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Fractional wave equation with irregular mass and dissipation

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Abstract. In this paper, we pursue our series of papers aiming to show the applicability of the concept of very weak solutions. We consider a wave model with irregular position-dependent mass and dissipation terms, in particular, allowing for δ-like coefficients and prove that the problem has a very weak solution. Furthermore, we prove the uniqueness in an appropriate sense and the coherence of the very weak solution concept with classical theory. A special case of the model considered here is the so-called telegraph equation.

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1. Introduction

The telegraph equations are a system of coupled linear equations governing voltage and current flow on a linear electrical line. For t denoting the time and x the distance from any fixed point, and ν, ζ the voltage and the current, respectively, the equations are as follows

$$\begin{cases} \partial_x \nu(t,x) = -L\partial_t \zeta(t,x) - R\zeta(t,x), \\ \partial_x \zeta(t,x) = -C\partial_t \nu(t,x) - G\nu(t,x), \end{cases}$$

where L is the inductance, C the capacitance, R the resistance and G stands for the conductance. When combined, a hyperbolic partial differential equation of the following form is obtained

$$\partial_x^2 u(t,x) - LC\partial_t^2 u(t,x) = (RC + GL)\partial_t u(t,x) + GRu(t,x), \tag{1.1}$$

where u represents either the voltage ν or the current ζ. For the derivation of equations, we refer the reader to [1] for more details. The form (1.1) can be regarded as a wave equation with additional mass and dissipation terms. This form is widely used in the literature to study wave propagation phenomena and random walk theory. See, for instance [2–6] and the references therein.

In the present paper, we consider the telegraph equation in a more general case. That is, we use the fractional Laplacian instead of the classical one and for fixed $T > 0$, we consider the Cauchy problem:

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^s u(t,x) + a(x)u(t,x) + b(x)u_t(t,x) = 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \end{cases} \tag{1.2}$$

where $(t,x) \in [0,T] \times \mathbb{R}^d$ and $s > 0$. Motivated by the fact that mechanical and physical properties of nowadays materials cannot be described by smooth functions due to the non-homogeneity of the material structure, the spatially dependent mass a and the dissipation coefficient b in (1.2) are assumed to be non-negative and singular, in particular to have δ-like behaviours. Our aim is to prove that this problem is well posed in the sense of the very weak solution concept introduced in [7] by Garetto and the first author in order to give a neat solution to the problem of multiplication that Schwartz theory of distributions

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is concerned with, see [8], and to provide a framework in which partial differential equations involving coefficients and data of low regularity can be rigorously studied. Let us give a brief literature review about this concept of solutions. After the original work of Garetto and Ruzhansky [7], many researchers started using this notion of solutions for different situations, either for abstract mathematical problems as [9–11] or for physical models as in [12–14] and [15–19] where it is shown that the concept of very weak solutions is very suitable for numerical modelling, and in [20] where the question of propagation of coefficients singularities of the very weak solution is studied. More recently, we cite [21–25].

The novelty of this work lies in the fact that we consider equations that cannot be formulated in the classical or the distributional sense. We employ the concept of very weak solutions which allows to overcome the problem of the impossibility of multiplication of distributions. Furthermore, the results obtained in this paper extend those of [16], firstly by incorporating a dissipation term, and secondly by relaxing the assumptions on the Cauchy data, allowing them to be as singular as the equation coefficients, whereas in [16] they were supposed to be smooth functions.

2. Preliminaries

For the reader's convenience, we review in this section notations and notions that are frequently used in the sequel.

2.1. Notation

- By the notation $f \lesssim g$, we mean that there exists a positive constant C, such that $f \leq Cg$ independently on f and g.
- We also define

$$\|u(t,\cdot)\|_1 := \|u(t,\cdot)\|_{L^2} + \|(-\Delta)^{\frac{s}{2}}u(t,\cdot)\|_{L^2} + \|u_t(t,\cdot)\|_{L^2},$$

  and

$$\|u(t,\cdot)\|_2 := \|u(t,\cdot)\|_{L^2} + \|(-\Delta)^{\frac{s}{2}}u(t,\cdot)\|_{L^2} + \|(-\Delta)^{s}u(t,\cdot)\|_{L^2} + \|u_t(t,\cdot)\|_{L^2}.$$

We also recall the well-known Hölder inequality.

Proposition 2.1. *Let $r \in (0,\infty)$ and $p,q \in (0,\infty)$ be such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Assume that $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, then, $fg \in L^r(\mathbb{R}^d)$ and we have*

$$\|fg\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}. \tag{2.1}$$

2.2. The fractional Sobolev space H^s and the fractional Laplacian

Definition 1. (**Fractional Sobolev space**) Given $s > 0$, the fractional Sobolev space is defined by

$$H^s(\mathbb{R}^d) = \big\{f \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} (1+|\xi|^{2s})|\widehat{f}(\xi)|^2 \mathrm{d}\xi < +\infty\big\},$$

where $\widehat{f}$ denotes the Fourier transform of f.

We note that, the fractional Sobolev space H^s endowed with the norm

$$\|f\|_{H^s} := \left(\int_{\mathbb{R}^d} (1+|\xi|^{2s})|\widehat{f}(\xi)|^2 \mathrm{d}\xi\right)^{\frac{1}{2}}, \quad \text{for } f \in H^s(\mathbb{R}^d), \tag{2.2}$$

is a Hilbert space.

Definition 2. (**Fractional Laplacian**) For $s>0$, $(-\Delta)^s$ denotes the fractional Laplacian defined by

$$(-\Delta)^s f = \mathcal{F}^{-1}(|\xi|^{2s}(\widehat{f})),$$

for all $\xi \in \mathbb{R}^d$.

In other words, the fractional Laplacian $(-\Delta)^s$ can be viewed as the pseudo-differential operator with symbol $|\xi|^{2s}$. With this definition and the Plancherel theorem, the fractional Sobolev space can be defined as:

$$H^s(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) : (-\Delta)^{\frac{s}{2}} f \in L^2(\mathbb{R}^d) \right\}, \tag{2.3}$$

moreover, the norm

$$\|f\|_{H^s} := \|f\|_{L^2} + \|(-\Delta)^{\frac{s}{2}} f\|_{L^2}, \tag{2.4}$$

is equivalent to the one defined in (2.2).

Remark 2.1. We note that the fractional Sobolev space $H^s(\mathbb{R}^d)$ can also be defined via the Gagliardo norm; however, we chose this approach, since it is valid for any real $s>0$, unlike the one via Gagliardo norm which is valid only for $s \in (0,1)$. We refer the reader to [26–28] for more details and alternative definitions.

Proposition 2.2. *(**Fractional Sobolev inequality**, e.g. Theorem 1.1. [29]) For $d \in \mathbb{N}_0$ and $s \in \mathbb{R}_+$, let $d > 2s$ and $q = \frac{2d}{d-2s}$. Then, the estimate*

$$\|f\|_{L^q} \le C(d,s)\|(-\Delta)^{\frac{s}{2}} f\|_{L^2}, \tag{2.5}$$

holds for all $f \in H^s(\mathbb{R}^d)$, where the constant C depends only on the dimension d and the order s.

2.3. Duhamel's principle

We prove the following special version of Duhamel's principle that will frequently be used throughout this paper. For more general versions of this principle, we refer the reader to [30]. Let us consider the following Cauchy problem,

$$\begin{cases} u_{tt}(t,x) + \lambda(x)u_t(t,x) + Lu(t,x) = f(t,x), \ (t,x) \in (0,\infty)\times\mathbb{R}^d, \\ u(0,x) = u_0(x), \ \ u_t(0,x) = u_1(x), \ \ x \in \mathbb{R}^d, \end{cases} \tag{2.6}$$

for a given function λ and L is a linear partial differential operator acting over the spatial variable.

Proposition 2.3. *The solution to the Cauchy problem (2.6) is given by*

$$u(t,x) = w(t,x) + \int_0^t v(t,x;\tau)\mathrm{d}\tau, \tag{2.7}$$

where $w(t,x)$ is the solution to the homogeneous problem

$$\begin{cases} w_{tt}(t,x) + \lambda(x)w_t(t,x) + Lw(t,x) = 0, \ (t,x) \in (0,\infty)\times\mathbb{R}^d, \\ w(0,x) = u_0(x), \ \ w_t(0,x) = u_1(x), \ \ x \in \mathbb{R}^d, \end{cases} \tag{2.8}$$

and $v(t,x;\tau)$ solves the auxiliary Cauchy problem

$$\begin{cases} v_{tt}(t,x;\tau) + \lambda(x)v_t(t,x;\tau) + Lv(t,x;\tau) = 0, \ (t,x) \in (\tau,\infty)\times\mathbb{R}^d, \\ v(\tau,x;\tau) = 0, \ \ v_t(\tau,x;\tau) = f(\tau,x), \ \ x \in \mathbb{R}^d, \end{cases} \tag{2.9}$$

where τ is a parameter varying over $(0,\infty)$.

