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Multiplicity and concentration behavior of solutions for magnetic Choquard equation with critical growth

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Abstract. In this paper, we consider the following nonlinear Choquard equation with magnetic field

$$\begin{cases} \left(\frac{\varepsilon}{i}\nabla - A(x)\right)^2 u + V(x)u = \varepsilon^{\mu-N}\left(\int\limits_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\mu} + F(|u(y)|^2)}{|x-y|^\mu} \mathrm{d}y\right)\left(|u|^{2^*_\mu-2}u + \frac{1}{2^*_\mu} f(|u|^2)u\right) \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N, \mathbb{C}) \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $N \geq 3$, $0 < \mu < N$, $2^*_\mu = \frac{2N-\mu}{N-2}$, $V(x) : \mathbb{R}^N \to \mathbb{R}^N$ and $A(x) : \mathbb{R}^N \to \mathbb{R}^N$ is a continuous potential, f is a continuous subcritical term, and F is the primitive function of f. Under a local assumption on the potential V, by the variational methods, the penalization techniques and the Ljusternik–Schnirelmann theory, we prove the multiplicity and concentration properties of nontrivial solutions of the above problem for $\varepsilon > 0$ small enough.

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1. Introduction and main results

In this paper, we study the following nonlinear Choquard equation with magnetic fields

$$\left(\frac{\varepsilon}{i}\nabla - A(x)\right)^2 u + V(x)u = \varepsilon^{\mu-N}\left(\int\limits_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\mu} + F(|u(y)|^2)}{|x-y|^\mu} \mathrm{d}y\right)\left(|u|^{2^*_\mu-2}u + \frac{1}{2^*_\mu} f(|u|^2)u\right), \quad (1.1)$$

where $x \in \mathbb{R}^N$, $\varepsilon > 0$ is a parameter, $N \geq 3$, $0 < \mu < N$, $2^*_\mu = \frac{2N-\mu}{N-2}$, $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ is an electric potential, and $A(x) \in C(\mathbb{R}^N, \mathbb{R}^N)$ is a magnetic potential. The operator $\left(\frac{1}{i}\nabla - A(x)\right)^2$, called magnetic Laplacian, is defined by

$$-\Delta_A u := \left(\frac{1}{i}\nabla - A(x)\right)^2 u = -\Delta u - \frac{2}{i}A(x)\cdot\nabla u + |A(x)|^2 u - \frac{1}{i} u \,\mathrm{div}(A(x)).$$

For problem (1.1), there are vast literature concerning the existence and multiplicity of nonlinear Choquard equation without magnetic field, namely $A \equiv 0$. Then, (1.1) turns to the Choquard equation

$$-\varepsilon^2\Delta u + V(x)u = \varepsilon^{\mu-N}\left(\int\limits_{\mathbb{R}^N} \frac{G(u(y))}{|x-y|^\mu}\mathrm{d}y\right) g(u) \text{ in } \mathbb{R}^N. \quad (1.2)$$

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When $\varepsilon = 1$, Eq. (1.2) reduces to the generalized Choquard equation:

$$-\Delta u + V(x)u = \Big(\frac{1}{|x|^{\mu}} * |u|^p\Big)|u|^{p-2}u \quad \text{in} \quad \mathbb{R}^3. \tag{1.3}$$

The case of (1.3) $N = 3, p = 2$, $V(x) = 1$ and $\mu = 1$ describes the quantum mechanics of a polaron at rest by Pekar [36]. In 1976, Choquard [26] described an approximation to Hartree–Fock theory of a one component plasma by (1.3). Thus, Eq. (1.3) is also called the nonlinear Schrödinger–Newton equation. By using critical point theory, Lions [28] obtained the existence of infinitely many radially symmetric solutions in $H^1(\mathbb{R}^N)$ and Ackermann [1] proved the existence of infinitely many geometrically distinct weak solutions for a general case.

Moroz and Van Schaftingen [31,33] eliminated this restriction and showed the regularity, positivity and radial symmetry of the ground states for the optimal range of parameters, and also derived that these solutions decay asymptotically at infinity.

On the other hand, the semiclassical limit problem attracts people's attention, namely $\varepsilon \to 0$ in (1.2). The question of the existence of semiclassical solutions for the nonlocal problem (1.2) has been posed in [6]. Moroz and Van Schaftingen [32] develop a novel nonlocal penalization by variational methods to show that equation (1.2) with $G(u) = |u|^q$ has a family of solutions concentrated at the local minimum of V, with V satisfying some additional assumptions at infinity. Other results see [5,12,25,30,43] and references therein.

The magnetic nonlinear Schrödinger equation (1.1) has been extensively investigated by many authors by applying suitable variational and topological methods (see [3,9,13,14,16,17,20,23,40,41] and references therein). To the best of our knowledge, the first result involving the magnetic field was obtained by Esteban and Lions [20]. They used the concentration–compactness principle and minimization arguments to obtain solutions for $\varepsilon > 0$ fixed and $N = 2, 3$. For the nonlinear magnetic Schrödinger equation

$$\Big(\frac{\varepsilon}{i}\nabla - A(x)\Big)^2 u + V(x)u = f(|u|^2)u, \quad x \in \mathbb{R}^N, \tag{1.4}$$

Alves [3] used the method of the Nehari manifold, the penalization method and the Ljusternik–Schnirelmann category theory to relate the number of solutions with the topology of the set for a subcritical nonlinearity $f \in C^1$. Ji and Rădulescu [23] showed that the arguments developed in [3] fail if f is only continuous, and they improved the methods to study the multiplicity and concentration results for magnetic Schrödinger equation in which the subcritical nonlinearity f is only continuous. Later, Ji and Rădulescu [24] used the same methods to study the multiplicity and concentration results for magnetic Schrödinger equation with critical growth.

About the Choquard equations with magnetic potential like

$$\Big(\frac{\varepsilon}{i}\nabla - A(x)\Big)^2 u + V(x)u = \varepsilon^{\mu-N}\Bigg(\int\limits_{\mathbb{R}^N} \frac{F(|u(y)|^2)}{|x-y|^{\mu}}\mathrm{d}y\Bigg) f(|u|^2)u. \tag{1.5}$$

For $\varepsilon = 1$, $\mu \in (0, N)$, $\frac{2N-\mu}{N} < p < \frac{2N-\mu}{N-2}$, $f(u) = |u|^p$, Cingolani et al. [14] established the existence of multiple complex-valued solutions. Alves et al. [4] proved the existence and multiplicity of solutions by using variational methods, penalization techniques and Ljusternik–Schnirelmann theory under the suitable conditions of V and f.

It is quite natural to consider the multiplicity and concentration phenomena of nontrivial solutions for problem (1.1) with critical growth. Inspired by [4,24], the main purpose of this paper is to investigate multiplicity and concentration of nontrivial solutions for problem (1.1) by proposing a local assumption on $V(x)$ and adapting the penalization technique and Ljusternik–Schnirelmann category theory.

To go on studying the problem (1.1), we recall an important inequality, namely:

Lemma 1.1. (Hardy–Littlewood–Sobolev inequality) [27] *Let $t, r > 1$ and $0 < \mu < N$ with $1/t + \mu/N + 1/r = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t, N, \mu, r)$, independent of f, h, such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^{\mu}} \mathrm{d}x\mathrm{d}y \leq C(t, N, \mu, r)|f|_t|h|_r. \tag{1.6}$$

If $t = r = 2N/(2N-\mu)$, then there is equality in (1.6) if and only if $f \equiv Ch$ and

$$h(x) = A(\gamma^2 + |x-a|^2)^{-(2N-\mu)/2}$$

for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^N$.

Notice that, by the Hardy–Littlewood–Sobolev inequality, the integral

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^q|u(y)|^q}{|x-y|^{\mu}} \mathrm{d}x\mathrm{d}y$$

is well defined if $|u|^q \in L^t(\mathbb{R}^N)$ for some $t > 1$ satisfying

$$\frac{2}{t} + \frac{\mu}{N} = 2.$$

Thus, for $u \in H^1(\mathbb{R}^N)$, by Sobolev embedding Theorems, we know

$$2 \leq tq \leq \frac{2N}{N-2},$$

that is

$$\frac{2N-\mu}{N} \leq q \leq \frac{2N-\mu}{N-2}.$$

Thus, $\frac{2N-\mu}{N}$ is called the lower critical exponent and $2^*_\mu = \frac{2N-\mu}{N-2}$ is the upper critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality and the Laplace operator.

Before giving our main result, we present the following hypothesis on the potential $V(x) \in C(\mathbb{R}^N, \mathbb{R})$:

(V_1) there exists $V_1 > 0$ such that $V_1 = \inf_{x\in\mathbb{R}^N} V(x)$;

(V_2) there exists a bounded open set $\Lambda \subset \mathbb{R}^N$such that

$$0 < V_0 = \inf_{x\in\Lambda} V(x) < \min_{x\in\partial\Lambda} V(x).$$

Observe that

$$M := \{x \in \Lambda : V(x) = V_0\} \neq \emptyset.$$

Moreover, the nonlinearity $f \in C(\mathbb{R}, \mathbb{R})$ be a function satisfying:

(f_1) $f(t) = 0$ if $t \leq 0$;

(f_2) there exists $p, q \in \left(\frac{2N-\mu}{N}, 2^*_\mu\right)$ and $\lambda > 0$ such that

$$f(t) \geq \lambda t^{(p-2)/2} \quad \forall t > 0, \qquad \lim_{t\to+\infty} \frac{f(t^2)t}{t^{q-1}} = 0;$$

(f_3) $f(t)$ is strictly increasing in $(0, \infty)$.

The main result of this paper is the following:

Theorem 1.2. *Suppose that condictions (V_1)–(V_2) and (f_1)–(f_3) are satisfied. Then, for any $\delta > 0$ such that*

$$M_\delta := \{x \in \mathbb{R}^N : \mathrm{dist}(x, M) < \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ such that, for any $0 < \varepsilon < \varepsilon_\delta$, problem (1.1) has at least $\operatorname{cat}_{M_\delta}(M)$ nontrivial solutions. Moreover, for every sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0^+$ as $n \to \infty$, if we denote by u_{ε_n} one of these solutions of (1.1) for $\varepsilon = \varepsilon_n$ and $\eta_{\varepsilon_n} \in \mathbb{R}^N$ the global maximum point of $|u_{\varepsilon_n}|$, then

$$\lim_n V\left(\eta_{\varepsilon_n}\right) = V_0.$$

We shall use the variational methods, penalization techniques and the Ljusternik–Schnirelmann theory to prove Theorem 1.2. There are three main difficulties to overcome. The first difficulty is the presence of the magnetic field $A(x)$, problem (1.1) became a complex-valued problem. We use the diamagnetic inequality to transform complex values into real values, while corresponding embedding theorem also holds. The second one is that the nonlocal terms appearing in the equation, the commonly used methods and techniques may be ineffective. We can use the Hardy–Littlewood–Sobolev inequality to overcome this difficulty. The last one is that critical exponent occurs the right of problem (1.1), we need more accurate estimate to overcome the lack of compactness.

The paper is organized as follows. In Sect. 2, we give some preliminaries. In Sect. 3, we introduce a modified problem. In Sect. 4, we study the associated autonomous problem. In Sect. 5, we obtain the multiplicity of solutions for problem (1.1). In Sect. 6, we complete the proof Theorem 1.2.

2. Preliminaries

In this section, we collect some notations and some useful preliminary lemmas. Recall that the Sobolev space with magnetic potential $H^1_A(\mathbb{R}^N, \mathbb{C})$ is defined by

$$H^1_A(\mathbb{R}^N, \mathbb{C}) = \{u \in L^2(\mathbb{R}^N, \mathbb{C}) : |\nabla_A u| \in L^2(\mathbb{R}^N, \mathbb{R})\},$$

where $\nabla_A u = (i^{-1}\nabla - A)u$. The space $H^1_A(\mathbb{R}^N, \mathbb{C})$ is an Hilbert space endowed with the scalar product

$$\langle u, v\rangle_\varepsilon = \Re\left(\int_{\mathbb{R}^N} \nabla_A u \overline{\nabla_A v} + u\overline{v}\mathrm{d}x\right),$$

where $\Re$ and the bar denote the real part of a complex number and the complex conjugation, respectively.

