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Global existence and exponential decay of strong solutions to the 3D nonhomogeneous nematic liquid crystal flows with density-dependent viscosity

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Abstract. In this paper, we consider an initial and boundary value problem to the three-dimensional (3D) nonhomogeneous nematic liquid crystal flows with density-dependent viscosity and vacuum. Combining delicate energy method with the structure of the system under consideration, the global well-posedness of strong solutions is established, provided that $\|\rho_0\|_{L^1} + \|\nabla \boldsymbol{d}_0\|_{L^2}$ is suitably small. In particular, the initial velocity can be arbitrarily large. Moreover, the exponential decay rates of the strong solution are also obtained.

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1. Introduction and main result

Let $\Omega \subset \mathbb{R}^3$ be a bounded smooth domain. The motion of the nonhomogeneous nematic liquid crystal flows is governed by the following simplified version of the Ericksen–Leslie equations with density-dependent viscosity in $\Omega \times (0, T]$:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) = 0, \\ \partial_t(\rho \boldsymbol{u}) + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \mathrm{div}(2\mu(\rho)\mathfrak{D}(\boldsymbol{u})) + \nabla P = -\mathrm{div}(\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d}), \\ \boldsymbol{d}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{d} = \Delta \boldsymbol{d} + |\nabla \boldsymbol{d}|^2 \boldsymbol{d}, \\ \mathrm{div} \boldsymbol{u} = 0, \quad |\boldsymbol{d}| = 1, \end{cases} \tag{1.1}$$

where $\rho, \boldsymbol{u}, \boldsymbol{d}$ and P are the density of the fluid, velocity, macroscopic average of the nematic liquid crystal orientation and pressure, respectively. The deformation tensor $\mathfrak{D}(\boldsymbol{u})$ is given by

$$\mathfrak{D}(\boldsymbol{u}) = \frac{1}{2}\left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right].$$

The viscosity coefficient $\mu = \mu(\rho)$ is a general function of density, which is assumed to satisfy

$$\mu \in C^1[0, \infty), \text{ and } \mu \geq \underline{\mu} > 0 \text{ on } [0, \infty) \tag{1.2}$$

for some positive constant $\underline{\mu}$. The notation $\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d}$ denotes the 3×3 matrix whose ij component is given by $\partial_i \boldsymbol{d} \cdot \partial_j \boldsymbol{d}, i, j = 1, 2, 3$.

We seek for the solutions to the system (1.1) with the following initial and boundary conditions:

$$(\rho, \rho \boldsymbol{u}, \boldsymbol{d})|_{t=0} = (\rho_0, \rho_0 \boldsymbol{u}_0, \boldsymbol{d}_0)(x), \qquad x \in \Omega, \tag{1.3}$$

$$(\boldsymbol{u}, \boldsymbol{d}) = (\boldsymbol{0}, \boldsymbol{d}_0), \qquad x \in \partial\Omega, t > 0. \tag{1.4}$$

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Here $\boldsymbol{d}_0 : \overline{\Omega} \to \mathbf{S}^2$ is a given vector satisfying $\nabla \boldsymbol{d}_0 = \mathbf{0}$ on the boundary $\partial\Omega$ (see more details of this fact in [9]).

The above system (1.1) describes the macroscopic evolution for the nematic liquid crystals. It is a simplified version of the Ericksen–Leslie model [4,11], but it still retains most important mathematical structures as well as most of the essential difficulties of the original Ericksen–Leslie model. For more details on the hydrodynamic continuum theory of liquid crystals, we refer the readers to the monographs [1,6]. Mathematically, system (1.1) is a coupling between the nonhomogeneous incompressible Navier–Stokes equations and the transported heat flows of harmonic map, and thus, its mathematical analysis is full of challenges.

When $\boldsymbol{d}$ is a constant vector satisfying $|\boldsymbol{d}| = 1$, system (1.1) reduces to the nonhomogeneous incompressible Navier–Stokes equations with density-dependent viscosity. As pointed out in many papers, the strong interaction between density and velocity will bring some difficulties in the mathematical analysis due to the density-dependent viscosity. When the initial vacuum is taken into account, Lions [17] established the global existence of weak solutions to the nonhomogeneous incompressible Navier–Stokes equations. Later, Cho and Kim [2] constructed a unique local strong solution by imposing the following compatibility condition on the initial data:

$$-\operatorname{div}(2\mu(\rho_0))\mathfrak{D}(\boldsymbol{u}_0)) + \nabla P_0 = \sqrt{\rho_0}\boldsymbol{g} \tag{1.5}$$

for some $(P_0, \boldsymbol{g}) \in H^1 \times L^2$. Recently, Huang and Wang [10], and independently by Zhang [29], obtained the global existence and uniqueness of strong solution of Navier–Stokes equations provided that $\|\nabla \boldsymbol{u}_0\|_{L^2}$ is suitably small.

Let us come back to the system (1.1). Compared with the nonhomogeneous incompressible Navier–Stokes equations, due to the strong coupling and interaction between the fluid motion and the macroscopic orientation vector, the mathematical analysis on the system (1.1) will become more subtle. When the viscosity μ is a positive constant, there is a huge literature on the studies of the well-posedness of solutions to (1.1). For the initial density away from vacuum, Wen and Ding [26] established the global existence and uniqueness of strong solution to the 2D problem with small initial energy $\|\sqrt{\rho_0}\boldsymbol{u}_0\|_{L^2}^2 + \|\nabla \boldsymbol{d}_0\|_{L^2}^2$. J. Li [12] obtained the same result of the 2D problem for large initial data under a geometric condition the initial direction field $\boldsymbol{d}_0 = (d_{01}, d_{02}, d_{03})$:

$$d_{03} \geq \varepsilon_0, \text{ for some positive } \varepsilon_0. \tag{1.6}$$

Meanwhile, X. Li and Wang [16] obtained the global strong solution for small initial data, and they also established the weak–strong uniqueness. On the other hand, if the initial density allows to vanish, Wen and Ding [26] established the local well-posedness of strong solution under the assumption that the initial data satisfy a similar compatibility condition as (1.5). Ding et. al [3] and J. Li [13] extended this local strong solution in 3D to global in time for some small initial data. Yu and Zhang [28] established the global well-posedness of strong solution in 3D for small initial energy with Nuemann boundary condition for the macroscopic orientation field. Recently, assuming that the initial orientation field satisfies a geometric condition (1.6), Liu and Zhang [19] and Liu et. al [18] established the global well-posedness of strong solution to the 2D Cauchy problem with large initial data for positive and zero far field density at space infinity, respectively. Meanwhile, Li et. al [14] obtained the same result under small initial data for the 2D Cauchy problem with zero far field density if the initial density decays not too slow at infinity. When the viscosity is a function of density, the analysis becomes more difficult due to the strong coupling of viscosity and velocity field. Gao et. al [7] established the local well-posedness of strong solution in a bounded domain under a compatibility condition on the initial data. In addition, they also obtained the Serrin-type blow-up criterion. Subsequently, Liu [20,21] proved the global existence and uniqueness of strong solution in 2D/3D for some small initial data (see also Liu and Zhong [22]). Very recently, Ye and Zhu [27] established the global well-posedness of strong solution under the initial norm $\|\boldsymbol{u}_0\|_{\dot{H}^\alpha} + \|\nabla \boldsymbol{d}_0\|_{\dot{H}^\alpha}(1/2 < \alpha \leq 1)$ being suitably small.

The purpose of this paper is to establish the global strong solutions to the 3D incompressible nematic liquid crystal flows with density-dependent viscosity, provided that $\|\rho_0\|_{L^1}+\|\nabla \boldsymbol{d}_0\|_{L^2}$ is suitably small allowing large oscillation of the velocity.

Before stating our main result, we first explain the notations and conventions used throughout this paper. For $p\in[1,\infty]$ and integer $k\in\mathbb{N}_+$, we use $L^p=L^p(\Omega)$ and $W^{k,p}=W^{k,p}(\Omega)$ to denote the standard Lebesgue and Sobolev spaces, respectively. When $p=2$, we use $H^k=W^{k,2}(\Omega)$. The space $H^1_{0,\sigma}$ stands for the closure in H^1 of the space $C^\infty_{0,\sigma}\triangleq\{\boldsymbol{\phi}\in C^\infty_0|\mathrm{div}\boldsymbol{\phi}=0\}$. And for two 3×3 matrices $A=(A_{ij})$ and $B=(B_{ij})$, we denote by

$$A:B=\sum_{i,j=1}^{3}A_{ij}B_{ij}.$$

Now we state our main result for the problem (1.1)-(1.4) as follows.

Theorem 1.1. *For $\bar\rho>0$ and $q\in(3,\infty)$, assume that the initial data $(\rho_0,\boldsymbol{u}_0,\boldsymbol{d}_0)$ satisfies*

$$\begin{cases} 0\le\rho_0\le\bar\rho,\rho_0\in H^1,\nabla\mu(\rho_0)\in L^q,\\ \boldsymbol{u}_0\in H^1_{0,\sigma},\boldsymbol{d}_0\in H^2,\|\nabla\boldsymbol{d}_0\|_{L^2}\le\|\rho_0\|_{L^1}^{\frac13}. \end{cases}\tag{1.7}$$

Then, there exists a small positive constant ε_0 depending only on $\Omega,\overline{\mu}\triangleq\sup\limits_{[0,\bar\rho]}\mu(\rho),\underline{\mu},\ q,\bar\rho,\|\nabla\mu(\rho_0)\|_{L^q}$, $\|\nabla\boldsymbol{u}_0\|_{L^2}$ and $\|\nabla^2\boldsymbol{d}_0\|_{L^2}$, such that if

$$\|\rho_0\|_{L^1}+\|\nabla\boldsymbol{d}_0\|_{L^2}\le\varepsilon_0,\tag{1.8}$$

the problem (1.1)-(1.4) admits a unique global strong solution $(\rho,\boldsymbol{u},\boldsymbol{d},P)$ satisfying, for any $3<r<\min\{6,q\}$ and $\tau>0$,

$$\begin{cases} \rho\in L^\infty(0,\infty;H^1)\cap C([0,\infty);H^1),\rho_t\in C([0,\infty);L^{\frac32}),\\ \boldsymbol{u}\in L^\infty(0,\infty;H^1)\cap L^\infty(\tau,\infty;H^2)\cap L^2(\tau,\infty;W^{2,r}),\\ \boldsymbol{d}\in L^\infty(0,\infty;H^2)\cap L^\infty(\tau,\infty;H^3)\cap L^2(\tau,\infty;H^4),\\ \nabla\boldsymbol{u},\nabla^2\boldsymbol{d}\in C([\tau,\infty);L^2),\rho\boldsymbol{u},\nabla\boldsymbol{d}\in C([0,\infty);L^2),\\ t\sqrt{\rho}\boldsymbol{u}_t,t\nabla\boldsymbol{d}_t\in L^\infty(0,\infty;L^2),t\nabla\boldsymbol{u}_t,t\nabla^2\boldsymbol{d}_t\in L^2(0,\infty;L^2). \end{cases}\tag{1.9}$$

Moreover, it holds that

$$\sup_{[0,T]}\|\nabla\mu(\rho)\|_{L^q}\le 2\|\nabla\mu(\rho_0)\|_{L^q},\tag{1.10}$$

and there exists some positive constant C depending only on $\Omega,\overline{\mu},\underline{\mu},q,\bar\rho,\ \|\nabla\mu(\rho_0)\|_{L^q},\|\nabla\boldsymbol{u}_0\|_{L^2}$ and $\|\nabla^2\boldsymbol{d}_0\|_{L^2}$, such that, for $t\ge1$,

$$\|\boldsymbol{u}(\cdot,t)\|_{H^2}^2+\|\nabla\boldsymbol{d}(\cdot,t)\|_{H^2}^2+\|\sqrt{\rho}\boldsymbol{u}_t(\cdot,t)\|_{L^2}^2+\|\nabla\boldsymbol{d}_t(\cdot,t)\|_{L^2}^2\le Ce^{-\sigma t}.\tag{1.11}$$

Here, $\sigma\triangleq\min\{\frac{\underline{\mu}}{\bar\rho l^2},\frac{1}{l^2}\}$ with l being the diameter of Ω.

Remark 1.1. Since Ω is a bounded smooth domain, we deduce from Hölder's inequality that

$$\|\rho_0\|_{L^1}+\|\nabla\boldsymbol{d}_0\|_{L^2}\le C(\Omega)(\|\rho_0\|_{L^\infty}+\|\nabla\boldsymbol{d}_0\|_{L^3}).$$

Thus, our Theorem 1.1 improves Liu's result [20]. Moreover, by modifying the proof of this paper slightly, similar result holds true for the case of 2D bounded domains. Hence, we also generalize Liu's result [21].

We mainly use the continuation argument to give a proof of Theorem 1.1. Since the local strong solution was obtained by Lemma 2.1, the key issue is to establish global *a priori* estimates on strong solutions to (1.1)-(1.4) in suitable higher-order norms. Due to the strong interaction between viscosity and velocity, the method used for the constant viscosity case cannot be applied here directly. Moreover, compared with the previous work on nonhomogeneous Navier–Stokes equations with density-dependent viscosity [28,31], the proof of Theorem 1.1 is much more involved due to the strong coupling between the velocity and the macroscopic orientation vector. Hence, some new ideas are needed to overcome these difficulties.