Proof. Firstly, we apply ∂_t to u in (2.7). We get

$$\partial_t u(t,x) = \partial_t w(t,x) + \int_0^t \partial_t v(t,x;\tau)\mathrm{d}\tau, \tag{2.10}$$

and accordingly

$$\lambda(x)\partial_t u(t,x) = \lambda(x)\partial_t w(t,x) + \int_0^t \lambda(x)\partial_t v(t,x;\tau)\mathrm{d}\tau, \tag{2.11}$$

where we used the fact that $v(t,x;t) = 0$ by the imposed initial condition in (2.9). We differentiate again (2.10) with respect to t to get

$$\partial_{tt} u(t,x) = \partial_{tt} w(t,x) + f(t,x) + \int_0^t \partial_{tt} v(t,x;\tau)\mathrm{d}\tau, \tag{2.12}$$

where we used that $\partial_t v(t,x;t) = f(t,x)$. Now, applying L to u in (2.7) gives

$$Lu(t,x) = Lw(t,x) + \int_0^t Lv(t,x;\tau)\mathrm{d}\tau. \tag{2.13}$$

By adding (2.12), (2.13) and (2.11), and by taking into consideration that w and v satisfy the equations in (2.8) and (2.9), we get

$$u_{tt}(t,x) + \lambda(x)u_t(t,x) + Lu(t,x) = f(t,x).$$

It remains to prove that u satisfy the initial conditions. Indeed, from (2.7) and (2.10), we have that $u(0,x) = w(0,x) = u_0(x)$ and that $u_t(0,x) = \partial_t w(0,x) = u_1(x)$. This concludes the proof. □

Remark 2.2. We note that the above statement of Duhamel's principle can be extended to differential operators of order $k \in \mathbb{N}$. Indeed, if we consider the Cauchy problem

$$\begin{cases} \partial_t^k u(t,x) + \sum_{j=1}^{k-1} \lambda_j(x)\partial_t^j u(t,x) + Lu(t,x) = f(t,x), \ (t,x) \in (0,\infty)\times\mathbb{R}^d, \\ \partial_t^j u(0,x) = u_j(x), \text{ for } \quad j = 0,\cdots,k-1, \quad x \in \mathbb{R}^d, \end{cases}$$

then, the solution is given by

$$u(t,x) = w(t,x) + \int_0^t v(t,\tau;\tau)\mathrm{d}\tau,$$

where $w(t,x)$ is the solution to the homogeneous problem

$$\begin{cases} \partial_t^k w(t,x) + \sum_{j=1}^{k-1} \lambda_j(x)\partial_t^j w(t,x) + Lw(t,x) = 0, \ (t,x) \in (0,\infty)\times\mathbb{R}^d, \\ \partial_t^j w(0,x) = u_j(x), \text{ for } \quad j = 0,\cdots,k-1, \quad x \in \mathbb{R}^d, \end{cases}$$

and $v(t,x;\tau)$ solves the auxiliary Cauchy problem

$$\begin{cases} \partial_t^k v(t,x;\tau) + \sum_{j=1}^{k-1} \lambda_j(x)\partial_t^j v_t(t,x;\tau) + Lv(t,x;\tau) = 0, \ (t,x) \in (\tau,\infty)\times\mathbb{R}^d, \\ \partial_t^j w(\tau,x;\tau) = 0, \text{ for } \quad j = 0,\cdots,k-2, \ \partial_t^{k-1} w(\tau,x;\tau) = f(\tau,x), \ \ x \in \mathbb{R}^d, \end{cases}$$

where $\tau \in (0,\infty)$.

2.4. Energy estimates for the classical solution

In order to prove existence and uniqueness of a very weak solution to the Cauchy problem (1.2) as well as the coherence with classical theory, we will often use the following lemmas that are stated in the case when the mass a and the dissipation coefficient b are regular functions. The statements of the lemmas are given under different assumptions on a and b.

Lemma 2.4. *Let $a, b \in L^\infty(\mathbb{R}^d)$ be non-negative and suppose that $u_0 \in H^s(\mathbb{R}^d)$ and $u_1 \in L^2(\mathbb{R}^d)$. Then the unique solution $u \in C([0,T]; H^s(\mathbb{R}^d)) \cap C^1([0,T]; L^2(\mathbb{R}^d))$ to the Cauchy problem (1.2) satisfies the estimate*

$$\|u(t,\cdot)\|_1 \lesssim \Big(1 + \|a\|_{L^\infty}\Big)\Big(1 + \|b\|_{L^\infty}\Big)\Big[\|u_0\|_{H^s} + \|u_1\|_{L^2}\Big], \tag{2.14}$$

for all $t \in [0,T]$.

Proof. Multiplying the equation in (1.2) by u_t and integrating with respect to the variable x over $\mathbb{R}^d$ and taking the real part, we get

$$\begin{aligned} Re\Bigg(&\langle u_{tt}(t,\cdot), u_t(t,\cdot)\rangle_{L^2} + \langle (-\Delta)^s u(t,\cdot), u_t(t,\cdot)\rangle_{L^2} \\ &+ \langle a(\cdot)u(t,\cdot), u_t(t,\cdot)\rangle_{L^2} + \langle b(\cdot)u_t(t,\cdot), u_t(t,\cdot)\rangle_{L^2}\Bigg) = 0. \end{aligned} \tag{2.15}$$

We easily see that

$$Re\langle u_{tt}(t,\cdot), u_t(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t \langle u_t(t,\cdot), u_t(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t \|u_t(t,\cdot)\|^2_{L^2}, \tag{2.16}$$

and

$$\begin{aligned} Re\langle (-\Delta)^s u(t,\cdot), u_t(t,\cdot)\rangle_{L^2} &= \frac{1}{2}\partial_t \langle (-\Delta)^{\frac{s}{2}} u(t,\cdot), (-\Delta)^{\frac{s}{2}} u(t,\cdot)\rangle_{L^2} \\ &= \frac{1}{2}\partial_t \|(-\Delta)^{\frac{s}{2}} u(t,\cdot)\|^2_{L^2}, \end{aligned} \tag{2.17}$$

where we used the self-adjointness of the operator $(-\Delta)^s$. For the remaining terms in (2.15), we have

$$Re\langle a(\cdot)u(t,\cdot), u_t(t,\cdot)\rangle_{L^2} = \frac{1}{2}\partial_t \|a^{\frac{1}{2}}(\cdot)u(t,\cdot)\|^2_{L^2}, \tag{2.18}$$

and

$$Re\langle b(\cdot)u_t(t,\cdot), u_t(t,\cdot)\rangle_{L^2} = \|b^{\frac{1}{2}}(\cdot)u_t(t,\cdot)\|^2_{L^2}. \tag{2.19}$$

By substituting (2.16),(2.17),(2.18) and (2.19) in (2.15), we get

$$\partial_t\Big[\|u_t(t,\cdot)\|^2_{L^2} + \|(-\Delta)^{\frac{s}{2}} u(t,\cdot)\|^2_{L^2} + \|a^{\frac{1}{2}}(\cdot)u(t,\cdot)\|^2_{L^2}\Big] = -2\|b^{\frac{1}{2}}(\cdot)u_t(t,\cdot)\|^2_{L^2}. \tag{2.20}$$

Let us denote

$$E(t) := \|u_t(t,\cdot)\|^2_{L^2} + \|(-\Delta)^{\frac{s}{2}} u(t,\cdot)\|^2_{L^2} + \|a^{\frac{1}{2}}(\cdot)u(t,\cdot)\|^2_{L^2}, \tag{2.21}$$

the energy function of the system (1.2). It follows from (2.20) that $\partial_t E(t) \leq 0$ and consequently that we have a decay of energy, that is: $E(t) \leq E(0)$ for all $t \in [0,T]$. By taking into consideration the estimate

$$\|a^{\frac{1}{2}}(\cdot)u_0\|^2_{L^2} \leq \|a\|_{L^\infty}\|u_0\|^2_{L^2}, \tag{2.22}$$

it follows that all terms in $E(t)$ satisfy the estimates:

$$\begin{aligned}\|a^{\frac{1}{2}}(\cdot)u(t,\cdot)\|^2_{L^2} &\lesssim \|u_1\|^2_{L^2} + \|(-\Delta)^{\frac{s}{2}}u_0\|^2_{L^2} + \|a\|_{L^\infty}\|u_0\|^2_{L^2} \\ &\lesssim \|u_1\|^2_{L^2} + \|u_0\|^2_{H^s} + \|a\|_{L^\infty}\|u_0\|^2_{H^s} \\ &\lesssim \left(1+\|a\|_{L^\infty}\right)\left[\|u_0\|^2_{H^s} + \|u_1\|^2_{L^2}\right] \\ &\lesssim \left(1+\|a\|_{L^\infty}\right)\left[\|u_0\|_{H^s} + \|u_1\|_{L^2}\right]^2, \end{aligned} \tag{2.23}$$

as well as

$$\left\{\|u_t(t,\cdot)\|^2_{L^2}, \|(-\Delta)^{\frac{s}{2}}u(t,\cdot)\|^2_{L^2}\right\} \lesssim \left(1+\|a\|_{L^\infty}\right)\left[\|u_0\|_{H^s} + \|u_1\|_{L^2}\right]^2, \tag{2.24}$$

uniformly in $t \in [0,T]$, where we use the fact that:

$$\left\{\|(-\Delta)^{\frac{s}{2}}u_0\|_{L^2}, \|u_0\|_{L^2}\right\} \leq \|u_0\|_{H^s}.$$

