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Boundedness and finite-time blow-up in a Keller–Segel chemotaxis-growth system with flux limitation

Chunmei Chen and Pan Zheng

Abstract. This paper deals with a parabolic–elliptic Keller–Segel chemotaxis-growth system with flux limitation

$$\begin{cases} u_t = \nabla\cdot((u+1)^{m-1}\nabla u) - \nabla\cdot(uf(|\nabla v|^2)\nabla v) + \lambda u - \mu u^k, & x\in\Omega, t>0,\\ 0 = \Delta v - M(t) + u, & x\in\Omega, t>0, \end{cases}$$

under homogeneous Neumann boundary conditions, where $\Omega\subset\mathbb{R}^N$ is a smoothly bounded domain, $m\in\mathbb{R}$, $\lambda>0, \mu>0$, $k>1$, $M(t) := \frac{1}{|\Omega|}\int\limits_\Omega u(x,t)dx$, $f\left(|\nabla v|^2\right) = (1+|\nabla v|^2)^{-\alpha}, \alpha\in\mathbb{R}$. In this framework, it is shown that when $N\geq 2, m+k>2, k>1, k\geq m$ and

$$\alpha > \frac{4N-(m+k)N-2}{4(N-1)},$$

then for all nonnegative initial data, the solution is global and bounded in time. Moreover, when $\Omega\subset\mathbb{R}^N$ $(N\geq 5)$ is a ball, if $1<m<\min\left\{\frac{2N-4}{N}, 1-\frac{1}{N}+\frac{1}{N}\sqrt{N^2-4N+1}\right\}$ and the parameters α and k satisfy suitable conditions, there exist some initial data u_0 such that the solution $u(x,t)$ blows up at finite time $T_{\max}$ in L^∞-norm sense.

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1. Introduction

In this paper, we consider the following Keller–Segel chemotaxis-growth system with flux limitation and nonlinear diffusion

$$\begin{cases} u_t = \nabla\cdot((u+1)^{m-1}\nabla u) - \nabla\cdot\left(uf\left(|\nabla v|^2\right)\nabla v\right) + \lambda u - \mu u^k, & x\in\Omega, t>0,\\ 0 = \Delta v - M(t) + u, & x\in\Omega, t>0,\\ \int\limits_\Omega v(x,t)dx = 0, & x\in\Omega, t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega, t>0,\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N (N\geq 2)$ is a smoothly bounded domain, $m\in\mathbb{R}$, $\lambda>0, \mu>0$, $k>1$, $M(t) := \frac{1}{|\Omega|}\int\limits_\Omega u(x,t)dx$ and

$$f\left(|\nabla v|^2\right) = (1+|\nabla v|^2)^{-\alpha} \tag{1.2}$$

with $\alpha\in\mathbb{R}$. System (1.1) can be an extension of the classical Keller–Segel model in [16–18] for chemotaxis processes. Next, let's introduce some research progress about (1.1) as follows.

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- Without the flux limitation (*i.e.*, $\alpha = 0$):
  When $\lambda = \mu = 0, m = 1$, it was showed in [13,25,26] that the corresponding initial-boundary value problems in the spatially two-dimensional setting indeed possess some solutions, which blow up in finite time provided that the initial mass is large enough and concentrated around some point to a suitable extent, whereas if the initial mass is small then solutions remain bounded in time. When $m \neq 1$, the scholars have obtained some interesting results addressing blow-up in [3,6,7]. Besides, when the second equation in (1.1) is replaced by $v_t = \Delta v - v + u$, please see the references in [11,28,41].
  On the other hand, for the case of $\lambda, \mu \neq 0$ and $m = 1$, under the assumptions $N \geq 5$ and $1 < k < \frac{3}{2} + \frac{1}{2(N-1)}$ in (1.1), Winkler [42] proved that radially symmetric solutions may blow up in finite time. Later, when $M(t)$ is replaced by the function $v(x,t)$ in the second equation, Winkler [45] proved finite-time blow-up of solutions in low-dimensional environments, especially in three dimensions, under the weaker condition of $1 < k < \frac{7}{6}$ if $N \in \{3,4\}$ or $1 < k < 1 + \frac{1}{2(N-1)}$ if $N \geq 5$. When the logistic source term is replaced by $\mu u(1 - \int_\Omega u^k \mathrm{d}x)$, Du and Liu [8] proved that the solution of this system blows up in finite time under the assumption $0 < k < \min\{2, \frac{N}{2}\}$. When $m \neq 1$, other types of logistic source term have also been studied by many authors [19,38,49]. When the second equation is replaced by $v_t = \Delta v - v + u$, more relevant results can refer to [37,43,51].
- With the flux limitation (*i.e.*, $\alpha \neq 0$):
  For the cases of $\lambda = \mu = 0, m = 1$, $f(|\nabla v|^2) = \chi |\nabla v|^{p-1}$ with $\chi > 0$ and

$$p \in \begin{cases} (1, \infty), & \text{if } N = 1, \\ \left(1, \frac{N}{N-1}\right), & \text{if } N \geq 2, \end{cases} \tag{1.3}$$

  Negreanu and Tello [27] obtained the uniform bounds in $L^\infty(\Omega)$ of global solutions and proved that in the one-dimensional case there exist infinitely many non-constant steady-states for $p \in (1,2)$ for a given positive mass. In particular, Winkler [39] proved that a global bounded classical solution exists if $\alpha > \frac{N-2}{2(N-1)}$, whereas finite-time blow-up occurs if $\alpha < \frac{N-2}{2(N-1)}$. Marras et.al. [22] proved that the solution blows up in finite time under the smallness conditions on α and k, and a lower bound of blow-up time is derived. In addition, they proved that the solution is global and bounded in time under the largeness conditions on α and k. Moreover, if $f(|\nabla v|^2) = \chi \frac{1}{\sqrt{1+|\nabla v|^2}}$ and $\nabla \cdot ((u+1)^{m-1} \nabla u)$ is replaced by $\nabla \cdot \left(\frac{u \nabla u}{\sqrt{u^2 + |\nabla u|^2}} \right)$, Bellomo and Winkler [1] asserted the existence of a unique classical solution for arbitrary positive radial initial data $u_0 \in C^3(\bar{\Omega})$ when either $N \geq 2$ and $\chi > 0$ or $N = 1, \chi > 0$ and $\int_\Omega u_0 \mathrm{d}x < m_c$, where

$$m_c := \begin{cases} \frac{1}{\sqrt{\chi^2 - 1}}, & \text{if } \chi > 1, \\ \infty, & \text{if } \chi \leq 1. \end{cases} \tag{1.4}$$

  In [2], Bellomo and Winkler showed that these above conditions are essentially optimal, if $\chi > 1$, then for any choice of

$$\int_\Omega u_0 \mathrm{d}x > \begin{cases} \frac{1}{\sqrt{\chi^2 - 1}}, & \text{if } N = 1, \\ 0, & \text{if } N \geq 2, \end{cases} \tag{1.5}$$

  there exist positive initial data $u_0 \in C^3(\bar{\Omega})$ such that the system possesses a uniquely determined classical solution that blows up in finite time. If $f(|\nabla v|^2) = \chi \frac{u^{q-1}}{\sqrt{1+|\nabla v|^2}}$ and $\nabla \cdot ((u+1)^{m-1} \nabla u)$ is replaced by $\nabla \cdot \left(\frac{u^p \nabla u}{\sqrt{u^2 + |\nabla u|^2}} \right)$, Mizukami et.al. [24] derived the local existence and extensibility criterion ruling out gradient blow-up when $p, q \geq 1$, and moreover showed global existence and boundedness of solutions when $p > q + 1 - \frac{1}{N}$ under no-flux boundary conditions, for a radially symmetric and positive initial data $u_0 \in C^3(\bar{\Omega})$, $\chi > 0$. Chiyoda et.al. [5] gave the existence of blow-up solutions under some condition for χ and u_0 when $1 \leq p \leq q$. Recently, when $m \neq 1$, Zhao and Yi [50] showed that the system has a unique global bounded classical solution under the conditions that $\alpha > \frac{2N-2-mN}{2(N-1)}$ and $m \geq 1$. Moreover, for other types of flux limitation, please read the references in [4,10,14,23,33,47].

In addition, for the case of $\lambda, \mu \neq 0, m = 1$, the flux limitation term is replaced by $f(|\nabla v|^2) = |\nabla v|^{p-2}$ and the logistic source term is replaced by $\mu u(1-u)$, Satre-Gomez and Tello [31] studied the global existence of solutions under the following assumptions

$$\begin{cases} p < 2, & N = 2, \\ p \in \left(1, \frac{3}{2}\right), & N \geq 3, \end{cases} \tag{1.6}$$

and $u_0 \in C^{2+\gamma}(\Omega), \gamma \in (0,1)$. When $f(|\nabla v|^2) = (1 + |\nabla v|^2)^{-\frac{\alpha}{2}}$ in system (1.1), Zhang [48] showed that the corresponding initial value problem possesses a global bounded classical solution for any $\alpha, \mu > 0$ and $N \leq 2$; if $k = 2$ and $\alpha = \frac{N-2}{2N}$, there exists $\mu_0 > 0$ such that for any $\mu \geq \mu_0$, a global bounded classical solution exists in the case $N \geq 3$. Furthermore, there are many similar models with flux limitation, which has been studied in previous works, such as chemotaxis-fluid models (see [30,44,46]), chemotaxis-haptotaxis models (see [15,34–36]), etc.