Firstly, motivated by the work of [8], we find that $\|\sqrt{\rho}\boldsymbol{u}\|_{L^2}^2$ and $\|\nabla \boldsymbol{d}\|_{L^2}^2$ decay at the rate of $e^{\sigma t}$ for some σ depending only on $\bar{\rho}, \underline{\mu}$ and Ω with the help of Poincaré's inequality and Sobolev's inequality (see (3.8)). Next, we attempt to obtain the uniform in time-weighted estimates of $\|\nabla \boldsymbol{u}\|_{L^2}^2$ and $\|\nabla^2 \boldsymbol{d}\|_{L^2}^2$. To overcome the difficulties caused by the density-dependent viscosity and strongly coupling between the velocity and macroscopic orientation vector, we assume the condition (3.2) holds. Moreover, regularity properties of the Stokes system and elliptic equations play important roles. Then we obtain the desired bounds of $\|\nabla \boldsymbol{u}\|_{L^2}^2$ and $\|\nabla^2 \boldsymbol{d}\|_{L^2}^2$(see Lemma 3.3). These bounds are crucial in deriving time-weighted estimates of $L^\infty(0,T;L^2)$-norms of $\sqrt{\rho}\boldsymbol{u}_t$ and $\nabla \boldsymbol{d}_t$. The next step is to show the quantity $\|\nabla\mu(\rho)\|_{L^q}$ is in fact less than $2\|\nabla\mu(\rho_0)\|_{L^q}$. To this end, it needs to deal with $\|\nabla \boldsymbol{u}\|_{L^1(0,T;L^\infty)}$. Based on the time-weighted estimates (Lemmas 3.1-3.4), we find that the uniform bound (with respect to time) on the $L^1(0,T;L^\infty)$-norm of $\nabla \boldsymbol{u}$ is bounded by the initial mass and L^2-norm of $\nabla \boldsymbol{d}_0$. This completes the proof of (3.3) provided that the assumption (1.8) stated in Theorem 1.1 holds. Finally, the higher-order estimates on solutions are obtained (see Lemmas 3.6-3.7).

The remaining parts of this paper are arranged as follows. In Section 2, we shall give some auxiliary lemmas which are useful in later analysis. In Section 3, we establish some necessary *a priori* estimates to extend the local strong solution. Finally, we give the proof of the main result Theorem 1.1 in Section 4.

2. Preliminaries

In this section, we shall recall some known facts and elementary inequalities that will be used extensively later.

We start with the local existence and uniqueness of strong solutions whose proof can be performed in a similar way as [15,24].

Lemma 2.1. *Assume that* $(\rho_0, \boldsymbol{u}_0, \boldsymbol{d}_0)$ *satisfies* (1.7)*. Then, there exist a small time* $T_0 > 0$ *and a unique strong solution* $(\rho, \boldsymbol{u}, \boldsymbol{d}, P)$ *to the problem* (1.1)-(1.4) *in* $\Omega \times (0, T_0]$.

Next, the following Gagliardo–Nirenberg inequality (see [23, Theorem 10.1, p.27]) will be useful in the next section.

Lemma 2.2. *For* $p \in [2,6], q \in (1,+\infty)$, *and* $r \in (3,+\infty)$, *there exists some generic constant* C *which may depend only on* p, q *and* r, *such that for* $f \in H_0^1, g \in L^q \cap D^{1,r}$, *the following inequalities hold.*

$$\|f\|_{L^p} \le C\|f\|_{L^2}^{\frac{6-p}{2p}} \|\nabla f\|_{L^2}^{\frac{3p-6}{2p}}, \tag{2.1}$$

and

$$\|g\|_{L^\infty} \le C\|g\|_{L^q} + C\|\nabla g\|_{L^r}. \tag{2.2}$$

Finally, the following regularity results for the Stokes system will be used frequently in deriving the higher-order estimates. Refer to [10, Lemma 2.1] for the proof.

Lemma 2.3. *For constants $q > 3, \underline{\mu}, \bar{\mu} > 0$, in addition to (1.2), the function μ satisfies*

$$\nabla \mu(\rho) \in L^q, \ \ \underline{\mu} \le \mu(\rho) \le \bar{\mu}.$$

Assume that $(\boldsymbol{u}, P) \in H^1_{0,\sigma} \times L^2$ is the unique weak solution to the following problem

$$\begin{cases} -\operatorname{div}(\mu(\rho)(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)) + \nabla P = \mathbf{F}, & x \in \Omega, \\ \operatorname{div} \boldsymbol{u} = 0, & x \in \Omega, \\ \boldsymbol{u} = 0, & x \in \partial\Omega. \end{cases} \tag{2.3}$$

Then, there exists a positive constant C depending only on $\Omega, \underline{\mu}$ and $\bar{\mu}$ such that the following regularity results hold true:

- *If $\mathbf{F} \in L^2$, then $(\boldsymbol{u}, P) \in H^2 \times H^1$ and*

$$\|\boldsymbol{u}\|_{H^2} + \|P/\mu(\rho)\|_{H^1} \le C\|\mathbf{F}\|_{L^2} \left(1 + \|\nabla \mu(\rho)\|_{L^q}^{\frac{q}{q-3}}\right). \tag{2.4}$$

- *If $\mathbf{F} \in L^r$ for some $r \in (2, q)$, then $(\boldsymbol{u}, P) \in W^{2,r} \times W^{1,r}$ and*

$$\|\boldsymbol{u}\|_{W^{2,r}} + \|P/\mu(\rho)\|_{W^{1,r}} \le C\|\mathbf{F}\|_{L^r} \left(1 + \|\nabla \mu(\rho)\|_{L^q}^{\frac{q(5r-6)}{2r(q-3)}}\right). \tag{2.5}$$

3. A priori estimates

In this section, we will establish some necessary *a priori* bounds for strong solution $(\rho, \boldsymbol{u}, \boldsymbol{d})$ of the problem (1.1)-(1.4) to extend the local strong solution guaranteed by Lemma 2.1. Thus, let $T > 0$ be a fixed time and $(\rho, \boldsymbol{u}, \boldsymbol{d})$ be the strong solution to (1.1)-(1.4) on $\Omega \times (0, T]$ with initial data $(\rho_0, \boldsymbol{u}_0, \boldsymbol{d}_0)$ satisfying (1.7). Before proceeding further, we rewrite another equivalent form of the system (1.1) as follows.

$$\begin{cases} \rho_t + \boldsymbol{u} \cdot \nabla \rho = 0, \\ \rho \boldsymbol{u}_t + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \operatorname{div}(2\mu(\rho)\mathfrak{D}(\boldsymbol{u})) + \nabla P = -\operatorname{div}(\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d}), \\ \boldsymbol{d}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{d} = \Delta \boldsymbol{d} + |\nabla \boldsymbol{d}|^2 \boldsymbol{d}, \\ \operatorname{div} \boldsymbol{u} = 0, \quad |\boldsymbol{d}| = 1. \end{cases} \tag{3.1}$$

In what follows, we denote by

$$\int \cdot dx = \int_\Omega \cdot dx.$$

We give the following key *a priori* estimates on $(\rho, \boldsymbol{u}, \boldsymbol{d}, P)$.

Proposition 3.1. *There exists some positive constant ε_0 depending only on $\Omega, \overline{\mu}, \underline{\mu}, q, \bar{\rho}$, $\|\nabla \mu(\rho_0)\|_{L^q}$, $\|\nabla \boldsymbol{u}_0\|_{L^2}$ and $\|\nabla^2 \boldsymbol{d}_0\|_{L^2}$ such that if $(\rho, \boldsymbol{u}, \boldsymbol{d})$ is a strong solution to (1.1)-(1.4) on $\overline{\Omega} \times (0, T]$ satisfying*

$$\begin{cases} \sup\limits_{[0,T]} \|\nabla \boldsymbol{d}\|_{L^3}^3 \le 2m_0^{\frac{1}{6}}, \quad \sup\limits_{[0,T]} \|\nabla \mu(\rho)\|_{L^q} \le 4\|\nabla \mu(\rho_0)\|_{L^q}, \\ \int_0^T (\|\nabla \boldsymbol{u}\|_{L^2}^4 + \|\nabla^2 \boldsymbol{d}\|_{L^2}^4) dt \le 2m_0^{\frac{1}{3}}, \end{cases} \tag{3.2}$$

then the following estimates hold

$$\begin{cases} \sup\limits_{[0,T]} \|\nabla \boldsymbol{d}\|_{L^3}^3 \le m_0^{\frac{1}{6}}, \quad \sup\limits_{[0,T]} \|\nabla \mu(\rho)\|_{L^q} \le 2\|\nabla \mu(\rho_0)\|_{L^q}, \\ \int_0^T (\|\nabla \boldsymbol{u}\|_{L^2}^4 + \|\nabla^2 \boldsymbol{d}\|_{L^2}^4) dt \le m_0^{\frac{1}{3}}, \end{cases} \tag{3.3}$$

provided that

$$m_0 \le \varepsilon_0. \tag{3.4}$$

Here, $m_0 \triangleq \|\rho_0\|_{L^1}$ *denotes the initial total mass.*

The proof of Proposition 3.1 consists of a series of lemmas. In the following, we will use the convention that C denotes some generic positive constant which may depend on $\Omega, \overline{\mu}, \underline{\mu}, q, \bar{\rho}$ and initial data.

First of all, due to the transport equation $(3.1)_1$, we have the following estimate on the $L^\infty(0,T;L^\infty)$-norm of the density, whose proof can be found in [17, Theorem 2.1].

Lemma 3.1. *It holds that for any* $t \in [0,T]$,

$$\|\rho(t)\|_{L^\infty} \le \|\rho_0\|_{L^\infty}. \tag{3.5}$$

Next, we give the following standard energy estimate of the system (3.1), which reads as follows.

Lemma 3.2. *Under the condition* (3.2), *it holds that*

$$\sup_{[0,T]} (\|\sqrt{\rho}\boldsymbol{u}\|_{L^2}^2 + \|\nabla \boldsymbol{d}\|_{L^2}^2) + \int_0^T \left(\underline{\mu}\|\nabla \boldsymbol{u}\|_{L^2}^2 + \|\Delta \boldsymbol{d}\|_{L^2}^2\right) dt \le C m_0^{\frac{2}{3}}, \tag{3.6}$$

provided that

$$m_0 \le \left(\frac{1}{2C_1}\right)^9, \tag{3.7}$$

where C_1 *is defined as in* (3.12) *depending only on* Ω. *Moreover, for* $\sigma \triangleq \min\left\{\frac{\underline{\mu}}{\bar{\rho} l^2}, \frac{1}{l^2}\right\}$ *with* l *being the diameter of* Ω, *one has that*

$$\sup_{[0,T]} [e^{\sigma t}(\|\sqrt{\rho}\boldsymbol{u}\|_{L^2}^2 + \|\nabla \boldsymbol{d}\|_{L^2}^2)] + \int_0^T e^{\sigma t} \left(\underline{\mu}\|\nabla \boldsymbol{u}\|_{L^2}^2 + \|\Delta \boldsymbol{d}\|_{L^2}^2\right) dt \le C m_0^{\frac{2}{3}}. \tag{3.8}$$

Proof. Multiplying $(3.1)_2$ by $\boldsymbol{u}$ and integrating over Ω, we deduce from integration by parts that

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}\boldsymbol{u}\|_{L^2}^2 + \int 2\mu(\rho)\mathfrak{D}(\boldsymbol{u}) : \nabla \boldsymbol{u} dx = -\int \boldsymbol{u} \cdot \nabla \boldsymbol{d} \cdot \Delta \boldsymbol{d} dx. \tag{3.9}$$

Multiplying $(3.1)_3$ by $-(\Delta \boldsymbol{d} + |\nabla \boldsymbol{d}|^2 \boldsymbol{d})$ and integrating over Ω yield

$$\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{d}\|_{L^2}^2 + \int |\Delta \boldsymbol{d} + |\nabla \boldsymbol{d}|^2 \boldsymbol{d}|^2 dx = \int \boldsymbol{u} \cdot \nabla \boldsymbol{d} \cdot \Delta \boldsymbol{d} dx. \tag{3.10}$$

Combining (3.10) and (3.9), we obtain that

$$\frac{1}{2}\frac{d}{dt}\left(\|\sqrt{\rho}\boldsymbol{u}\|_{L^2}^2 + \|\nabla \boldsymbol{d}\|_{L^2}^2\right) + 2\int \mu(\rho)\mathfrak{D}(\boldsymbol{u}) : \nabla \boldsymbol{u} dx + \|\Delta \boldsymbol{d} + |\nabla \boldsymbol{d}|^2 \boldsymbol{d}\|_{L^2}^2 = 0. \tag{3.11}$$