Using the change of variable $x \to \varepsilon x$, problem (1.1) is equivalent to the following one

$$\begin{cases} \left(\dfrac{1}{i}\nabla - A_\varepsilon(x)\right)^2 u + V_\varepsilon(x)u = \left(\displaystyle\int_{\mathbb{R}^N} \frac{|u(y)|^{2^*_\mu} + F(|u(y)|^2)}{|x-y|^\mu}\mathrm{d}y\right)\left(|u|^{2^*_\mu-2}u + \dfrac{1}{2^*_\mu}f(|u|^2)u\right) \text{ in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N, \mathbb{C}) \end{cases} \tag{2.1}$$

where $A_\varepsilon(x) = A(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$.

Owing to the presence of potential $V_\varepsilon(x)$, we introduce the subspace

$$H_\varepsilon = \{u \in H^1_A(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} V_\varepsilon(x)|u|^2\mathrm{d}x < \infty\}.$$

The space H_ε is an Hilbert space endowed with the scalar product

$$(u, v)_\varepsilon = \Re\left(\int_{\mathbb{R}^N} \nabla_\varepsilon u \overline{\nabla_\varepsilon v} + V(\varepsilon x)u\overline{v}\mathrm{d}x\right),$$

where $\nabla_\varepsilon u = (D^1_\varepsilon u, \ldots, D^N_\varepsilon u)$ and $D^j_\varepsilon u = i^{-1}\partial_j u - A_j(\varepsilon x)u$ for $j = 1, \ldots, N$. The norm induced by this inner product is given by

$$\|u\|_\varepsilon = \left(\int_{\mathbb{R}^N} |\nabla_\varepsilon u|^2 + V(\varepsilon x)|u|^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

For readers' convenience, we recall some useful lemmas.

Lemma 2.1. *(Diamagnetic Inequality)* [20] *For any $u \in H_\varepsilon$, we get $|u| \in H^1(\mathbb{R}^N, \mathbb{R})$ and it holds*

$$|\nabla |u|| \le |\nabla_\varepsilon u|. \tag{2.2}$$

Lemma 2.2. [20] *The space H_ε is continuously embedded into $L^r\left(\mathbb{R}^N, \mathbb{C}\right)$ for any $r \in [2, 2^*]$ and compactly embedded into $L^r_{loc}\left(\mathbb{R}^N, \mathbb{C}\right)$ for any $r \in [1, 2^*)$.*

From Gao and Yang [21], we denote by

$$\begin{aligned} S_{H,L} :&= \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \right)^{\frac{N-2}{2N-\mu}}} \\ &= \inf_{u \in D^{1,2}_A(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla_A u|^2 \mathrm{d}x}{\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)|^{2^*_\mu}|u(y)|^{2^*_\mu}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \right)^{\frac{N-2}{2N-\mu}}} := S_A, \end{aligned} \tag{2.3}$$

where $D^{1,2}_A(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N, \mathbb{C}) : \nabla_A u \in L^2(\mathbb{R}^N, \mathbb{C})\}$. The equality between $S_{H,L}$ and S_A was proved in Mukherjee and Sreenadh [34]. We remark that S_A is attained if and only if rot $A = 0$, see [34, Theorem 4.1].

Lemma 2.3. [21] *The constant $S_{H,L}$ defined in (2.3) is achieved if and only if*

$$u = C\left(\frac{b}{b^2 + |x-a|^2}\right)^{\frac{N-2}{2}},$$

where $C > 0$ is a fixed constant, $a \in \mathbb{R}^N$ and $b \in (0, \infty)$ are parameters. Furthermore,

$$S_{H,L} = \frac{S}{C(N,\mu)^{\frac{N-2}{2N-\mu}}},$$

where S is the best Sobolev constant of the immersion $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ and $C(N,\mu)$ depends on N and μ.

If we consider the minimizer for S given by $U(x) := \frac{[N(N-2)]^{\frac{N-2}{4}}}{(|1+|x|^2)^{\frac{N-2}{2}}}$ (see [42, Theorem 1.42]), then

$$\bar{U}(x) = S^{\frac{(N-\mu)(2-\mu)}{4(N+2-\mu)}} C(N,\mu)^{\frac{2-N}{2(N+2-\mu)}} \frac{[N(N-2)]^{\frac{N-2}{4}}}{(|1+|x|^2)^{\frac{N-2}{2}}}$$

is the unique minimizer for $S_{H,L}$ that satisfies

$$-\Delta u = \left(\int_{\mathbb{R}^N} \frac{|u|^{2^*_\mu}}{|x-y|^\mu} \mathrm{d}y \right) |u|^{2^*_\mu - 2} u \quad \text{in } \mathbb{R}^N,$$

with

$$\int_{\mathbb{R}^N} |\nabla \bar{U}|^2 \mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\bar{U}(x)|^{2_\mu^*}|\bar{U}(y)|^{2_\mu^*}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y = S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}.$$

Lemma 2.4. [29] *Let $N \geq 3$ and $r \in [2, 2^*)$. If $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\mathbb{R}^N, \mathbb{R})$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^r \, \mathrm{d}x = 0,$$

where $R > 0$, then $u_n \to 0$ in $L^t(\mathbb{R}^N, \mathbb{R})$ for all $t \in (2, 2^*)$.

3. The modified problem

We will adapt an argument explored by the penalization method, which introduced by del Pino and Felmer [19] to overcome the lack of compactness. More precisely, let $K > 0$ determined later, there exists a unique number $a > 0$ satisfying

$$f(a) + a^{(2_\mu^*-2)/2} = V_0/K$$

by (f_3), where V_0 is given in (V_2). Suppose that

$$\widetilde{f}(t) := \begin{cases} f(t) + (t^+)^{(2_\mu^*-2)/2}, & t \leq a, \\ V_0/K, & t > a, \end{cases}$$

and introduce the penalized nonlinearity

$$g(x,t) := \chi_\Lambda(x)(f(t) + (t^+)^{2_\mu^*}) + (1-\chi_\Lambda(x))\widetilde{f}(t),$$

where χ_Λ is the characteristic function on Λ and $G(x,t) := \int_0^t f(s)\mathrm{d}s$.

From assumptions (f_1)–(f_3), it follows that g satisfies the following properties:

(g_1) $g(x,t) = 0$ for each $t \leq 0$ and $\lim_{t\to 0^+} g(x,t) = 0$ uniformly in $x \in \mathbb{R}^N$;
(g_2) $g(x,t) \leq f(t) + (t^+)^{2_\mu^*}$ for all $t \geq 0$ uniformly in $x \in \mathbb{R}^N$;
(g_3) $0 < G(t) \leq g(x,t)t$, for each $x \in \Lambda, t > 0$;
(g_4) $0 < G(t) \leq g(x,t)t \leq V_0t/K$, for each $x \in \Lambda^c, t > 0$;
(g_5) $t \mapsto g(x,t)$ and $t \mapsto \frac{G(x,t)}{t}$ are increasing for all $x \in \mathbb{R}^N$ and $t \in (0,\infty)$.

Then, we consider the following modified problem

$$\left(\frac{1}{i}\nabla - A_\varepsilon(x)\right)^2 u + V_\varepsilon(x)u = \left(\frac{1}{|x|^\mu} * G(\varepsilon x, |u|^2)\right) g(\varepsilon x, |u|^2)u, \quad x \in \mathbb{R}^N. \tag{3.1}$$

Observe that if u is a nontrivial solution of problem (3.1) with

$$|u(x)|^2 \leq a, \ \forall x \in \Lambda_\varepsilon^c,$$

where $\Lambda_\varepsilon = \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}$, then u is a nontrivial solution of problem (2.1).

The energy function associated to problem (3.1) is

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_\varepsilon u|^2 + V_\varepsilon(x)|u|^2 \mathrm{d}x - \frac{1}{2\cdot 2_\mu^*}\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, |u|^2)\right) G(\varepsilon x, |u|^2)\mathrm{d}x.$$

In the sequel, set $p_0 = \frac{2N}{2N-\mu}$, then for any $u \in H_\varepsilon$, we have

$$|F(u)|_{p_0} \leq C\left(|u|_{2p_0}^2 + |u|_{qp_0}^q\right) \tag{3.2}$$

and

$$|G(u)|_{p_0} \leq C\left(|u|_{2p_0}^2 + |u|_{qp_0}^q + |u|_{2^*}^{2\cdot 2_\mu^*}\right). \tag{3.3}$$

Therefore, the Hardy–Littlewood–Sobolev inequality implies that

$$\int\limits_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, |u|^2)\right) G(\varepsilon x, |u|^2)\mathrm{d}x \leq C|G(u)|_{p_0}^2 \leq C\left(|u|_{2p_0}^4 + |u|_{qp_0}^{2q} + |u|_{2^*}^{2\cdot 2_\mu^*}\right) \tag{3.4}$$

and

$$\int\limits_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, |u|^2)\right) g(\varepsilon x, |u|^2)|u|^2\mathrm{d}x \leq C\left(|u|_{2p_0}^4 + |u|_{qp_0}^{2q} + |u|_{2^*}^{2\cdot 2_\mu^*}\right). \tag{3.5}$$

As the argument in the proof of [5, Theorem 1.3], we have the following lemma.

Lemma 3.1. *Suppose that* (V_1) *and* $(f_1) - (f_3)$ *are satisfied, then* J_ε' *is weakly sequentially continuous. Namely, if* $u_n \rightharpoonup u$ *in* H_ε*, then* $J_\varepsilon'(u_n) \rightharpoonup J_\varepsilon'(u)$ *in* $(H_\varepsilon)^*$.

Proof. Observe that

$$\langle J_\varepsilon'(u), v\rangle = \langle u, v\rangle_\varepsilon - \frac{1}{2_\mu^*}\int\limits_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G\left(\varepsilon x, |u|^2\right)\right) g\left(\varepsilon x, |u|^2\right) u\bar{v}\mathrm{d}x.$$

Since $f(t)$ has subcritical growth in the sense of the Hardy–Littlewood–Sobolev inequality and the definition of $g(x,t)$, to prove that J_ε' is weakly sequentially continuous, we only need to check that if $u_n \rightharpoonup u$ in H_ε, then

$$\iint_{\mathbb{R}^{2N}} \frac{|u_n(y)|^{2_\mu^*}|u_n(x)|^{2_\mu^*-2}u_n(x)\overline{v(x)}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \longrightarrow \iint_{\mathbb{R}^{2N}} \frac{|u(y)|^{2_\mu^*}|u(x)|^{2_\mu^*-2}u(x)\overline{v(x)}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \tag{3.6}$$

for any $v \in H_\varepsilon$ as $n \to \infty$. Indeed, by the Hardy–Littlewood–Sobolev inequality, the Riesz potential defines a linear continuous map from $L^{\frac{2N}{2N-\mu}}\left(\mathbb{R}^N\right)$ to $L^{\frac{2N}{\mu}}\left(\mathbb{R}^N\right)$, so we know

$$\int\limits_{\mathbb{R}^N} \frac{|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}y \rightharpoonup \int\limits_{\mathbb{R}^N} \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}y \quad \text{in} \quad L^{\frac{2N}{\mu}}\left(\mathbb{R}^N\right).$$

Since $|u_n|^{2_\mu^*-2}u_n \rightharpoonup |u|^{2_\mu^*-2}u$ in $L^{\frac{2N}{N-\mu+2}}\left(\mathbb{R}^N\right)$ as $n \to \infty$, we have

$$\left(\int\limits_{\mathbb{R}^N} \frac{|u_n(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}y\right)|u_n(x)|^{2_\mu^*-2}u_n(x) \rightharpoonup \left(\int\limits_{\mathbb{R}^N} \frac{|u(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}y\right)|u(x)|^{2_\mu^*-2}u(x) \quad \text{in} \quad L^{\frac{2N}{N+2}}\left(\mathbb{R}^N\right)$$

as $n \to \infty$. Hence (3.6) holds. □

By a similar argument to Proposition 3.2 in [33], we have the following results.