Inspired by the works in [22,39,50], we extend their approaches to study the global boundedness and finite-time blow-up of solutions in system (1.1). The present work is addressed to concern with the interplay of the nonlinear diffusivity $(u+1)^{m-1}$, flux limitation $f(|\nabla v|^2) = (1+|\nabla v|^2)^{-\alpha}$ and generalized logistic source $\lambda u - \mu u^k$ in (1.1). Our main results of this paper are as follows. Firstly, we consider the global existence and boundedness of solutions for system (1.1).

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N, N \geq 2$ be a bounded domain with smooth boundary. Assume that $m \in \mathbb{R}, m + k > 2, k > 1, k \geq m, \lambda, \mu > 0$ and f satisfies (1.2) with*

$$\alpha > \frac{4N - (m+k)N - 2}{4(N-1)}, \tag{1.7}$$

then for all nonnegative initial data $u_0 \in C^0(\bar{\Omega})$, the system (1.1) possesses a unique global bounded classical solution (u, v) in $\Omega \times (0, \infty)$.

Remark 1.1. In contrast to [50], under the influence of a source term, the range of m can be expanded, which means that the logistic source plays an important role in (1.1).

The second purpose of this paper is to study finite-time blow-up of radially symmetric solutions of (1.1) under some suitable conditions, when $\Omega = B_R(0) \subset \mathbb{R}^N$ is a ball, which is centered at the origin with radius $R > 0$.

Theorem 1.2. *Let $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \geq 5$ be a ball. Suppose that $\lambda, \mu > 0$, $1 < m < \min \left\{\frac{2N-4}{N}, 1 - \frac{1}{N} + \frac{1}{N}\sqrt{N^2 - 4N + 1}\right\}$, f satisfies (1.2) with*

$$\frac{2N - 4 - mN}{(2N-2)m} < \alpha < \frac{2N - 2 - mN}{2N - 2} \tag{1.8}$$

and

$$k \in (1, \min\{2, k_1, k_2\}), \tag{1.9}$$

where

$$k_1 = \frac{\left(\frac{\left(2-\frac{2}{N}\right)(\alpha - \alpha m + 1) - m}{\frac{2}{N} + \left(2 - \frac{2}{N}\right)\alpha}\right)^2 - \left(2 - \frac{2}{N}\right)\alpha \frac{\left(2-\frac{2}{N}\right)(\alpha - \alpha m + 1) - m}{\frac{2}{N} + \left(2-\frac{2}{N}\right)\alpha}}{\left(2 - \frac{2}{N}\right)\alpha + 1} + 1 \tag{1.10}$$

and

$$k_2 = \frac{\left(\frac{\left(2-\frac{2}{N}\right)\alpha\left(1-\frac{1}{m}\right)+2+\left(\frac{2}{N}-1\right)\frac{1}{m}}{\left(2-\frac{2}{N}\right)\alpha-\frac{2}{mN}}\right)^2 - \left(2-\frac{2}{N}\right)\alpha\frac{\left(2-\frac{2}{N}\right)\alpha\left(1-\frac{1}{m}\right)+2+\left(\frac{2}{N}-1\right)\frac{1}{m}}{\left(2-\frac{2}{N}\right)\alpha-\frac{2}{mN}}}{\left(2-\frac{2}{N}\right)\alpha+1} + 1. \tag{1.11}$$

Then for all $m_0 > 0$, there exist positive radially symmetric nonincreasing initial data

$$u_0 \in C^2(\bar{\Omega}) \text{ with } \frac{\partial u_0}{\partial \nu} = 0 \text{ on } \partial\Omega \tag{1.12}$$

fulfilling $\frac{1}{|\Omega|}\int_\Omega u_0 dx = m_0$, such that (1.1) possesses a unique classical solution (u, v) in $\Omega \times (0, T_{\max})$ with some $T_{\max} \in (0, \infty)$, which satisfies

$$\lim_{t\nearrow T_{\max}} \sup \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{1.13}$$

Remark 1.2. If $N \geq 5$, $1 < m < \min\left\{\frac{2N-4}{N}, 1 - \frac{1}{N} + \frac{1}{N}\sqrt{N^2 - 4N + 1}\right\}$ and α fulfills (1.8), it is easy to see that $k_1, k_2 > 1$, which implies that (1.9) makes sense. Moreover, it follows from Theorem 1.2 that the logistic source cannot completely suppress the occurrence of finite-time blow-up of solutions in (1.1).

Remark 1.3. For the particular cases of $m = 1$ and $\alpha = 0$, Theorem 1.2 can extend the previous results in [42] into more complex situations.

2. Preliminaries

In this section, we present some preliminary lemmas, which shall be used in the proof of our main results. The first lemma concerns with the local-in-time existence of classical solution to system (1.1).

Lemma 2.1. *Let $N \geq 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Assume that the function f satisfies (1.2) and $u_0 \in C^0(\bar{\Omega})$ is a nonnegative initial function. Then there exist $T_{\max} \in (0, \infty]$ and a unique pair*

$$(u, v) \in \left((C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max}))\right)^2,$$

which solves (1.1) in the classical sense in $\Omega \times (0, T_{\max})$. Moreover, if $T_{\max} < +\infty$, then

$$\lim_{t\nearrow T_{\max}} \sup ||u(\cdot, t)||_{L^\infty(\Omega)} = \infty.$$

Proof. The local existence of classical solution of system (1.1) is established by a fixed point theorem in the context of Keller–Segel-type chemotaxis systems. We refer the readers to [7,12,42] for detailed reasonings in closely related situations. □

The next result is the standard Gagliardo–Nirenberg inequality, referring to [38] for the details.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^N$ be a smoothly bounded domain. Assume that $l \in (0, p)$ and $\Phi \in W^{1,2}(\Omega) \cap L^l(\Omega)$, then there exists a positive constant $C_{GN}(\Omega, p, l)$ such that*

$$||\Phi||_{L^p(\Omega)} \leq C_{GN}(||\nabla\Phi||^r_{L^2(\Omega)}||\Phi||^{1-r}_{L^l(\Omega)} + ||\Phi||_{L^l(\Omega)}), \tag{2.1}$$

where $r \in (0, 1)$ fulfills

$$\frac{1}{p} = r\left(\frac{1}{2} - \frac{1}{N}\right) + (1 - r)\frac{1}{l},$$

namely

$$r = \frac{\frac{1}{l} - \frac{1}{p}}{\frac{1}{l} + \frac{1}{N} - \frac{1}{2}}.$$

Lemma 2.3. (See Lemma 2.3 of [22]). *Let* $\Omega \subset \mathbb{R}^N$, $N \geq 1$ *be a bounded and smooth domain, and* $\lambda > 0$, $\mu > 0$, $k > 1$. *Then for a solution* (u, v) *of* (1.1), *we have*

$$\int_\Omega u\, dx \leq \bar{m} \quad \text{for all } t \in (0, T_{\max}), \tag{2.2}$$

where

$$\bar{m} = \max\left\{ \int_\Omega u_0\, dx,\ \left(\frac{\lambda}{\mu}\, |\Omega|^{k-1}\right)^{\frac{1}{k-1}} \right\}. \tag{2.3}$$

Lemma 2.4. (See Lemma 2.4 of [42]). *Let* $\theta > 0$, $\delta > 0$, $\gamma > 0$ *and suppose that for some* $T > 0$, $y \in C^0([0, T])$ *is a nonnegative function satisfying*

$$y(t) \geq \theta + \delta \int_0^t y^{1+\gamma}(\tau) d\tau \quad \forall t \in (0, T).$$

Then $T \leq \frac{1}{\gamma\delta\theta^\gamma}$.

3. Boundedness

This section mainly discusses the boundedness of solutions for (1.1) through the following estimates.