Noting that

$$\begin{aligned} 2\int \mu(\rho)\mathfrak{D}(\boldsymbol{u}) : \nabla \boldsymbol{u} dx &= \int \mu(\rho)(\partial_i u_j + \partial_j u_i)\partial_i u_j dx \\ &= \frac{1}{2}\int \mu(\rho)(\partial_i u_j + \partial_j u_i)^2 dx \\ &= 2\int \mu(\rho)|\mathfrak{D}(\boldsymbol{u})|^2 dx, \\ 2\int |\mathfrak{D}(\boldsymbol{u})|^2 dx &= \frac{1}{2}\int (\partial_i u_j + \partial_j u_i)(\partial_i u_j + \partial_j u_i) dx \\ &= \int |\nabla \boldsymbol{u}|^2 dx + \int \partial_i u_j \partial_j u_i dx = \int |\nabla \boldsymbol{u}|^2 dx, \end{aligned}$$

and

$$\begin{aligned} \|\Delta \boldsymbol{d} + |\nabla \boldsymbol{d}|^2 \boldsymbol{d}\|_{L^2}^2 &= \int \left(|\Delta \boldsymbol{d}|^2 + |\nabla \boldsymbol{d}|^4 - 2|\nabla \boldsymbol{d}|^2 \Delta \boldsymbol{d} \cdot \boldsymbol{d}\right) dx \\ &= \int \left(|\Delta \boldsymbol{d}|^2 + |\nabla \boldsymbol{d}|^4 - 2|\nabla \boldsymbol{d}|^4\right) dx \\ &= \|\Delta \boldsymbol{d}\|_{L^2}^2 - \|\nabla \boldsymbol{d}\|_{L^4}^4, \end{aligned}$$

we thus deduce from (3.11), (1.2), (3.2) and the Gagliardo–Nirenberg inequality that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\sqrt{\rho}\boldsymbol{u}\|_{L^2}^2 + \|\nabla \boldsymbol{d}\|_{L^2}^2) + \underline{\mu}\|\nabla \boldsymbol{u}\|_{L^2}^2 + \|\Delta \boldsymbol{d}\|_{L^2}^2 \\ \le &\|\nabla \boldsymbol{d}\|_{L^4}^4 \le C\|\nabla \boldsymbol{d}\|_{L^3}^2 \|\Delta \boldsymbol{d}\|_{L^2}^2 \le C_1 m_0^{\frac{1}{9}} \|\Delta \boldsymbol{d}\|_{L^2}^2 \end{aligned} \tag{3.12}$$

for some constant C_1 depending only on Ω. Thus, we obtain that

$$\frac{d}{dt}(\|\sqrt{\rho}\boldsymbol{u}\|_{L^2}^2 + \|\nabla \boldsymbol{d}\|_{L^2}^2) + \underline{\mu}\|\nabla \boldsymbol{u}\|_{L^2}^2 + \|\Delta \boldsymbol{d}\|_{L^2}^2 \le 0, \tag{3.13}$$

provided that

$$m_0 \le \left(\frac{1}{2C_1}\right)^9. \tag{3.14}$$

Integrating (3.13) over $[0, T]$ implies

$$\begin{aligned} &\sup_{[0,T]} \left(\|\sqrt{\rho}\boldsymbol{u}\|_{L^2}^2 + \|\nabla \boldsymbol{d}\|_{L^2}^2\right) + \int_0^T \left(\underline{\mu}\|\nabla \boldsymbol{u}\|_{L^2}^2 + \|\Delta \boldsymbol{d}\|_{L^2}^2\right) dt \\ \le &\|\sqrt{\rho_0}\boldsymbol{u}_0\|_{L^2}^2 + \|\nabla \boldsymbol{d}_0\|_{L^2}^2 \\ \le &\|\sqrt{\rho_0}\|_{L^3}^2 \|\boldsymbol{u}_0\|_{L^6}^2 + \|\nabla \boldsymbol{d}_0\|_{L^2}^2 \\ \le &\|\rho_0\|_{L^1}^{\frac{2}{3}} \|\rho_0\|_{L^\infty}^{\frac{1}{3}} \|\nabla \boldsymbol{u}_0\|_{L^2}^2 + \|\rho_0\|_{L^1}^{\frac{2}{3}} \\ \le &C\|\rho_0\|_{L^1}^{\frac{2}{3}} \end{aligned} \tag{3.15}$$

due to the Gagliardo–Nirenberg inequality and (1.7).

It follows from Poincaré's inequality (see [25, (A.3), p.266] that

$$\|\sqrt{\rho}\boldsymbol{u}\|_{L^2}^2 \le \bar{\rho}\|\boldsymbol{u}\|_{L^2}^2 \le \bar{\rho} l^2 \|\nabla \boldsymbol{u}\|_{L^2}^2, \quad \|\nabla \boldsymbol{d}\|_{L^2}^2 \le l^2 \|\Delta \boldsymbol{d}\|_{L^2}^2, \tag{3.16}$$

where l is the diameter of Ω. Hence, we get

$$\frac{1}{\bar{\rho} l^2}\|\sqrt{\rho}\boldsymbol{u}\|_{L^2}^2 \le \|\nabla \boldsymbol{u}\|_{L^2}^2, \quad \frac{1}{l^2}\|\nabla \boldsymbol{d}\|_{L^2}^2 \le \|\Delta \boldsymbol{d}\|_{L^2}^2. \tag{3.17}$$

Consequently, letting $\sigma \triangleq \min\left\{\frac{\underline{\mu}}{2\bar{\rho}l^2}, \frac{1}{2\,l^2}\right\}$, then we derive from (3.13) and (3.17) that

$$\frac{d}{dt}(\|\sqrt{\rho}\boldsymbol{u}\|_{L^2}^2 + \|\nabla\boldsymbol{d}\|_{L^2}^2) + \frac{\underline{\mu}}{2}\|\nabla\boldsymbol{u}\|_{L^2}^2 + \frac{1}{2}\|\Delta\boldsymbol{d}\|_{L^2}^2 + \sigma(\|\sqrt{\rho}\boldsymbol{u}\|_{L^2} + \|\nabla\boldsymbol{d}\|_{L^2}^2) \le 0. \tag{3.18}$$

Multiplying (3.18) by $e^{\sigma t}$, one has

$$\frac{d}{dt}[e^{\sigma t}(\|\sqrt{\rho}\boldsymbol{u}\|_{L^2}^2 + \|\nabla\boldsymbol{d}\|_{L^2}^2)] + \frac{e^{\sigma t}}{2}(\underline{\mu}\|\nabla\boldsymbol{u}\|_{L^2}^2 + \|\Delta\boldsymbol{d}\|_{L^2}^2) \le 0. \tag{3.19}$$

Thus, integrating (3.19) with respect to t gives (3.8), which combined with (3.15) completes the proof of Lemma 3.2. □

Next, we will derive important (time-weighted) estimates on the spatial gradients of the strong solution $(\boldsymbol{u}, \nabla\boldsymbol{d})$.

Lemma 3.3. *Under the conditions* (3.2) *and* (3.7), *if*

$$m_0 \le \left(\frac{1}{8C_2}\right)^9, \tag{3.20}$$

where C_2 is defined as in (3.35) *depending only on Ω and $\underline{\mu}$, then*

$$\sup_{[0,T]}(\|\nabla\boldsymbol{u}\|_{L^2}^2 + \|\Delta\boldsymbol{d}\|_{L^2}^2) + \int_0^T(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2 + \|\nabla\boldsymbol{d}_t\|_{L^2}^2)dt \le C. \tag{3.21}$$

Furthermore, for $i = \{1, 2\}$ and σ as in Lemma 3.2, one has that

$$\sup_{[0,T]}[t^i(\|\nabla\boldsymbol{u}\|_{L^2}^2 + \|\Delta\boldsymbol{d}\|_{L^2}^2)] + \int_0^T t^i(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2 + \|\nabla\boldsymbol{d}_t\|_{L^2}^2)dt \le Cm_0^{\frac{2}{3}}, \tag{3.22}$$

$$\sup_{[0,T]}[e^{\sigma t}(\|\nabla\boldsymbol{u}\|_{L^2}^2 + \|\Delta\boldsymbol{d}\|_{L^2}^2)] + \int_0^T e^{\sigma t}(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2 + \|\nabla\boldsymbol{d}_t\|_{L^2}^2)dt \le C. \tag{3.23}$$

Proof. 1. Since $\mu(\rho)$ is a continuously differentiable function, we deduce from $(3.1)_1$ that

$$\mu(\rho)_t + \boldsymbol{u}\cdot\nabla\mu(\rho) = 0. \tag{3.24}$$

Multiplying $(3.1)_2$ by $\boldsymbol{u}_t$, and integrating by parts over Ω, we have

$$\begin{aligned} 2\int \mu(\rho)\mathfrak{D}(\boldsymbol{u}) : \nabla\boldsymbol{u}_t dx + \int \rho|\boldsymbol{u}_t|^2 dx = \\ -\int \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}_t dx - \int \operatorname{div}(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})\cdot\boldsymbol{u}_t dx. \end{aligned} \tag{3.25}$$

First, we obtain from (3.24) that

$$\begin{aligned} 2\int \mu(\rho)\mathfrak{D}(\boldsymbol{u}) : \nabla\boldsymbol{u}_t dx &= \frac{d}{dt}\int \mu(\rho)|\mathfrak{D}(\boldsymbol{u})|^2 dx - \int \mu(\rho)_t|\mathfrak{D}(\boldsymbol{u})|^2 dx \\ &= \frac{d}{dt}\int \mu(\rho)|\mathfrak{D}(\boldsymbol{u})|^2 dx + \int \boldsymbol{u}\cdot\nabla\mu(\rho)|\mathfrak{D}(\boldsymbol{u})|^2 dx. \end{aligned} \tag{3.26}$$

Then, inserting (3.26) into (3.25), it follows from integration by parts that

$$\begin{aligned}
&\frac{d}{dt}\int \mu(\rho)|\mathfrak{D}(\boldsymbol{u})|^2dx + \int \rho|\boldsymbol{u}_t|^2dx\\
=&-\int \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}_tdx - \int \boldsymbol{u}\cdot\nabla\mu(\rho)|\mathfrak{D}(\boldsymbol{u})|^2dx - \int \operatorname{div}(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})\cdot\boldsymbol{u}_tdx\\
=&\int(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d}):\nabla\boldsymbol{u}_tdx - \int \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}_tdx - \int \boldsymbol{u}\cdot\nabla\mu(\rho)|\mathfrak{D}(\boldsymbol{u})|^2dx\\
=&\frac{d}{dt}\int(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d}):\nabla\boldsymbol{u}dx - \int(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})_t:\nabla\boldsymbol{u}dx - \int \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}_tdx\\
&-\int \boldsymbol{u}\cdot\nabla\mu(\rho)|\mathfrak{D}(\boldsymbol{u})|^2dx\\
\triangleq&\frac{d}{dt}\int(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d}):\nabla\boldsymbol{u}dx + I_1 + I_2 + I_3.
\end{aligned}\tag{3.27}$$

Now, we are ready to estimate terms I_1-I_3. By Hölder's inequality and the Gagliardo–Nirenberg inequality, we get

$$\begin{aligned}
I_1 \le& C\|\nabla\boldsymbol{d}\|_{L^6}\|\nabla\boldsymbol{d}_t\|_{L^2}\|\nabla\boldsymbol{u}\|_{L^3}\\
\le& C\|\nabla\boldsymbol{d}_t\|_{L^2}\|\nabla^2\boldsymbol{d}\|_{L^2}\|\nabla\boldsymbol{u}\|_{L^2}^{\frac12}\|\nabla\boldsymbol{u}\|_{H^1}^{\frac12}\\
\le& \frac14\|\nabla\boldsymbol{d}_t\|_{L^2}^2 + C\|\nabla^2\boldsymbol{d}\|_{L^2}^2\|\nabla\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{u}\|_{H^1}.
\end{aligned}\tag{3.28}$$

By Hölder's inequality, the Gagliardo–Nirenberg inequality and (3.5), we obtain

$$\begin{aligned}
I_2 =& \left|-\int \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}_tdx\right|\\
\le& \|\sqrt{\rho}\|_{L^\infty}\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}\|\boldsymbol{u}\|_{L^6}\|\nabla\boldsymbol{u}\|_{L^3}\\
\le& C\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}\|\nabla\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{u}\|_{L^2}^{\frac12}\|\nabla\boldsymbol{u}\|_{H^1}^{\frac12}\\
\le& \frac12\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2 + C\|\nabla\boldsymbol{u}\|_{L^2}^3\|\nabla\boldsymbol{u}\|_{H^1}.
\end{aligned}\tag{3.29}$$

By Sobolev's inequality and (3.2), we have

$$\begin{aligned}
I_3 =& \left|\int \boldsymbol{u}\cdot\nabla\mu(\rho)|\mathfrak{D}(\boldsymbol{u})|^2dx\right|\\
\le& C\|\boldsymbol{u}\|_{L^{\frac{2q}{q-2}}}\|\nabla\mu(\rho)\|_{L^q}\|\nabla\boldsymbol{u}\|_{L^4}^2\\
\le& C\|\nabla\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{u}\|_{L^2}^{\frac12}\|\nabla\boldsymbol{u}\|_{L^6}^{\frac32}\\
\le& C\|\nabla\boldsymbol{u}\|_{L^2}^{\frac32}\|\nabla\boldsymbol{u}\|_{H^1}^{\frac32}.
\end{aligned}\tag{3.30}$$

Substituting (3.28)-(3.30) into (3.27) leads to

$$\begin{aligned}
&\frac{d}{dt}\int\left[\mu(\rho)|\mathfrak{D}(\boldsymbol{u})|^2 - (\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d}):\nabla\boldsymbol{u}\right]dx + \frac12\int\rho|\boldsymbol{u}_t|^2dx\\
\le& \frac14\|\nabla\boldsymbol{d}_t\|_{L^2}^2 + C\|\nabla\boldsymbol{u}\|_{L^2}^3\|\nabla\boldsymbol{u}\|_{H^1} + C\|\nabla\boldsymbol{u}\|_{L^2}^{\frac32}\|\nabla\boldsymbol{u}\|_{H^1}^{\frac32} + C\|\nabla^2\boldsymbol{d}\|_{L^2}^2\|\nabla\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{u}\|_{H^1}.
\end{aligned}\tag{3.31}$$