Lemma 3.2. *Let* $N \geq 3$ *and* $\mu \in (0, N)$*. If* $H, K \in L^{\frac{2N}{N-\mu+2}}(\mathbb{R}^N) + L^{\frac{2N}{N-\mu}}(\mathbb{R}^N)$*, and* $u \in H_\varepsilon$ *satisfies*

$$\left(\frac{1}{i}\nabla - A_\varepsilon(x)\right)^2 u + u = \left(|x|^{-\mu} * (Hu)\right)K.$$

Then, $u \in L^r(\mathbb{R}^N)$ *for some* $r \in \left[2, \frac{N}{N-\mu}\frac{2N}{N-2}\right)$*. Moreover, there exists a positive constant* $C(r)$ *independent of* u *such that*

$$\|u\|_r \leq C(r)\|u\|_2.$$

Now, let us define

$$\mathcal{B} = \left\{u \in H_\varepsilon\left(\mathbb{R}^N\right) : \|u\|_\varepsilon \leq d\right\}$$

where $d > 0$ is a constant, and we set

$$K_\varepsilon(u)(x) = \frac{1}{|x|^\mu} * G\left(\varepsilon x, |u|^2\right).$$

The next lemma implies that we can treat the convolution term as a bounded term.

Lemma 3.3. *Suppose that $(f_1) - (f_3)$ hold. Then, there exists $C_0 > 0$ such that*

$$\sup_{u\in\mathcal{B}} |K_\varepsilon(u)(x)|_{L^\infty(\mathbb{R}^N)} < C_0. \tag{3.7}$$

Proof. Observe that

$$|G(x, |s|^2)| \leq |F(|s|^2)| + |s|^{2^*_\mu}, \quad \forall s \in \mathbb{R}.$$

Thereby,

$$\begin{aligned} |K_\varepsilon(u)(x)| &\leq \left|\int_{\mathbb{R}^N} \frac{F(|u|^2)}{|x-y|^\mu} \mathrm{d}y\right| + \left|\int_{\mathbb{R}^N} \frac{|u|^{\frac{2N-\mu}{N-2}}}{|x-y|^\mu} \mathrm{d}y\right| \\ &\leq \left|\int_{\mathbb{R}^N} \frac{F(|u|^2)}{|x-y|^\mu} \mathrm{d}y\right| + \left|\int_{|x-y|\leq 1} \frac{|u|^{\frac{2N-\mu}{N-2}}}{|x-y|^\mu} \mathrm{d}y\right| + \left|\int_{|x-y|>1} \frac{|u|^{\frac{2N-\mu}{N-2}}}{|x-y|^\mu} \mathrm{d}y\right| \\ &:= K_1 + K_2 + K_3. \end{aligned}$$

From [4, Lemma 2.5], we know there exist a constant C independent of x and ε such that

$$|K_1| \leq C. \tag{3.8}$$

Let $H(u) = K(u) = |u|^{\frac{N-\mu+2}{N-2}}$, then it is easy to verify that

$$H(u), K(u) \in L^{\frac{2N}{N-\mu+2}}\left(\mathbb{R}^N\right) + L^{\frac{2N}{N-\mu}}(\mathbb{R}^N).$$

Using Lemma 3.2, we obtain $u \in L^r(\mathbb{R}^N)$ for $r \in \left[2, \frac{N}{N-\mu}\frac{2N}{N-2}\right)$. Moreover,

$$|u|_r \leq C(r)|u|_2 \leq C$$

for some constant $C > 0$ independent of ε.
It follows from $\mu < N$ that

$$\frac{2N^2}{3N\mu - \mu^2} < \frac{N}{\mu}.$$

Choosing $t > \frac{N}{\mu}$ and $\frac{2N^2}{3N\mu-\mu^2} < s < \frac{N}{\mu}$, then

$$\frac{t}{t-1}\frac{2N-\mu}{N-2} < \frac{N}{N-\mu}\frac{2N}{N-2}, \quad \frac{s}{s-1}\frac{2N-\mu}{N-2} < \frac{N}{N-\mu}\frac{2N}{N-2}.$$

By the Hölder inequality, we have

$$K_2 \leq \left(\int_{|x-y|\leq 1} \frac{1}{|x-y|^{\mu s}} \mathrm{d}y\right)^{\frac{1}{s}} \left(\int_{|x-y|\leq 1} |u|^{\frac{s}{s-1}\frac{2N-\mu}{N-2}} \mathrm{d}y\right)^{\frac{s-1}{s}} \leq C,$$

and

$$K_3 \le \left(\int\limits_{|x-y|\ge 1} \frac{1}{|x-y|^{\mu t}} \mathrm{d}y \right)^{\frac{1}{t}} \left(\int\limits_{|x-y|\ge 1} |u|^{\frac{t}{t-1}\frac{2N-\mu}{N-2}} \mathrm{d}y \right)^{\frac{t-1}{t}} \le C,$$

where the constant C is independent of x and ε.

The above equalities with (3.8) imply (3.7). Then, we can find $C_0 > 0$ such that

$$\sup_{u\in\mathcal{B}} |K_\varepsilon(u)(x)|_{L^\infty(\mathbb{R}^N)} < C_0.$$

□

From Lemma 3.3, we know there exists k_0 such that

$$\frac{\sup_{u\in\mathcal{B}} |K_\varepsilon(u)(x)|_{L^\infty(\mathbb{R}^N)}}{k_0} < \frac{1}{2},$$

hence we can choose $K \ge k_0$.

We denote by $\mathcal{N}_\varepsilon$ the Nehari manifold of J_ε, that is,

$$\mathcal{N}_\varepsilon = \{u \in H_\varepsilon \setminus \{0\} : \langle J'_\varepsilon(u), u\rangle = 0\},$$

and define the number c_ε by

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u).$$

Let H_ε^+ be the open subset of H_ε given by

$$H_\varepsilon^+ = \{u \in H_\varepsilon : |\operatorname{supp}(u) \cap \Lambda_\varepsilon| > 0\},$$

and $S_\varepsilon^+ = S_\varepsilon \cap H_\varepsilon^+$, where S_ε is the unit sphere of H_ε. Observe that S_ε^+ is a non-complete $C^{1,1}$-manifold of codimension 1, modeled on H_ε and contained in H_ε^+. Hence, $H_\varepsilon = T_u S_\varepsilon^+ \oplus \mathbb{R}u$ for each $u \in S_\varepsilon^+$, where $T_u S_\varepsilon^+ = \{v \in H_\varepsilon : \langle u, v\rangle_\varepsilon = 0\}$.

The functional J_ε satisfies the mountain pass geometry [42].

Lemma 3.4. *For any fixed $\varepsilon > 0$, the functional J_ε satisfies the following properties:*
(a) there exist $\alpha, \rho > 0$ such that $J_\varepsilon(u) \ge \alpha$ with $\|u\|_\varepsilon = \rho$;
(b) there exists $e \in H_\varepsilon$ such that $\|e\|_\varepsilon \ge \rho$ and $J_\varepsilon(e) < 0$.

Proof. (a) For any $u \in H_\varepsilon \setminus \{0\}$, it follows from (3.4) and Lemma 2.2 that

$$J_\varepsilon(u) = \frac{1}{2}\|u\|_\varepsilon^2 - \frac{1}{2\cdot 2_\mu^*} \int\limits_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, |u|^2)\right) G(\varepsilon x, |u|^2)\mathrm{d}x$$
$$\ge \frac{1}{2}\|u\|_\varepsilon^2 - C\left(\|u\|_\varepsilon^4 - \|u\|_\varepsilon^{2q} - \|u\|_\varepsilon^{2\cdot 2_\mu^*}\right).$$

Hence, we can find $\alpha, \rho > 0$ such that $J_\varepsilon \ge \alpha$ with $\|u\|_\varepsilon = \rho$.

(b) Fix a positive function $u_0 \in H_\varepsilon^+$ with $\operatorname{supp}(u_0) \subset \Lambda_\varepsilon$, and observe

$$G(\varepsilon x, |u_0|^2) = F(|u_0|^2).$$

Set

$$\alpha(t) := P\left(\frac{tu_0}{\|u_0\|_\varepsilon}\right) > 0 \ \text{ for } \ t > 0,$$

where

$$P(u) = \frac{1}{2\cdot 2_\mu^*} \int\limits_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, |u|^2)\right) G(\varepsilon x, |u|^2)\mathrm{d}x.$$

By assumption (f_3), we have

$$F(t) \leq f(t)t, \quad \text{for } t > 0.$$

Therefore, we deduce that

$$\alpha'(t) \geq \frac{2 \cdot 2^*_\mu}{t} \alpha(t). \tag{3.9}$$

Integrating (3.9) on $[1, t\|u_0\|_\varepsilon]$ with $t > \frac{1}{\|u_0\|_\varepsilon}$, we have

$$P(tu_0) \geq P\left(\frac{u_0}{\|u_0\|_\varepsilon}\right) \|u_0\|_\varepsilon^{2\cdot 2^*_\mu} t^{2\cdot 2^*_\mu}.$$

Hence, we have

$$J_\varepsilon(tu_0) = \frac{t^2}{2}\|u_0\|^2_\varepsilon - P(tu_0) \leq C_1 t^2 - C_2 t^{2\cdot 2^*_\mu} \quad \text{for } t > \frac{1}{\|u_0\|_\varepsilon}.$$

Taking $e = tu_0$ with t sufficiently large, we can see (b) holds. □

Since f is only continuous, we can apply the next two conclusions to overcome the non-differentiability of $\mathcal{N}_\varepsilon$ and the incompleteness of S^+_ε.

Lemma 3.5. *Suppose that the function V satisfies (V_1)–(V_2) and f satisfies (f_1)–(f_3), then the following properties hold:*

(A_1) *For each $u \in H^+_\varepsilon$, let $\varphi_u : \mathbb{R}^+ \to \mathbb{R}$ be given by $\varphi_u(t) = J_\varepsilon(tu)$. Then, there exists a unique $t_u > 0$ such that $\varphi'_u(t) > 0$ in $(0, t_u)$ and $\varphi'_u(t) < 0$ in (t_u, ∞).*

(A_2) *There is a $\sigma > 0$ independent on u such that $t_u > \sigma$ for all $u \in S^+_\varepsilon$. Furthermore, for each compact set $\mathcal{W} \subset S^+_\varepsilon$, there is $C_{\mathcal{W}} > 0$ such that $t_u \leq C_{\mathcal{W}}$ for all $u \in \mathcal{W}$.*

(A_3) *The map $\widehat{m}_\varepsilon : H^+_\varepsilon \to \mathcal{N}_\varepsilon$ given by $\widehat{m}_\varepsilon(u) = t_u u$ is continuous and $m_\varepsilon = \widehat{m}_\varepsilon|_{S^+_\varepsilon}$ is a homeomorphism between S^+_ε and $\mathcal{N}_\varepsilon$. Moreover, $m^{-1}_\varepsilon = \frac{u}{\|u\|_\varepsilon}$.*

(A_4) *If there is a sequence $\{u_n\} \subset S^+_\varepsilon$ such that $\operatorname{dist}(u_n, \partial S^+_\varepsilon) \to 0$, then $\|m_\varepsilon(u_n)\|_\varepsilon \to \infty$ and $J_\varepsilon(m_\varepsilon(u_n)) \to \infty$.*

Proof. (A_1) We notice from the proof of Lemma 3.4 that $\varphi_u(0) = 0$, $\varphi_u(t) > 0$ for $t > 0$ small and $\varphi_u(t) < 0$ for $t > 0$ large. Therefore, $\max_{t\geq 0} \varphi_u(tu)$ is achieved at a global maximum point $t = t_u$ verifying $\varphi'_u(t_u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$. From (f_3), the definition of φ and $|\operatorname{supp}(u) \cap \Lambda_\varepsilon| > 0$, we may obtain the uniqueness of t_u. Therefore, $\max_{t\geq 0} \varphi_u(tu)$ is achieved at a unique $t = t_u$ such that $\varphi'_u(t_u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$.