Lemma 3.1. *Assume that the conditions of Theorem* 1.1 *hold, then for all* $p > 1$, *there exists a positive constant* $C > 0$ *such that*

$$||u(\cdot, t)||_{L^p(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}).$$

Proof. Letting

$$p > \max\left\{1, 1 - k + \frac{(m+k-2)N}{(2-4\alpha)(N-1)}, 2 - \frac{2}{N} - m, p_1, p_2\right\}, \tag{3.1}$$

where $p_1 = \frac{(N-m-k)(2-4\alpha)-(m+k-2)N+\sqrt{[(N-m+k-2)(2-4\alpha)-(m+k-2)N]^2-2(2-4\alpha)(m+k-2)N(2m-2k)}}{2(2-4\alpha)}$ and $p_2 = \frac{(2-4\alpha)(k-1)(N-1)-(m+\frac{2}{N}-1)(m+k-2)N}{(m+k-2)N-(2-4\alpha)(N-1)}$, multiplying the first equation in (1.1) by u^{p-1}, integrating by parts and using Young's inequality, we get

$$\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p &= \int_\Omega u^{p-1}\nabla\cdot\left((u+1)^{m-1}\nabla u\right) - \int_\Omega u^{p-1}\nabla\cdot\left(\frac{u\nabla v}{(1+|\nabla v|^2)^\alpha}\right) \\
&\quad + \lambda\int_\Omega u^p - \mu\int_\Omega u^{p+k-1} \\
&= -(p-1)\int_\Omega u^{p-2}(u+1)^{m-1}|\nabla u|^2 + (p-1)\int_\Omega u^{p-1}\frac{\nabla u\cdot\nabla v}{(1+|\nabla v|^2)^\alpha} \\
&\quad + \lambda\int_\Omega u^p - \mu\int_\Omega u^{p+k-1} \\
&\leq -\frac{p-1}{2}\int_\Omega u^{p+m-3}|\nabla u|^2 + \frac{p-1}{2}\int_\Omega u^{p+1-m}|\nabla v|^{2-4\alpha} - \frac{\mu}{2}\int_\Omega u^{p+k-1} + c_1
\end{aligned} \tag{3.2}$$

for all $t \in (0, T_{\max})$, where $c_1 > 0$. For the case $\alpha < \frac{1}{2}$, using Young's inequality with $m+k > 2$, we have

$$\frac{p-1}{2}\int_\Omega u^{p+1-m}|\nabla v|^{2-4\alpha} \le c_2 \int_\Omega |\nabla v|^{(2-4\alpha)\frac{p+k-1}{m+k-2}} + \frac{\mu}{4}\int_\Omega u^{p+k-1}, \tag{3.3}$$

for all $t \in (0, T_{\max})$, where $c_2 > 0$. Combining (3.2) with (3.3) and applying Young's inequality, we obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \int_\Omega u^p \le -\frac{2(p-1)}{(p+m-1)^2}\int_\Omega \left|\nabla u^{\frac{p+m-1}{2}}\right|^2 + c_2\int_\Omega |\nabla v|^{(2-4\alpha)\frac{p+k-1}{m+k-2}} + c_3, \tag{3.4}$$

for all $t \in (0, T_{\max})$, where $c_3 > 0$. Then applying the standard Sobolev inequality and using the second equation of (1.1), one can find $c_4 = c_4(p, k, m, \alpha, \Omega)$ fulfilling

$$\begin{aligned}\int_\Omega |\nabla v|^{(2-4\alpha)\frac{p+k-1}{m+k-2}} &= ||\nabla v||^{(2-4\alpha)\frac{p+k-1}{m+k-2}}_{L^{(2-4\alpha)\frac{p+k-1}{m+k-2}}(\Omega)} \\ &\le c_4 ||u||^{(2-4\alpha)\frac{p+k-1}{m+k-2}}_{L^{\frac{(2-4\alpha)(p+k-1)N}{(2-4\alpha)(p+k-1)+(m+k-2)N}}(\Omega)} \\ &= c_4\left\|u^{\frac{p+m-1}{2}}\right\|^{\frac{2}{p+m-1}(2-4\alpha)\frac{p+k-1}{m+k-2}}_{L^{\frac{2}{p+m-1}\frac{(2-4\alpha)(p+k-1)N}{(2-4\alpha)(p+k-1)+(m+k-2)N}}(\Omega)}.\end{aligned}$$

Making use of the Gagliardo–Nirenberg inequality, there exists a positive constant c_5 such that

$$\begin{aligned}\left\|u^{\frac{p+m-1}{2}}\right\|^{\frac{2}{p+m-1}(2-4\alpha)\frac{p+k-1}{m+k-2}}_{L^{\frac{2}{p+m-1}\frac{(2-4\alpha)(p+k-1)N}{(2-4\alpha)(p+k-1)+(m+k-2)N}}(\Omega)} &\le c_5\left(\left\|\nabla u^{\frac{p+m-1}{2}}\right\|^{a_1}_{L^2(\Omega)}\left\|u^{\frac{p+m-1}{2}}\right\|^{1-a_1}_{L^{\frac{2}{p+m-1}}(\Omega)}\right. \\ &\quad \left. + \left\|u^{\frac{p+m-1}{2}}\right\|_{L^{\frac{2}{p+m-1}}(\Omega)}\right)^{\frac{2}{p+m-1}(2-4\alpha)\frac{p+k-1}{m+k-2}} \\ &\le c_5\left(\bar{m}^{\frac{(1-a_1)(p+m-1)}{2}}\left\|\nabla u^{\frac{p+m-1}{2}}\right|^{a_1}_{L^2(\Omega)} + \right. \\ &\quad \left. \bar{m}^{\frac{p+m-1}{2}}\right)^{\frac{2}{p+m-1}(2-4\alpha)\frac{p+k-1}{m+k-2}},\end{aligned} \tag{3.5}$$

where $\bar{m}$ is given in (2.3) and

$$a_1 = \frac{\frac{p+m-1}{2} - \frac{(2-4\alpha)(p+k-1)+(m+k-2)}{(2-4\alpha)(p+k-1)N}}{\frac{p+m-1}{2} + \frac{1}{N} - \frac{1}{2}} \in (0,1),$$

by selecting

$$p > \max\left\{1, 1-k+\frac{(m+k-2)N}{(2-4\alpha)(N-1)}, 2-\frac{2}{N}-m, p_1\right\},$$

where p_1 is given in (3.1), $\alpha < \frac{1}{2}$ and $k \ge m$. Next combining (3.4) and (3.5), there exists a positive constant c_6 such that

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}||u||^p_{L^p(\Omega)} + ||u||^p_{L^p(\Omega)} &\le -\frac{2(p-1)}{(p+m-1)^2}\left\|\nabla u^{\frac{p+m-1}{2}}\right\|^2_{L^2(\Omega)} \\ &\quad + c_6\left(\bar{m}^{\frac{(1-a_1)(p+m-1)}{2}}\left\|\nabla u^{\frac{p+m-1}{2}}\right\|^{a_1}_{L^2(\Omega)} + \bar{m}^{\frac{p+m-1}{2}}\right)^{\frac{2}{p+m-1}(2-4\alpha)\frac{p+k-1}{m+k-2}} + c_5,\end{aligned}$$

for all $t \in (0, T_{\max})$. When $m+k-2 > 0, k > 1, k \ge m$ and $\frac{4N-(m+k)N-2}{4(N-1)} < \alpha < \frac{1}{2}$, we conclude

$$a_1(2-4\alpha)\frac{2}{p+m-1}\frac{p+k-1}{m+k-2} = \frac{(2-4\alpha)\frac{p+k-1}{m+k-2} - \frac{(2-4\alpha)(p+k-1)}{(m+k-2)N} - 1}{\frac{p+m-1}{2} + \frac{1}{N} - \frac{1}{2}} < 2,$$

where p satisfies (3.1). In view of Young's inequality, there exists a positive constant c_7 such that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \int_\Omega u^p \le c_7.$$

For the case $\alpha \ge \frac{1}{2}$, using (3.2) and applying Young's inequality with $m+k>2$, we obtain

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p &\le -\frac{2(p-1)}{(p+m-1)^2}\int_\Omega \left|\nabla u^{\frac{p+m-1}{2}}\right|^2 + \frac{p-1}{2}\int_\Omega u^{p+1-m} - \frac{\mu}{2}\int_\Omega u^{p+k-1} + c_1 \\ &\le -\frac{2(p-1)}{(p+m-1)^2}\int_\Omega \left|\nabla u^{\frac{p+m-1}{2}}\right|^2 - \frac{\mu}{4}\int_\Omega u^{p+k-1} + c_8,\end{aligned} \tag{3.6}$$

where $c_8>0$. Adding $\int_\Omega u^p$ to both sides of (3.6), applying Young's inequality and neglecting the negative term $-\frac{2(p-1)}{(p+m-1)^2}\int_\Omega |\nabla u^{\frac{p+m-1}{2}}|^2$, there exists $c_9>0$ such that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p + \int_\Omega u^p \le c_9. \tag{3.7}$$

In light of Young's inequality and the ODE comparison, there is a positive constant c_{10} such that

$$\int_\Omega u^p \le c_{10}.$$

The proof of Lemma 3.1 is completed. □

Proof of Theorem 1.1. In view of Lemma 3.1 and the elliptic regularity theory applied to the second equation in system (1.1), there exists $c_{11}>0$ such that

$$\sup_{0<t<T_{\max}} ||v(\cdot,t)||_{W^{2,p}(\Omega)} \le c_{11} \quad \text{for all } p>1.$$

It follows from the Sobolev embedding theorem that

$$\sup_{0<t<T_{\max}} ||\nabla v(\cdot,t)||_{L^\infty(\Omega)} \le c_{12},$$

where $c_{12}>0$. Following the steps in the proof of Lemma A.1 in [32], there exists a positive constant c_{13} independent of t such that

$$||u(\cdot,t)||_{L^\infty(\Omega)} \le c_{13} \quad \text{for all } t\in(0,T_{\max}),$$

which along with Lemma 2.1 shows that $T_{\max}=\infty$. The proof of Theorem 1.1 is completed. □