We now need to estimate u. For this purpose, we apply the Fourier transform to (1.2) with respect to the variable x to get the non-homogeneous ordinary differential equation

$$\widehat{u}_{tt}(t,\xi) + |\xi|^{2s}\widehat{u}(t,\xi) = \widehat{f}(t,\xi), \quad (t,\xi) \in [0,T]\times\mathbb{R}^d, \tag{2.25}$$

with the initial conditions $\widehat{u}(0,\xi) = \widehat{u}_0(\xi)$ and $\widehat{u}_t(0,\xi) = \widehat{u}_1(\xi)$. Here $\widehat{f}$, $\widehat{u}$ denote the Fourier transform of f and u, respectively, where $f(t,x) := -a(x)u(t,x) - b(x)u_t(t,x)$. Treating $\widehat{f}(t,\xi)$ as a source term and using Duhamel's principle (Proposition 2.3 with $\lambda \equiv 0$) to solve (2.25), we derive the following representation of the solution,

$$\widehat{u}(t,\xi) = \cos(t|\xi|^s)\widehat{u}_0(\xi) + \frac{\sin(t|\xi|^s)}{|\xi|^s}\widehat{u}_1(\xi) + \int_0^t \frac{\sin((t-\tau)|\xi|^s)}{|\xi|^s}\widehat{f}(\tau,\xi)\mathrm{d}\tau. \tag{2.26}$$

Taking the L^2 norm in (2.26) and using the estimates:

1. $|\cos(t|\xi|^s)| \leq 1$, for $t \in [0,T]$ and $\xi \in \mathbb{R}^d$,
2. $|\sin(t|\xi|^s)| \leq 1$, for large frequencies and $t \in [0,T]$ and
3. $|\sin(t|\xi|^s)| \leq t|\xi|^s \leq T|\xi|^s$, for small frequencies and $t \in [0,T]$,

leads to

$$\|\widehat{u}(t,\cdot)\|^2_{L^2} \lesssim \|\widehat{u}_0\|^2_{L^2} + \|\widehat{u}_1\|^2_{L^2} + \int_0^t \|\widehat{f}(\tau,\cdot)\|^2_{L^2}\mathrm{d}\tau,$$

and by using the Parseval–Plancherel identity, we get

$$\|u(t,\cdot)\|^2_{L^2} \lesssim \|u_0\|^2_{L^2} + \|u_1\|^2_{L^2} + \int_0^t \|f(\tau,\cdot)\|^2_{L^2}\mathrm{d}\tau, \tag{2.27}$$

for all $t \in [0,T]$. To estimate $\|f(\tau,\cdot)\|_{L^2}$, the last term in the above inequality, we use the triangle inequality and the estimates

$$\begin{aligned}\|a(\cdot)u(\tau,\cdot)\|_{L^2} &\leq \|a\|^{\frac{1}{2}}_{L^\infty}\|a^{\frac{1}{2}}(\cdot)u(\tau,\cdot)\|_{L^2} \\ &\lesssim \|a\|^{\frac{1}{2}}_{L^\infty}\left(1+\|a\|_{L^\infty}\right)^{\frac{1}{2}}\left[\|u_0\|_{H^s} + \|u_1\|_{L^2}\right] \\ &\lesssim \left(1+\|a\|_{L^\infty}\right)\left[\|u_0\|_{H^s} + \|u_1\|_{L^2}\right], \end{aligned} \tag{2.28}$$

resulting from (2.23), and similarly

$$\begin{aligned}\|b(\cdot)u_t(\tau,\cdot)\|_{L^2} &\leq \|b\|_{L^\infty}\|u_t(\tau,\cdot)\|_{L^2}\\ &\lesssim \|b\|_{L^\infty}\big(1+\|a\|_{L^\infty}\big)\big[\|u_0\|_{H^s}+\|u_1\|_{L^2}\big],\end{aligned} \tag{2.29}$$

resulting from (2.24), to get

$$\|f(\tau,\cdot)\|_{L^2} \lesssim \big(1+\|a\|_{L^\infty}\big)\big(1+\|b\|_{L^\infty}\big)\big[\|u_0\|_{H^s}+\|u_1\|_{L^2}\big]. \tag{2.30}$$

The desired estimate for u follows by substituting (2.30) into (2.27), finishing the proof. □

Lemma 2.5. *Let $d > 2s$. Assume that $a \in L^{\frac{d}{s}}(\mathbb{R}^d) \cap L^{\frac{d}{2s}}(\mathbb{R}^d)$ and $b \in L^{\frac{d}{s}}(\mathbb{R}^d)$ be non-negative. If $u_0 \in H^{2s}(\mathbb{R}^d)$ and $u_1 \in H^s(\mathbb{R}^d)$, then, there is a unique solution $u \in C([0,T]; H^{2s}(\mathbb{R}^d)) \cap C^1([0,T]; H^s(\mathbb{R}^d))$ to (1.2) and it satisfies the estimate*

$$\|u(t,\cdot)\|_2 \lesssim \Big(1+\|a\|_{L^{\frac{d}{s}}}\Big)\Big(1+\|a\|_{L^{\frac{d}{2s}}}\Big)\Big(1+\|b\|_{L^{\frac{d}{s}}}\Big)^2\Big[\|u_0\|_{H^{2s}}+\|u_1\|_{H^s}\Big], \tag{2.31}$$

uniformly in $t \in [0,T]$.

Proof. Proceeding as in the proof of Lemma 2.4, we get

$$\partial_t E(t) = -2\|b^{\frac{1}{2}}(\cdot)u_t(t,\cdot)\|_{L^2}^2 \leq 0, \tag{2.32}$$

for the energy function of the system defined by

$$E(t) = \|u_t(t,\cdot)\|_{L^2}^2 + \|(-\Delta)^{\frac{s}{2}}u(t,\cdot)\|_{L^2}^2 + \|a^{\frac{1}{2}}(\cdot)u(t,\cdot)\|_{L^2}^2, \tag{2.33}$$

which implies the decay of the energy over t. That is

$$E(t) \leq \|u_1\|_{L^2}^2 + \|(-\Delta)^{\frac{s}{2}}u_0\|_{L^2}^2 + \|a^{\frac{1}{2}}(\cdot)u_0\|_{L^2}^2, \tag{2.34}$$

for all $t \in [0,T]$. Using Hölder's inequality (see Proposition 2.1) for the last term in (2.34) together with $\|a^{\frac{1}{2}}\|_{L^p}^2 = \|a\|_{L^{\frac{p}{2}}}$, gives

$$\|a^{\frac{1}{2}}(\cdot)u_0(\cdot)\|_{L^2}^2 \leq \|a\|_{L^{\frac{p}{2}}}\|u_0\|_{L^q}^2, \tag{2.35}$$

for $1 < p, q < \infty$, satisfying $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$. Now, if we choose $q = \frac{2d}{d-2s}$ and consequently $p = \frac{d}{s}$, it follows from Proposition 2.2 that

$$\|u_0\|_{L^q} \lesssim \|(-\Delta)^{\frac{s}{2}}u_0(\cdot)\|_{L^2} \leq \|u_0\|_{H^s}, \tag{2.36}$$

and thus

$$\|a^{\frac{1}{2}}(\cdot)u_0(\cdot)\|_{L^2}^2 \lesssim \|a\|_{L^{\frac{d}{2s}}}\|u_0\|_{H^s}^2. \tag{2.37}$$

Substituting (2.37) in (2.34), we get the estimates

$$\Big\{\|u_t(t,\cdot)\|_{L^2}^2, \|(-\Delta)^{\frac{s}{2}}u(t,\cdot)\|_{L^2}^2, \|a^{\frac{1}{2}}(\cdot)u(t,\cdot)\|_{L^2}^2\Big\} \lesssim \big(1+\|a\|_{L^{\frac{d}{2s}}}\big)\big[\|u_0\|_{H^s}+\|u_1\|_{L^2}\big]^2, \tag{2.38}$$

uniformly in $t \in [0,T]$. To prove the estimate for the solution u, we argue as in the proof of Lemma 2.4 to get

$$\|u(t,\cdot)\|_{L^2}^2 \lesssim \|u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2 + \int_0^t \|f(\tau,\cdot)\|_{L^2}^2 \mathrm{d}\tau, \tag{2.39}$$

for all $t \in [0,T]$, with $f(t,x) := -a(x)u(t,x) - b(x)u_t(t,x)$. In order to estimate $\|f(\tau,\cdot)\|_{L^2}$, we use the triangle inequality to get

$$\|f(t,\cdot)\|_{L^2} \leq \|a(\cdot)u(t,\cdot)\|_{L^2} + \|b(\cdot)u_t(t,\cdot)\|_{L^2}. \tag{2.40}$$