(A_2) Suppose $u \in S^+_\varepsilon$, then from (3.5) we have

$$t^2_u = \frac{1}{2^*_\mu} \int\limits_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, |u|^2)\right) g(\varepsilon x, |t_u u|^2)|t_u u|^2 \mathrm{d}x \leq C\left(t^4_u + t^{2q}_u + t_u^{2\cdot 2^*_\mu}\right).$$

which implies that $t_u > \sigma$ for some $\sigma > 0$.

If $\mathcal{W} \subset S^+_\varepsilon$ is compact, and suppose by contradiction that there is $\{u_n\} \subset \mathcal{W}$ such that $t_n := t_{u_n} \to \infty$. Since $\mathcal{W}$ is compact, there is $u \in \mathcal{W}$ such that $u_n \to u$ in H_ε. Then, $u \in \mathcal{W} \subset S^+_\varepsilon$. Moreover, using the proof of Lemma 3.4 (b), we have $J_\varepsilon(t_n u_n) \to -\infty$.

On the other hand, using the proof of lemma 3.4 (a), there exists $\rho > 0$ such that $\inf_{\|u\|_\varepsilon = \rho} J_\varepsilon(u) > 0$. Then combining this with (A_1), we have $c_\varepsilon = \inf_{\mathcal{N}_\varepsilon} J_\varepsilon(u) \geq \inf_{\|u\|_\varepsilon=\rho} J_\varepsilon(u) > 0$, which yields a contradiction. Hence (A_2) is true.

(A_3) Firstly, we observe that m_ε, $\widehat{m}_\varepsilon$ and m^{-1}_ε are well defined. In fact, by (A_1), for each $u \in H^+_\varepsilon$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}_\varepsilon$, hence there is a unique $\widehat{m}_\varepsilon(u) = t_u u \in \mathcal{N}_\varepsilon$. On the other hand, if $u \in \mathcal{N}_\varepsilon$, then $u \in H_\varepsilon^+$. Therefore, $m_\varepsilon^{-1} = \frac{u}{\|u\|_\varepsilon} \in S_\varepsilon^+$, is well defined and it is a continuous function. Since

$$m_\varepsilon^{-1}(m_\varepsilon(u)) = m_\varepsilon^{-1}(t_u u) = \frac{t_u u}{\|t_u u\|_\varepsilon} = u, \quad \forall u \in S_\varepsilon^+,$$

we conclude that m_ε is a bijection. To prove that $\widehat{m}_\varepsilon : H_\varepsilon^+ \to \mathcal{N}_\varepsilon$ is continuous, let $\{u_n\} \subset H_\varepsilon^+$ and $u \in H_\varepsilon^+$ satisfy $u_n \to u$ in H_ε^+. By (A_2), there is a $t_0 > 0$ such that $t_n := t_{u_n} \to t_0$. From $t_n u_n \in \mathcal{N}_\varepsilon$, we obtain

$$t_n^2 \|u_n\|_\varepsilon^2 = \frac{1}{2_\mu^*} \int\limits_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, |t_n u_n|^2) \right) g(\varepsilon x, |t_n u_n|^2) |t_u u_n|^2 \mathrm{d}x.$$

By Lemma 3.1 and passing to the limit as $n \to \infty$, it follows that

$$t_0^2 \|u\|_\varepsilon^2 = \frac{1}{2_\mu^*} \int\limits_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, |t_0 u|^2) \right) g(\varepsilon x, |t_0 u|^2) |t_0 u|^2 \mathrm{d}x$$

which means that $t_0 u \in \mathcal{N}_\varepsilon$ and $t_u = t_0$. This proves $\widehat{m}_\varepsilon(u_n) \to \widehat{m}_\varepsilon(u)$ in H_ε^+. So, $\widehat{m}_\varepsilon$, m_ε are continuous functions and (A_3) is proved.

(A_4) Let $\{u_n\} \subset S_\varepsilon^+$ be a subsequence such that $\mathrm{dist}(u_n, \partial S_\varepsilon^+) \to 0$, then for each $v \in \partial S_\varepsilon^+$and $n \in \mathbb{N}$, we have $|u_n| = |u_n - v|$ a.e. in Λ_ε. Therefore, by $(V_1), (V_2)$ and the Sobolev embedding, for any $t \in [2, 2^*]$, there exists a constant $C_t > 0$ such that

$$\begin{aligned} \|u_n\|_{L^t(\Lambda_\varepsilon)} &\le \inf_{v \in \partial S_\varepsilon^+} \|u_n - v\|_{L^t(\Lambda_\varepsilon)} \\ &\le C_t \left(\inf_{v \in \partial S_\varepsilon^+} \int\limits_{\Lambda_\varepsilon} \left(|\nabla_\varepsilon (u_n - v)|^2 + V_\varepsilon(x) |u_n - v|^2 \right) \mathrm{d}x \right)^{\frac{1}{2}} \\ &\le C_t \,\mathrm{dist}\left(u_n, \partial S_\varepsilon^+\right) \end{aligned}$$

for all $n \in \mathbb{N}$. By (g_2) and (g_4), for each $t > 0$, we have

$$\begin{aligned} &\int\limits_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, |t u_n|^2) \right) G(\varepsilon x, |t u_n|^2) \mathrm{d}x \\ &= \int\limits_{\Lambda_\varepsilon} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, |t u_n|^2) \right) G(\varepsilon x, |t u_n|^2) \mathrm{d}x + \int\limits_{\Lambda_\varepsilon^c} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, |t u_n|^2) \right) G(\varepsilon x, |t u_n|^2) \mathrm{d}x \\ &\le C \left(t^4 |u_n|_{L^{2p_0}(\Lambda_\varepsilon)}^4 + t^{2q} |u_n|_{L^{qp_0}(\Lambda_\varepsilon)}^{2q} + t^{2\cdot 2_\mu^*} |u_n|_{L^{2^*}(\Lambda_\varepsilon)}^{2\cdot 2_\mu^*} \right) + \frac{k_0 t^2}{2K} \int\limits_{\Lambda_\varepsilon^c} V(\varepsilon x) |u_n|^2 \,\mathrm{d}x \\ &\le C \left(t^4 |u_n|_{L^{2p_0}(\Lambda_\varepsilon)}^4 + t^{qp_0} |u_n|_{L^{2q}(\Lambda_\varepsilon)}^{2q} + t^{2\cdot 2_\mu^*} |u_n|_{L^{2^*}(\Lambda_\varepsilon)}^{2\cdot 2_\mu^*} \right) + \frac{k_0 t^2}{2K} \|u_n\|_\varepsilon^2 \\ &\le C \left(t^4 \,\mathrm{dist}(u_n, \partial S_\varepsilon^+)^4 + t^{2q} \,\mathrm{dist}(u_n, \partial S_\varepsilon^+)^{2q} + t^{2\cdot 2_\mu^*} \,\mathrm{dist}(u_n, \partial S_\varepsilon^+)^{2\cdot 2_\mu^*} \right) + \frac{k_0 t^2}{2K}. \end{aligned}$$

Therefore,

$$\limsup_n \int\limits_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, |t u_n|^2) \right) G(\varepsilon x, |t u_n|^2) \mathrm{d}x \le \frac{k_0 t^2}{2K}, \quad \forall t > 0.$$

On the other hand, from the definition of m_ε and the last inequality, for all $t>0$, we obtain

$$\begin{aligned}\liminf_n J_\varepsilon(m_\varepsilon(u_n)) &\geq \liminf_n J_\varepsilon(tu_n)\\ &\geq \liminf_n \frac{t^2}{2}\|u_n\|_\varepsilon^2 - \frac{k_0t^2}{2K}\\ &= \frac{K-k_0}{2K}t^2.\end{aligned}$$

This implies that

$$\liminf_n \frac{1}{2}\|m_\varepsilon(u_n)\|_\varepsilon^2 \geq \liminf_n J_\varepsilon(m_\varepsilon(u_n)) \geq \frac{K-k_0}{2K}t^2, \quad \forall t>0.$$

From the arbitrariness of $t>0$, we see that

$$\|m_\varepsilon(u_n)\|_\varepsilon \to \infty \quad \text{and} \quad J_\varepsilon(m_\varepsilon(u_n)) \to \infty \quad \text{as } n\to\infty.$$

This completes the proof of Lemma 3.5. □

Now we define the function

$$\widehat{\Psi}_\varepsilon : H_\varepsilon^+ \to \mathbb{R}$$

by $\widehat{\Psi}_\varepsilon(u) = J_\varepsilon(\widehat{m}_\varepsilon(u))$ and denote by $\Psi_\varepsilon(u) = (\widehat{\Psi}_\varepsilon(u))|_{S_\varepsilon^+}$.

From Lemma 3.2, by a similar argument to that of [39, Corollary 10], we have the following results.

Lemma 3.6. *Assume that* (V_1)–(V_2) *and* (f_1)–(f_3) *are satisfied, then*

(B_1) $\widehat{\Psi}_\varepsilon \in C^1(H_\varepsilon^+,\mathbb{R})$ *and*

$$\langle \widehat{\Psi}_\varepsilon'(u), v\rangle = \frac{\|\widehat{m}_\varepsilon(u)\|_\varepsilon}{\|u\|_\varepsilon}\langle J_\varepsilon'(\widehat{m}_\varepsilon(u)), v\rangle, \quad \forall u\in H_\varepsilon^+ \text{ and } \forall v \in H_\varepsilon.$$

(B_2) $\Psi_\varepsilon \in C^1(S_\varepsilon^+,\mathbb{R})$ *and*

$$\langle \Psi_\varepsilon'(u), v\rangle = \|m_\varepsilon(u)\|_\varepsilon\langle J_\varepsilon'(\widehat{m}_\varepsilon(u)), v\rangle, \quad \forall v\in T_uS_\varepsilon^+.$$

(B_3) *If* $\{u_n\}$ *is a* $(PS)_c$ *sequence of* Ψ_ε, *then* $\{m_\varepsilon(u_n)\}$ *is a* $(PS)_c$ *sequence of* J_ε. *If* $\{u_n\}\subset\mathcal{N}_\varepsilon$ *is a bounded* $(PS)_c$ *sequence of* J_ε, *then* $\{m_\varepsilon^{-1}(u_n)\}$ *is a* $(PS)_c$ *sequence of* Ψ_ε.