4. Blow-up in L^∞-norm

In this section, the aim is to prove Theorem 1.2. To this end, we show that the radially symmetric solutions blow up in finite time under some suitable conditions. Assume that $\Omega = B_R(0)\subset\mathbb{R}^N$ is a ball with $R>0$, u_0 satisfies (1.12) and is radially symmetric with respect to $x=0$. If (u,v) is the corresponding radial solution in $\Omega\times(0,T_{\max})$ asserted by Lemma 2.1, we write $u=u(r,t)$ and $v=v(r,t)$ with $r=|x|\in[0,R]$. Following [13], we introduce the mass accumulation function

$$w(s,t) := \int_0^{s^{\frac{1}{N}}} \rho^{N-1}u(\rho,t)\mathrm{d}\rho, s=r^N\in\left[0,R^N\right], \quad t\in[0,T_{\max}), \tag{4.1}$$

then

$$w_s(s,t) = \frac{1}{N}u\left(s^{\frac{1}{N}},t\right) \geq 0, \quad w_{ss}(s,t) = \frac{1}{N^2}s^{\frac{1}{N}-1}u_r\left(s^{\frac{1}{N}},t\right).$$

From the second equation in (1.1), we deduce

$$\frac{1}{r^{N-1}}\left(r^{N-1}v_r(r,t)\right)_r = M(t) - u$$

and

$$r^{N-1}v_r(r,t) = M(t)\int_0^r \rho^{N-1}d\rho - \int_0^r \rho^{N-1}u(\rho,t)d\rho = \frac{M(t)r^N}{N} - \int_0^r \rho^{N-1}u(\rho,t)\mathrm{d}\rho. \tag{4.2}$$

Using (1.1), we obtain

$$\begin{aligned}
w_t(s,t) &= \int_0^{s^{\frac{1}{N}}} \rho^{N-1}u_t(\rho,t)\mathrm{d}\rho \\
&= \int_0^{s^{\frac{1}{N}}} \left(\rho^{N-1}(u+1)^{m-1}u_r\right)_r(\rho,t)\mathrm{d}\rho - \int_0^{s^{\frac{1}{N}}} \left(\rho^{N-1}u(\rho,t)v_r f\left(v_r^2\right)\right)_r \mathrm{d}\rho \\
&\quad + \lambda\int_0^{s^{\frac{1}{N}}} \rho^{N-1}u(\rho,t)\mathrm{d}\rho - \mu\int_0^{s^{\frac{1}{N}}} \rho^{N-1}u^k(\rho,t)\mathrm{d}\rho \\
&= s^{1-\frac{1}{N}}(u+1)^{m-1}u_r\left(s^{\frac{1}{N}},t\right) - s^{1-\frac{1}{N}}uv_r f\left(v_r^2\left(s^{\frac{1}{N}},t\right)\right) \\
&\quad + \lambda\int_0^{s^{\frac{1}{N}}} \rho^{N-1}u(\rho,t)\mathrm{d}\rho - \mu\int_0^{s^{\frac{1}{N}}} u^k(\rho,t)\mathrm{d}\rho \\
&= N^2 s^{2-\frac{2}{N}}w_{ss}(Nw_s+1)^{m-1} + Nw_s\left(w - \frac{M(t)}{N}s\right)f\left(s^{\frac{2}{N}-2}\left(w - \frac{M(t)}{N}s\right)^2\right) \\
&\quad + \lambda w - \mu N^{k-1}\int_0^s w_s^k(\sigma,t)\mathrm{d}\sigma.
\end{aligned}$$

Thus, we have

$$\begin{cases}
w_t = N^2 s^{2-\frac{2}{N}}w_{ss}(Nw_s+1)^{m-1} + N\left(w - \frac{mM(t)}{N}s\right)w_s f\left(s^{\frac{2}{N}-2}\left(w - \frac{M(t)}{N}s\right)^2\right) \\
\qquad +\lambda w - \mu N^{k-1}\int_0^s w_s^k(\sigma,t)d\sigma, \quad s \in \left(0,R^N\right), t \in (0,T_{\max}), \\
w(0,t) = 0, \quad w\left(R^N,t\right) = \frac{M(t)R^N}{N}, \quad t \in (0,T_{\max}), \\
w(s,0) = w_0(s), \quad s \in \left(0,R^N\right),
\end{cases} \tag{4.3}$$

where $w_0(s) = \int_0^{s^{\frac{1}{N}}} \rho^{N-1}u_0(\rho)\mathrm{d}\rho, s \in \left[0,R^N\right]$. Our aim is to prove that the functional $y(t) := \int_0^{R^N} s^{-a}w^b(s,t)\mathrm{d}s$ with suitable parameters $a,b \in (0,1)$ blows up in finite time.

Lemma 4.1. (See Lemma 3.4 of [39]). *Let $\eta \in \mathbb{R}$ and $\beta \in (0,1]$. Then*

$$(1+\xi)^{-\eta} \ge 1 - \frac{\eta_+}{\beta}\xi^{\beta} \quad \textit{for all } \xi \ge 0,$$

where $\eta_+ := \max\{\eta, 0\}$.

Lemma 4.2. *Assume that the nonnegative initial data u_0 satisfies (1.12) and is radially symmetric and nonincreasing with respect to $|x|$, then for all $s \in [0, R^N]$ and $t \in (0, T_{\max})$,*

$$w_s(s,t) \le \frac{w(s,t)}{s} \le w_s(0,t) \tag{4.4}$$

holds.

Proof. Under the condition of (1.12), it follows from the well-known theory on the higher regularity in scalar parabolic equations in [20,21] that $u_r \in C^0([0,R]\times[0,T_{\max})) \cap C^{2,1}((0,R)\times(0,T_{\max}))$. By the similar way as in Lemma 2.2 of [40] (see Lemma 2.3 of [1] or Lemma 3.7 of [9]), we have

$$\begin{aligned} u_t = {} & \frac{N-1}{r}(u+1)^{m-1}u_r + (m-1)(u+1)^{m-2}u_r^2 + (u+1)^{m-1}u_{rr} \\ & - \frac{u_r v_r}{(1+|v_r|^2)^{\alpha}} + \frac{2\alpha u v_r^2 v_{rr}}{(1+|v_r|^2)^{\alpha+1}} - \frac{uM(t)}{(1+|v_r|^2)^{\alpha}} + \frac{u^2}{(1+|v_r|^2)^{\alpha}} + g(u) \end{aligned}$$

where $g(u) = \lambda u - \mu u^k$ for $u \ge 0$. Then we get

$$\begin{aligned} u_{rt} = {} & -\frac{N-1}{r^2}(u+1)^{m-1}u_r + \frac{N-1}{r}(m-1)(u+1)^{m-2}u_r^2 + \frac{N-1}{r}(u+1)^{m-1}u_{rr} \\ & + (m-1)(m-2)(u+1)^{m-3}u_r^3 + 3(m-1)(u+1)^{m-2}u_r u_{rr} \\ & + (u+1)^{m-1}u_{rrr} - \frac{u_{rr}v_r}{(1+|v_r|^2)^{\alpha}} - \frac{u_r v_{rr}}{(1+|v_r|^2)^{\alpha}} \\ & + \frac{4\alpha u_r v_r^2 v_{rr}}{(1+|v_r|^2)^{\alpha+1}} + \frac{4\alpha u v_r v_{rr}^2}{(1+|v_r|^2)^{\alpha+1}} + \frac{2\alpha u v_r^2 v_{rrr}}{(1+|v_r|^2)^{\alpha+1}} - \frac{4(\alpha+1)\alpha u v_r^3 v_{rr}^2}{(1+|v_r|^2)^{\alpha+2}} \\ & - \frac{u_r M(t)}{(1+|v_r|^2)^{\alpha}} + \frac{2\alpha u M(t) v_r v_{rr}}{(1+|v_r|^2)^{\alpha+1}} + \frac{2uu_r}{(1+|v_r|^2)^{\alpha}} - \frac{2\alpha u^2 v_r v_{rr}}{(1+|v_r|^2)^{\alpha+1}} + g'(u)u_r \\ = {} & a(r,t)u_{rrr} + b(r,t)u_{rr} + c(r,t)u_r + d(r,t)u, \end{aligned}$$

where

$$\begin{aligned} a(r,t) &:= (u+1)^{m-1} \\ b(r,t) &:= \frac{N-1}{r}(u+1)^{m-1} + 3(m-1)(u+1)^{m-2}u_r - \frac{v_r}{(1+|v_r|^2)^{\alpha}} \\ c(r,t) &:= -\frac{N-1}{r^2}(u+1)^{m-1} + \frac{N-1}{r}(m-1)(u+1)^{m-2}u_r \\ & \quad + (m-1)(m-2)(u+1)^{m-3}u_r^2 - \frac{v_{rr}}{(1+|v_r|^2)^{\alpha}} + \frac{4\alpha v_r^2 v_{rr}}{(1+|v_r|^2)^{\alpha+1}} \\ & \quad - \frac{M(t)}{(1+|v_r|^2)^{\alpha}} + \frac{2u}{(1+|v_r|^2)^{\alpha}} + g'(u) \\ d(r,t) &:= \frac{4\alpha v_r v_{rr}^2}{(1+|v_r|^2)^{\alpha+1}} + \frac{2\alpha v_r^2 v_{rrr}}{(1+|v_r|^2)^{\alpha+1}} - \frac{4(\alpha+1)\alpha v_r^3 v_{rr}^2}{(1+|v_r|^2)^{\alpha+2}} + \frac{2\alpha M(t) v_r v_{rr}}{(1+|v_r|^2)^{\alpha+1}} - \frac{2\alpha u v_r v_{rr}}{(1+|v_r|^2)^{\alpha+1}}. \end{aligned}$$