2. Multiplying $(3.1)_3$ by $\Delta \boldsymbol{d}_t$ and integrating the resulting equation over Ω, it follows from the Gagliardo–Nirenberg inequality, Hölder's inequality and (1.4) that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta \boldsymbol{d}\|_{L^2}^2+\|\nabla \boldsymbol{d}_t\|_{L^2}^2 \\
=&\int \nabla(\boldsymbol{u}\cdot\nabla \boldsymbol{d}):\nabla \boldsymbol{d}_t dx-\int \nabla(|\nabla \boldsymbol{d}|^2\boldsymbol{d}):\nabla \boldsymbol{d}_t dx \\
\le&\frac{1}{4}\int |\nabla \boldsymbol{d}_t|^2dx+C\int |\nabla \boldsymbol{u}|^2|\nabla \boldsymbol{d}|^2dx+C\int |\boldsymbol{u}|^2|\nabla^2 \boldsymbol{d}|^2dx \\
&+C\int |\nabla \boldsymbol{d}|^6dx+C\int |\nabla \boldsymbol{d}|^2|\nabla^2\boldsymbol{d}|^2dx \\
\le&\frac{1}{4}\|\nabla \boldsymbol{d}_t\|_{L^2}^2+C\|\nabla \boldsymbol{u}\|_{L^3}^2\|\nabla \boldsymbol{d}\|_{L^6}^2+C\|\boldsymbol{u}\|_{L^6}^2\|\nabla^2 \boldsymbol{d}\|_{L^3}^2 \\
&+C\|\nabla^2 \boldsymbol{d}\|_{L^2}^6+C\|\nabla^2 \boldsymbol{d}\|_{L^3}^2\|\nabla \boldsymbol{d}\|_{L^6}^2 \\
\le&\frac{1}{4}\|\nabla \boldsymbol{d}_t\|_{L^2}^2+C\|\nabla \boldsymbol{u}\|_{L^2}\|\nabla^2 \boldsymbol{d}\|_{L^2}^2\|\nabla \boldsymbol{u}\|_{H^1}+C\|\nabla \boldsymbol{u}\|_{L^2}^2\|\nabla^2 \boldsymbol{d}\|_{L^2}\|\nabla^2 \boldsymbol{d}\|_{H^1} \\
&+C\|\nabla^2 \boldsymbol{d}\|_{L^2}^6+C\|\nabla^2 \boldsymbol{d}\|_{L^2}^3\|\nabla^2 \boldsymbol{d}\|_{H^1},
\end{aligned} \tag{3.32}$$

which together with (3.31) gives rise to

$$\begin{aligned}
&\frac{d}{dt}B(t)+\frac{1}{2}\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\frac{1}{2}\|\nabla \boldsymbol{d}_t\|_{L^2}^2 \\
\le& C\|\nabla \boldsymbol{u}\|_{L^2}^3\|\nabla \boldsymbol{u}\|_{H^1}+C\|\nabla \boldsymbol{u}\|_{L^2}^{\frac{3}{2}}\|\nabla \boldsymbol{u}\|_{H^1}^{\frac{3}{2}}+C\|\nabla \boldsymbol{u}\|_{L^2}\|\nabla^2 \boldsymbol{d}\|_{L^2}^2\|\nabla \boldsymbol{u}\|_{H^1} \\
&+C\|\nabla \boldsymbol{u}\|_{L^2}^2\|\nabla^2 \boldsymbol{d}\|_{L^2}\|\nabla^2 \boldsymbol{d}\|_{H^1}+C\|\nabla^2 \boldsymbol{d}\|_{L^2}^6+C\|\nabla^2 \boldsymbol{d}\|_{L^2}^3\|\nabla^2 \boldsymbol{d}\|_{H^1},
\end{aligned} \tag{3.33}$$

where

$$B(t)\triangleq \int (\mu(\rho)|\mathfrak{D}(\boldsymbol{u})|^2+\frac{1}{2}|\Delta \boldsymbol{d}|^2-\nabla \boldsymbol{d}\odot\nabla \boldsymbol{d}:\nabla \boldsymbol{u})dx. \tag{3.34}$$

By Hölder's inequality and Sobolev's inequality, we have

$$\begin{aligned}
&\left|\int (\nabla \boldsymbol{d}\odot\nabla \boldsymbol{d}):\nabla \boldsymbol{u}dx\right| \\
\le&\|\nabla \boldsymbol{u}\|_{L^2}\|\nabla \boldsymbol{d}\|_{L^3}\|\nabla \boldsymbol{d}\|_{L^6} \\
\le& C\|\nabla \boldsymbol{u}\|_{L^2}\|\nabla \boldsymbol{d}\|_{L^3}\|\nabla^2 \boldsymbol{d}\|_{L^2} \\
\le&\frac{\underline{\mu}}{4}\|\nabla \boldsymbol{u}\|_{L^2}^2+C_2\|\nabla \boldsymbol{d}\|_{L^3}^2\|\Delta \boldsymbol{d}\|_{L^2}^2 \\
\le&\frac{\underline{\mu}}{4}\|\nabla \boldsymbol{u}\|_{L^2}^2+2C_2m_0^{\frac{1}{9}}\|\Delta \boldsymbol{d}\|_{L^2}^2
\end{aligned} \tag{3.35}$$

for some positive constant C_2 depending only on Ω and $\underline{\mu}$. Thus, we obtain that

$$\frac{\underline{\mu}}{4}\|\nabla \boldsymbol{u}\|_{L^2}^2+\frac{1}{4}\|\Delta \boldsymbol{d}\|_{L^2}^2\le B(t)\le C\|\nabla \boldsymbol{u}\|_{L^2}^2+C\|\Delta \boldsymbol{d}\|_{L^2}^2, \tag{3.36}$$

provided that

$$m_0\le\left(\frac{1}{8C_2}\right)^9. \tag{3.37}$$

3. Recall that $(\boldsymbol{u}, P)$ satisfies the following density-dependent Stokes system:

$$\begin{cases} -\operatorname{div}(2\mu(\rho)\mathfrak{D}(\boldsymbol{u})) + \nabla P = -\rho \boldsymbol{u}_t - \rho \boldsymbol{u}\cdot\nabla \boldsymbol{u} - \operatorname{div}(\nabla \boldsymbol{d}\odot\nabla \boldsymbol{d}), & x\in\Omega,\\ \operatorname{div}\boldsymbol{u} = 0, & x\in\Omega,\\ \boldsymbol{u} = 0, & x\in\partial\Omega. \end{cases} \tag{3.38}$$

Applying Lemma 2.3 with $\mathbf{F} = -\rho\boldsymbol{u}_t - \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} - \operatorname{div}(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})$, we obtain from (3.2), (3.5) and the Gagliardo–Nirenberg inequality that

$$\begin{aligned} &\|\boldsymbol{u}\|_{H^2} + \|\nabla P\|_{H^1}\\ \le & C(\|\rho\boldsymbol{u}_t\|_{L^2} + \|\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{L^2} + \|\operatorname{div}(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})\|_{L^2})(1+\|\nabla\mu(\rho)\|_{L^q}^{\frac{q}{q-3}})\\ \le & C\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2} + C\|\boldsymbol{u}\|_{L^6}\|\nabla\boldsymbol{u}\|_{L^3} + C\|\nabla\boldsymbol{d}\|_{L^6}\|\nabla^2\boldsymbol{d}\|_{L^3}\\ \le & C\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2} + C\|\nabla\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{u}\|_{L^2}^{\frac12}\|\nabla\boldsymbol{u}\|_{H^1}^{\frac12} + C\|\nabla^2\boldsymbol{d}\|_{L^2}\|\nabla^2\boldsymbol{d}\|_{L^2}^{\frac12}\|\nabla^2\boldsymbol{d}\|_{H^1}^{\frac12}\\ \le & C\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2} + C\|\nabla\boldsymbol{u}\|_{L^2}^3 + \frac12\|\nabla\boldsymbol{u}\|_{H^1} + C\|\nabla^2\boldsymbol{d}\|_{L^2}^{\frac32}\|\nabla^2\boldsymbol{d}\|_{H^1}^{\frac12}, \end{aligned} \tag{3.39}$$

which implies that

$$\|\boldsymbol{u}\|_{H^2} + \|\nabla P\|_{H^1} \le C\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2} + C\|\nabla\boldsymbol{u}\|_{L^2}^3 + C\|\nabla^2\boldsymbol{d}\|_{L^2}^{\frac32}\|\nabla^2\boldsymbol{d}\|_{H^1}^{\frac12}. \tag{3.40}$$

Taking ∇ operator to the equation $(3.1)_3$, one has

$$\nabla\boldsymbol{d}_t - \Delta\nabla\boldsymbol{d} = -\nabla(\boldsymbol{u}\cdot\nabla\boldsymbol{d}) + \nabla(|\nabla\boldsymbol{d}|^2\boldsymbol{d}). \tag{3.41}$$

It follows from L^2 estimates of the elliptic system (3.41), it is easy to deduce from $(3.1)_3$ that

$$\begin{aligned} \|\nabla^2\boldsymbol{d}\|_{H^1} \le & C\|\nabla\boldsymbol{d}_t\|_{L^2} + C\|\nabla(\boldsymbol{u}\cdot\nabla\boldsymbol{d})\|_{L^2} + C\|\nabla(|\nabla\boldsymbol{d}|^2\boldsymbol{d})\|_{L^2} + C\|\nabla^2\boldsymbol{d}\|_{L^2}\\ \le & C\|\nabla\boldsymbol{d}_t\|_{L^2} + C\|\boldsymbol{u}\|_{L^6}\|\nabla^2\boldsymbol{d}\|_{L^3} + C\|\nabla\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{d}\|_{L^\infty}\\ & + C\|\nabla\boldsymbol{d}\|_{L^6}^3 + C\|\nabla\boldsymbol{d}\|_{L^6}\|\nabla^2\boldsymbol{d}\|_{L^3} + C\|\nabla^2\boldsymbol{d}\|_{L^2}\\ \le & C\|\nabla\boldsymbol{d}_t\|_{L^2} + C\|\nabla\boldsymbol{u}\|_{L^2}\|\nabla^2\boldsymbol{d}\|_{L^2}^{\frac12}\|\nabla^2\boldsymbol{d}\|_{H^1}^{\frac12} + C\|\nabla\boldsymbol{u}\|_{L^2}\|\nabla^2\boldsymbol{d}\|_{L^2}^{\frac12}\|\nabla^2\boldsymbol{d}\|_{H^1}^{\frac12}\\ & + C\|\nabla^2\boldsymbol{d}\|_{L^2}^3 + C\|\nabla^2\boldsymbol{d}\|_{L^2}\|\nabla^2\boldsymbol{d}\|_{L^2}^{\frac12}\|\nabla^2\boldsymbol{d}\|_{H^1}^{\frac12} + C\|\nabla^2\boldsymbol{d}\|_{L^2}\\ \le & C\|\nabla\boldsymbol{d}_t\|_{L^2} + C\|\nabla\boldsymbol{u}\|_{L^2}^3 + C\|\nabla^2\boldsymbol{d}\|_{L^2}^3 + C\|\nabla^2\boldsymbol{d}\|_{L^2} + \frac12\|\nabla^2\boldsymbol{d}\|_{H^1}, \end{aligned} \tag{3.42}$$

which gives

$$\|\nabla^2\boldsymbol{d}\|_{H^1} \le C\|\nabla\boldsymbol{d}_t\|_{L^2} + C\|\nabla\boldsymbol{u}\|_{L^2}^3 + C\|\nabla^2\boldsymbol{d}\|_{L^2}^3 + C\|\nabla^2\boldsymbol{d}\|_{L^2}. \tag{3.43}$$

This along with (3.40) leads to

$$\|\boldsymbol{u}\|_{H^2} + \|\nabla P\|_{H^1} \le C\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2} + C\|\nabla\boldsymbol{d}_t\|_{L^2} + C\|\nabla\boldsymbol{u}\|_{L^2}^3 + C\|\nabla^2\boldsymbol{d}\|_{L^2}^3 + C\|\nabla^2\boldsymbol{d}\|_{L^2}^2. \tag{3.44}$$

Inserting (3.43) and (3.44) into (3.33), and applying Young's inequality, we deduce that

$$\begin{aligned} & B'(t) + \frac14\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2 + \frac14\|\nabla\boldsymbol{d}_t\|_{L^2}^2\\ \le & C\left(\|\nabla\boldsymbol{u}\|_{L^2}^4 + \|\nabla^2\boldsymbol{d}\|_{L^2}^4 + \|\nabla\boldsymbol{u}\|_{L^2}^2 + \|\nabla^2\boldsymbol{d}\|_{L^2}^2\right)\left(\|\nabla\boldsymbol{u}\|_{L^2}^2 + \|\nabla^2\boldsymbol{d}\|_{L^2}^2\right). \end{aligned} \tag{3.45}$$

Applying Gronwall's inequality, (3.2), (3.6) and (3.36) leads to

$$\sup_{[0,T]}(\|\nabla\boldsymbol{u}\|_{L^2}^2 + \|\nabla^2\boldsymbol{d}\|_{L^2}^2) + \int_0^T (\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2 + \|\nabla\Delta\boldsymbol{d}\|_{L^2}^2)dt \le C. \tag{3.46}$$