(B_4) *u is a critical point of* Ψ_ε *if and only if* $m_\varepsilon(u)$ *is a critical point of* J_ε. *Moreover, the corresponding critical values coincide and*

$$\inf_{u\in S_\varepsilon^+}\Psi_\varepsilon(u) = \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u).$$

As in [39], we have the following variational characterization of the infimum of J_ε over $\mathcal{N}_\varepsilon$:

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u) = \inf_{u\in H_\varepsilon^+}\sup_{t>0} J_\varepsilon(tu) = \inf_{u\in S_\varepsilon^+}\sup_{t>0} J_\varepsilon(tu). \tag{3.10}$$

Lemma 3.7. *Suppose that* (f_1)–(f_3) *hold. Assume that* $\{u_n\}\subset\mathcal{N}_\varepsilon$ *is a* $(PS)_c$ *sequence with*

$$0 < c_\varepsilon \leq c < \frac{N+2-\mu}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}},$$

then $\{u_n\}$ *is bounded in* H_ε. *Moreover,* $\{u_n\}$ *cannot be vanishing, namely there exist* $r,\delta>0$ *and a sequence* $\{y_n\}\subset\mathbb{R}^N$ *such that*

$$\liminf_{n\to\infty}\int_{B_r(y_n)} |u_n|^2 \mathrm{d}x \geq \delta.$$

Proof. Firstly, we prove the boundedness of $\{u_n\}$. Arguing by contradiction, we suppose that $\{u_n\}$ is unbounded in H_ε. Without loss of generality, we assume that $\|u_n\|_\varepsilon \to \infty$. Let $v_n := \frac{u_n}{\|u_n\|_\varepsilon}$, up to a subsequence, then there exists $v \in H_\varepsilon$ such that

$$\begin{aligned}&v_n \rightharpoonup v \text{ weakly in } H_\varepsilon,\\&v_n \to v \text{ strongly in } L^r_{\text{loc}}(\mathbb{R}^N), 2 \le r < 2^*,\\&v_n(x) \to v(x) \text{ a.e. in } \mathbb{R}^N.\end{aligned}$$

If v_n is vanishing, i.e.,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int\limits_{B_r(y)} |v_n|^2(x)\mathrm{d}x = 0,$$

then Lemma 2.4 implies that $v_n \to 0$ in $L^t(\mathbb{R}^N, \mathbb{R})$ for all $t \in (2, 2^*)$. By (3.2) and (3.3), we have

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(|tv_n(x)|^2)F(|tv_n(y)|^2)}{|x-y|^\mu}\mathrm{d}y\mathrm{d}x \to 0, \quad \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(|tv_n(y)|^2)}{|x-y|^\mu}|tv_n(x)|^{2^*_\mu}\mathrm{d}y\mathrm{d}x \to 0. \tag{3.11}$$

Hence for sufficiently large n, we have

$$\begin{aligned}c_\varepsilon + o_n(1) = I_\varepsilon(u_n) &\ge \sup_{t\ge 0} I_\varepsilon(tv_n)\\&\ge \sup_{t\ge0}\left(\frac{t^2}{2} - \frac{t^{2\cdot 2^*_\mu}}{2\cdot 2^*_\mu} S_{H,L}^{-2^*_\mu}\right) + o_n(1)\\&= \frac{N+2-\mu}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}} + o_n(1),\end{aligned}$$

which is a contradiction. Therefore, $\{v_n\}$ is nonvanishing, namely there exists $y_n \in \mathbb{R}^N$ and $\delta > 0$ such that

$$\int\limits_{B_r(y_n)} |v_n|^2(x)\mathrm{d}x > \delta. \tag{3.12}$$

Denote $\tilde{v}_n(\cdot) = v_n(\cdot + y_n)$, then we can suppose that

$$\begin{aligned}&\tilde{v}_n \rightharpoonup \tilde{v} \text{ weakly in } H_\varepsilon,\\&\tilde{v}_n \to \tilde{v} \text{ strongly in } L^r_{\text{loc}}\left(\mathbb{R}^N\right), 2 \le r < 2^*,\\&\tilde{v}_n(x) \to \tilde{v}(x) \text{ a.e. in } \mathbb{R}^N.\end{aligned}$$

By (3.12), we have $\tilde{v} \ne 0$. Hence, there exists a measure set E such that $\tilde{v}(x) \ne 0$ for $x \in E$. Let $|\hat{u}_n| := |\tilde{v}_n|\|u_n\|_\varepsilon$. Then, $|\hat{u}_n(x)| \to +\infty$ for $x \in E$. By (f_3), we have

$$\int\limits_E\int\limits_E \frac{1}{|x-y|^\mu} \frac{F(|\hat{u}_n(y)|^2)}{|\hat{u}(y)|^2}|\tilde{v}_n(y)|^2 \frac{F(|\hat{u}_n(x)|^2)}{|\hat{u}(x)|^2}|\tilde{v_n}(x)|^2\mathrm{d}y\mathrm{d}x \to +\infty.$$

Hence, we know

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{1}{|x-y|^\mu} \frac{F(|\hat{u}_n(y)|^2)}{|\hat{u}_n(y)|^2}|\tilde{v}_n(y)|^2 \frac{F(|\hat{u}_n(x)|^2)}{|\hat{u}_n(x)|^2}|\tilde{v}_n(x)|^2\mathrm{d}y\mathrm{d}x = +\infty,$$

namely

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{1}{|x-y|^\mu} \frac{F(|u_n(y)|^2)}{\|u_n\|^2_\varepsilon} \frac{F(|u_n(x)|^2)}{\|u_n\|^2_\varepsilon}\mathrm{d}y\mathrm{d}x = +\infty.$$

Then, we have

$$\frac{c_{\varepsilon}}{\|u_n\|_{\varepsilon}^4}+o_n(1)=\frac{I_{\varepsilon}\left(u_n\right)}{\|u_n\|_{\varepsilon}^4}\to-\infty,$$

which is a contradiction. Therefore, $\{u_n\}$ is bounded in H_{ε}.

Next we prove the second conclusion. By contradiction, if $\{u_n\}$ is vanishing, then similar to (3.12), we have

$$c_{\varepsilon}+o_n(1)=J_{\varepsilon}\left(u_n\right)=\frac{1}{2}\left\|u_n\right\|_{\varepsilon}^2-\frac{1}{2\cdot 2_{\mu}^*}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{\left|u_n(y)\right|^{2_{\mu}^*}\left|u_n(x)\right|^{2_{\mu}^*}}{|x-y|^{\mu}}\mathrm{d}y\mathrm{d}x+o_n(1) \tag{3.13}$$

and

$$0=\left\|u_n\right\|_{\varepsilon}^2-\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{\left|u_n(y)\right|^{2_{\mu}^*}\left|u_n(x)\right|^{2_{\mu}^*}}{|x-y|^{\mu}}\mathrm{d}y\mathrm{d}x+o_n(1). \tag{3.14}$$

If $\left\|u_n\right\|_{\varepsilon}\to 0$, then it follows from (3.13) and (3.14) that $c_{\varepsilon}=0$, which is impossible. Then, $\left\|u_n\right\|_{\varepsilon}\nrightarrow 0$ and by virtue of (3.14) we get

$$\left\|u_n\right\|_{\varepsilon}^2\leq S_{H,L}^{2_{\mu}^*}\left\|u_n\right\|_{\varepsilon}^{2\cdot 2_{\mu}^*}+o_n(1).$$

Hence,

$$\liminf_{n\to\infty}\left\|u_n\right\|_{\varepsilon}^2\geq S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}.$$

From (3.13) and (3.14), we deduce that

$$c_{\varepsilon}+o_n(1)=I_{\varepsilon}\left(u_n\right)\geq\frac{N+2-\mu}{2(2N-\mu)}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}},$$

which yields a contradiction. Therefore, $\{u_n\}$ is nonvanishing. □

Lemma 3.8. *Assume* (V_1)–(V_2) *and* (f_1)–(f_3) *hold, the functional* J_{ε} *satisfies the* $(PS)_c$ *condiction for all* $c\in\left[c_{\varepsilon},\frac{N+2-\mu}{2(2N-\mu)}S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}\right)$.

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence for J_{ε}. By Lemma 3.7, $\{u_n\}$ is bounded in H_{ε}. Therefore, up to a subsequence, $u_n\rightharpoonup u$ in H_{ε} and $u_n\to u$ in $L_{loc}^r(\mathbb{R}^N)$ for any $r\in[2,2^*)$.

Let $R>0$ be such that $\Lambda_{\varepsilon}\subset B_R(0)$. We show that for each $\xi>0$ and R large enough, it holds that

$$\limsup_{n\to\infty}\int\limits_{B_R^c}|\nabla_{\varepsilon}u_n|^2+V_{\varepsilon}|u_n|^2\mathrm{d}x<\xi. \tag{3.15}$$

Let us consider a cut-off function $\eta_R\in C^{\infty}(\mathbb{R}^N,\mathbb{R})$ defined as

$$\eta_R(x)=\begin{cases}0 & \text{if } x\in B_R,\\ 1 & \text{if } x\notin B_{2R},\end{cases}$$

and $|\nabla\eta_R|_{\infty}\leq C/R$, where $C>0$ is a constant independent of R. Since $\{u_n\eta_R\}$ is bounded in H_{ε}, it follows that

$$\langle J_{\varepsilon}'(u_n),u_n\eta_R\rangle=o_n(1),$$

therefore,

$$\Re\left(\int_{\mathbb{R}^N} |\nabla_\varepsilon u_n \overline{\nabla_\varepsilon (u_n\eta_R)}\mathrm{d}x\right) + \int_{\mathbb{R}^N} V_\varepsilon |u_n|^2\eta_R \mathrm{d}x$$
$$= \frac{1}{2_\mu^*}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * G(\varepsilon x, |u_n|^2)\right) g(\varepsilon x, |u_n|^2)|u_n|^2\eta_R \mathrm{d}x + o_n(1).$$

Since $\overline{\nabla_\varepsilon (u_n\eta_R)} = i\bar{u}_n\nabla\eta_R + \eta_R\overline{\nabla_\varepsilon u_n}$, using (g_4), we obtain

$$\int_{\mathbb{R}^N}\left(|\nabla_\varepsilon u_n|^2 + V_\varepsilon |u_n|^2\right)\eta_R \mathrm{d}x$$
$$=\frac{1}{2_\mu^*}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * G(\varepsilon x, |u_n|^2)\right) g(\varepsilon x, |u_n|^2)|u_n|^2\eta_R \mathrm{d}x - \Re\left(\int_{\mathbb{R}^N} i\bar{u}_n\nabla_\varepsilon u_n\nabla\eta_R \mathrm{d}x\right) + o_n(1)$$
$$\le\frac{k_0}{2\cdot 2_\mu^* K}\int_{\mathbb{R}^N} V_\varepsilon(x)|u(x)|^2\mathrm{d}x\mathrm{d}y - \Re\left(\int_{\mathbb{R}^N} i\bar{u}_n\nabla_\varepsilon u_n\nabla\eta_R \mathrm{d}x\right) + o_n(1).$$

By the Hölder inequality and the boundedness of $\{u_n\}$ in H_ε, we deduce that

$$\left(1 - \frac{k_0}{2\cdot 2_\mu^* K}\right)\left(\int_{\mathbb{R}^N}(|\nabla_\varepsilon u_n|^2 + V_\varepsilon|u_n|^2)\mathrm{d}x\right) \le \frac{C}{R}|\bar{u}_n|_2|\nabla_\varepsilon u_n|_2 + o_n(1)$$
$$\le \frac{C_1}{R} + o_n(1),$$

and so (3.15) holds.

Next we prove that $u_n \to u$ in H_ε as $n\to\infty$. Setting $\omega_n = \|u_n - u\|_\varepsilon^2$, we have

$$\omega_n = \langle J_\varepsilon'(u_n), u_n\rangle - \langle J_\varepsilon'(u_n), u\rangle + \frac{1}{2_\mu^*}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * G(\varepsilon x, |u_n|^2)\right) g(\varepsilon x, |u_n|^2)(|u_n|^2 - |u|^2)\mathrm{d}x + o_n(1).$$

Observe that $\langle J_\varepsilon'(u_n), u_n\rangle = \langle J_\varepsilon'(u_n), u\rangle = o_n(1)$, so we only need to prove that

$$\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * G(\varepsilon x, |u_n|^2)\right) g(\varepsilon x, |u_n|^2)(|u_n|^2 - |u|^2)\mathrm{d}x = o_n(1) \text{ as } n\to\infty.$$

By the [21, lemma 2.4], we have

$$\lim_{n\to\infty}\int_{B_R}\left(\frac{1}{|x|^\mu} * G(\varepsilon x, |u_n|^2)\right) g(\varepsilon x, |u_n|^2)(|u_n|^2 - |u|^2)\mathrm{d}x = 0.$$

Now, by (g_4) and (3.15) we obtain

$$\int_{B_R^c}\left(\frac{1}{|x|^\mu} * G(\varepsilon x, |u_n|^2)\right) g(\varepsilon x, |u_n|^2)(|u_n|^2 - |u|^2)\mathrm{d}x \le \frac{k_0}{K}\int_{B_R^c}|\nabla_\varepsilon u_n|^2 + V_\varepsilon|u_n|^2\mathrm{d}x < \frac{k_0}{K}\xi.$$

Therefore, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * G(\varepsilon x, |u_n|^2)\right) g(\varepsilon x, |u_n|^2)(|u_n|^2 - |u|^2)\mathrm{d}x = 0.$$

□

Since f is only assumed to be continuous, the following Lemma is required for the multiplicity result in the next section.