To estimate the first term in (2.40), we first use Hölder's inequality together with $\|a^2\|_{L^{\frac{p}{2}}} = \|a\|_{L^p}^2$, to get

$$\|a(\cdot)u(t,\cdot)\|_{L^2} \leq \|a\|_{L^p}\|u(t,\cdot)\|_{L^q}, \tag{2.41}$$

for $1 < p, q < \infty$, satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, and we choose $q = \frac{2d}{d-2s}$ and consequently $p = \frac{d}{s}$, in order to get (from Proposition 2.2)

$$\|u(t,\cdot)\|_{L^q} \lesssim \|(-\Delta)^{\frac{s}{2}}u(t,\cdot)\|_{L^2}, \tag{2.42}$$

and thus

$$\|a(\cdot)u(t,\cdot)\|_{L^2} \lesssim \|a\|_{L^{\frac{d}{s}}}\|(-\Delta)^{\frac{s}{2}}u(t,\cdot)\|_{L^2}, \tag{2.43}$$

for all $t \in [0,T]$. Using the estimate (2.38), we arrive at

$$\begin{aligned}\|a(\cdot)u(t,\cdot)\|_{L^2} &\lesssim \|a\|_{L^{\frac{d}{s}}}\big(1+\|a\|_{L^{\frac{d}{2s}}}\big)^{\frac{1}{2}}\big[\|u_0\|_{H^s}+\|u_1\|_{L^2}\big]\\ &\lesssim \|a\|_{L^{\frac{d}{s}}}\big(1+\|a\|_{L^{\frac{d}{2s}}}\big)\big[\|u_0\|_{H^s}+\|u_1\|_{L^2}\big].\end{aligned} \tag{2.44}$$

For the second term in (2.40), we argue as above, to get

$$\|b(\cdot)u_t(t,\cdot)\|_{L^2} \lesssim \|b\|_{L^{\frac{d}{s}}}\|(-\Delta)^{\frac{s}{2}}u_t(t,\cdot)\|_{L^2}, \tag{2.45}$$

for all $t \in [0,T]$. We need now to estimate $\|(-\Delta)^{\frac{s}{2}}u_t(t,\cdot)\|_{L^2}$. For this, we note that if u solves the Cauchy problem

$$\begin{cases} u_{tt}(t,x)+(-\Delta)^s u(t,x)+a(x)u(t,x)+b(x)u_t(t,x)=0, & (t,x)\in[0,T]\times\mathbb{R}^d,\\ u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x), \quad x\in\mathbb{R}^d, \end{cases}$$

then u_t solves

$$\begin{cases} (u_t)_{tt}(t,x)+(-\Delta)^s u_t(t,x)+a(x)u_t(t,x)+b(x)(u_t)_t(t,x)=0, & (t,x)\in[0,T]\times\mathbb{R}^d,\\ u_t(0,x)=u_1(x), \quad u_{tt}(0,x)=-(-\Delta)^s u_0(x)-a(x)u_0(x)-b(x)u_1(x), \quad x\in\mathbb{R}^d. \end{cases}$$

Thanks to (2.43) and (2.45), one has

$$\|a(\cdot)u_0(\cdot)\|_{L^2} \lesssim \|a\|_{L^{\frac{d}{s}}}\|u_0\|_{H^s}, \quad \|b(\cdot)u_1(\cdot)\|_{L^2} \lesssim \|b\|_{L^{\frac{d}{s}}}\|u_1\|_{H^s}. \tag{2.46}$$

The estimate for $\|(-\Delta)^{\frac{s}{2}}u_t(t,\cdot)\|_{L^2}$ follows by using (2.38) applied to the problem (2.4), to get

$$\begin{aligned}\|(-\Delta)^{\frac{s}{2}}u_t(t,\cdot)\|_{L^2} &\lesssim \big(1+\|a\|_{L^{\frac{d}{2s}}}\big)^{\frac{1}{2}}\big[\|u_1\|_{H^s}+\|u_{tt}(0,\cdot)\|_{L^2}\big]\\ &\lesssim \big(1+\|a\|_{L^{\frac{d}{2s}}}\big)^{\frac{1}{2}}\big[\|u_1\|_{H^s}+\|u_0\|_{H^{2s}}+\|a\|_{L^{\frac{d}{s}}}\|u_0\|_{H^s}+\|b\|_{L^{\frac{d}{s}}}\|u_1\|_{H^s}\big]\\ &\lesssim \big(1+\|a\|_{L^{\frac{d}{2s}}}\big)\big(1+\|a\|_{L^{\frac{d}{s}}}\big)\big(1+\|b\|_{L^{\frac{d}{s}}}\big)\big[\|u_0\|_{H^{2s}}+\|u_1\|_{H^s}\big].\end{aligned} \tag{2.47}$$

By substituting (2.47) in (2.45), we get

$$\begin{aligned}\|b(\cdot)u_t(t,\cdot)\|_{L^2} &\lesssim \|b\|_{L^{\frac{d}{s}}}\big(1+\|a\|_{L^{\frac{d}{2s}}}\big)\big(1+\|a\|_{L^{\frac{d}{s}}}\big)\big(1+\|b\|_{L^{\frac{d}{s}}}\big)\big[\|u_0\|_{H^{2s}}+\|u_1\|_{H^s}\big]\\ &\lesssim \big(1+\|a\|_{L^{\frac{d}{2s}}}\big)\big(1+\|a\|_{L^{\frac{d}{s}}}\big)\big(1+\|b\|_{L^{\frac{d}{s}}}\big)^2\big[\|u_0\|_{H^{2s}}+\|u_1\|_{H^s}\big],\end{aligned} \tag{2.48}$$

and the estimate for $\|f(t,\cdot)\|_{L^2}$ follows from (2.30) and (2.44) with (2.48), yielding

$$\|f(t,\cdot)\|_{L^2} \lesssim \big(1+\|a\|_{L^{\frac{d}{2s}}}\big)\big(1+\|a\|_{L^{\frac{d}{s}}}\big)\big(1+\|b\|_{L^{\frac{d}{s}}}\big)^2\big[\|u_0\|_{H^{2s}}+\|u_1\|_{H^s}\big]. \tag{2.49}$$

Combining these estimates, we get the estimate for the solution u. Now, to estimate $\|(-\Delta)^s u\|_{L^2}$, we need first to estimate u_{tt}. Reasoning as in (2.47), the first estimate for u_t in (2.38), when applied to u_t

the solution to (2.4) instead of u, gives

$$
\begin{aligned}
\|u_{tt}(t,\cdot)\|_{L^2} &\lesssim \left(1+\|a\|_{L^{\frac{d}{2s}}}\right)^{\frac{1}{2}}\left[\|u_1\|_{H^s}+\|u_{tt}(0,\cdot)\|_{L^2}\right]\\
&\lesssim \left(1+\|a\|_{L^{\frac{d}{2s}}}\right)^{\frac{1}{2}}\left[\|u_1\|_{H^s}+\|u_0\|_{H^{2s}}+\|a\|_{L^{\frac{d}{s}}}\|u_0\|_{H^s}+\|b\|_{L^{\frac{d}{s}}}\|u_1\|_{H^s}\right]\\
&\lesssim \left(1+\|a\|_{L^{\frac{d}{2s}}}\right)\left(1+\|a\|_{L^{\frac{d}{s}}}\right)\left(1+\|b\|_{L^{\frac{d}{s}}}\right)\left[\|u_0\|_{H^{2s}}+\|u_1\|_{H^s}\right].
\end{aligned} \tag{2.50}
$$

The estimate for $\|(-\Delta)^s u\|_{L^2}$ follows by taking the L^2 norm in the equality

$$
(-\Delta)^s u(t,x) = -u_{tt}(t,x) - a(x)u(t,x) - b(x)u_t(t,x),
$$

and using the triangle inequality in the right-hand side and by taking into consideration the so far obtained estimates (2.44), (2.48) and (2.50). This completes the proof. □

3. Very weak well-posedness

Here and in the sequel, we consider the case when the equation coefficients a, b and the Cauchy data u_0 and u_1 are irregular (functions) and prove that the Cauchy problem

$$
\begin{cases} u_{tt}(t,x)+(-\Delta)^s u(t,x)+a(x)u(t,x)+b(x)u_t(t,x)=0,\\ u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x), \end{cases} \tag{3.1}
$$

for $(t,x)\in[0,T]\times\mathbb{R}^d$, has a unique very weak solution. We have in mind "functions" having δ or δ^2-like behaviours. We note that we understand a multiplication of distributions as multiplication of approximating families, in particular the multiplication of their representatives in Colombeau algebra.

3.1. Existence of very weak solutions

In order to prove existence of very weak solutions to (3.1), we need the following definitions.

Definition 3. (**Friedrichs mollifier**) A function $\psi\in C_0^\infty(\mathbb{R}^d)$ is said to be a Friedrichs mollifier if ψ is non-negative and $\int_{\mathbb{R}^d}\psi(x)\mathrm{d}x=1$.