For $i=\{1,2,3\}$, multiplying (3.45) by t^i, we obtain from (3.46) and (3.36) that

$$\begin{aligned}\frac{d}{dt}(t^iB(t))+\frac{1}{4}t^i(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\nabla\Delta\boldsymbol{d}\|_{L^2}^2)&\\ \le Ct^i(\|\nabla\boldsymbol{u}\|_{L^2}^2+\|\nabla^2\boldsymbol{d}\|_{L^2}^2)+Ct^{i-1}(\|\nabla\boldsymbol{u}\|_{L^2}^2+\|\nabla^2\boldsymbol{d}\|_{L^2}^2).&\end{aligned}\tag{3.47}$$

For σ as in Lemma 3.2, and any $k\in\mathbb{N}$, we derive from (3.8) that

$$\int_0^T t^k(\|\nabla\boldsymbol{u}\|_{L^2}^2+\|\nabla^2\boldsymbol{d}\|_{L^2}^2)dt\le\sup_{[0,T]}\{t^ke^{-\sigma t}\}\int_0^T e^{\sigma t}(\|\nabla\boldsymbol{u}\|_{L^2}^2+\|\nabla^2\boldsymbol{d}\|_{L^2}^2)dt\le Cm_0^{\frac{2}{3}}.\tag{3.48}$$

Integrating (3.47) over $[0,T]$ together with (3.48) leads to (3.22). Moreover, multiplying (3.45) by $e^{\sigma t}$, we deduce from (3.46) and (3.36) that

$$\frac{d}{dt}(e^{\sigma t}B(t))+e^{\sigma t}(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\nabla\Delta\boldsymbol{d}\|_{L^2}^2)\le Ce^{\sigma t}(\|\nabla\boldsymbol{u}\|_{L^2}^2+\|\nabla^2\boldsymbol{d}\|_{L^2}^2).\tag{3.49}$$

Integrating the above inequality over $[0,T]$ together with (3.8) and (3.36) gives (3.23). Therefore, the proof of lemma 3.3 is completed. □

Remark 3.1. Combining (3.43) and (3.44), we have

$$\|\boldsymbol{u}\|_{H^2}+\|\nabla\boldsymbol{d}\|_{H^2}+\|\nabla P\|_{H^1}\le C\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}+C\|\nabla\boldsymbol{d}_t\|_{L^2}+C\|\nabla\boldsymbol{u}\|_{L^2}^3+C\|\nabla^2\boldsymbol{d}\|_{L^2}^3+C\|\nabla^2\boldsymbol{d}\|_{L^2}\tag{3.50}$$

And it follows from $(3.1)_3$, (3.21) and the Gagliardo–Nirenberg inequality that

$$\begin{aligned}\|\boldsymbol{d}_t\|_{L^2}=&\|-\boldsymbol{u}\cdot\nabla\boldsymbol{d}+\Delta\boldsymbol{d}+|\nabla\boldsymbol{d}|^2\boldsymbol{d}\|_{L^2}\\ \le&\|\boldsymbol{u}\|_{L^6}\|\nabla\boldsymbol{d}\|_{L^3}+\|\nabla^2\boldsymbol{d}\|_{L^2}+\|\nabla\boldsymbol{d}\|_{L^4}^2\\ \le&C\|\nabla\boldsymbol{u}\|_{L^2}+C\|\nabla^2\boldsymbol{d}\|_{L^2},\end{aligned}\tag{3.51}$$

which together with (3.21), (3.6) and (3.50) implies, for $i=\{1,2\}$

$$\int_0^T t^i\left(\|\boldsymbol{u}\|_{H^2}^2+\|\nabla\boldsymbol{d}\|_{H^2}^2+\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\boldsymbol{d}_t\|_{H^1}^2\right)dt\le Cm_0^{\frac{2}{3}}.\tag{3.52}$$

Lemma 3.4. *Under the conditions (3.2), (3.7) and (3.20), for $i\in\{1,2\}$, it holds that*

$$\sup_{[0,T]}[t^i(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\nabla\boldsymbol{d}_t\|_{L^2}^2)]+\int_0^T t^i(\|\nabla\boldsymbol{u}_t\|_{L^2}^2+\|\nabla^2\boldsymbol{d}_t\|_{L^2}^2)dt\le Cm_0^{\frac{2}{3}}.\tag{3.53}$$

Moreover, for σ as in Lemma 3.2 and $\zeta(t)\triangleq\min\{1,t\}$, one has that

$$\sup_{[\zeta(T),T]}\left[e^{\sigma t}(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\nabla\boldsymbol{d}_t\|_{L^2}^2)\right]+\int_{\zeta(T)}^T e^{\sigma t}(\|\nabla\boldsymbol{u}_t\|_{L^2}^2+\|\nabla^2\boldsymbol{d}_t\|_{L^2}^2)dt\le C.\tag{3.54}$$

Proof. 1. Differentiating $(3.1)_2$ with respect to time variable t gives

$$\begin{aligned}\rho\boldsymbol{u}_{tt}+\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}_t-\operatorname{div}(2\mu(\rho)\mathfrak{D}(\boldsymbol{u})_t)+\nabla P_t=&-\rho_t(\boldsymbol{u}_t+\boldsymbol{u}\cdot\nabla\boldsymbol{u})\\ &-\rho\boldsymbol{u}_t\cdot\nabla\boldsymbol{u}+\operatorname{div}(2\mu(\rho)_t\mathfrak{D}(\boldsymbol{u}))-[\operatorname{div}(\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})]_t.\end{aligned}\tag{3.55}$$

Multiplying (3.55) by $\boldsymbol{u}_t$ and integrating the resulting equality by parts over Ω, we deduce from $(3.1)_1$ that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|\boldsymbol{u}_t|^2dx + \int 2\mu(\rho)\mathfrak{D}(\boldsymbol{u})_t : \nabla \boldsymbol{u}_t dx \\
=&\int \operatorname{div}(\rho\boldsymbol{u})|\boldsymbol{u}_t|^2dx + \int \operatorname{div}(\rho\boldsymbol{u})(\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}_t)dx - \int \rho\boldsymbol{u}_t\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}_t dx \\
&-2\int \mu(\rho)_t\mathfrak{D}(\boldsymbol{u}) : \nabla\boldsymbol{u}_t dx + \int (\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})_t : \nabla\boldsymbol{u}_t dx \\
\triangleq&\sum_{i=1}^{5} J_i.
\end{aligned} \tag{3.56}$$

By Hölder's inequality, Sobolev's inequality, the Gagliardo–Nirenberg inequality, (3.5), (3.21) and (3.24), we obtain that

$$\begin{aligned}
J_1 \le& \left|\int \operatorname{div}(\rho\boldsymbol{u})|\boldsymbol{u}_t|^2dx\right| \\
=&\left|2\int \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}_t\cdot\boldsymbol{u}_t dx\right| \\
\le& 2\|\rho\|_{L^\infty}\|\boldsymbol{u}\|_{L^6}\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^3}\|\nabla\boldsymbol{u}_t\|_{L^2} \\
\le& C\|\nabla\boldsymbol{u}\|_{L^2}\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^{\frac{1}{2}}\|\nabla\boldsymbol{u}_t\|_{L^2}^{\frac{3}{2}} \\
\le& \frac{\mu}{10}\|\nabla\boldsymbol{u}_t\|_{L^2}^2 + C\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2, \\
J_2 \le& \left|\int \operatorname{div}(\rho\boldsymbol{u})(\boldsymbol{u}\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}_t)dx\right| \\
\le& \int \rho|\boldsymbol{u}||\nabla\boldsymbol{u}|^2|\boldsymbol{u}_t|dx + \int \rho|\boldsymbol{u}|^2|\nabla^2\boldsymbol{u}||\boldsymbol{u}_t|dx \\
&+\int \rho|\boldsymbol{u}|^2|\nabla\boldsymbol{u}||\nabla\boldsymbol{u}_t|dx \\
\le& \|\rho\|_{L^\infty}\|\boldsymbol{u}\|_{L^6}\|\nabla\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{u}\|_{L^6}\|\boldsymbol{u}_t\|_{L^6} \\
&+\|\rho\|_{L^\infty}\|\boldsymbol{u}\|_{L^6}^2\|\nabla^2\boldsymbol{u}\|_{L^2}\|\boldsymbol{u}_t\|_{L^6} \\
&+\|\rho\|_{L^\infty}\|\boldsymbol{u}\|_{L^6}^2\|\nabla\boldsymbol{u}\|_{L^6}\|\nabla\boldsymbol{u}_t\|_{L^2} \\
\le& C\|\nabla\boldsymbol{u}\|_{L^2}^2\|\nabla\boldsymbol{u}\|_{H^1}\|\nabla\boldsymbol{u}_t\|_{L^2} \\
\le& \frac{\mu}{10}\|\nabla\boldsymbol{u}_t\|_{L^2}^2 + C\|\nabla\boldsymbol{u}\|_{H^1}^2, \\
J_3 \le& \left|-\int \rho\boldsymbol{u}_t\cdot\nabla\boldsymbol{u}\cdot\boldsymbol{u}_t dx\right| \\
\le& \|\sqrt{\rho}\boldsymbol{u}_t\|_{L^4}^2\|\nabla\boldsymbol{u}\|_{L^2} \\
\le& C\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^6}^{\frac{3}{2}}\|\nabla\boldsymbol{u}\|_{L^2} \\
\le& C\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^{\frac{1}{2}}\|\nabla\boldsymbol{u}_t\|_{L^2}^{\frac{3}{2}}\|\nabla\boldsymbol{u}\|_{L^2} \\
\le& \frac{\mu}{10}\|\nabla\boldsymbol{u}_t\|_{L^2}^2 + C\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}J_4 \le& \left|-2\int \mu(\rho)_t \mathfrak{D}(\boldsymbol{u}) : \nabla \boldsymbol{u}_t dx\right|\\ =&\left|2\int \boldsymbol{u}\cdot\nabla\mu(\rho)\mathfrak{D}(\boldsymbol{u}) : \nabla \boldsymbol{u}_t dx\right|\\ \le& C\|\nabla\mu(\rho)\|_{L^q}\|\boldsymbol{u}\|_{L^\infty}\|\nabla\boldsymbol{u}\|_{L^{\frac{2q}{q-2}}}\|\nabla\boldsymbol{u}_t\|_{L^2}\\ \le& C\|\nabla\boldsymbol{u}\|_{L^2}^{\frac12}\|\nabla\boldsymbol{u}\|_{H^1}^{\frac32}\|\nabla\boldsymbol{u}_t\|_{L^2}\\ \le& \frac{\underline{\mu}}{10}\|\nabla\boldsymbol{u}_t\|_{L^2}^2 + C\|\nabla\boldsymbol{u}\|_{H^1}^3,\\ J_5 \le& \left|\int (\nabla\boldsymbol{d}\odot\nabla\boldsymbol{d})_t : \nabla\boldsymbol{u}_t dx\right|\\ \le& C\|\nabla\boldsymbol{d}\|_{L^4}\|\nabla\boldsymbol{d}_t\|_{L^4}\|\nabla\boldsymbol{u}_t\|_{L^2}\\ \le& C\|\nabla\boldsymbol{d}_t\|_{L^2}^{\frac14}\|\nabla^2\boldsymbol{d}_t\|_{L^2}^{\frac34}\|\nabla\boldsymbol{u}_t\|_{L^2}\\ \le& \frac{\underline{\mu}}{10}\|\nabla\boldsymbol{u}_t\|_{L^2}^2 + C\|\nabla\boldsymbol{d}_t\|_{L^2}^{\frac12}\|\nabla^2\boldsymbol{d}_t\|_{L^2}^{\frac32}\\ \le& \frac{\underline{\mu}}{10}\|\nabla\boldsymbol{u}_t\|_{L^2}^2 + \delta\|\nabla^2\boldsymbol{d}_t\|_{L^2}^2 + C\|\nabla\boldsymbol{d}_t\|_{L^2}^2.\end{aligned}$$

Substituting the above estimates of J_1-J_5 into (3.56) and noting that

$$2\int \mu(\rho)\mathfrak{D}(\boldsymbol{u})_t : \nabla\boldsymbol{u}_t dx \ge \underline{\mu}\|\nabla\boldsymbol{u}_t\|_{L^2}^2, \tag{3.57}$$

we obtain from (3.50), (3.21) and Young's inequality that

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2 + \underline{\mu}\|\nabla\boldsymbol{u}_t\|_{L^2}^2\\ \le& C(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2 + \|\nabla\boldsymbol{d}_t\|_{L^2}^2 + \|\nabla\boldsymbol{u}\|_{L^2}^2 + \|\nabla^2\boldsymbol{d}\|_{L^2}^2)\\ &+ C\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^4 + C\|\nabla\boldsymbol{d}_t\|_{L^2}^4 + \delta\|\nabla^2\boldsymbol{d}_t\|_{L^2}^2.\end{aligned} \tag{3.58}$$