Due to Lemma 3.6 of [9], we have $-v_{rr} \le u$, so that for fixed $T \in (0, T_{\max})$, $1 < m < 2$, we can obtain

$$\sup_{r\in(0,R),t\in(0,T)} c(r,t) \le 3||u||_{L^\infty((0,R)\times(0,T))} + ||g'||_{L^\infty(0,||u||_{L^\infty((0,R)\times(0,T))})} < \infty,$$

which implies that the maximum principle in Proposition 52.4 of [29] becomes applicable and yields $u_r \le 0$ in $(0,R)\times(0,T)$, which upon taking $T \nearrow T_{\max}$ implies the statement. Next following the steps in [10,40], we arrive to (4.4). □

Lemma 4.3. *Assume that u_0 satisfies (1.12) and (u,v) denotes the solution of (1.1) in $\Omega\times(0,T_{\max})$. Then for all $\beta\in(0,1), \alpha>0, a\in\left(\left(2-\frac{2}{N}\right)\alpha,\left(2-\frac{2}{N}\right)(\alpha+\beta)\right)$ and $b\in(0,1)$, the function w satisfies*

$$\begin{aligned}\frac{1}{b}\int_0^{R^N} s^{-a}w^b(s,t)\mathrm{d}s \ge{}& \frac{1}{b}\int_0^{R^N} s^{-a}w_0^b(s)\mathrm{d}s + c_1\int_0^t\int_0^{R^N} s^{2-\frac{2}{N}-a}w^{b-2}w_s^{m+1}\mathrm{d}s\mathrm{d}\tau\\ &+c_1\int_0^t\int_0^{R^N} s^{(2-\frac{2}{N})\alpha-a-1}w^{b+1}\mathrm{d}s\mathrm{d}\tau - 2N^m(N-1)\int_0^t\int_0^{R^N} s^{1-\frac{2}{N}-a}w^{b-1}w_s^m\mathrm{d}s\mathrm{d}\tau\\ &-\bar{m}|\Omega|^{-1}\int_0^t\int_0^{R^N} s^{1-a}w^{b-1}w_s\mathrm{d}s\mathrm{d}\tau - \mu N^{k-1}\int_0^t\int_0^{R^N} s^{-a}w^{b-1}\left(\int_0^s w_s^k d\sigma\right)\mathrm{d}s\mathrm{d}\tau\\ ={}& H_1+H_2+H_3-H_4-H_5-H_6 \end{aligned} \tag{4.5}$$

for all $t\in(0,T_{\max})$, where $c_1 := \min\left\{\frac{1}{m}N^{m+1}(1-b), \frac{N\bar{M}^{-\alpha}}{b+1}\left[a-(2+\frac{2}{N})\alpha\right]\right\}$, $\bar{M} := \frac{2|\Omega|\bar{m}^2R^{2N}}{N^2}$ and $\bar{m}$ is defined by (2.3).

Proof. Multiplying the first equation in (4.3) by $(s+\epsilon)^{-a}w^{b-1}$ with $\epsilon>0$, following the steps of [52] and integrating over $s\in(0,R^N)$, we have

$$\begin{aligned}\frac{1}{b}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{R^N}(s+\epsilon)^{-a}w^b(s,t)\mathrm{d}s \ge{}& N^2\int_0^{R^N} s^{2-\frac{2}{N}}(s+\epsilon)^{-a}w^{b-1}w_{ss}(Nw_s+1)^{m-1}\mathrm{d}s\\ &+N\int_0^{R^N}(s+\epsilon)^{-a}w^{b-1}w_s\left(w-\frac{M(t)s}{N}\right)f\left(s^{\frac{2}{N}-2}\left(w-\frac{M(t)s}{N}\right)^2\right)\mathrm{d}s\\ &-\mu N^{k-1}\int_0^{R^N}(s+\epsilon)^{-a}w^{b-1}\left(\int_0^s w_s^k d\sigma\right)\mathrm{d}s\\ ={}& J_1+J_2+J_3. \end{aligned}\tag{4.6}$$

Integrating by parts, we obtain

$$\begin{aligned} J_1 &= N^2\int_0^{R^N} s^{2-\frac{2}{N}}(s+\epsilon)^{-a}w^{b-1}w_{ss}(Nw_s+1)^{m-1}\mathrm{d}s\\ &\ge \frac{1}{m}N^{m+1}\int_0^{R^N} s^{2-\frac{2}{N}}(s+\epsilon)^{-a}w^{b-1}(w_s^m)_s\mathrm{d}s\\ &= \frac{1}{m}N^{m+1}s^{2-\frac{2}{N}}(s+\epsilon)^{-a}w^{b-1}w_s^m\Big|_0^{R^N}\\ &\quad -\frac{1}{m}N^{m+1}(b-1)\int_0^{R^N} s^{2-\frac{2}{N}}(s+\epsilon)^{-a}w^{b-2}w_s^{m+1}\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{m}N^{m+1}\int_0^{R^N}\frac{\mathrm{d}}{\mathrm{d}s}(s^{2-\frac{2}{N}}(s+\epsilon)^{-a})w^{b-1}w_s^m\mathrm{d}s \\
&\geq \frac{1}{m}N^{m+1}(1-b)\int_0^{R^N}s^{2-\frac{2}{N}}(s+\epsilon)^{-a}w^{b-2}w_s^{m+1}\mathrm{d}s \\
&\quad -2\frac{1}{m}N^m(N-1)\int_0^{R^N}s^{1-\frac{2}{N}}(s+\epsilon)^{-a}w^{b-1}w_s^m\mathrm{d}s,
\end{aligned}
\tag{4.7}
$$

where we have used the fact that $m>1, N\geq 2, b\in(0,1)$ and

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\left(s^{2-\frac{2}{N}}(s+\epsilon)^{-a}\right) &= \left(2-\frac{2}{N}\right)s^{1-\frac{2}{N}}(s+\epsilon)^{-a} - as^{2-\frac{2}{N}}(s+\epsilon)^{-a-1} \\
&\leq \left(2-\frac{2}{N}\right)s^{1-\frac{2}{N}}(s+\epsilon)^{-a}.
\end{aligned}
$$

As for J_2, we have

$$
\begin{aligned}
J_2 &= N\int_0^{R^N}(s+\epsilon)^{-a}w^{b-1}w_s\left(w-\frac{M(t)s}{N}\right)f\left(s^{\frac{2}{N}-2}\left(w-\frac{M(t)s}{N}\right)^2\right)\mathrm{d}s \\
&= N\int_0^{R^N}(s+\epsilon)^{-a}w^b w_s f\left(s^{\frac{2}{N}-2}\left(w-\frac{M(t)s}{N}\right)^2\right)\mathrm{d}s \\
&\quad -\int_0^{R^N}s(s+\epsilon)^{-a}w^{b-1}w_s M(t)f\left(s^{\frac{2}{N}-2}\left(w-\frac{M(t)s}{N}\right)^2\right)\mathrm{d}s \\
&= J_{21}+J_{22}.
\end{aligned}
\tag{4.8}
$$

By the strong maximum principle, we have $u\geq 0$ in $\bar{\Omega}\times(0,T_{\max})$. Thus, it follows from $w_s(s,t)=\frac{1}{N}u(s^{\frac{1}{N}},t)\geq 0$ and the boundary condition at $s=R^N$ that $w(s,t)\leq\frac{M(t)R^N}{N}$ for all $s\in[0,R^N]$ and $t\in(0,T_{\max})$. Using $w(s,t)\leq\frac{M(t)R^N}{N}$ and $s\leq R^N$, we arrive that

$$
\left(w-\frac{M(t)s}{N}\right)^2 \leq \frac{M(t)^2s^2}{N^2}+w^2 \leq 2\frac{M(t)^2R^{2N}}{N^2} \leq 2\frac{|\Omega|^2\bar{m}^2R^{2N}}{N^2} := \bar{M},
\tag{4.9}
$$

where $\bar{m}$ is defined by (2.3). Next by means of Lemma 4.1, we can estimate

$$
\begin{aligned}
\left(1+s^{\frac{2}{N}-2}\left(\frac{M(t)s}{N}-w\right)^2\right)^{-\alpha} &\geq s^{\left(2-\frac{2}{N}\right)\alpha}\bar{M}^{-\alpha}\cdot\left(1+s^{2-\frac{2}{N}}\bar{M}^{-1}\right)^{-\alpha} \\
&\geq s^{\left(2-\frac{2}{N}\right)\alpha}\bar{M}^{-\alpha}\cdot\left(1-\frac{\alpha}{\beta}\left(s^{2-\frac{2}{N}}\bar{M}^{-1}\right)^{\beta}\right) \\
&= s^{\left(2-\frac{2}{N}\right)\alpha}\bar{M}^{-\alpha}-\frac{\alpha}{\beta}s^{\left(2-\frac{2}{N}\right)(\alpha+\beta)}\bar{M}^{-(\alpha+\beta)}.
\end{aligned}
\tag{4.10}
$$

