully parabolic attraction or repulsion chemotaxis system with nonlinear signal production. Discrete Contin. Dyn. Syst. Ser. B. **27**(12), 7227–7244 (2022)

[12] Li, X.: Global existence and boundedness of a chemotaxis model with indirect production and general kinetic function. Z. Angew. Math. Phys. **71**, 1–22 (2020)

[13] Liu, D., Tao, Y.: Boundedness in a chemotaxis system with nonlinear signal production. Appl. Math. J. Chin. Univ. Ser. B **31**(4), 379–388 (2016)

[14] Mizukami, M.: Boundedness and asymptotic stability in a two-species chemotaxis-competition model with signal-dependent sensitivity. Discrete Contin. Dyn. Syst. Ser. B **22**, 2301–2319 (2017)

[15] Mizukami, M.: Boundedness and stabilization in a two-species chemotaxis-competition system of parabolic-parabolic-elliptic type. Math. Methods Appl. Sci. **41**, 234–249 (2018)

[16] Mizukami, M.: Improvement of conditions for asymptotic stability in a two-species chemotaxis competition model with signal-dependent sensitivity. Discrete Contin. Dyn. Syst. Ser. S. **13**, 269–278 (2020)

[17] Negreanu, M., Tello, J.I.: On a two species chemotaxis model with slow chemical diffusion. SIAM J. Math. Anal. **46**, 3761–3781 (2014)

[18] Negreanu, M., Tello, J.I.: Asymptotic stability of a two species chemotaxis system with non-diffusive chemoattractant. J. Differ. Equ. **258**, 1592–1617 (2015)

[19] Nirenberg, L.: An extended interpolation inequality. Ann. Scuola Norm-Sci. **20**, 733–737 (1966)

[20] Pan, X., Mu, C., Tao, W.: On the strongly competitive case in a fully parabolic two-species chemotaxis system with Lotka–Volterra competitive kinetics. J. Differ. Equ. **354**, 90–132 (2023)

[21] Pan, X., Wang, L.: Boundedness and asymptotic stability in a quasilinear two-species chemotaxis system with nonlinear signal production. Commun. Pure Appl. Anal. **20**(6), 2211–2236 (2021)

[22] Pan, X., Wang, L.: Improvement of conditions for boundedness in a fully parabolic chemotaxis system with nonlinear signal production. C. R. Math. **359**(2), 161–168 (2021)

[23] Shan, W., Zheng, P.: Boundedness and asymptotic behavior in a quasilinear chemotaxis system for alopecia areata. Nonlinear Anal. RWA **72**, 103858 (2023)

[24] Shan, W., Zheng, P.: Global boundedness of the immune chemotaxis system with general kinetic functions. NODEA-Nonlinear Differ. **30**, 29 (2023)

[25] Stinner, C., Tello, J.I., Winkler, M.: Competitive exclusion in a two-species chemotaxis model. J. Math. Biol. **68**, 1607–1626 (2014)

[26] Tao, X., Zhou, S., Ding, M.: Boundedness of solutions to a quasilinear parabolic–parabolic chemotaxis model with nonlinear signal production. J. Math. Anal. Appl. **474**(1), 733–747 (2019)

[27] Tao, Y., Winkler, M.: Boundedness in a quasilinear parabolic-parabolic Keller–Segel system with subcritical sensitivity. J. Differ. Equ. **252**, 692–715 (2012)

[28] Tello, J.I., Winkler, M.: A chemotaxis system with logistic source. Commun. Partial Differ. Equ. **32**, 849–877 (2007)

[29] Tello, J.I., Winkler, M.: Stabilization in a two-species chemotaxis system with a logistic source. Nonlinearity **25**, 1413–1425 (2012)

[30] Temam, R.: Infinite-Dimensional Dynamical Systems in Mechanics and Physics, vol. 68, 2nd edn. Springer, New York (1997)

[31] Tian, M., Zheng, S.: Global boundedness versus finite-time blow-up of solutions to a quasilinear fully parabolic Keller–Segel system of two species. Commun. Pure Appl. Anal. **15**, 243–260 (2016)
[32] Tu, X., Mu, C., Zheng, P., Lin, K.: Global dynamics in a two-species chemotaxis-competition system with two signals. Discrete Contin. Dyn. Syst. **38**, 3617–3636 (2018)
[33] Viglialoro, G., Woolley, T.E.: Solvability of a Keller–Segel system with signal-dependent sensitivity and essentially sublinear production. Appl. Anal. **99**(14), 2507–2525 (2020)
[34] Wang, W., Zhang, M., Zheng, S.: Positive effects of repulsion on boundedness in a fully parabolic attraction–repulsion chemotaxis system with logistic source. J. Differ. Equ. **264**, 2011–2027 (2018)
[35] Wang, X., Xiang, T., Zhang, N.: Dynamics in a quasilinear parabolic–elliptic Keller–Segel system with generalized logistic source and nonlinear secretion. In: Proceedings of the First International Forum on Financial Mathematics and Financial Technology, pp. 177–206. Springer (2021)
[36] Winkler, M.: A critical blow-up exponent in a chemotaxis system with nonlinear signal production. Nonlinearity **31**(5), 2031–2056 (2018)
[37] Winkler, M.: A family of mass-critical Keller–Segel systems. Proc. Lond. Math. Soc. **124**(3), 133–181 (2022)
[38] Winkler, M.: Aggregation vs. global diffusive behavior in the higher-dimensional Keller–Segel model. J. Differ. Equ. **248**, 2889–2905 (2010)
[39] Winkler, M.: Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source. Commun. Partial Differ. Equ. **35**, 1516–1537 (2010)
[40] Winkler, M.: Does a 'volume-filling effect' always prevent chemotactic collapse? Math. Methods Appl. Sci. **33**(1), 12 (2010)
[41] Winkler, M.: Global classical solvability and generic infinite-time blow-up in quasilinear Keller–Segel systems with bounded sensitivities. J. Differ. Equ. **266**, 8034–8066 (2019)
[42] Xiang, T.: Sub-logistic source can prevent blow-up in the 2D minimal Keller–Segel chemotaxis system. J. Math. Phys. **59**, 081502 (2018)
[43] Yu, H., Wang, W., Zheng, S.: Criteria on global boundedness versus finite time blow-up to a two-species chemotaxis system with two chemicals. Nonlinearity **31**, 502–514 (2018)
[44] Zhang, Q.: Competitive exclusion for a two-species chemotaxis system with two chemicals. Appl. Math. Lett. **83**, 27–32 (2018)
[45] Zheng, J.: Boundedness of solutions to a quasilinear parabolic–elliptic Keller–Segel system with logistic source. J. Differ. Equ. **259**, 120–140 (2015)
[46] Zheng, P., Hu, R., Shan, W.: On a two-species attraction–repulsion chemotaxis system with nonlocal terms. J. Nonlinear Sci. **33**, 57 (2023)
[47] Zheng, P., Shan, W.: Global boundedness and stability analysis of the quasilinear immune chemotaxis system. J. Differ. Equ. **344**, 556–607 (2023)

Yifeng Huili
College of Science
Chongqing University of Posts and Telecommunications
Chongqing 400065
People's Republic of China
e-mail: 1543206819@qq.com

(Received: April 9, 2024; revised: August 31, 2024; accepted: September 4, 2024)