2. Differentiating $(3.1)_3$ with respect to t, and multiplying the resulting equality by $\boldsymbol{d}_t$, we obtain from integration by parts over Ω that

$$\begin{aligned}\frac12\frac{d}{dt}\|\boldsymbol{d}_t\|_{L^2}^2 + \|\nabla\boldsymbol{d}_t\|_{L^2}^2 \le& C\int|\boldsymbol{u}_t||\nabla\boldsymbol{d}||\boldsymbol{d}_t|dx + C\int|\nabla\boldsymbol{d}_t||\nabla\boldsymbol{d}||\boldsymbol{d}_t|dx\\ &+ C\int|\nabla\boldsymbol{d}|^2|\boldsymbol{d}_t|^2dx \triangleq M_1+M_2+M_3.\end{aligned} \tag{3.59}$$

By Hölder's inequality, Sobolev's inequality, the Gagliardo–Nirenberg inequality, (3.5) and (3.21), we obtain that

$$\begin{aligned}M_1 \le& C\|\boldsymbol{u}_t\|_{L^4}\|\nabla\boldsymbol{d}\|_{L^2}\|\boldsymbol{d}_t\|_{L^4}\\ \le& C\|\nabla\boldsymbol{u}_t\|_{L^2}\|\nabla\boldsymbol{d}\|_{L^2}\|\boldsymbol{d}_t\|_{H^1}\\ \le& \frac{\underline{\mu}}{4}\|\nabla\boldsymbol{u}_t\|_{L^2}^2 + C\|\boldsymbol{d}_t\|_{H^1}^2,\\ M_2+M_3 \le& C\|\nabla\boldsymbol{d}_t\|_{L^2}\|\nabla\boldsymbol{d}\|_{L^4}\|\boldsymbol{d}_t\|_{L^4} + C\|\nabla\boldsymbol{d}\|_{L^4}^2\|\boldsymbol{d}_t\|_{L^4}^2\\ \le& C\|\nabla\boldsymbol{d}_t\|_{L^2}\|\nabla^2\boldsymbol{d}\|_{L^2}\|\boldsymbol{d}_t\|_{H^1} + C\|\nabla^2\boldsymbol{d}\|_{L^2}^2\|\boldsymbol{d}_t\|_{H^1}^2\\ \le& C\|\boldsymbol{d}_t\|_{H^1}^2.\end{aligned}$$

Hence,

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{d}_t\|_{L^2}^2+\|\nabla\boldsymbol{d}_t\|_{L^2}^2\le\frac{\mu}{4}\|\nabla\boldsymbol{u}_t\|_{L^2}^2+C\|\boldsymbol{d}_t\|_{H^1}^2. \tag{3.60}$$

Differentiating (3.41) with respect to t gives

$$\nabla\boldsymbol{d}_{tt}-\nabla\Delta\boldsymbol{d}_t=-\nabla(\boldsymbol{u}\cdot\nabla\boldsymbol{d})_t+\nabla(|\nabla\boldsymbol{d}|^2\boldsymbol{d})_t. \tag{3.61}$$

Multiplying (3.61) by $\nabla\boldsymbol{d}_t$, we obtain from integration by parts that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{d}_t\|_{L^2}^2+\|\nabla^2\boldsymbol{d}_t\|_{L^2}^2\le&C\int|\nabla\boldsymbol{u}_t||\nabla\boldsymbol{d}||\nabla\boldsymbol{d}_t|dx+C\int|\nabla\boldsymbol{u}||\nabla\boldsymbol{d}_t|^2dx\\&+C\int|\boldsymbol{u}_t||\nabla^2d||\nabla\boldsymbol{d}_t|dx+C\int|\nabla\boldsymbol{d}|^2|\boldsymbol{d}_t||\nabla^2\boldsymbol{d}_t|dx\\&+C\int|\nabla\boldsymbol{d}||\nabla\boldsymbol{d}_t||\nabla^2\boldsymbol{d}_t|dx\triangleq\sum_{i=1}^{5}K_i.\end{aligned} \tag{3.62}$$

By Hölder's inequality, Sobolev's inequality and the Gagliardo–Nirenberg inequality, we obtain from (3.5) and (3.21) that

$$\begin{aligned}K_1\le&C\|\nabla\boldsymbol{u}_t\|_{L^2}\|\nabla\boldsymbol{d}_t\|_{L^4}\|\nabla\boldsymbol{d}\|_{L^4}\\\le&C\|\nabla\boldsymbol{u}_t\|_{L^2}\|\nabla\boldsymbol{d}_t\|_{L^2}^{\frac{1}{4}}\|\nabla^2\boldsymbol{d}_t\|_{L^2}^{\frac{3}{4}}\\\le&\frac{\mu}{4}\|\nabla\boldsymbol{u}_t\|_{L^2}^2+\|\nabla\boldsymbol{d}_t\|_{L^2}^{\frac{1}{4}}\|\nabla^2\boldsymbol{d}_t\|_{L^2}^{\frac{3}{2}}\\\le&\frac{\mu}{2}\|\nabla\boldsymbol{u}_t\|_{L^2}^2+\frac{1}{14}\|\nabla^2\boldsymbol{d}_t\|_{L^2}^2+C\|\nabla\boldsymbol{d}_t\|_{L^2}^2.\\K_2\le&C\|\nabla\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{d}_t\|_{L^4}^2\\\le&C\|\nabla\boldsymbol{d}_t\|_{L^2}^{\frac{1}{2}}\|\nabla^2\boldsymbol{d}_t\|_{L^2}^{\frac{3}{2}}\\\le&\frac{1}{14}\|\nabla^2\boldsymbol{d}_t\|_{L^2}^2+C\|\nabla\boldsymbol{d}_t\|_{L^2}^2.\\K_3\le&C\|\boldsymbol{u}_t\|_{L^6}\|\Delta\boldsymbol{d}\|_{L^2}\|\nabla\boldsymbol{d}_t\|_{L^3}\\\le&C\|\nabla\boldsymbol{u}_t\|_{L^2}\|\nabla^2\boldsymbol{d}_t\|_{L^2}^{\frac{1}{2}}\|\nabla\boldsymbol{d}_t\|_{L^2}^{\frac{1}{2}}\\\le&\frac{\mu}{12}\|\nabla\boldsymbol{u}_t\|_{L^2}^2+C\|\nabla^2\boldsymbol{d}_t\|_{L^2}\|\nabla\boldsymbol{d}_t\|_{L^2}\\\le&\frac{\mu}{12}\|\nabla\boldsymbol{u}_t\|_{L^2}^2+\frac{1}{14}\|\nabla^2\boldsymbol{d}_t\|_{L^2}^2+C\|\nabla\boldsymbol{d}_t\|_{L^2}^2.\\K_4\le&C\|\nabla\boldsymbol{d}\|_{L^6}^2\|\boldsymbol{d}_t\|_{L^6}\|\nabla^2\boldsymbol{d}_t\|_{L^2}\\\le&C\|\nabla^2\boldsymbol{d}\|_{L^2}^2\|\boldsymbol{d}_t\|_{L^6}\|\nabla^2\boldsymbol{d}_t\|_{L^2}\\\le&C\|\boldsymbol{d}_t\|_{H^1}\|\nabla^2\boldsymbol{d}_t\|_{L^2}\\\le&\frac{1}{14}\|\nabla^2\boldsymbol{d}_t\|_{L^2}^2+C\|\boldsymbol{d}_t\|_{H^1}^2.\\K_5\le&C\|\nabla\boldsymbol{d}\|_{L^4}\|\nabla\boldsymbol{d}_t\|_{L^4}\|\nabla^2\boldsymbol{d}_t\|_{L^2}\\\le&C\|\nabla\boldsymbol{d}_t\|_{L^2}^{\frac{1}{4}}\|\nabla^2\boldsymbol{d}_t\|_{L^2}^{\frac{3}{4}}\|\nabla^2\boldsymbol{d}_t\|_{L^2}\\\le&\frac{1}{14}\|\nabla^2\boldsymbol{d}_t\|_{L^2}^2+C\|\nabla\boldsymbol{d}_t\|_{L^2}^2.\end{aligned}$$

Substituting the above estimates of K_1-K_5 into (3.62), we arrive at

$$\frac{d}{dt}\|\nabla \boldsymbol{d}_t\|_{L^2}^2+\|\nabla^2 \boldsymbol{d}_t\|_{L^2}^2 \le \frac{\mu}{4}\|\nabla \boldsymbol{u}_t\|_{L^2}^2+C\|\boldsymbol{d}_t\|_{H^1}^2. \tag{3.63}$$

Adding the resulting inequality with (3.58) and (3.60), and choosing δ suitably small, we deduce that

$$\begin{aligned}&\frac{d}{dt}(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\boldsymbol{d}_t\|_{H^1}^2)+\|\nabla \boldsymbol{u}_t\|_{L^2}^2+\|\nabla \boldsymbol{d}_t\|_{H^1}^2\\ \le& C(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\boldsymbol{d}_t\|_{H^1}^2+\|\nabla \boldsymbol{u}\|_{L^2}^2+\|\nabla^2 \boldsymbol{d}\|_{L^2}^2)\\ &+C(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\boldsymbol{d}_t\|_{H^1}^2)(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\boldsymbol{d}_t\|_{H^1}^2).\end{aligned} \tag{3.64}$$

Multiplying (3.64) by $t^i(i\in\{1,2,3\})$ gives that

$$\begin{aligned}&\frac{d}{dt}t^i(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\boldsymbol{d}_t\|_{H^1}^2)+t^i(\|\nabla \boldsymbol{u}_t\|_{L^2}^2+\|\nabla \boldsymbol{d}_t\|_{H^1}^2)\\ \le& C(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\boldsymbol{d}_t\|_{H^1}^2)[t^i(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\boldsymbol{d}_t\|_{H^1}^2)]\\ &+Ct^i(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\boldsymbol{d}_t\|_{H^1}^2+\|\nabla \boldsymbol{u}\|_{L^2}^2+\|\nabla^2 \boldsymbol{d}\|_{L^2}^2)+Ct^{i-1}(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\boldsymbol{d}_t\|_{H^1}^2),\end{aligned} \tag{3.65}$$

which together with Gronwall's inequality, (3.48), (3.21), and (3.22) leads to (3.53). Furthermore, multiplying (3.64) by $e^{\sigma t}$ gives that

$$\begin{aligned}&\frac{d}{dt}[e^{\sigma t}(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\boldsymbol{d}_t\|_{H^1}^2)]+e^{\sigma t}(\|\nabla \boldsymbol{u}_t\|_{L^2}^2+\|\nabla \boldsymbol{d}_t\|_{H^1}^2)\\ \le& C(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\boldsymbol{d}_t\|_{H^1}^2)[e^{\sigma t}(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\boldsymbol{d}_t\|_{H^1}^2)]\\ &+e^{\sigma t}(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\boldsymbol{d}_t\|_{H^1}^2+\|\nabla \boldsymbol{u}\|_{L^2}^2+\|\nabla^2 \boldsymbol{d}\|_{L^2}^2)\\ &+\sigma e^{\sigma t}(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\boldsymbol{d}_t\|_{H^1}^2),\end{aligned} \tag{3.66}$$

which combined with Gronwall's inequality (3.21), (3.8) and (3.23) implies (3.54). □

Lemma 3.5. *Under the conditions (3.2), (3.7) and (3.20), there exists a positive constant C depending only on $\Omega, q, \bar{\rho}, \underline{\mu}, \bar{\mu}, \|\nabla\mu(\rho_0)\|_{L^q}, \|\nabla \boldsymbol{u}_0\|_{L^2}$ and $\|\nabla^2 \boldsymbol{d}_0\|_{L^2}$ such that*

$$\int_0^T \|\nabla \boldsymbol{u}\|_{L^\infty}dt \le C\left(m_0^{\frac{1}{3}}+m_0^{\frac{1}{6}}\right). \tag{3.67}$$

Proof. For $3<r<\min\{q,6\}$, we deduce from Lemma 2.3, (3.2), (3.5), Hölder's inequality, Sobolev's inequality and the Gagliardo–Nirenberg inequality that

$$\begin{aligned}&\|\nabla \boldsymbol{u}\|_{W^{1,r}}+\|P\|_{W^{1,r}}\\ \le& C\|\rho \boldsymbol{u}_t\|_{L^r}+C\|\rho \boldsymbol{u}\cdot\nabla \boldsymbol{u}\|_{L^r}+C\||\nabla \boldsymbol{d}||\nabla^2 \boldsymbol{d}|\|_{L^r}\\ \le& C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^{\frac{6-r}{2r}}\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^6}^{\frac{3r-6}{2r}}+C\|\rho\|_{L^{\frac{6r}{6-r}}}\|\boldsymbol{u}\|_{L^\infty}\|\nabla \boldsymbol{u}\|_{L^6}+C\|\nabla \boldsymbol{d}\|_{L^\infty}\|\nabla^2 \boldsymbol{d}\|_{L^6}\\ \le& C\bar{\rho}^{\frac{5r-6}{4r}}\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla \boldsymbol{u}_t\|_{L^2}^{\frac{3r-6}{2r}}+C\|\nabla \boldsymbol{u}\|_{L^2}^{\frac{1}{2}}\|\nabla \boldsymbol{u}\|_{H^1}^{\frac{1}{2}}\|\nabla \boldsymbol{u}\|_{H^1}+C\|\nabla^2 \boldsymbol{d}\|_{L^2}^{\frac{1}{2}}\|\nabla^2 \boldsymbol{d}\|_{H^1}^{\frac{1}{2}}\|\nabla^2 \boldsymbol{d}\|_{H^1}\\ \le& C\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla \boldsymbol{u}_t\|_{L^2}^{\frac{3r-6}{2r}}+C\|\nabla \boldsymbol{u}\|_{L^2}^{\frac{1}{2}}\|\nabla \boldsymbol{u}\|_{H^1}^{\frac{3}{2}}+C\|\nabla^2 \boldsymbol{d}\|_{L^2}^{\frac{1}{2}}\|\nabla^2 \boldsymbol{d}\|_{H^1}^{\frac{3}{2}},\end{aligned} \tag{3.68}$$

which combined with Sobolev's inequality gives that

$$\begin{aligned}\|\nabla \boldsymbol{u}\|_{L^\infty}\le& C\|\nabla \boldsymbol{u}\|_{W^{1,r}}\\ \le& C\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla \boldsymbol{u}_t\|_{L^2}^{\frac{3r-6}{2r}}+C\|\nabla \boldsymbol{u}\|_{L^2}^{\frac{1}{2}}\|\nabla \boldsymbol{u}\|_{H^1}^{\frac{3}{2}}+C\|\nabla^2 \boldsymbol{d}\|_{L^2}^{\frac{1}{2}}\|\nabla^2 \boldsymbol{d}\|_{H^1}^{\frac{3}{2}}.\end{aligned} \tag{3.69}$$