Combining (4.8)–(4.10), we derive

$$\begin{aligned} J_{21} &= N\int_0^{R^N}(s+\epsilon)^{-a}w^b w_s f\left(s^{\frac{2}{N}-2}\left(w-\frac{M(t)s}{N}\right)^2\right)\mathrm{d}s \\ &\geq N\bar{M}^{-\alpha}\int_0^{R^N}(s+\epsilon)^{-a}w^b w_s s^{\left(2-\frac{2}{N}\right)\alpha}\mathrm{d}s - \frac{\alpha N\bar{M}^{-(\alpha+\beta)}}{\beta}\int_0^{R^N}(s+\epsilon)^{-a}w^b w_s s^{\left(2-\frac{2}{N}\right)(\alpha+\beta)}\mathrm{d}s \\ &= I_1 - I_2. \end{aligned} \tag{4.11}$$

As for I_1, integrating by parts, we have

$$\begin{aligned} I_1 &= N\bar{M}^{-\alpha}\int_0^{R^N}(s+\epsilon)^{-a}w^b w_s s^{\left(2-\frac{2}{N}\right)\alpha}\mathrm{d}s \\ &= -\frac{N\bar{M}^{-\alpha}}{b+1}\int_0^{R^N}\partial_s\left\{(s+\epsilon)^{-a}s^{\left(2-\frac{2}{N}\right)\alpha}\right\}w^{b+1}\mathrm{d}s \\ &= \frac{aN\bar{M}^{-\alpha}}{b+1}\int_0^{R^N}(s+\epsilon)^{-a-1}s^{\left(2-\frac{2}{N}\right)\alpha}w^{b+1}\mathrm{d}s - \frac{N\bar{M}^{-\alpha}}{b+1}\left(2-\frac{2}{N}\right)\alpha\int_0^{R^N}(s+\epsilon)^{-a}s^{\left(2-\frac{2}{N}\right)\alpha-1}w^{b+1}\mathrm{d}s. \end{aligned} \tag{4.12}$$

As for I_2, once more integrating by parts, we compute

$$\begin{aligned} I_2 &= \frac{\alpha N\bar{M}^{-(\alpha+\beta)}}{\beta}\int_0^{R^N}(s+\epsilon)^{-a}w^b w_s s^{\left(2-\frac{2}{N}\right)(\alpha+\beta)}\mathrm{d}s \\ &= -\frac{\alpha N\bar{M}^{-(\alpha+\beta)}}{\beta(b+1)}\int_0^{R^N}\partial_s\left\{(s+\epsilon)^{-a}s^{\left(2-\frac{2}{N}\right)(\alpha+\beta)}\right\}w^{b+1}\mathrm{d}s \\ &= \frac{a\alpha N\bar{M}^{-(\alpha+\beta)}}{\beta(b+1)}\int_0^{R^N}(s+\epsilon)^{-a-1}s^{\left(2-\frac{2}{N}\right)(\alpha+\beta)}w^{b+1}\mathrm{d}s \\ &\quad - \frac{\alpha N\bar{M}^{-(\alpha+\beta)}}{\beta(b+1)}\left(2-\frac{2}{N}\right)(\alpha+\beta)\int_0^{R^N}(s+\epsilon)^{-a}s^{\left(2-\frac{2}{N}\right)(\alpha+\beta)-1}w^{b+1}\mathrm{d}s. \end{aligned} \tag{4.13}$$

Since $\frac{1}{\left[1+s^{\frac{2}{N}-2}\left(\frac{M(t)s}{N}-w\right)^2\right]^\alpha}\leq 1$, it follows that

$$\begin{aligned} J_{22} &= -\int_0^{R^N}s(s+\epsilon)^{-a}w^{b-1}w_s M(t) f\left(s^{\frac{2}{N}-2}\left(w-\frac{M(t)s}{N}\right)^2\right)\mathrm{d}s \\ &= -\int_0^{R^N}s(s+\epsilon)^{-a}w^{b-1}w_s M(t)\frac{1}{\left[1+s^{\frac{2}{N}-2}\left(\frac{M(t)s}{N}-w\right)^2\right]^\alpha}\mathrm{d}s \end{aligned} \tag{4.14}$$

$$\geq -\int_0^{R^N} s(s+\epsilon)^{-a} w^{b-1} w_s M(t) \mathrm{d}s$$
$$\geq -\bar{m}|\Omega|^{-1} \int_0^{R^N} s(s+\epsilon)^{-a} w^{b-1} w_s \mathrm{d}s,$$

where $\bar{m}$ is defined by (2.3). Replacing (4.7), (4.8) and (4.11)–(4.14) in (4.6) and integrating over $(0, t)$, we obtain

$$\begin{aligned}
\frac{1}{b}\int_0^{R^N} (s+\epsilon)^{-a} w^b(s,t) \mathrm{d}s \geq{} & \frac{1}{b}\int_0^{R^N} (s+\epsilon)^{-a} w_0^b(s) \mathrm{d}s \\
& + \frac{1}{m} N^{m+1}(1-b) \int_0^t \int_0^{R^N} s^{2-\frac{2}{N}} (s+\epsilon)^{-a} w^{b-2} w_s^{m+1} \mathrm{d}s\mathrm{d}\tau \\
& - 2\frac{1}{m} N^m (N-1) \int_0^t \int_0^{R^N} s^{1-\frac{2}{N}} (s+\epsilon)^{-a} w^{b-1} w_s^m \mathrm{d}s\mathrm{d}\tau \\
& + \frac{aN\bar{M}^{-\alpha}}{b+1} \int_0^t \int_0^{R^N} (s+\epsilon)^{-a-1} s^{\left(2-\frac{2}{N}\right)\alpha} w^{b+1} \mathrm{d}s\mathrm{d}\tau \\
& - \frac{N\bar{M}^{-\alpha}}{b+1} \left(2-\frac{2}{N}\right) \alpha \int_0^t \int_0^{R^N} (s+\epsilon)^{-a} s^{\left(2-\frac{2}{N}\right)\alpha - 1} w^{b+1} \mathrm{d}s\mathrm{d}\tau \\
& + \frac{a\alpha N\bar{M}^{-(\alpha+\beta)}}{\beta(b+1)} \int_0^t \int_0^{R^N} (s+\epsilon)^{-a-1} s^{\left(2-\frac{2}{N}\right)(\alpha+\beta)} w^{b+1} ds\mathrm{d}\tau \\
& - \frac{\alpha N\bar{M}^{-(\alpha+\beta)}\left(2-\frac{2}{N}\right)(\alpha+\beta)}{\beta(b+1)} \int_0^t \int_0^{R^N} (s+\epsilon)^{-a} s^{\left(2-\frac{2}{N}\right)(\alpha+\beta)-1} w^{b+1} \mathrm{d}s\mathrm{d}\tau \\
& - \bar{m}|\Omega|^{-1} \int_0^t \int_0^{R^N} s(s+\epsilon)^{-a} w^{b-1} w_s \mathrm{d}s\mathrm{d}\tau \\
& - \mu N^{k-1} \int_0^t \int_0^{R^N} (s+\epsilon)^{-a} w^{b-1} \left(\int_0^s w_s^k \mathrm{d}\sigma\right) \mathrm{d}s d\tau.
\end{aligned} \tag{4.15}$$

Taking $\epsilon \searrow 0$ in (4.15), applying the monotone convergence theorem and neglecting the positive term I_2 in (4.13) by selecting $a \in ((2-\frac{2}{N})\alpha, (2-\frac{2}{N})(\alpha+\beta))$ where $\alpha > 0, \beta \in (0,1)$, then we can arrive at (4.5). □

Next we estimate H_4 and H_5, which are defined in Lemma 4.3.