Lemma 3.9. *The functional Ψ_ε satisfies the $(PS)_c$ condition in S_ε^+ for all $c \in \left[c_\varepsilon, \frac{N+2-\mu}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}\right)$.*

Proof. Let $\{u_n\} \subset S_\varepsilon^+$ be a $(PS)_c$ sequence for Ψ_ε. Thus, $\Psi_\varepsilon(u_n) \to c$ and $\|\Psi_\varepsilon'\|_* \to 0$, where $\|\cdot\|_*$ is the norm in the dual space $(T_{u_n} S_\varepsilon^+)^*$. It follows from Lemma 3.6 (B_3) that $\{m_\varepsilon(u_n)\}$ is a $(PS)_c$ sequence for J_ε in H_ε. From Lemma 3.8, we see that there exists a $u \in S_\varepsilon^+$ such that $m_\varepsilon(u_n) \to m_\varepsilon(u)$ in H_ε. From Lemma 3.5 (A_3), we conclude that

$$u_n \to u \text{ in } S_\varepsilon^+,$$

and the proof is complete. □

4. The autonomous problem

For our scope, we start by considering the limit problem associated to (2.1), namely the problem

$$-\Delta u + V_0 u = \left(\frac{|u(y)|^{2_\mu^*} + F(|u(y)|^2)}{|x-y|^\mu}\right)\left(|u|^{2_\mu^*-2}u + \frac{1}{2_\mu^*} f(|u|^2)u\right) \quad \text{in } \mathbb{R}^N, \tag{4.1}$$

which has the following associated functional

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V_0|u|^2)\mathrm{d}x - \frac{1}{2\cdot 2_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(|u(x)|^{2_\mu^*} + F(|u(x)|^2))(|u(y)|^{2_\mu^*} + F(|u(y)|^2))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y.$$

The functional I_0 is well defined on the Hilbert space $H_0 = H^1(\mathbb{R}^N, \mathbb{R})$ with the inner product

$$\langle u, v\rangle_{V_0} = \int_{\mathbb{R}^N} (\nabla u \nabla v + V_0 uv)\mathrm{d}x,$$

and the norm

$$\|u\|_{V_0} = \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + V_0|u|^2)\mathrm{d}x\right)^{\frac{1}{2}}.$$

We denote the Nehari manifold associated to I_0 by

$$\mathcal{N}_0 = \{u \in H_0 \setminus \{0\} : \langle I_0'(u), u\rangle = 0\}.$$

and the least energy on $\mathcal{N}_0$ is defined by

$$c_{V_0} = \inf_{u \in \mathcal{N}_0} I_0(u).$$

Let H_0^+ be the open subset of H_0 given by

$$H_0^+ = \{u \in H_0 : |\operatorname{supp}(u) \cap \Lambda_\varepsilon| > 0\},$$

and $S_0^+ = S_0 \cap H_0^+$, where S_0 is the unit sphere of H_0. Note that S_0^+ is a non-complete $C^{1,1}$-manifold of codimension 1, modeled on H_0 and contained in H_0^+. Therefore, $H_0 = T_u S_0^+ \oplus \mathbb{R}u$ for each $u \in S_0^+$, where $T_u S_0^+ = \{v \in H_0 : \langle u, v\rangle_{V_0} = 0\}$.

Next we have the following Lemmas and the proofs that follow from a similar argument used in the proofs of Lemmas 3.5 and 3.6.

Lemma 4.1. *Let V_0 be given in (V_1) and suppose that f satisfies (f_1)–(f_3). Then, the following properties hold:*

(a_1) *For each* $u \in H_0 \setminus \{0\}$, *let* $\phi_u : \mathbb{R}^+ \to \mathbb{R}$ *be given by* $\phi_u(t) = I_0(tu)$. *Then, there exists a unique* $t_u > 0$ *such that* $\phi_u'(t) > 0$ *in* $(0, t_u)$ *and* $\phi_u'(t) < 0$ *in* (t_u, ∞).
(a_2) *There is a* $\sigma > 0$ *independent on* u *such that* $t_u > \sigma$ *for all* $u \in S_0^+$. *Moreover, for each compact set* $\mathcal{W} \subset S_0^+$, *there is* $C_{\mathcal{W}} > 0$ *such that* $t_u \le C_{\mathcal{W}}$ *for all* $u \in \mathcal{W}$.
(a_3) *The map* $\widehat{m} : H_0 \setminus \{0\} \to \mathcal{N}_0$ *given by* $\widehat{m}(u) = t_u u$ *is continuous and* $m = \widehat{m}|_{S_0^+}$ *is a homeomorphism between* S_0^+ *and* $\mathcal{N}_0$. *Moreover,* $m^{-1}(u) = \frac{u}{\|u\|_0}$.
(a_4) *If there is a sequence* $\{u_n\} \subset S_\varepsilon^+$ *such that* $\operatorname{dist}(u_n, \partial S_\varepsilon^+) \to 0$, *then we have* $\|m_\varepsilon(u_n)\|_\varepsilon \to \infty$ *and* $I_0(m_\varepsilon(u_n)) \to \infty$.

We shall consider the functional defined by

$$\widehat{\Psi}_0(u) = I_0(\widehat{m}(u)) \qquad \text{and} \qquad \Psi_0 = (\widehat{\Psi}_0)|_{S_0^+}.$$

Lemma 4.2. *Let* V_0 *be given in* (V_1) *and suppose that* f *satisfies* (f_1)-(f_3), *then*
(b_1) $\widehat{\Psi}_0 \in C^1(H_0^+, \mathbb{R})$ *and*

$$\langle \widehat{\Psi}_0'(u), v\rangle = \frac{\|\widehat{m}(u)\|_0}{\|u\|_0} \langle I_0'(\widehat{m}(u)), v\rangle, \quad \forall u \in H_0^+ \text{ and } \quad \forall v \in H_0.$$

(b_2) $\Psi_0 \in C^1(S_0^+, \mathbb{R})$ *and*

$$\langle \Psi_0'(u), v\rangle = \|m(u)\|_0 \langle I_0'(m(u)), v\rangle, \forall v \in TS_0^+.$$

(b_3) *If* $\{u_n\}$ *is a* $(PS)_c$ *sequence of* Ψ_0, *then* $\{m(u_n)\}$ *is a* $(PS)_c$ *sequence of* I_0. *If* $\{u_n\} \subset \mathcal{N}_0$ *is a bounded* $(PS)_c$ *sequence of* I_0, *then* $\{m^{-1}(u_n)\}$ *is a* $(PS)_c$ *sequence of* Ψ_0.
(b_4) u *is a critical point of* Ψ_0 *if and only if* $m(u)$ *is a critical point of* I_0. *Moreover, the corresponding critical values coincide and*

$$\inf_{u \in S_0^+} \Psi_\varepsilon(u) = \inf_{u \in \mathcal{N}_0} I_0(u).$$

Similar to the previous argument, we have the following variational characterization of the infimum of I_0 over $\mathcal{N}_0$:

$$c_{V_0} = \inf_{u \in \mathcal{N}_0} I_0(u) = \inf_{u \in H_0^+} \sup_{t>0} I_0(tu) = \inf_{u \in S_0^+} \sup_{t>0} I_0(tu).$$

The next result is useful in later arguments.

Lemma 4.3. *Let* $\{u_n\} \in H_0$ *be a* $(PS)_c$ *sequence with* $c \in \left[c_{V_0}, \frac{N+2-\mu}{2(2N-\mu)} S_{H,L}^{\frac{2N-\mu}{N+2-\mu}}\right)$ *for* I_0 *such that* $u_n \rightharpoonup 0$. *Then, only one of the following conclusions holds.*
(i) $u_n \to 0$ *in* H_0 *as* $n \to \infty$;
(ii) *there exists* $R, \beta > 0$ *and* $\{y_n\} \subset \mathbb{R}^N$ *such that*

$$\int_{B_R(y_n)} |u_n|^2 \ge \beta.$$

Proof. Suppose that (ii) does not hold. Then, for any $R > 0$, we have

$$\lim_n \sup_{y \in \mathbb{R}^N} \int_{B_R(y_n)} |u_n|^2 = 0.$$

Similarly to Lemma 3.7, $\{u_n\}$ is bounded in H_0. Then, by Lemma 2.4, we know

$$u_n \to 0 \ \text{ in } \ L^r(\mathbb{R}^N), r \in (2, 2^*).$$

Thus, by $(f2)$ we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(|u_n(x)|^{2^*_\mu} + F(|u_n(x)|^2))(|u_n(y)|^{2^*_\mu} + F(|u_n(y)|^2))}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y = o_n(1).$$

From $\langle I_0'(u_n), u_n\rangle \to 0$, we get

$$\|u_n\|_0^2 = o_n(1).$$

Therefore the conclusion follows. □

From Lemma 4.3, we can see that, if u is the weak limit of a $(PS)_{c_{V_0}}$ sequence $\{u_n\}$ for the functional I_0, then we can assume $u_n \neq 0$. Otherwise we would have $u_n \rightharpoonup 0$ and once it does not occur that $u_n \to 0$, we conclude from Lemma 4.3 that there exist $\{y_n\} \subset \mathbb{R}^N$ and $R, \beta > 0$ such that

$$\liminf_{n\to+\infty} \int_{B_R(y_n)} |u_n|^2 \mathrm{d}x \geq \beta > 0.$$

Then set $v_n(x) = u_n(x + y_n)$, making a change of variable, we can prove that $\{v_n\}$ is also a $(PS)_{c_{V_0}}$ sequence for the functional I_0, it is bounded in H_0 and there exists $v \in H_0$ such that $v_n \to v$ in H_0 with $v \neq 0$.

Next we devote to estimating the least energy c_{V_0}.

Lemma 4.4. *There exists u_ε such that*

$$\sup_{t\geq 0} I_0(tu_\varepsilon) < \frac{N+2-\mu}{2(2N-\mu)} (S_{H,L})^{\frac{2N-\mu}{N+2-\mu}},$$

provided that one of the following conditions holds:

(*i*) $\frac{N+2-\mu}{N-2} < p < 2^*_\mu, N = 3, 4$ *and* $\lambda > 0$*;*
(*ii*) $\frac{2N-\mu}{N} < p \leq \frac{N+2-\mu}{N-2}, N = 3, 4$ *and* λ *sufficiently large;*
(*iii*) $\frac{2N+2-\mu}{N-2} < p < 2^*_\mu, N \geq 5$ *and* $\lambda > 0$*;*
(*iv*) $\frac{2N-\mu}{N} < p \leq \frac{2N-2-\mu}{N-2}, N \geq 5$ *and* λ *sufficiently large.*

Proof. We consider $\psi \in C_0^\infty\left(\mathbb{R}^N\right)$ such that

$$\psi(x) = \begin{cases} 1, & \text{if } x \in B_\delta, \\ 0, & \text{if } x \in \mathbb{R}^N \backslash B_{2\delta}, \end{cases} \quad 0 \leq |\psi(x)| \leq 1, \quad |D\psi(x)| \leq C,$$

where $C > 0$ is a constant. For $\varepsilon > 0$, we define

$$U_\varepsilon(x) := \varepsilon^{\frac{2-N}{2}} U\left(\frac{x}{\varepsilon}\right),$$
$$u_\varepsilon(x) := \psi(x) U_\varepsilon(x).$$

From [21], we have the estimates

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \mathrm{d}x = C(N,\mu)^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}} S_{H,L}^{\frac{N}{2}} + O\left(\varepsilon^{N-2}\right) \tag{4.2}$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\varepsilon(x)|^{2^*_\mu}|u_\varepsilon(y)|^{2^*_\mu}}{|x-y|^\mu} \mathrm{d}x\mathrm{d}y \geq C(N,\mu)^{\frac{N}{2}} S_{H,L}^{\frac{2N-\mu}{2}} - O\left(\varepsilon^{N-\frac{\mu}{2}}\right). \tag{4.3}$$