If $0 < T \le 1$, we derive from (3.53) and Hölder's inequality that

$$\begin{aligned}
&\int_0^T \|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^{\frac{6-r}{2r}} \|\nabla \boldsymbol{u}_t\|_{L^2}^{\frac{3r-6}{2r}}\, dt \\
\le &\int_0^T \left(t\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2\right)^{\frac{6-r}{4r}} \left(t\|\nabla \boldsymbol{u}_t\|_{L^2}^2\right)^{\frac{3r-6}{4r}} t^{-\frac{1}{2}} dt \\
\le &C \sup_{[0,T]} \left(t\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2\right)^{\frac{6-r}{4r}} \left(\int_0^T t\|\nabla \boldsymbol{u}_t\|_{L^2}^2\, dt\right)^{\frac{3r-6}{4r}} \left(\int_0^T t^{-\frac{1}{2}\cdot\frac{4r}{6+r}} dt\right)^{\frac{6+r}{4r}} \\
\le &Cm_0^{\frac{1}{3}}.
\end{aligned} \tag{3.70}$$

If $T > 1$, due to $3 < r < \min\{6, q\}$, we deduce from (3.53) and Hölder's inequality that

$$\begin{aligned}
&\int_1^T \|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^{\frac{6-r}{2r}} \|\nabla \boldsymbol{u}_t\|_{L^2}^{\frac{3r-6}{2r}}\, dt \\
\le &\int_1^T \left(t^2\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2\right)^{\frac{6-r}{4r}} \left(t^2\|\nabla \boldsymbol{u}_t\|_{L^2}^2\right)^{\frac{3r-6}{4r}} t^{-1} dt \\
\le &C \sup_{[1,T]} \left(t^2\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2\right)^{\frac{6-r}{4r}} \left(\int_1^T t^2\|\nabla \boldsymbol{u}_t\|_{L^2}^2\, dt\right)^{\frac{3r-6}{4r}} \left(\int_1^T t^{-\frac{4r}{6+r}} dt\right)^{\frac{6+r}{4r}} \\
\le &Cm_0^{\frac{1}{3}},
\end{aligned} \tag{3.71}$$

which along with (3.70) yields that, for any $T > 0$,

$$\int_0^T \|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^{\frac{6-r}{2r}} \|\nabla \boldsymbol{u}_t\|_{L^2}^{\frac{3r-6}{2r}}\, dt \le Cm_0^{\frac{1}{3}}. \tag{3.72}$$

It follows from (3.40), (3.2), (3.21) and (3.6) that

$$\int_0^T \|\nabla \boldsymbol{u}\|_{L^2}^{\frac{1}{2}} \|\nabla \boldsymbol{u}\|_{H^1}^{\frac{3}{2}} dt \le \left(\int_0^T \|\nabla \boldsymbol{u}\|_{L^2}^2 dt\right)^{\frac{1}{4}} \left(\int_0^T \|\nabla \boldsymbol{u}\|_{H^1}^2 dt\right)^{\frac{3}{4}} \le Cm_0^{\frac{1}{6}}, \tag{3.73}$$

and

$$\int_0^T \|\nabla^2 \boldsymbol{d}\|_{L^2}^{\frac{1}{2}} \|\nabla^2 \boldsymbol{d}\|_{H^1}^{\frac{3}{2}} dt \le \left(\int_0^T \|\nabla^2 \boldsymbol{d}\|_{L^2}^2 dt\right)^{\frac{1}{4}} \left(\int_0^T \|\nabla^2 \boldsymbol{d}\|_{H^1}^2 dt\right)^{\frac{3}{4}} \le Cm_0^{\frac{1}{6}}. \tag{3.74}$$

With the estimates (3.72)-(3.74), integrating (3.69) on $[0, T]$ gives

$$\int_0^T \|\nabla \boldsymbol{u}\|_{L^\infty} dt \le C(m_0^{\frac{1}{3}} + m_0^{\frac{1}{6}}). \tag{3.75}$$

This completes the proof of Lemma 3.5. □

Proof of Proposition 3.1. Taking the spatial gradient operator ∇ on the transport equation (3.24) implies

$$(\nabla\mu(\rho))_t + \boldsymbol{u}\cdot\nabla^2\mu(\rho) + \nabla\boldsymbol{u}\cdot\nabla\mu(\rho) = \mathbf{0}. \tag{3.76}$$

Multiplying (3.76) by $q|\nabla\mu(\rho)|^{q-2}\nabla\mu(\rho)$ and integrating the resulting equation over Ω give

$$\frac{d}{dt}\|\nabla\mu(\rho)\|_{L^q} \le \|\nabla\boldsymbol{u}\|_{L^\infty}\|\nabla\mu(\rho)\|_{L^q}, \tag{3.77}$$

which combined with Gronwall's inequality and (3.67) leads to

$$\begin{aligned}\sup_{[0,T]}\|\nabla\mu(\rho)\|_{L^q} &\le \|\nabla\mu(\rho_0)\|_{L^q}\exp\left\{\int_0^T\|\nabla\boldsymbol{u}\|_{L^\infty}dt\right\}\\ &\le \|\nabla\mu(\rho_0)\|_{L^q}\exp\left\{C_3(m_0^{\frac{1}{3}}+m_0^{\frac{1}{6}})\right\}\end{aligned} \tag{3.78}$$

for some constant C_3 depending only on $\Omega, q, \bar{\rho}, \underline{\mu}, \bar{\mu}, \|\nabla\mu(\rho_0)\|_{L^q}, \|\nabla\boldsymbol{u}_0\|_{L^2}$ and $\|\nabla^2\boldsymbol{d}_0\|_{L^2}$. This implies that

$$\sup_{[0,T]}\|\nabla\mu(\rho)\|_{L^q} \le 2\|\nabla\mu(\rho_0)\|_{L^q}, \tag{3.79}$$

provided $m_0 \le \varepsilon_1 \triangleq \min\left\{1, \left(\frac{1}{2C_1}\right)^9, \left(\frac{1}{8C_2}\right)^9, \left(\frac{\log 2}{2C_3}\right)^6\right\}$.

Next, it follows from (3.6) and (3.21) that

$$\begin{aligned}\int_0^T(\|\nabla\boldsymbol{u}\|_{L^2}^4+\|\nabla^2\boldsymbol{d}\|_{L^2}^4)dt &\le \sup_{[0,T]}(\|\nabla\boldsymbol{u}\|_{L^2}^2+\|\nabla^2\boldsymbol{d}\|_{L^2}^2)\int_0^T(\|\nabla\boldsymbol{u}\|_{L^2}^2+\|\nabla^2\boldsymbol{d}\|_{L^2}^2)dt\\ &\le C_4 m_0^{\frac{2}{3}}\end{aligned} \tag{3.80}$$

for some constant C_4 depending only on $\Omega, q, \bar{\rho}, \underline{\mu}, \bar{\mu}, \|\nabla\mu(\rho_0)\|_{L^q}, \|\nabla\boldsymbol{u}_0\|_{L^2}$ and $\|\nabla^2\boldsymbol{d}_0\|_{L^2}$. This yields that

$$\int_0^T\left(\|\nabla\boldsymbol{u}\|_{L^2}^4+\|\nabla\boldsymbol{u}\|_{L^2}^4\right)dt \le m_0^{\frac{1}{3}}, \tag{3.81}$$

provided $m_0 \le \varepsilon_2 \triangleq \min\left\{\left(\frac{1}{2C_1}\right)^9, \left(\frac{1}{8C_2}\right)^9, (\frac{1}{C_4})^3\right\}$.

Finally, multiplying (3.41) by $4|\nabla\boldsymbol{d}|^2\nabla\boldsymbol{d}$ and integrating by parts over Ω give rise to

$$\begin{aligned}&\frac{d}{dt}\|\nabla\boldsymbol{d}\|_{L^4}^4 + 4\||\nabla\boldsymbol{d}||\nabla^2\boldsymbol{d}|\|_{L^2}^2 + 2\|\nabla|\nabla\boldsymbol{d}|^2\|_{L^2}^2\\ \le& 4\int|\nabla\boldsymbol{u}||\nabla\boldsymbol{d}|^4dx + 4\int|\nabla\boldsymbol{d}|^6dx + 4\int|\nabla^2\boldsymbol{d}||\nabla\boldsymbol{d}|^4dx\\ \le& C\|\nabla\boldsymbol{u}\|_{L^2}\||\nabla\boldsymbol{d}|^2\|_{L^4}^2 + C\||\nabla\boldsymbol{d}||\nabla\boldsymbol{d}|^2\|_{L^2}^2 + C\|\nabla^2\boldsymbol{d}\|_{L^2}\||\nabla\boldsymbol{d}|^2\|_{L^4}^2\\ \le& C\|\nabla\boldsymbol{u}\|_{L^2}\||\nabla\boldsymbol{d}|^2\|_{L^2}^{\frac{1}{2}}\|\nabla|\nabla\boldsymbol{d}|^2\|_{L^2}^{\frac{3}{2}} + C\|\nabla\boldsymbol{d}\|_{L^6}^2\||\nabla\boldsymbol{d}|^2\|_{L^3}^2\\ &+ C\|\nabla^2\boldsymbol{d}\|_{L^2}\||\nabla\boldsymbol{d}|^2\|_{L^2}^{\frac{1}{2}}\|\nabla|\nabla\boldsymbol{d}|^2\|_{L^2}^{\frac{3}{2}}\\ \le& \|\nabla|\nabla\boldsymbol{d}|^2\|_{L^2}^2 + C\|\nabla\boldsymbol{u}\|_{L^2}^4\|\nabla\boldsymbol{d}\|_{L^4}^4 + C\|\nabla^2\boldsymbol{d}\|_{L^2}^4\||\nabla\boldsymbol{d}|^2\|_{L^2}^2\\ \le& \|\nabla|\nabla\boldsymbol{d}|^2\|_{L^2}^2 + C(\|\nabla\boldsymbol{u}\|_{L^2}^4 + \|\nabla^2\boldsymbol{d}\|_{L^2}^4\|)\|\nabla\boldsymbol{d}\|_{L^4}^4,\end{aligned} \tag{3.82}$$

which implies

$$\frac{d}{dt}\|\nabla \boldsymbol{d}\|_{L^4}^4+\||\nabla \boldsymbol{d}||\nabla^2 \boldsymbol{d}|\|_{L^2}^2+\|\nabla|\nabla \boldsymbol{d}|^2\|_{L^2}^2\le C(\|\nabla \boldsymbol{u}\|_{L^2}^4+\|\nabla^2 \boldsymbol{d}\|_{L^2}^4)\|\nabla \boldsymbol{d}\|_{L^4}^4. \tag{3.83}$$

This along with Gronwall's inequality and (3.2) yields that

$$\sup_{[0,T]}\|\nabla \boldsymbol{d}\|_{L^4}^4+\int_0^T\||\nabla \boldsymbol{d}||\nabla^2 \boldsymbol{d}|\|_{L^2}^2dt\le\|\nabla \boldsymbol{d}_0\|_{L^4}^4\exp\int_0^T(\|\nabla \boldsymbol{u}\|_{L^2}^4+\|\nabla^2 \boldsymbol{d}\|_{L^2}^4)dt\le C. \tag{3.84}$$

Subsequently, it follows from (3.6), (3.84) and Hölder's inequality that

$$\sup_{[0,T]}\|\nabla \boldsymbol{d}\|_{L^3}^3\le\sup_{[0,T]}\left(\|\nabla \boldsymbol{d}\|_{L^2}\|\nabla \boldsymbol{d}\|_{L^4}^2\right)\le C\sup_{[0,T]}\|\nabla \boldsymbol{d}\|_{L^2}\le C_5 m_0^{\frac13} \tag{3.85}$$

for some constant C_5 depending only on $\Omega, q, \bar{\rho}, \underline{\mu}, \bar{\mu}, \|\nabla\mu(\rho_0)\|_{L^q}, \|\nabla \boldsymbol{u}_0\|_{L^2}$ and $\|\nabla^2 \boldsymbol{d}_0\|_{L^2}$. This yields that

$$\sup_{[0,T]}\|\nabla \boldsymbol{d}\|_{L^3}^3\le m_0^{\frac16} \tag{3.86}$$

provided $m_0\le\varepsilon_3\triangleq\min\left\{\left(\frac{1}{2C_1}\right)^9,\left(\frac{1}{8C_2}\right)^9,(\frac{1}{C_5})^6\right\}$.
As a consequence, if

$$m_0\le\varepsilon_0\triangleq\min\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}=\min\left\{1,\left(\frac{1}{2C_1}\right)^9,\left(\frac{1}{8C_2}\right)^9,\left(\frac{\log 2}{2C_3}\right)^6,\left(\frac{1}{C_4}\right)^3,\left(\frac{1}{C_5}\right)^6\right\},$$

we derive (3.2) from (3.79), (3.75) and (3.85). Therefore, the proof of Proposition 3.1 is complete. □