Example 3.1. An example of a Friedrichs mollifier is given by:

$$
\psi(x)=\begin{cases}\alpha \mathrm{e}^{-\frac{1}{1-|x|^2}} & |x|<1,\\ 0 & |x|\geq 1,\end{cases}
$$

where the constant α is choosed in such way that $\int_{\mathbb{R}^d}\psi(x)\mathrm{d}x=1$.

Assume now ψ as defined above a Friedrichs mollifier.

Definition 4. (**Mollifying net**) For $\varepsilon\in(0,1]$, and $x\in\mathbb{R}^d$, a net of functions $(\psi_\varepsilon)_{\varepsilon\in(0,1]}$ is called a mollifying net if

$$
\psi_\varepsilon(x)=\omega(\varepsilon)^{-1}\psi\left(x/\omega(\varepsilon)\right),
$$

where $\omega(\varepsilon)$ is a positive function converging to 0 as $\varepsilon\to 0$ and ψ is a Friedrichs mollifier. In particular, if we take $\omega(\varepsilon)=\varepsilon$, then, we get

$$
\psi_\varepsilon(x)=\varepsilon^{-1}\psi\left(x/\varepsilon\right).
$$

Given a function (distribution) f, regularising f by convolution with a mollifying net $(\psi_\varepsilon)_{\varepsilon\in(0,1]}$, yields a net of smooth functions, namely

$$
(f_\varepsilon)_{\varepsilon\in(0,1]}=(f*\psi_\varepsilon)_{\varepsilon\in(0,1]}. \tag{3.2}
$$

Remark 3.2. The term "regularisation" of a function or distribution f, when used, will be viewed as a net of smooth functions $(f_\varepsilon)_{\varepsilon\in(0,1]}$ arising from convolution with a mollifying net (as in Definition 4). However, the term "approximation" is more general in the sense that approximations are not necessarily arising from convolution with molliffying nets. For instance, if we consider $(f_\varepsilon)_{\varepsilon\in(0,1]}$ a regularisation of f, then the net of functions $(\tilde{f}_\varepsilon)_{\varepsilon\in(0,1]}$ defined by

$$\tilde{f}_\varepsilon = f_\varepsilon + \mathrm{e}^{-\frac{1}{\varepsilon}}, \tag{3.3}$$

is an approximation of f but not resulting from regularisation.

Now, for a function (distribution) f, let $(f_\varepsilon)_{\varepsilon\in(0,1]}$ be a net of smooth functions approximating f, not necessarily coming from regularisation.

Definition 5. (**Moderateness**) Let X be a normed space of functions on $\mathbb{R}^d$ endowed with the norm $\|\cdot\|_X$.

1. A net of functions $(f_\varepsilon)_{\varepsilon\in(0,1]}$ from X is said to be X-moderate, if there exist $N\in\mathbb{N}_0$ such that
$$\|f_\varepsilon\|_X \lesssim \omega(\varepsilon)^{-N}. \tag{3.4}$$
2. For $T>0$. A net of functions $(u_\varepsilon(\cdot,\cdot))_{\varepsilon\in(0,1]}$ from $C\big([0,T];H^s(\mathbb{R}^d)\big)\cap C^1\big([0,T];L^2(\mathbb{R}^d)\big)$ is said to be $C\big([0,T];H^s(\mathbb{R}^d)\big)\cap C^1\big([0,T];L^2(\mathbb{R}^d)\big)$-moderate, if there exist $N\in\mathbb{N}_0$ such that
$$\sup_{t\in[0,T]}\|u_\varepsilon(t,\cdot)\|_1 \lesssim \omega(\varepsilon)^{-N}. \tag{3.5}$$
3. For $T>0$. A net of functions $(u_\varepsilon(\cdot,\cdot))_{\varepsilon\in(0,1]}$ from $C\big([0,T];H^{2s}(\mathbb{R}^d)\big)\cap C^1\big([0,T];H^s(\mathbb{R}^d)\big)$ is said to be $C\big([0,T];H^{2s}(\mathbb{R}^d)\big)\cap C^1\big([0,T];H^s(\mathbb{R}^d)\big)$-moderate, if there exist $N\in\mathbb{N}_0$ such that
$$\sup_{t\in[0,T]}\|u_\varepsilon(t,\cdot)\|_2 \lesssim \omega(\varepsilon)^{-N}. \tag{3.6}$$

For the second and the third definitions of moderateness, we will shortly write C_1-moderate and C_2-moderate.

The following proposition states that moderateness as defined above is a natural assumption for compactly supported distributions. Indeed, we have:

Proposition 3.1. *Let $f\in\mathcal{E}'(\mathbb{R}^d)$ and let $(f_\varepsilon)_{\varepsilon\in(0,1]}$ be regularisation of f obtained via convolution with a mollifying net $(\psi_\varepsilon)_{\varepsilon\in(0,1]}$ (see Definition 4). Then, the net $(f_\varepsilon)_{\varepsilon\in(0,1]}$ is $L^p(\mathbb{R}^d)$-moderate for any $1\le p\le\infty$.*

Proof. Fix $p\in[1,\infty]$ and let $f\in\mathcal{E}'(\mathbb{R}^d)$. By the structure theorems for distributions (see [31, Corollary 5.4.1]), there exists $n\in\mathbb{N}$ and compactly supported functions $f_\alpha\in C(\mathbb{R}^d)$ such that

$$f=\sum_{|\alpha|\le n}\partial^\alpha f_\alpha,$$

where $|\alpha|$ is the length of the multi-index α. The convolution of f with a mollifying net $(\psi_\varepsilon)_{\varepsilon\in(0,1]}$ yields

$$f*\psi_\varepsilon=\sum_{|\alpha|\le n}\partial^\alpha f_\alpha*\psi_\varepsilon=\sum_{|\alpha|\le n}f_\alpha*\partial^\alpha\psi_\varepsilon=\sum_{|\alpha|\le n}\varepsilon^{-d-|\alpha|}f_\alpha*\partial^\alpha\psi(x/\varepsilon). \tag{3.7}$$

Taking the L^p norm in (3.7) gives

$$\|f*\psi_\varepsilon\|_{L^p}\le\sum_{|\alpha|\le n}\varepsilon^{-d-|\alpha|}\|f_\alpha*\partial^\alpha\psi(x/\varepsilon)\|_{L^p}. \tag{3.8}$$

Since f_α and ψ are compactly supported then, Young's inequality applies for any $p_1, p_2 \in [1,\infty]$, provided that $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p}$. That is

$$\|f_\alpha * \partial^\alpha\psi(x/\varepsilon)\|_{L^p} \le \|f_\alpha\|_{L^{p_1}}\|\partial^\alpha\psi(x/\varepsilon)\|_{L^{p_2}} < \infty.$$

It follows from (3.8) that $(f_\varepsilon)_{\varepsilon\in(0,1]}$ is $L^p(\mathbb{R}^d)$-moderate. □

Example 3.3. Let $(\psi_\varepsilon)_\varepsilon$ be a mollifying net such that $\psi_\varepsilon(x)=\varepsilon^{-1}\psi(\varepsilon^{-1}x)$. Since ψ is compactly supported, then,

(1) For $f(x)=\delta_0(x)$, we have $f_\varepsilon(x)=\varepsilon^{-1}\psi(\varepsilon^{-1}x)\le C\varepsilon^{-1}$.
(2) For $f(x)=\delta_0^2(x)$, we can take $f_\varepsilon(x)=\varepsilon^{-2}\psi^2(\varepsilon^{-1}x)\le C\varepsilon^{-2}$.

Now, we are ready to introduce the notion of very weak solutions adapted to our problem. Here and in the sequel, we consider $\omega(\varepsilon)=\varepsilon$, in all the above definitions.

Definition 6. (**Very weak solution**) A net of functions $(u_\varepsilon)_\varepsilon \in C([0,T];H^s(\mathbb{R}^d))\cap C^1([0,T];L^2(\mathbb{R}^d))$ is said to be a very weak solution to the Cauchy problem (3.1), if there exist

- $L^\infty(\mathbb{R}^d)$-moderate approximations $(a_\varepsilon)_\varepsilon$ and $(b_\varepsilon)_\varepsilon$ to a and b, with $a_\varepsilon\ge 0$ and $b_\varepsilon\ge 0$,
- $H^s(\mathbb{R}^d)$-moderate approximation $(u_{0,\varepsilon})_\varepsilon$ to u_0,
- $L^2(\mathbb{R}^d)$-moderate approximation $(u_{1,\varepsilon})_\varepsilon$ to u_1,

such that, $(u_\varepsilon)_\varepsilon$ solves the approximating problems

$$\begin{cases}\partial_t^2 u_\varepsilon(t,x)+(-\Delta)^s u_\varepsilon(t,x)+a_\varepsilon(x)u_\varepsilon(t,x)+b_\varepsilon(x)\partial_t u_\varepsilon(t,x)=0, & (t,x)\in[0,T]\times\mathbb{R}^d,\\ u_\varepsilon(0,x)=u_{0,\varepsilon}(x), \quad \partial_t u_\varepsilon(0,x)=u_{1,\varepsilon}(x), \quad x\in\mathbb{R}^d,\end{cases} \tag{3.9}$$

for all $\varepsilon\in(0,1]$, and is C_1-moderate.