By direct computation, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)|^p|u_\varepsilon(y)|^p}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|\psi(x)U_\varepsilon(x)|^p|\psi(y)U_\varepsilon(y)|^p}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\\
&=\int_{B_{2\delta}}\int_{B_{2\delta}}\frac{|\psi(x)U_\varepsilon(x)|^p|\psi(y)U_\varepsilon(y)|^p}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\geq\int_{B_\delta}\int_{B_\delta}\frac{|U_\varepsilon(x)|^p|U_\varepsilon(y)|^p}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\\
&=[N(N-2)]^{(N-2)p}\varepsilon^{(2-N)p}\int_{B_\delta}\int_{B_\delta}\frac{1}{\left(1+\left|\frac{x}{\varepsilon}\right|^2\right)^{\frac{(N-2)p}{2}}|x-y|^\mu\left(1+\left|\frac{y}{\varepsilon}\right|^2\right)^{\frac{(N-2)p}{2}}}\mathrm{d}x\mathrm{d}y \qquad (4.4)\\
&=[N(N-2)]^{\frac{(N-2)p}{2}}\varepsilon^{2N-\mu-(N-2)p}\int_{B_\delta}\int_{B_\delta}\frac{1}{(1+|x|^2)^{\frac{(N-2)p}{2}}|x-y|^\mu(1+|y|^2)^{\frac{(N-2)p}{2}}}\mathrm{d}x\mathrm{d}y\\
&=C_3\varepsilon^{2N-\mu-(N-2)p}.
\end{aligned}$$

Case 1. $\frac{N+2-\mu}{N-2}<p<2_\mu^*$ and $N=3,4$ or $\frac{2N-2-\mu}{N-2}<p<2_\mu^*$ and $N\geq 5$.
From Lemma 4.1, there exists unique $t_\varepsilon>0$ such that $t_\varepsilon u_\varepsilon\in\mathcal{N}_0$. Now we estimate $I_0(t_\varepsilon u_\varepsilon)$. Notice that

$$\begin{aligned}
I_0(t_\varepsilon u_\varepsilon)=&\frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N}(|\nabla u_\varepsilon|^2+V_0|u_\varepsilon|^2)\mathrm{d}x\\
&-\frac{1}{2\cdot 2_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(|t_\varepsilon u_\varepsilon(x)|^{2_\mu^*}+F(|t_\varepsilon u_\varepsilon(x)|^2))(|t_\varepsilon u_\varepsilon(y)|^{2_\mu^*}+F(|t_\varepsilon u_\varepsilon(y)|^2))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\\
\leq&\frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N}(|\nabla u_\varepsilon|^2+V_0|u_\varepsilon|^2)\mathrm{d}x-\frac{t_\varepsilon^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\\
&-\frac{1}{2\cdot 2_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{F(|t_\varepsilon u_\varepsilon(x)|^2)F(|t_\varepsilon u_\varepsilon(y)|^2)}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\\
\leq&\left(\frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\mathrm{d}x-\frac{t_\varepsilon^{2\cdot 2_\mu^*}}{2\cdot 2_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)\\
&+\left(\frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^N}V_0|u_\varepsilon|^2\mathrm{d}x-\frac{\lambda^2}{2\cdot 2_\mu^*}t_\varepsilon^{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)|^p|u_\varepsilon(y)|^p}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)\\
:=&I_1+I_2.
\end{aligned}$$

For I_1, we may assume that

$$\begin{aligned}
E&:=\int_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\mathrm{d}x,\\
F&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y,
\end{aligned}$$

and consider the function $\theta:[0,\infty)\to\mathbb{R}$ defined by

$$\theta(t)=\frac{1}{2}Et^2-\frac{1}{2\cdot 2_\mu^*}Ft^{2\cdot 2_\mu^*}.$$

We have that $t_0=\left(\frac{E}{F}\right)^{\frac{1}{2\cdot 2_\mu^*-2}}$ is a maximum point of θ and

$$\theta(t_0)=\frac{2_\mu^*-1}{2\cdot 2_\mu^*}E^{\frac{2_\mu^*}{2_\mu^*-1}}\frac{1}{F^{1-2_\mu^*}}.$$

Hence, we have

$$\begin{aligned}
I_1 &\le \frac{N+2-\mu}{2(2N-\mu)}\left(\frac{\int\limits_{\mathbb{R}^N}|\nabla u_\varepsilon|^2\mathrm{d}x}{\left(\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{|u_\varepsilon(x)|^{2_\mu^*}|u_\varepsilon(y)|^{2_\mu^*}}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y\right)^{\frac{N-2}{2N-\mu}}}\right)^{\frac{2N-\mu}{N+2-\mu}}\\
&\le \frac{N+2-\mu}{2(2N-\mu)}\left(\frac{(C(N,\mu))^{\frac{N-2}{2N-\mu}\cdot\frac{N}{2}}S_{H,L}^{\frac{N}{2}}+O\left(\varepsilon^{N-2}\right)}{\left(C(N,\mu)^{\frac{N}{2}}S_{H,L}^{\frac{2N-\mu}{2}}-O\left(\varepsilon^{\frac{2N-\mu}{2}}\right)\right)^{\frac{N-2}{2N-\mu}}}\right)^{\frac{2N-\mu}{N+2-\mu}}\\
&= \frac{N+2-\mu}{2(2N-\mu)}\left(S_{H,L}\right)^{\frac{2N-\mu}{N+2-\mu}}\left(\frac{1+O\left(\varepsilon^{N-2}\right)}{\left(1-O\left(\varepsilon^{\frac{2N-\mu}{2}}\right)\right)^{\frac{2N-\mu}{N+2-\mu}}}\right)^{\frac{2N-\mu}{N+2-\mu}}\\
&\le \frac{N+2-\mu}{2(2N-\mu)}\left(S_{H,L}\right)^{\frac{2N-\mu}{N+2-\mu}}\left(1+C(N,\mu)\frac{O\left(\varepsilon^{N-2}\right)+O\left(\varepsilon^{\frac{2N-\mu}{2}}\right)}{\left(1-O\left(\varepsilon^{\frac{2N-\mu}{2}}\right)\right)^{\frac{N-2}{2N-\mu}}}\right).
\end{aligned}$$

Observing that for $\varepsilon>0$ sufficiently small, it holds

$$\left(1-O\left(\varepsilon^{\frac{N-2}{2N-\mu}}\right)\right)^{\frac{N-2}{2N-\mu}}\ge\frac{1}{2}.$$

Therefore, we conclude that for any $\varepsilon>0$ sufficiently small, we have

$$I_1\left(t_\varepsilon u_\varepsilon\right)\le\frac{N+2-\mu}{2(2N-\mu)}\left(S_{H,L}\right)^{\frac{2N-\mu}{N+2-\mu}}+O\left(\varepsilon^{\min\left\{N-2,\frac{2N-\mu}{2}\right\}}\right). \tag{4.5}$$

We claim that there exists $C_0>0$ such that for all $\varepsilon>0$,

$$t_\varepsilon^{2p}\ge C_0.$$

In fact, assume that there exists a sequence $\{\varepsilon_n\}$ and $\varepsilon_n\to 0$ as $n\to\infty$, such that $t_{\varepsilon_n}\to 0$ as $n\to\infty$. Thus,

$$0<c_{V_0}\le\sup_{t\ge 0}I_0(tu_{\varepsilon_n})=I_0(t_{\varepsilon_n}u_{\varepsilon_n}).$$

Since $u_{\varepsilon_n}\in H_0$ is bounded and $t_{\varepsilon_n}\to 0$ as $n\to\infty$, we have $t_{\varepsilon_n}u_{\varepsilon_n}\to 0$ in H_0.

The continuity of I_0 implies that $I_0(t_{\varepsilon_n}u_{\varepsilon_n})\to I_0(0)=0$. Therefore,

$$0<c_{V_0}\le\lim_{n\to\infty}I_0(t_{\varepsilon_n}u_{\varepsilon_n})=0,$$

which is a contradiction. The claim is holds.

From (4.5) and the above claim, we have

$$I_0(t_\varepsilon u_\varepsilon)<\frac{N+2-\mu}{2(N-\mu)}(S_{H,L})^{\frac{2N-\mu}{N+2-\mu}}+O\left(\varepsilon^\eta\right)+C_2\int\limits_{\mathbb{R}^N}|u_\varepsilon(x)|^2\mathrm{d}x-C_3\varepsilon^{2N-\mu-(N-2)p}. \tag{4.6}$$

We want to obtain that

$$\lim_{\varepsilon\to 0}\varepsilon^{-\eta}(C_2\int\limits_{\mathbb{R}^N}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\mu-(N-2)p}) = -\infty. \tag{4.7}$$

In order to do that, it suffices to show that

$$\lim_{\varepsilon\to 0}\varepsilon^{-\eta}(C_2\int\limits_{B_\delta}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\mu-(N-2)p}) = -\infty \tag{4.8}$$

and

$$C_2\int\limits_{B_{2\delta}\setminus B_\delta}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\mu-(N-2)p} = O\left(\varepsilon^\eta\right). \tag{4.9}$$

Suppose that (4.8) and (4.9) hold, let us continue to complete our proof. Since

$$\begin{aligned} O\left(\varepsilon^\eta\right) &+ C_2\int\limits_{\mathbb{R}^N}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\mu-(N-2)p} \\ &= \varepsilon^\eta\left[\frac{O\left(\varepsilon^\eta\right)}{\varepsilon^\eta} + \varepsilon^{-\eta}\left(C_2\int\limits_{\mathbb{R}^N}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\mu-(N-2)p}\right)\right], \end{aligned}$$

from (4.7), we have

$$O\left(\varepsilon^\eta\right) + C_2\int\limits_{\mathbb{R}^N}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\mu-(N-2)p} < 0 \tag{4.10}$$

for $\varepsilon > 0$ sufficiently small. Thus, (4.6) and (4.10) guarantee

$$\sup_{t\geq 0} I_0(tu_\varepsilon) < \frac{N+2-\mu}{2(2N-\mu)}\left(S_{H,L}\right)^{\frac{2N-\mu}{N+2-\mu}}$$

for $\varepsilon > 0$ sufficiently small and fixed. Once (4.8) and (4.9) are verified, the proof of Case 1 is complete.

Now we prove (4.8). By direct computation, we know

$$\int\limits_{B_\delta}|u_\varepsilon(x)|^2\mathrm{d}x = N\omega_N[N(N-2)]^{\frac{N-2}{2}}\varepsilon^2\int\limits_0^{\frac{\delta}{\varepsilon}}\frac{r^{N-1}}{\left(1+r^2\right)^{N-2}}\mathrm{d}r,$$

where ω_N denotes the volume of the unit ball in $\mathbb{R}^N$.

Let

$$\begin{aligned} I_\varepsilon :&= \varepsilon^{-\eta}\left(C_2\int\limits_{B_\delta}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\mu-(N-2)p}\right) \\ &= \varepsilon^{-\eta}\left(C_4\varepsilon^2\int\limits_0^{\frac{\delta}{\varepsilon}}\frac{r^{N-1}}{\left(1+r^2\right)^{N-2}}\mathrm{d}r - C_3\varepsilon^{2N-\mu-(N-2)p}\right). \end{aligned}$$

The Case N=3. In this case, we have $5-\mu < p < 2^*_\mu$, thus $5-\mu-p < 0$. And we know $\min\left\{N-2, \frac{2N-\mu}{2}\right\} = N-2 = 1$ by $0 < \mu < N$. It is easy to show that

$$\varepsilon^2 \int_0^{\frac{\delta}{\varepsilon}} \frac{r^2}{1+r^2} \mathrm{d}r = \varepsilon \left(\delta - \varepsilon \arctan\left(\frac{\delta}{\varepsilon}\right)\right).$$

Therefore,

$$I_\varepsilon = C_4 \left(\delta - \varepsilon \arctan\left(\frac{\delta}{\varepsilon}\right)\right) - C_3 \varepsilon^{5-\mu-p}.$$

Our claim follows.