Lemma 4.4. *Assume that* $N \geq 2$, $1 < m < \frac{2N-2}{N}$, $0 < \alpha < \frac{2N-2-mN}{2N-2}$, *then there exist*

$$b \in (\max\{0, B_1, B_2\}, 1),$$

where $B_1 = \frac{(2-\frac{2}{N})\alpha(2-m)+m-2+\frac{2}{N}}{(2-\frac{2}{N})(1-\alpha)-m}$, $B_2 = \frac{m-2+\frac{2}{N}}{(2-\frac{2}{N})(1-\alpha)-m}$ *and*

$$0 < a < \min\{A_1, A_2\}, \tag{4.16}$$

with

$$A_1 = \frac{\left(2-\frac{2}{N}\right)\alpha + \left[\left(2-\frac{2}{N}\right)(1-\alpha)-m\right](b+1)}{2-m}, \tag{4.17}$$

and

$$A_2 = \frac{\left(2-\frac{2}{N}\right)\alpha\left(2-\frac{1}{m}\right) + \left[2+\left(\frac{2}{N}-1\right)\frac{1}{m} - \left(2-\frac{2}{N}\right)\alpha\right](b+1)}{2-\frac{1}{m}}, \tag{4.18}$$

such that

$$H_4 \leq \frac{1}{2}H_2 + \frac{1}{4}H_3 + c_2 t \tag{4.19}$$

and

$$H_5 \leq \frac{1}{2}H_2 + \frac{1}{4}H_3 + c_3 t \tag{4.20}$$

for all $t \in (0, T_{\max})$, *where* c_2, $c_3 > 0$ *and* H_2, H_3, H_4, H_5 *are defined in Lemma* 4.3.

Proof. Using Young's inequality, we have

$$\begin{aligned}
H_4 &= 2N^m(N-1)\int_0^t\int_0^{R^N} s^{1-\frac{2}{N}-a}w^{b-1}w_s^m \mathrm{d}s\mathrm{d}\tau \\
&\leq \frac{c_1}{2}\int_0^t\int_0^{R^N} s^{2-\frac{2}{N}-a}w^{b-2}w_s^{m+1}\mathrm{d}sd\tau + c_4\int_0^t\int_0^{R^N} s^{2-a-\frac{2}{N}}w^{b-2}(s^{-1}w)^{m+1}dsd\tau \\
&= \frac{1}{2}H_2 + c_4\int_0^t\int_0^{R^N} s^{1-a-m-\frac{2}{N}}w^{b+m-1}\mathrm{d}sd\tau \\
&\leq \frac{1}{2}H_2 + \frac{1}{4}c_1\int_0^t\int_0^{R^N} s^{\left(2-\frac{2}{N}\right)\alpha-a-1}w^{b+1}\mathrm{d}s\mathrm{d}\tau + c_5\int_0^t\int_0^{R^N} s^{\left(2-\frac{2}{N}\right)\alpha-a-1}s^{\left[\left(2-\frac{2}{N}\right)(1-\alpha)-m\right]\frac{b+1}{2-m}}\mathrm{d}s\mathrm{d}\tau \\
&\leq \frac{1}{2}H_2 + \frac{1}{4}H_3 + c_2 t,
\end{aligned}$$

where $c_1, c_2, c_4, c_5 > 0$ and we have used the fact that $(2-\frac{2}{N})\alpha - a - 1 + [(2-\frac{2}{N})(1-\alpha) - m]\frac{b+1}{2-m} > -1$ due to $0 < a < A_1$. It follows from Young's inequality that

$$
\begin{aligned}
H_5 &= \bar{m}|\Omega|^{-1}\int_0^t\int_0^{R^N} s^{1-a}w^{b-1}w_s \mathrm{d}s\mathrm{d}\tau\\
&\le \frac{1}{2}c_1\int_0^t\int_0^{R^N} s^{2-\frac{2}{N}-a}w^{b-2}w_s^{m+1}\mathrm{d}s\mathrm{d}\tau + c_6\int_0^t\int_0^{R^N} s^{2-\frac{2}{N}-a}w^{b-2}(s^{\frac{2}{N}-1}w)^{\frac{m+1}{m}}\mathrm{d}s\mathrm{d}\tau\\
&= \frac{1}{2}H_2 + c_6\int_0^t\int_0^{R^N} s^{1-a+\left(\frac{2}{N}-1\right)\frac{1}{m}}w^{b-1+\frac{1}{m}}\mathrm{d}s\mathrm{d}\tau\\
&\le \frac{1}{2}H_2 + \frac{1}{4}c_1\int_0^t\int_0^{R^N} s^{\left(2-\frac{2}{N}\right)\alpha-a-1}w^{b+1}\mathrm{d}s\mathrm{d}\tau + c_7\int_0^t\int_0^{R^N} s^{\left(2-\frac{2}{N}\right)\alpha-a-1}s^{\left[2+\left(\frac{2}{N}-1\right)\frac{1}{m}-\left(2-\frac{2}{N}\right)\alpha\right]\frac{b+1}{2-\frac{1}{m}}}\mathrm{d}s\mathrm{d}\tau\\
&\le \frac{1}{2}H_2 + \frac{1}{4}H_3 + c_3 t,
\end{aligned}
$$

for all $t \in (0, T_{\max})$, where $c_1, c_3, c_6, c_7 > 0$ and we have used the fact that $(2-\frac{2}{N})\alpha - a - 1 + [2 + (\frac{2}{N} - 1)\frac{1}{m} - (2-\frac{2}{N})\alpha]\frac{b+1}{2-\frac{1}{m}} > -1$ due to $0 < a < A_2$. □

Now, we shall estimate the term H_6 in Lemma 4.3.

Lemma 4.5. *Let $N \ge 5$ and suppose that $1 < m < \min\left\{\frac{2N-4}{N}, 1 - \frac{1}{N} + \frac{1}{N}\sqrt{N^2-4N+1}\right\}$, $\frac{2N-4-mN}{(2N-2)m} < \alpha < \frac{2N-2-mN}{2N-2}$ and*

$$k \in (1, \min\{2, k_1, k_2\}), \tag{4.21}$$

where

$$k_1 = \frac{\left(\frac{\left(2-\frac{2}{N}\right)(\alpha-\alpha m+1)-m}{\frac{2}{N}+\left(2-\frac{2}{N}\right)\alpha}\right)^2 - \left(2-\frac{2}{N}\right)\alpha\frac{\left(2-\frac{2}{N}\right)(\alpha-\alpha m+1)-m}{\frac{2}{N}+\left(2-\frac{2}{N}\right)\alpha}}{\left(2-\frac{2}{N}\right)\alpha+1} + 1 \tag{4.22}$$

and

$$k_2 = \frac{\left(\frac{\left(2-\frac{2}{N}\right)\alpha\left(1-\frac{1}{m}\right)+2+\left(\frac{2}{N}-1\right)\frac{1}{m}}{\left(2-\frac{2}{N}\right)\alpha-\frac{2}{mN}}\right)^2 - \left(2-\frac{2}{N}\right)\alpha\frac{\left(2-\frac{2}{N}\right)\alpha\left(1-\frac{1}{m}\right)+2+\left(\frac{2}{N}-1\right)\frac{1}{m}}{\left(2-\frac{2}{N}\right)\alpha-\frac{2}{mN}}}{\left(2-\frac{2}{N}\right)\alpha+1} + 1. \tag{4.23}$$

Then we can find $a = b \in (0,1)$ fulfilling (4.16) such that

$$H_6 \le \frac{1}{4}H_3 + c_8 t, \tag{4.24}$$

for all $t \in (0, T_{\max})$, where $c_8 > 0$ and H_3, H_6 are defined in Lemma 4.3.

Proof. By Fubini's theorem, we obtain

$$
\begin{aligned}
H_6 &= \mu N^{k-1}\int_0^t\int_0^{R^N} s^{-a}w^{b-1}\left(\int_0^s w_s^k d\sigma\right)\mathrm{d}s\mathrm{d}\tau\\
&= \mu N^{k-1}\int_0^t\int_0^{R^N}\left(\int_\sigma^{R^N} s^{-a}w^{b-1}ds\right)w_s^k(\sigma)\mathrm{d}\sigma\mathrm{d}\tau.
\end{aligned}
$$

Since $b \in (0,1)$ and $w_s \geq 0$, then $w^{b-1}(s)$ decreases in s. Thus,

$$H_6 \leq \mu N^{k-1} \int_0^t \int_0^{R^N} \left(\int_\sigma^{R^N} s^{-a} ds \right) w^{b-1}(\sigma) w_s^k(\sigma) \mathrm{d}\sigma \mathrm{d}\tau$$
$$= \frac{1}{1-a} \mu N^{k-1} \int_0^t \int_0^{R^N} \left(R^{N(1-a)} - \sigma^{1-a} \right) w^{b-1}(\sigma) w_s^k(\sigma) \mathrm{d}\sigma \mathrm{d}\tau.$$

Since $a \in (0,1)$, we neglect the negative term to derive

$$H_6 \leq \frac{\mu N^{k-1}}{1-a} R^{N(1-a)} \int_0^t \int_0^{R^N} w^{b-1}(s) w_s^k(s) \mathrm{d}s \mathrm{d}\tau.$$