We have also the following alternative definition of a very weak solution to (3.1), under the assumptions of Lemma 2.5.

Definition 7. Let $d>2s$. A net of functions $(u_\varepsilon)_\varepsilon\in C([0,T];H^{2s}(\mathbb{R}^d))\cap C^1([0,T];H^s(\mathbb{R}^d))$ is said to be a very weak solution to the Cauchy problem (3.1), if there exist

- $(L^{\frac{d}{s}}(\mathbb{R}^d)\cap L^{\frac{d}{2s}}(\mathbb{R}^d))$-moderate approximation $(a_\varepsilon)_\varepsilon$ to a, with $a_\varepsilon\ge 0$,
- $L^{\frac{d}{s}}(\mathbb{R}^d)$-moderate approximation $(b_\varepsilon)_\varepsilon$ to b, with $b_\varepsilon\ge 0$,
- $H^{2s}(\mathbb{R}^d)$-moderate approximation $(u_{0,\varepsilon})_\varepsilon$ to u_0,
- $H^s(\mathbb{R}^d)$-moderate approximation $(u_{1,\varepsilon})_\varepsilon$ to u_1,

such that, $(u_\varepsilon)_\varepsilon$ solves the approximating problems (as in Definition 6) for all $\varepsilon\in(0,1]$, and is C_2-moderate.

Now, under the assumptions in Definition 6 and Definition 7, the existence of a very weak solution is straightforward.

Theorem 3.2. *Assume that there exist $\{L^\infty(\mathbb{R}^d), L^\infty(\mathbb{R}^d), H^s(\mathbb{R}^d), L^2(\mathbb{R}^d)\}$-moderate approximations to a, b, u_0 and u_1, respectively, with $a_\varepsilon\ge 0$ and $b_\varepsilon\ge 0$. Then, the Cauchy problem (3.1) has a very weak solution.*

Proof. Let a, b, u_0 and u_1 as in assumptions. Then, there exists $N_1, N_2, N_3, N_4\in\mathbb{N}$, such that

$$\|a_\varepsilon\|_{L^\infty}\lesssim\varepsilon^{-N_1}, \quad \|b_\varepsilon\|_{L^\infty}\lesssim\varepsilon^{-N_2},$$

and

$$\|u_{0,\varepsilon}\|_{H^s}\lesssim\varepsilon^{-N_3}, \quad \|u_{1,\varepsilon}\|_{H^s}\lesssim\varepsilon^{-N_4}.$$

It follows from the energy estimate (2.14), that

$$\|u_\varepsilon(t,\cdot)\|_1\lesssim\varepsilon^{-N_1-N_2-\max\{N_3,N_4\}},$$

uniformly in $t\in[0,T]$, which means that the net $(u_\varepsilon)_\varepsilon$ is C_1-moderate. This concludes the proof. □

As an alternative to Theorem 3.2 in the case when $d > 2s$ and the equation coefficients and data satisfy the hypothesis of Definition 7, we have the following theorem for which we do not give the proof, since it is similar to the one of Theorem 3.2.

Theorem 3.3. *Assume that there exist* $\{(L^{\frac{d}{s}}(\mathbb{R}^d)\cap L^{\frac{d}{2s}}(\mathbb{R}^d)), L^{\frac{d}{s}}(\mathbb{R}^d), H^{2s}(\mathbb{R}^d), H^s(\mathbb{R}^d)\}$*-moderate approximations to* a, b, u_0 *and* u_1*, respectively, with* $a_\varepsilon \geq 0$ *and* $b_\varepsilon \geq 0$*. Then, the Cauchy problem (3.1) has a very weak solution.*

3.2. Uniqueness

In what follows we want to prove the uniqueness of the very weak solution to the Cauchy problem (3.1) in both situations, either in the case when very weak solutions exist with the assumptions of Theorem 3.2 or in the case of Theorem 3.3. We need the following definition.

Definition 8. (**Negligibility**) Let X be a normed space endowed with the norm $\|\cdot\|_X$. A net of functions $(f_\varepsilon)_{\varepsilon\in(0,1]}$ from X is said to be X-negligible, if the estimate

$$\|f_\varepsilon\|_X \lesssim \varepsilon^k, \tag{3.10}$$

is valid for all $k > 0$.

Roughly speaking, we understand the uniqueness of the very weak solution to the Cauchy problem (3.1), in the sense that negligible changes in the approximations of the equation coefficients and initial data lead to negligible changes in the corresponding very weak solutions. More precisely,

Definition 9. (**Uniqueness**) We say that the Cauchy problem (3.1) has a unique very weak solution, if for all families of approximations $(a_\varepsilon)_\varepsilon$, $(\tilde{a}_\varepsilon)_\varepsilon$ and $(b_\varepsilon)_\varepsilon$, $(\tilde{b}_\varepsilon)_\varepsilon$ for the equation coefficients a and b, and families of approximations $(u_{0,\varepsilon})_\varepsilon$, $(\tilde{u}_{0,\varepsilon})_\varepsilon$ and $(u_{1,\varepsilon})_\varepsilon$, $(\tilde{u}_{1,\varepsilon})_\varepsilon$ for the Cauchy data u_0 and u_1, such that the nets $(a_\varepsilon - \tilde{a}_\varepsilon)_\varepsilon$, $(b_\varepsilon - \tilde{b}_\varepsilon)_\varepsilon$, $(u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})_\varepsilon$ and $(u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon})_\varepsilon$ are $\{L^\infty(\mathbb{R}^d), L^\infty(\mathbb{R}^d), H^s(\mathbb{R}^d), L^2(\mathbb{R}^d)\}$-negligible, it follows that the net

$$\big(u_\varepsilon(t,\cdot) - \tilde{u}_\varepsilon(t,\cdot)\big)_\varepsilon$$

is $L^2(\mathbb{R}^d)$-negligible for all $t \in [0, t]$, where $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ are the families of solutions to the approximating Cauchy problems

$$\begin{cases} \partial_t^2 u_\varepsilon(t,x) + (-\Delta)^s u_\varepsilon(t,x) + a_\varepsilon(x)u_\varepsilon(t,x) + b_\varepsilon(x)\partial_t u_\varepsilon(t,x) = 0, \ (t,x) \in [0,T]\times\mathbb{R}^d, \\ u_\varepsilon(0,x) = u_{0,\varepsilon}(x), \quad \partial_t u_\varepsilon(0,x) = u_{1,\varepsilon}(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{3.11}$$

and

$$\begin{cases} \partial_t^2 \tilde{u}_\varepsilon(t,x) + (-\Delta)^s \tilde{u}_\varepsilon(t,x) + \tilde{a}_\varepsilon(x)\tilde{u}_\varepsilon(t,x) + \tilde{b}_\varepsilon(x)\partial_t \tilde{u}_\varepsilon(t,x) = 0, \ (t,x) \in [0,T]\times\mathbb{R}^d, \\ \tilde{u}_\varepsilon(0,x) = \tilde{u}_{0,\varepsilon}(x), \quad \partial_t \tilde{u}_\varepsilon(0,x) = \tilde{u}_{1,\varepsilon}(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{3.12}$$

respectively.

Theorem 3.4. *Assume that* $a, b \geq 0$*, in the sense that their approximating nets are non-negative. Under the conditions of Theorem 3.2, the very weak solution to the Cauchy problem (3.1) is unique.*

Proof. Let $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ be the families of solutions to (3.11) and (3.12) and assume that the nets $(a_\varepsilon - \tilde{a}_\varepsilon)_\varepsilon$, $(b_\varepsilon - \tilde{b}_\varepsilon)_\varepsilon$, $(u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})_\varepsilon$ and $(u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon})_\varepsilon$ are $L^\infty(\mathbb{R}^d)$, $L^\infty(\mathbb{R}^d)$, $H^s(\mathbb{R}^d)$, $L^2(\mathbb{R}^d)$-negligible, respectively. The function $U_\varepsilon(t,x)$ defined by

$$U_\varepsilon(t,x) := u_\varepsilon(t,x) - \tilde{u}_\varepsilon(t,x),$$

satisfies

$$\begin{cases} \partial_t^2 U_\varepsilon(t,x) + (-\Delta)^s U_\varepsilon(t,x) + a_\varepsilon(x)U_\varepsilon(t,x) + b_\varepsilon(x)\partial_t U_\varepsilon(t,x) = f_\varepsilon(t,x), \\ U_\varepsilon(0,x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x), \quad \partial_t U_\varepsilon(0,x) = (u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon})(x), \end{cases} \tag{3.13}$$

for $(t,x) \in [0,T] \times \mathbb{R}^d$, where,

$$f_\varepsilon(t,x) := \big(\tilde{a}_\varepsilon(x) - a_\varepsilon(x)\big)\tilde{u}_\varepsilon(t,x) + \big(\tilde{b}_\varepsilon(x) - b_\varepsilon(x)\big)\partial_t \tilde{u}_\varepsilon(t,x).$$