Lemma 3.6. *Under the conditions* (3.2) *and* (3.20)*, there exists a positive* C *depending only on* $\Omega, q, \overline{\mu}, \underline{\mu}, \bar{\rho}, \|\nabla \boldsymbol{u}_0\|_{L^2}, \|\nabla^2 \boldsymbol{d}_0\|_{L^2}$ *and* $\|\nabla\mu(\rho_0)\|_{L^q}$ *such that*

$$\sup_{[0,T]}(\|\nabla\rho\|_{L^2}+\|\rho_t\|_{L^{\frac32}})\le C. \tag{3.87}$$

Proof. By an argument similar to the one used in (3.79), we obtain from (1.7) that

$$\sup_{[0,T]}\|\nabla\rho\|_{L^2}\le C. \tag{3.88}$$

It follows from $(3.1)_1$, Hölder's inequality and Sobolev's inequality that

$$\|\rho_t\|_{L^{\frac32}}=\|\boldsymbol{u}\cdot\nabla\rho\|_{L^{\frac32}}\le\|\boldsymbol{u}\|_{L^6}\|\nabla\rho\|_{L^2}\le C\|\nabla \boldsymbol{u}\|_{L^2}\|\nabla\rho\|_{L^2}, \tag{3.89}$$

which together with (3.89) and (3.21) yields that

$$\sup_{[0,T]}\|\rho_t\|_{L^{\frac32}}\le C. \tag{3.90}$$

This completes the proof of Lemma 3.6. □

Lemma 3.7. *Under the conditions* (3.2),(3.7) *and* (3.20)*, there exists a positive constant* C *depending on* $\Omega, q, \overline{\mu}, \underline{\mu}, \bar{\rho}, \|\nabla \boldsymbol{u}_0\|_{L^2}, \|\nabla^2 \boldsymbol{d}_0\|_{L^2}$ *and* $\|\nabla\mu(\rho_0)\|_{L^q}$ *such that for* $r\in(3,\min\{q,6\})$*,*

$$\sup_{[0,T]}[t(\|\boldsymbol{u}\|_{H^2}^2+\|\nabla \boldsymbol{d}\|_{H^2}^2)]+\int_0^T t(\|\boldsymbol{u}\|_{W^{2,r}}^2+\|\nabla P\|_{L^r}^2+\|\nabla \boldsymbol{d}\|_{H^3}^2)dt\le C. \tag{3.91}$$

Furthermore, for σ *as in Lemma 3.2 and* $\zeta(t)$ *as in Lemma 3.4, one has that*

$$\sup_{[\zeta(T),T]}[e^{\sigma t}(\|\boldsymbol{u}\|_{H^2}^2+\|\nabla P\|_{L^2}^2+\|\nabla \boldsymbol{d}\|_{H^2}^2)]\le C. \tag{3.92}$$

Proof. We obtain from (3.50) and (3.21) that

$$\|\boldsymbol{u}\|_{H^2}^2+\|\nabla \boldsymbol{d}\|_{H^2}^2 \le C(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\nabla \boldsymbol{d}_t\|_{L^2}^2+\|\nabla \boldsymbol{u}\|_{L^2}^2+\|\nabla^2 \boldsymbol{d}\|_{L^2}^2), \tag{3.93}$$

which together with (3.22) and (3.53) implies

$$\sup_{[0,T]}[t(\|\boldsymbol{u}\|_{H^2}^2+\|\nabla \boldsymbol{d}\|_{H^2}^2)] \le C. \tag{3.94}$$

And it follows from (3.93), (3.53) and (3.23) that

$$\sup_{[0,T]}[e^{\sigma t}(\|\boldsymbol{u}\|_{H^2}^2+\|\nabla \boldsymbol{d}\|_{H^2}^2)] \le C. \tag{3.95}$$

For $3<r<\min\{q,6\}$, we get from (3.68) and (3.93) that

$$\begin{aligned}
&\|\boldsymbol{u}\|_{W^{2,r}}^2+\|\nabla P\|_{L^r}^2\\
\le& C(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^r}^2+\|\rho\boldsymbol{u}\cdot\nabla\boldsymbol{u}\|_{L^r}^2+\||\nabla\boldsymbol{d}||\nabla^2\boldsymbol{d}|\|_{L^r}^2)\\
\le& C\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^{\frac{6-r}{r}}\|\nabla\boldsymbol{u}_t\|_{L^2}^{\frac{3r-6}{r}}+C\|\nabla\boldsymbol{u}\|_{L^2}\|\nabla\boldsymbol{u}\|_{H^1}^3+C\|\nabla^2\boldsymbol{d}\|_{L^2}\|\nabla^2\boldsymbol{d}\|_{H^1}^3\\
\le& C\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+C\|\nabla\boldsymbol{u}_t\|_{L^2}^2+C(\|\sqrt{\rho}\boldsymbol{u}_t\|_{L^2}^2+\|\nabla\boldsymbol{d}_t\|_{L^2}^2+\|\nabla\boldsymbol{u}\|_{L^2}^2+\|\nabla^2\boldsymbol{d}\|_{L^2}^2)^2,
\end{aligned} \tag{3.96}$$

which combined with (3.6), (3.21), (3.22) and (3.53) leads to

$$\int_0^T t(\|\boldsymbol{u}\|_{W^{2,r}}^2+\|\nabla P\|_{L^r}^2)dt \le C. \tag{3.97}$$

Finally, it follows from L^2-theory of elliptic equations, (3.21) and Sobolev's inequality that

$$\begin{aligned}
\|\nabla\boldsymbol{d}\|_{H^3}^2 \le& C\|\nabla\boldsymbol{d}_t\|_{H^1}^2+C\|\nabla(\boldsymbol{u}\cdot\nabla\boldsymbol{d})\|_{H^1}^2+C\|\nabla(|\nabla\boldsymbol{d}|^2\boldsymbol{d})\|_{H^1}^2+C\|\nabla\boldsymbol{d}\|_{H^2}^2\\
\le& C\|\nabla\boldsymbol{d}_t\|_{L^2}^2+C\|\nabla^2\boldsymbol{d}_t\|_{L^2}^2+C\|\boldsymbol{u}\cdot\nabla^2\boldsymbol{d}\|_{L^2}^2+C\|\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{d}\|_{L^2}^2\\
&+C\|\boldsymbol{u}\cdot\nabla^3\boldsymbol{d}\|_{L^2}^2+C\|\nabla\boldsymbol{u}\cdot\nabla^2\boldsymbol{d}\|_{L^2}^2+C\|\nabla^2\boldsymbol{u}\cdot\nabla\boldsymbol{d}\|_{L^2}^2+C\|\nabla\boldsymbol{d}\cdot\nabla^2\boldsymbol{d}\|_{L^2}^2\\
&+C\||\nabla\boldsymbol{d}|^3\|_{L^2}^2+C\|\nabla\boldsymbol{d}\cdot\nabla^3\boldsymbol{d}\|_{L^2}^3+C\||\nabla\boldsymbol{d}|^2|\nabla^2\boldsymbol{d}|\|_{L^2}^2+C\|\nabla^2\boldsymbol{d}\|_{H^1}^2\\
\le& C\|\nabla\boldsymbol{d}_t\|_{L^2}^2+C\|\nabla^2\boldsymbol{d}_t\|_{L^2}^2+C\|\nabla\boldsymbol{u}\|_{L^3}^2\|\nabla^2\boldsymbol{d}\|_{L^6}^2+C\|\nabla^2\boldsymbol{d}\|_{H^1}^2\\
&+C(\|\boldsymbol{u}\|_{L^\infty}^2+\|\nabla\boldsymbol{d}\|_{L^\infty}^2)(\|\boldsymbol{u}\|_{H^2}^2+\|\nabla\boldsymbol{d}\|_{H^2}^2)\\
\le& C\|\nabla\boldsymbol{d}_t\|_{L^2}^2+C\|\nabla^2\boldsymbol{d}_t\|_{L^2}^2+C\|\nabla^2\boldsymbol{d}\|_{H^1}^2+C\|\boldsymbol{u}\|_{H^2}^4+\|\nabla\boldsymbol{d}\|_{H^2}^4,
\end{aligned} \tag{3.98}$$

which along with (3.22), (3.53), (3.48), (3.94) and (3.52) implies that

$$\int_0^T t\|\nabla\boldsymbol{d}\|_{H^3}^2 dt \le C. \tag{3.99}$$

Therefore, the proof of Lemma 3.7 is complete. □

4. proof of Theorem 1.1

With all the *a priori* estimates obtained in Sect. 2 at hand, we are now in a position to give a proof of Theorem 1.1.

Proof of Theorem 1.1. First, by Lemma 2.1, there exists a $T_*>0$ such that the initial and boundary value problem (1.1)-(1.4) admits a unique local strong solution $(\rho,\boldsymbol{u},\boldsymbol{d},P)$ on $\Omega\times(0,T_*]$. It follows from (1.7) that there exists a $T_1\in(0,T_*]$ such that (3.2) holds for $T=T_1$.

Next, set

$$T_1^* \triangleq \sup\{T > 0 | (\rho, \boldsymbol{u}, \boldsymbol{d}, P) \text{ is a strong solution on } \Omega \times (0, T] \text{ and (3.2) holds}\}, \tag{4.1}$$

and

$$T^* \triangleq \sup\{T > 0 | (\rho, \boldsymbol{u}, \boldsymbol{d}, P) \text{ is a strong solution on } \Omega \times (0, T]\}. \tag{4.2}$$

Then $T_1^* \geq T_1 > 0$. In particular, Proposition 3.1 together with continuity argument implies that (3.2) in fact holds on $(0, T^*)$. Thus,

$$T_1^* = T^*, \tag{4.3}$$

provided that $m_0 < \varepsilon_0$ as assumed.

Moreover, for any $0 < \tau < T \leq T^*$ with T finite, one deduces from standard embedding that

$$\nabla \boldsymbol{d} \in L^\infty(\tau, T; H^2) \cap H^1(\tau, T; H^2) \hookrightarrow C([\tau, T]; H^2). \tag{4.4}$$

Combining (3.53) and (3.91) gives for any $0 < \tau < T \leq T^*$,

$$\nabla \boldsymbol{u}, P \in C([\tau, T]; L^2) \cap C(\overline{\Omega} \times [\tau, T]), \tag{4.5}$$

where one has used the standard embedding

$$L^\infty(\tau, T; H^1 \cap W^{1,r}) \cap H^1(\tau, T; L^2) \hookrightarrow C([\tau, T]; L^2) \cap C(\overline{\Omega} \times [\tau, T]).$$

Moreover, it follows from (3.2), (3.5), (3.87) and [17, Lemma 2.3] that

$$\rho \in C([0, T]; H^1), \ \nabla \mu(\rho) \in C([0, T]; L^q). \tag{4.6}$$

Thanks to (3.23) and (3.91), the standard arguments yield that

$$\rho \boldsymbol{u}_t \in H^1(\tau, T; L^2) \hookrightarrow C([\tau, T]; L^2), \tag{4.7}$$

which together with (4.5) and (4.6) gives

$$\rho \boldsymbol{u}_t + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \operatorname{div}(\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d}) \in C([\tau, T]; L^2). \tag{4.8}$$

Since $(\rho, \boldsymbol{u})$ satisfies (3.38) with $\mathbf{F} = -\rho \boldsymbol{u}_t - \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \operatorname{div}(\nabla \boldsymbol{d} \odot \nabla \boldsymbol{d})$, we deduce from $(3.1)_2$, (4.5), (4.6), (4.8) and (3.91) that

$$\nabla \boldsymbol{u}, P \in C([\tau, T]; H^1 \cap W^{1,r}). \tag{4.9}$$

Now, we claim that

$$T^* = \infty. \tag{4.10}$$

Otherwise, $T^* < \infty$. Proposition 3.1 implies that (3.3) holds at $T = T^*$. It follows from (3.87), (3.79) and (3.21) that

$$(\rho^*, \boldsymbol{u}^*, \boldsymbol{d}^*)(x) \triangleq (\rho, \boldsymbol{u}, \boldsymbol{d})(x, T^*) = \lim_{t \to T^*} (\rho, \boldsymbol{u}, \boldsymbol{d})(x, t)$$

satisfies

$$\rho^* \in H^1, \ \nabla \mu(\rho^*) \in L^q, \ \boldsymbol{u}^*, \nabla \boldsymbol{d}^* \in H_0^1.$$

Therefore, one can take $(\rho^*, \boldsymbol{u}^*, \boldsymbol{d}^*)$ as the initial data and apply Lemma 2.1 again to extend the local strong solution beyond T^*. This contradicts the assumption of T^* in (4.2). Hence, $T^* = \infty$. We thus complete the proof of Theorem 1.1 since exponential decay of solution (1.11) follows directly from (3.92) and (3.54).

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