Fixed $b = a \in (0,1)$, then we have $\frac{(2-\frac{2}{N})(\alpha-\alpha m+1)-m}{\frac{2}{N}+(2-\frac{2}{N})\alpha} \in (0,1)$ and $\frac{(2-\frac{2}{N})\alpha(1-\frac{1}{m})+2+(\frac{2}{N}-1)\frac{1}{m}}{(2-\frac{2}{N})\alpha-\frac{2}{mN}} \in (0,1)$ fulfilling (4.16), thanks to the facts that $1 < m < \min\left\{\frac{2N-4}{N}, 1-\frac{1}{N}+\frac{1}{N}\sqrt{N^2-4N+1}\right\}$, $N \geq 5$ and $\frac{2N-4-mN}{(2N-2)m} < \alpha < \frac{2N-2-mN}{2N-2}$. Then by selecting $a = b$, using Young's inequality and applying Lemma 4.2 we have

$$H_6 \leq \frac{\mu N^{k-1}}{1-a} R^{N(1-a)} \int_0^t \int_0^{R^N} w^{b-1} w_s^k \mathrm{d}s \mathrm{d}\tau$$
$$\leq \frac{\mu N^{k-1}}{1-a} R^{N(1-a)} \int_0^t \int_0^{R^N} w^{k+a-1} s^{-k} \mathrm{d}s \mathrm{d}\tau$$
$$\leq \frac{1}{4} c_1 \int_0^t \int_0^{R^N} s^{(2-\frac{2}{N})\alpha-a-1} w^{b+1} \mathrm{d}s \mathrm{d}\tau + c_9 \int_0^t \int_0^{R^N} s^{(2-\frac{2}{N})\alpha-a-1} s^{\left[-k+a+1-\left(2-\frac{2}{N}\right)\alpha\right]\frac{b+1}{2-k}} \mathrm{d}s \mathrm{d}\tau$$
$$= \frac{1}{4} H_3 + c_8 t,$$

for all $t \in (0, T_{\max})$, where $c_8, c_9 > 0$ and we have used the fact that $(2-\frac{2}{N})\alpha - a - 1 + [-k+a+1-(2-\frac{2}{N})\alpha]\frac{b+1}{2-k} > -1$ due to (4.21). □

Taking into account of Lemmas 4.3–4.5, we obtain an integral inequality for the functional $y(t) = \int_0^{R^N} s^{-a} w^b(s) ds$.

Lemma 4.6. *Assume that the conditions of Theorem 1.2 hold. Then there exist $a, b \in (0,1)$, $\delta > 0$ and $C > 0$ such that*

$$\int_0^{R^N} s^{-a} w^b(s,t) \mathrm{d}s \geq \int_0^{R^N} s^{-a} w_0^b(s) \mathrm{d}s + \delta \int_0^t \left(\int_0^{R^N} s^{-a} w^b(s,\tau) \mathrm{d}s \right)^{\frac{b+1}{b}} d\tau - Ct \tag{4.25}$$

for all $t \in (0, T_{\max})$.

Proof. Collecting (4.19), (4.20) and (4.24) in (4.5) and selecting

$$a = b \in \left(0, \min\left\{ \frac{\left(2-\frac{2}{N}\right)(\alpha-\alpha m+1)-m}{\frac{2}{N}+\left(2-\frac{2}{N}\right)\alpha}, \frac{\left(2-\frac{2}{N}\right)\alpha\left(1-\frac{1}{m}\right)+2+\left(\frac{2}{N}-1\right)\frac{1}{m}}{\left(2-\frac{2}{N}\right)\alpha-\frac{2}{mN}} \right\}\right),$$

then we have

$$\int_0^{R^N} s^{-a}w^b(s,t)\mathrm{d}s \ge \int_0^{R^N} s^{-a}w_0^b(s)\mathrm{d}s + \frac{b}{4}c_1\int_0^t\int_0^{R^N} s^{\left(2-\frac{2}{N}\right)\alpha-a-1}w^{b+1}\mathrm{d}s\mathrm{d}\tau - Ct \tag{4.26}$$

for all $t\in(0,T_{\max})$.

Using the Hölder inequality, we obtain

$$\begin{aligned}\int_0^{R^N} s^{-a}w^b(s,t)\mathrm{d}s &= \int_0^{R^N}\left(s^{\left(2-\frac{2}{N}\right)\alpha-a-1}w^{b+1}\right)^{\frac{b}{b+1}} s^{-a-\frac{\left[\left(2-\frac{2}{N}\right)\alpha-a-1\right]b}{b+1}}\mathrm{d}s\\ &\le \left(\int_0^{R^N} s^{\left(2-\frac{2}{N}\right)\alpha-a-1}w^{b+1}\mathrm{d}s\right)^{\frac{b}{b+1}}\left(\int_0^{R^N} s^{-a-\frac{\left[\left(2-\frac{2}{N}\right)\alpha-a-1\right]b}{b+1}}ds\right)^{\frac{1}{b+1}},\end{aligned}$$

which implies

$$\int_0^{R^N} s^{(2-\frac{2}{N})\alpha-a-1}w^{b+1}ds \ge c_{10}\left(\int_0^{R^N} s^{-a}w^b ds\right)^{\frac{b+1}{b}} \tag{4.27}$$

where $c_{10} = \left(\frac{1-a-\frac{[(2-\frac{2}{N})\alpha-a-1]b}{b+1}}{R^{N\left(1-a-\frac{[(2-\frac{2}{N})\alpha-a-1]b}{b+1}\right)}}\right)^{\frac{1}{b}}$ and we have used the fact $-a-\frac{[(2-\frac{2}{N})\alpha-a-1]b}{b+1} > -1$ due to $a<1<1+[2-(2-\frac{2}{N})\alpha]b$. Then replacing (4.27) into (4.26) we arrive at (4.25) with $\delta=\frac{1}{4}bc_1c_{10}$. □

Proof of Theorem 1.2.. We fix $N\ge 5$ and may assume that $\Omega=B_R(0)\subset\mathbb{R}^N$ with some $R>0$. Then for given $1<m<\min\left\{\frac{2N-4}{N}, 1-\frac{1}{N}+\frac{1}{N}\sqrt{N^2-4N+1}\right\}$, $\frac{2N-4-mN}{(2N-2)m}<\alpha<\frac{2N-2-mN}{2N-2}$, $k\in(1,\min\{2,k_1,k_2\})$, where k_1 and k_2 are defined by (4.22) and (4.23) and $m_0>0$, we let $a,b\in(0,1)$, $\delta>0$ and $C>0$ be as provided by Lemma 4.6. Now for fixed $T>0$, we pick $\theta>0$ large such that

$$\theta > \left(\frac{b}{\delta T}\right)^b. \tag{4.28}$$

Next, following the steps in the proof of Theorem 0.1 in [42], let

$$\phi_\epsilon(s) := \frac{m_0}{N}\cdot\frac{R^N+\epsilon}{s+\epsilon}\cdot s,\quad s\in[0,R^N], \epsilon>0, \tag{4.29}$$

then $\phi_\epsilon(s)$ is nonnegative and satisfies

$$\phi_\epsilon(s)\nearrow\frac{m_0R^N}{N}\ \text{ for all } s\in[0,R^N] \text{ as } \epsilon\searrow 0. \tag{4.30}$$

By the monotone convergence theorem, it asserts

$$\int_0^{R^N} s^{-a}\phi_\epsilon(s)^b(s)\mathrm{d}s \to +\infty\ \text{ as } \epsilon\searrow 0. \tag{4.31}$$

Thus, we can find some sufficiently small $\epsilon>0$ such that

$$\int_0^{R^N} s^{-a}\phi_\epsilon^b(s)(s)\mathrm{d}s \ge \theta + CT. \tag{4.32}$$

With this value of ϵ fixed henceforth, we let

$$w_0(s) := \phi_\epsilon(s), \quad s \in [0, R^N], \tag{4.33}$$

and then it is obvious to see that w_0 belongs to $C^\infty([0, R^N])$ and satisfies $w_0 = 0, w_0(R^N) = \frac{m_0 R^N}{N}$ and $w_{0\,s}(s) > 0$ for all $s \in [0, R^N]$. Therefore, the function u_0 defined by $u_0(x) := N w_{0s}(|x|^N)$ for $x \in \bar{\Omega}$ is radially symmetric, smooth and positive in $\bar{\Omega}$ with $\frac{1}{|\Omega|}\int\limits_{\Omega} u_0(x)dx = m_0$. Next, we claim that the maximal existence time $T_{\max}$ of the corresponding solution (u, v) of system (1.1) satisfies $T_{\max} < T$. Let

$$y(t) := \int\limits_0^{R^N} s^{-a} w^b(s)ds, \quad t \in (0, T_{\max}), \tag{4.34}$$

then it follows from (4.32), (4.33), (4.34) and Lemma 4.6 that

$$y(t) \geq \theta + \delta \int\limits_0^t y^{1+\frac{1}{b}}(\tau)d\tau \ \text{ for all } t \in (0, T_{\max}). \tag{4.35}$$

By Lemma 2.4, it is easy to see that

$$T_{\max} \leq \frac{1}{\frac{1}{b}\delta\theta^{\frac{1}{b}}}.$$

In conjunction with (4.28), this entails that indeed $T_{\max} < T$. The proof of Theorem 1.2 is complete. □

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Declarations

Conflict of interest The authors declare that this work does not have any Conflict of interest.

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Chunmei Chen and Pan Zheng
School of Science
Chongqing University of Posts and Telecommunications
Chongqing 400065
People's Republic of China
e-mail: zhengpan52@sina.com

Chunmei Chen
e-mail: chenchunmei225@163.com

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