According to Duhamel's principle (see Proposition 2.3), the solution to (3.13) has the following representation

$$U_\varepsilon(t,x) = W_\varepsilon(t,x) + \int_0^t V_\varepsilon(t,x;\tau)\mathrm{d}\tau, \tag{3.14}$$

where $W_\varepsilon(t,x)$ is the solution to the homogeneous problem

$$\begin{cases} \partial_t^2 W_\varepsilon(t,x) + (-\Delta)^s W_\varepsilon(t,x) + a_\varepsilon(x) W_\varepsilon(t,x) + b_\varepsilon(x)\partial_t W_\varepsilon(t,x) = 0, \\ W_\varepsilon(0,x) = (u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})(x), \quad \partial_t W_\varepsilon(0,x) = (u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon})(x), \end{cases} \tag{3.15}$$

for $(t,x) \in [0,T] \times \mathbb{R}^d$, and $V_\varepsilon(t,x;\tau)$ solves

$$\begin{cases} \partial_t^2 V_\varepsilon(t,x;\tau) + (-\Delta)^s V_\varepsilon(t,x;\tau) + a_\varepsilon(x) V_\varepsilon(t,x;\tau) + b_\varepsilon(x)\partial_t V_\varepsilon(t,x;\tau) = 0, \\ V_\varepsilon(\tau,x;\tau) = 0, \quad \partial_t V_\varepsilon(\tau,x;\tau) = f_\varepsilon(\tau,x), \end{cases} \tag{3.16}$$

for $(t,x) \in [\tau,T] \times \mathbb{R}^d$ and $\tau \in [0,T]$. By taking the L^2-norm on both sides of (3.14) and using Minkowski's integral inequality, we get

$$\|U_\varepsilon(t,\cdot)\|_{L^2} \le \|W_\varepsilon(t,\cdot)\|_{L^2} + \int_0^t \|V_\varepsilon(t,\cdot;\tau)\|_{L^2}\mathrm{d}\tau. \tag{3.17}$$

The energy estimate (2.14) allows us to control $\|W_\varepsilon(t,\cdot)\|_{L^2}$ and $\|V_\varepsilon(t,\cdot;\tau)\|_{L^2}$ to get

$$\|W_\varepsilon(t,\cdot)\|_{L^2} \lesssim \Big(1 + \|a_\varepsilon\|_{L^\infty}\Big)\Big(1 + \|b_\varepsilon\|_{L^\infty}\Big)\Big[\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{H^s} + \|u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon}\|_{L^2}\Big],$$

and

$$\|V_\varepsilon(t,\cdot;\tau)\|_{L^2} \lesssim \Big(1 + \|a_\varepsilon\|_{L^\infty}\Big)\Big(1 + \|b_\varepsilon\|_{L^\infty}\Big)\Big[\|f_\varepsilon(\tau,\cdot)\|_{L^2}\Big].$$

By taking into consideration that $t \in [0,T]$, it follows from (3.17) that

$$\|U_\varepsilon(t,\cdot)\|_{L^2} \lesssim \Big(1 + \|a_\varepsilon\|_{L^\infty}\Big)\Big(1 + \|b_\varepsilon\|_{L^\infty}\Big)\Big[\|u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon}\|_{H^s} + \|u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon}\|_{L^2} + \int_0^T \|f_\varepsilon(\tau,\cdot)\|_{L^2}\mathrm{d}\tau\Big], \tag{3.18}$$

where $\|f_\varepsilon(\tau,\cdot)\|_{L^2}$ is estimated as follows,

$$\begin{aligned} \|f_\varepsilon(\tau,\cdot)\|_{L^2} &\le \|(\tilde{a}_\varepsilon(\cdot) - a_\varepsilon(\cdot))\tilde{u}_\varepsilon(\tau,\cdot)\|_{L^2} + \|(\tilde{b}_\varepsilon(\cdot) - b_\varepsilon(\cdot))\partial_t\tilde{u}_\varepsilon(\tau,\cdot)\|_{L^2} \\ &\le \|\tilde{a}_\varepsilon - a_\varepsilon\|_{L^\infty}\|\tilde{u}_\varepsilon(\tau,\cdot)\|_{L^2} + \|\tilde{b}_\varepsilon - b_\varepsilon\|_{L^\infty}\|\partial_t\tilde{u}_\varepsilon(\tau,\cdot)\|_{L^2}. \end{aligned} \tag{3.19}$$

On the one hand, the nets $(a_\varepsilon)_\varepsilon$ and $(b_\varepsilon)_\varepsilon$ are L^∞-moderate by assumption, and the net $(\tilde{u}_\varepsilon)_\varepsilon$ is C_1-moderate being a very weak solution to (3.12). On the other hand, the nets $(a_\varepsilon - \tilde{a}_\varepsilon)_\varepsilon$, $(b_\varepsilon - \tilde{b}_\varepsilon)_\varepsilon$, $(u_{0,\varepsilon} - \tilde{u}_{0,\varepsilon})_\varepsilon$ and $(u_{1,\varepsilon} - \tilde{u}_{1,\varepsilon})_\varepsilon$ are $L^\infty(\mathbb{R}^d)$, $L^\infty(\mathbb{R}^d)$, $H^s(\mathbb{R}^d)$, $L^2(\mathbb{R}^d)$-negligible. It follows from (3.18) combined with (3.19) that

$$\|U_\varepsilon(t,\cdot)\|_{L^2} \lesssim \varepsilon^k,$$

for all $k > 0$, showing the uniqueness of the very weak solution. □

The analogue to Definition 9 and Theorem 3.4 in the case when $d > 2s$ with Theorem 3.3's background, read:

Definition 10. We say that the Cauchy problem (3.1) has a unique very weak solution, if for all families of approximations $(a_\varepsilon)_\varepsilon$, $(\tilde{a}_\varepsilon)_\varepsilon$ and $(b_\varepsilon)_\varepsilon$, $(\tilde{b}_\varepsilon)_\varepsilon$ for the equation coefficients a and b, and families of approximations $(u_{0,\varepsilon})_\varepsilon$, $(\tilde{u}_{0,\varepsilon})_\varepsilon$ and $(u_{1,\varepsilon})_\varepsilon$, $(\tilde{u}_{1,\varepsilon})_\varepsilon$ for the Cauchy data u_0 and u_1, such that the nets $(a_\varepsilon-\tilde{a}_\varepsilon)_\varepsilon$, $(b_\varepsilon-\tilde{b}_\varepsilon)_\varepsilon$, $(u_{0,\varepsilon}-\tilde{u}_{0,\varepsilon})_\varepsilon$ and $(u_{1,\varepsilon}-\tilde{u}_{1,\varepsilon})_\varepsilon$ are $\{(L^{\frac{d}{s}}(\mathbb{R}^d)\cap L^{\frac{d}{2s}}(\mathbb{R}^d)), L^{\frac{d}{s}}(\mathbb{R}^d), H^{2s}(\mathbb{R}^d), H^s(\mathbb{R}^d)\}$-negligible, it follows that the net $\big(u_\varepsilon(t,\cdot)-\tilde{u}_\varepsilon(t,\cdot)\big)_{\varepsilon\in(0,1]}$, is $L^2(\mathbb{R}^d)$-negligible for all $t\in[0,T]$, where $(u_\varepsilon)_\varepsilon$ and $(\tilde{u}_\varepsilon)_\varepsilon$ are the families of solutions to the corresponding approximating Cauchy problems.

Theorem 3.5. *Let $d>2s$ and assume that $a,b\geq 0$, in the sense that there approximating nets are non-negative. With the assumptions of Theorem 3.3, the very weak solution to the Cauchy problem (3.1) is unique.*

4. Coherence with classical theory

The question to be answered here is that, in the case when $a,b\in L^\infty(\mathbb{R}^d)$, $u_0\in H^s(\mathbb{R}^d)$ and $u_1\in L^2(\mathbb{R}^d)$ or alternatively when $(a,b)\in(L^{\frac{d}{s}}(\mathbb{R}^d)\cap L^{\frac{d}{2s}}(\mathbb{R}^d))\times L^{\frac{d}{s}}(\mathbb{R}^d)$, $u_0\in H^{2s}(\mathbb{R}^d)$ and $u_1\in H^s(\mathbb{R}^d)$ and a classical solution to the Cauchy problem

$$\begin{cases} u_{tt}(t,x)+(-\Delta)^s u(t,x)+a(x)u(t,x)+b(x)u_t(t,x)=0,\ (t,x)\in[0,T]\times\mathbb{R}^d,\\ u(0,x)=u_0(x),\quad u_t(0,x)=u_1(x),\quad x\in\mathbb{R}^d,\end{cases}\tag{4.1}$$

exists, does the very weak solution obtained via regularisation techniques recapture it?

Theorem 4.1. *Let ψ be a Friedrichs mollifier. Assume $a,b\in L^\infty(\mathbb{R}^d)$ be non-negative and suppose that $u_0\in H^s(\mathbb{R}^d)$ and $u_1\in L^2(\mathbb{R}^d)$. Then, for any regularising families $(a_\varepsilon)_\varepsilon=(a*\psi_\varepsilon)_\varepsilon$ and $(b_\varepsilon)_\varepsilon=(b*\psi_\varepsilon)_\varepsilon$ for the equation coefficients, satisfying*

$$\|a_\varepsilon-a\|_{L^\infty}\to 0,\quad \text{and}\quad \|b_\varepsilon-b\|_{L^\infty}\to 0,\tag{4.2}$$

and any regularising families $(u_{0,\varepsilon})_\varepsilon=(u_0*\psi_\varepsilon)_\varepsilon$ and $(u_{1,\varepsilon})_\varepsilon=(u_1*\psi_\varepsilon)_\varepsilon$ for the initial data, the net $(u_\varepsilon)_\varepsilon$ converges to the classical solution (given by Lemma 2.4) of the Cauchy problem (4.1) in L^2 as $\varepsilon\to 0$.

Proof. Let $(u_\varepsilon)_\varepsilon$ be the very weak solution given by Theorem 3.2 and u the classical one, as in Lemma 2.4. The classical solution satisfies

$$\begin{cases} u_{tt}(t,x)+(-\Delta)^s u(t,x)+a(x)u(t,x)+b(x)u_t(t,x)=0,\ (t,x)\in[0,T]\times\mathbb{R}^d,\\ u(0,x)=u_0(x),\quad u_t(0,x)=u_1(x),\quad x\in\mathbb{R}^d,\end{cases}\tag{4.3}$$

and $(u_\varepsilon)_\varepsilon$ solves

$$\begin{cases} \partial_t^2 u_\varepsilon(t,x)+(-\Delta)^s u_\varepsilon(t,x)+a_\varepsilon(x)u_\varepsilon(t,x)+b_\varepsilon(x)\partial_t u_\varepsilon(t,x)=0,\ (t,x)\in[0,T]\times\mathbb{R}^d,\\ u_\varepsilon(0,x)=u_{0,\varepsilon}(x),\quad \partial_t u_\varepsilon(0,x)=u_{1,\varepsilon}(x),\quad x\in\mathbb{R}^d.\end{cases}\tag{4.4}$$

Denoting $U_\varepsilon(t,x):=u_\varepsilon(t,x)-u(t,x)$, we have that U_ε solves the Cauchy problem

$$\begin{cases} \partial_t^2 U_\varepsilon(t,x)+(-\Delta)^s U_\varepsilon(t,x)+a_\varepsilon(x)U_\varepsilon(t,x)+b_\varepsilon(x)\partial_t U_\varepsilon(t,x)=\Theta_\varepsilon(t,x),\\ U_\varepsilon(0,x)=(u_{0,\varepsilon}-u_0)(x),\quad \partial_t U_\varepsilon(0,x)=(u_{1,\varepsilon}-u_1)(x),\end{cases}\tag{4.5}$$

where

$$\Theta_\varepsilon(t,x):=-\big(a_\varepsilon(x)-a(x)\big)u(t,x)-\big(b_\varepsilon(x)-b(x)\big)\partial_t u(t,x).\tag{4.6}$$

Thanks to Duhamel's principle, U_ε can be represented by

$$U_\varepsilon(t,x)=W_\varepsilon(t,x)+\int_0^t V_\varepsilon(t,x;\tau)\mathrm{d}\tau,\tag{4.7}$$

where $W_\varepsilon(t,x)$ is the solution to the homogeneous problem

$$\begin{cases} \partial_t^2 W_\varepsilon(t,x) + (-\Delta)^s W_\varepsilon(t,x) + a_\varepsilon(x) W_\varepsilon(t,x) + b_\varepsilon(x)\partial_t W_\varepsilon(t,x) = 0, \\ W_\varepsilon(0,x) = (u_{0,\varepsilon} - u_0)(x), \quad \partial_t W_\varepsilon(0,x) = (u_{1,\varepsilon} - u_1)(x), \end{cases} \tag{4.8}$$

for $(t,x) \in [0,T] \times \mathbb{R}^d$, and $V_\varepsilon(t,x;\tau)$ solves

$$\begin{cases} \partial_t^2 V_\varepsilon(t,x;\tau) + (-\Delta)^s V_\varepsilon(t,x;\tau) + a_\varepsilon(x) V_\varepsilon(t,x;\tau) + b_\varepsilon(x)\partial_t V_\varepsilon(t,x;\tau) = 0, \\ V_\varepsilon(\tau,x;\tau) = 0, \quad \partial_t V_\varepsilon(\tau,x;\tau) = \Theta_\varepsilon(\tau,x), \end{cases} \tag{4.9}$$

for $(t,x) \in [\tau,T] \times \mathbb{R}^d$ and $\tau \in [0,T]$. We take the L^2-norm in (4.7) and we argue as in the proof of Theorem 3.4. We obtain

$$\|U_\varepsilon(t,\cdot)\|_{L^2} \le \|W_\varepsilon(t,\cdot)\|_{L^2} + \int_0^t \|V_\varepsilon(t,\cdot;\tau)\|_{L^2} \mathrm{d}\tau, \tag{4.10}$$

where

$$\|W_\varepsilon(t,\cdot)\|_{L^2} \lesssim \Big(1 + \|a_\varepsilon\|_{L^\infty}\Big)\Big(1 + \|b_\varepsilon\|_{L^\infty}\Big)\Big[\|u_{0,\varepsilon} - u_0\|_{H^s} + \|u_{1,\varepsilon} - u_{1,}\|_{L^2}\Big],$$

and

$$\|V_\varepsilon(t,\cdot;\tau)\|_{L^2} \lesssim \Big(1 + \|a_\varepsilon\|_{L^\infty}\Big)\Big(1 + \|b_\varepsilon\|_{L^\infty}\Big)\Big[\|\Theta_\varepsilon(\tau,\cdot)\|_{L^2}\Big],$$

by the energy estimate from Lemma 2.4, and Θ_ε is estimated by

$$\|\Theta_\varepsilon(\tau,\cdot)\|_{L^2} \le \|a_\varepsilon - a\|_{L^\infty}\|u(\tau,\cdot)\|_{L^2} + \|b_\varepsilon - b\|_{L^\infty}\|\partial_t u(\tau,\cdot)\|_{L^2}. \tag{4.11}$$

First, one observes that $\|a_\varepsilon\|_{L^\infty} < \infty$ and $\|b_\varepsilon\|_{L^\infty} < \infty$ uniformly in ε by the fact that $a, b \in L^\infty(\mathbb{R}^d)$ and $\|u(\tau,\cdot)\|_{L^2}$ and $\|\partial_t u(\tau,\cdot)\|_{L^2}$ are bounded as well, since u is a classical solution to (4.1). This, together with

$$\|a_\varepsilon - a\|_{L^\infty} \to 0, \quad \text{and} \quad \|b_\varepsilon - b\|_{L^\infty} \to 0, \quad \text{as } \varepsilon \to 0,$$

from the assumptions, and

$$\|u_{0,\varepsilon} - u_0\|_{H^s} \to 0, \quad \|u_{1,\varepsilon} - u_{1,}\|_{L^2} \to 0, \quad \text{as } \varepsilon \to 0,$$

shows that

$$\|U_\varepsilon(t,\cdot)\|_{L^2} \to 0, \quad \text{as } \varepsilon \to 0,$$

uniformly in $t \in [0,T]$, and this finishes the proof. □

In the case when a classical solution exists in the sense of Lemma 2.5, the coherence theorem reads as follows. We avoid giving the proof since it is similar to the proof of Theorem 4.1.

Theorem 4.2. *Let ψ be a Friedrichs mollifier. Assume $(a,b) \in (L^{\frac{d}{s}}(\mathbb{R}^d) \cap L^{\frac{d}{2s}}(\mathbb{R}^d)) \times L^{\frac{d}{s}}(\mathbb{R}^d)$ be non-negative and suppose that $u_0 \in H^{2s}(\mathbb{R}^d)$ and $u_1 \in H^s(\mathbb{R}^d)$. Then, for any regularising families $(a_\varepsilon)_\varepsilon = (a*\psi_\varepsilon)_\varepsilon$ and $(b_\varepsilon)_\varepsilon = (b*\psi_\varepsilon)_\varepsilon$ for the equation coefficients, and any regularising families $(u_{0,\varepsilon})_\varepsilon = (u_0*\psi_\varepsilon)_\varepsilon$ and $(u_{1,\varepsilon})_\varepsilon = (u_1*\psi_\varepsilon)_\varepsilon$ for the initial data, the net $(u_\varepsilon)_\varepsilon$ converges to the classical solution (given by Lemma 2.5) of the Cauchy problem (4.1) in L^2 as $\varepsilon \to 0$.*

Remark 4.1. In Theorem 4.1, we proved the coherence result, provided that

$$\|a_\varepsilon - a\|_{L^\infty} \to 0, \quad \text{and} \quad \|b_\varepsilon - b\|_{L^\infty} \to 0,$$

as $\varepsilon \to 0$. This is in particular true if we consider coefficients from $C_0(\mathbb{R}^d)$, the space of continuous functions on $\mathbb{R}^d$ vanishing at infinity which is a Banach space when endowed with the L^∞-norm. For more details, see Section 3.1.10 in [32].

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