The Case N=4. In this case, $\frac{6-\mu}{2} < p < 2^*_\mu$ implies $6-\mu-2p < 0$ and $\min\left\{N-2, \frac{2N-\mu}{2}\right\} = N-2 = 2$ if $0 < \mu < 4$. We also have

$$\varepsilon^2 \int_0^{\frac{\delta}{\varepsilon}} \frac{r^3}{(1+r^2)^2} \, \mathrm{d}r = \frac{\varepsilon^2}{2} \left[\ln\left(1 + \frac{\delta^2}{\varepsilon^2}\right) + \frac{\varepsilon^2}{\varepsilon^2+\delta^2} - 1\right].$$

So,

$$I_\varepsilon = \frac{C_4}{2} \left(\ln\left(1 + \frac{\delta^2}{\varepsilon^2}\right) + \frac{\varepsilon^2}{\varepsilon^2+\delta^2} - 1\right) - C_3 \varepsilon^{6-\mu-2p}.$$

Our claim follows.

The Case $N \geq 5$. We have

$$I_\varepsilon = \varepsilon^{2-\min\left\{N-2, \frac{2N-\mu}{2}\right\}} \left(C_4 \int_0^{\frac{\delta}{\varepsilon}} \frac{r^{N-1}}{(1+r^2)^{N-2}} \, \mathrm{d}r - C_3 \varepsilon^{2N-\mu-(N-2)p-2} \right).$$

It is easy to show that if $N \geq 5$, then the integral

$$\lim_{\varepsilon \to 0} \int_0^{\frac{\delta}{\varepsilon}} \frac{r^{N-1}}{(1+r^2)^{N-2}} \, \mathrm{d}r$$

converges.

There are two cases left to be considered:

- $0 < \mu < 4$ and $N \geq 5$;
- $\mu \geq 4$ and $N \geq 5$.

Assume that $0 < \mu < 4$ and $N \geq 5$, we have

$$2 - \eta = 2 - \min\left\{N-2, \frac{2N-\mu}{2}\right\} = -N + 4 < 0.$$

Also $\frac{2N-\mu-2}{N-2} < p < \frac{2N-\mu}{N-2}$ implies $2N-\mu-(N-2)p-2 < 0$. Therefore, $I_\varepsilon \to -\infty$ as $\varepsilon \to 0$. Now we consider the case $\mu \geq 4$ and $N \geq 5$. We have $N-2 \geq \frac{2N-\mu}{2}$ and therefore

$$2 - \eta = 2 - \min\left\{N-2, \frac{2N-\mu}{2}\right\} = 2 - N + \frac{\mu}{2} < 0.$$

Since

$$I_\varepsilon = \varepsilon^{2-N+\frac{\mu}{2}} \left[C_4 \int_0^{\frac{\delta}{\varepsilon}} \frac{r^{N-1}}{(1+r^2)^{N-2}} \mathrm{d}r - C_3 \varepsilon^{2N-\mu-(N-2)p-2} \right],$$

we conclude that $I_\varepsilon \to -\infty$ as $\varepsilon \to 0$. We are done.

Now we prove (4.9).

For $\delta > 0$ sufficiently large, we have $U_\varepsilon^2(x) \le \varepsilon^{1+\eta}$ if $|x| \ge \delta$. Since

$$\frac{1}{\varepsilon^\eta}\left[C_2\int\limits_{B_{2\delta}\setminus B_\delta}|u_\varepsilon(x)|^2\mathrm{d}x - C_3\varepsilon^{2N-\mu-(N-2)p}\right] < \frac{C_2}{\varepsilon^\eta}\int\limits_{B_{2\delta}\setminus B_\delta}\psi^2(x)U_\varepsilon^2(x)\mathrm{d}x \le C_2\varepsilon\|\psi\|_2 \le C_1\varepsilon\|\psi\|_\varepsilon,$$

our proof is complete.

Case 2. For λ sufficiently large, $\frac{2N-\mu}{N} < p \le \frac{N+2-\mu}{N-2}$ and $N = 3, 4$ or $\frac{2N-\mu}{N} < p \le \frac{2N-2-\mu}{N-2}$ and $N \ge 5$. From lemma 4.1, we know that $\max_{t\ge 0} I_0(tu_\varepsilon)$ is attained at some $t_\lambda > 0$. Since $I_0'(t_\lambda u_\varepsilon) = 0$, we have

$$\begin{aligned}\int\limits_{\mathbb{R}^N}\left(|\nabla u_\varepsilon|^2 + V_0|u_\varepsilon|^2\right)\mathrm{d}x &= \frac{1}{2_\mu^* t_\lambda^2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{(|u_\varepsilon(x)|^{2_\mu^*} + F(|u_\varepsilon(x)|^2))(|u_\varepsilon(y)|^{2_\mu^*} + F(|u_\varepsilon(y)|^2))}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y \\ &\ge \frac{1}{2_\mu^*}\lambda^2 t_\lambda^{2p-2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{|u_\varepsilon(x)|^p|u_\varepsilon(y)|^p}{|x-y|^\mu}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Thus, $t_\lambda \to 0$ as $\lambda \to +\infty$ and

$$\begin{aligned}\max_{t\ge 0} I_0(tu_\varepsilon) &= \frac{t_\lambda^2}{2}\int\limits_{\mathbb{R}^N}\left(|\nabla u_\varepsilon|^2 + V_0|u_\varepsilon|^2\right)\mathrm{d}x - \frac{1}{2\cdot 2_\mu^*}\int\limits_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * G(\varepsilon x, |t_\lambda u_\varepsilon|^2)\right)G(\varepsilon x, |t_\lambda u_\varepsilon|^2)\mathrm{d}x \\ &< \frac{t_\lambda^2}{2}\int\limits_{\mathbb{R}^N}\left(|\nabla u_\varepsilon|^2 + V_0|u_\varepsilon|^2\right)\mathrm{d}x.\end{aligned}$$

Since $t_\lambda \to 0$ as $\lambda \to +\infty$ and $\frac{N+2-\mu}{2(N-\mu)}(S_{H,L})^{\frac{2N-\mu}{N+2-\mu}} > 0$, we conclude that

$$\frac{t_\lambda^2}{2}\int\limits_{\mathbb{R}^N}\left(|\nabla u_\varepsilon|^2 + V_0|u_\varepsilon|^2\right)\mathrm{d}x < \frac{N+2-\mu}{2(2N-\mu)}(S_{H,L})^{\frac{2N-\mu}{N+2-\mu}}$$

for $\lambda > 0$ sufficiently large.

Therefore,

$$\sup_{t\ge 0} I_0(tu_\varepsilon) < \frac{N+2-\mu}{2(2N-\mu)}(S_{H,L})^{\frac{2N-\mu}{N+2-\mu}}$$

for $\lambda > 0$ sufficiently large. □

Theorem 4.1. *Assume that (f_1)–(f_3) hold. Then, autonomous problem (4.1) has a positive ground state solution u with $I_0(u) = c_{V_0}$.*

Proof. By Lemma 3.4 with $V(x) = V_0$ and the mountain pass theorem without (PS) condition, there exists a $(PS)_{c_{V_0}}$ sequence $\{u_n\} \subset H_0$ of I_0 with

$$c_{V_0} < \frac{N-\mu+2}{2(2N-\mu)}(S_{H,L})^{\frac{2N-\mu}{N+2-\mu}}.$$

If $u_0 \in \mathcal{N}_0$ satisfies $I_0(u_0) = c_{V_0}$, then $m^{-1}(u_0) \in S_0$ is a minimizer of Ψ_0, so that u_0 is a critical point of I_0 by Lemma 4.2. Now, we show that there exists a minimizer $u \in \mathcal{N}_0$ of $I_0|_{\mathcal{N}_0}$. Since $\inf_{S_0}\Psi_0 = \inf_{\mathcal{N}_0} I_0 = c_{V_0}$ and S_0 is a C^1 manifold, by Ekeland's variational principle, there exists a sequence $\omega_n \subset S_0$ with $\Psi_0(\omega_n) \to c_{V_0}$ and $\Psi_0'(\omega_n) \to 0$ as $n \to \infty$. Put $u_n = m(\omega_n) \in \mathcal{N}_0$ for $n \in \mathbb{N}$. Then by Lemma 4.2 (b_3), we have $I_0(u_n) \to c_{V_0}$ and $I_0'(u_n) \to 0$ as $n \to \infty$. Similar to the proof of Lemma 3.7, it is easy to check that $\{u_n\}$ is bounded in H_0. Thus, we have $u_n \rightharpoonup u$ in H_0, $u_n \to u$ in $L_{loc}^r(\mathbb{R}^N), 1 \le r < 2^*$ and

$u_n \to u$ a.e. in $\mathbb{R}^N$. Similar to the proof of Lemma 3.1, we have $I_0'(u) = 0$. From Lemma 4.3, we know that $u \neq 0$. Moreover,

$$\begin{aligned}
c_{V_0} &\le I_0(u) = I_0(u) - \frac{1}{2}\left\langle I_0'(u), u\right\rangle \\
&= \frac{1}{2\cdot 2_\mu^*}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * G(\varepsilon x, |u|^2)\right)\left(g(\varepsilon x, |u|^2) - G(\varepsilon x, |u|^2)\right)\mathrm{d}x \\
&\le \frac{1}{2\cdot 2_\mu^*}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * G(\varepsilon x, |u_n|^2)\right)\left(g(\varepsilon x, |u_n|^2) - G(\varepsilon x, |u_n|^2)\right)\mathrm{d}x \\
&= \liminf_{n\to\infty}\left(I_0(u_n) - \frac{1}{2}\langle I_0'(u_n), u_n\rangle\right) \\
&= c_{V_0},
\end{aligned}$$

where we used Fatou's lemma. Therefore, $I_0(u) = c_{V_0}$, which means that u is a ground state solution for (4.1). From the assumption of f, by [11, Propositions 6 and 7], we know that $u(x) > 0$ for $x \in \mathbb{R}^N$. The proof is complete. □

The next result is a compactness result on autonomous problem which we will use later.

Lemma 4.5. *Let $(u_n) \subset \mathcal{N}_0$ be a sequence such that $I_0(u_n) \to c_{V_0}$. Then, $\{u_n\}$ has a convergent subsequence in H_0.*

Proof. Since $\{u_n\} \subset \mathcal{N}_0$, it follows from Lemma 4.1 (a_3), Lemma 4.2 (b_4) and the definition of c_{V_0} that

$$v_n = m^{-1}(u_n) = \frac{u_n}{\|u_n\|_{V_0}} \in S_0^+, \quad \forall n \in \mathbb{N}$$

and

$$\Psi_0(v_n) = I_0(u_n) \to c_{V_0} = \inf_{S_0^+}\Psi_0(u).$$

Although S_0^+ is not a complete C^1 manifold, we still can apply Ekeland's variational principle to the functional $\mathcal{E}_0 : H \to \mathbb{R}\cup\{\infty\}$ defined by

$$\mathcal{E}_0(u) := \widehat{\Psi}_0(u) \quad \text{ if } u \in S_0^+$$

and

$$\mathcal{E}_0(u) := \infty \quad \text{ if } u \in \partial S_0^+,$$

where $H = \overline{S_0^+}$ is the complete metric space equipped with the metric

$$d(u, v) := \|u - v\|_0.$$

In fact, by Lemma 4.1 (a_4), $\mathcal{E}_0 \in C(H, \mathbb{R}\cup\{\infty\})$, and from Lemma 4.2 (b_4), $\mathcal{E}_0$ is bounded from below. Therefore, there exists a sequence $\{\tilde{v}_n\} \subset S_0^+$ such that $\{\tilde{v}_n\}$ is a $(PS)_{c_{V_0}}$ sequence for Ψ_0 on S_0^+ and

$$\|\tilde{v}_n - v_n\|_0 = o_n(1).$$

Arguing as in Lemma 3.8, we obtain the conclusion of this lemma. □

5. Solutions for the penalized problem

This subsection is devoted to proving a multiplicity result for the modified problem (3.1) by applying the Ljusternik–Schnirelmann category theory.

Let $\delta > 0$ be such that $M_\delta \subset \Lambda$, $\omega \in H^1(\mathbb{R}^N, \mathbb{R})$ be a positive ground state solution of the limit problem (4.1), and $\eta \in C^\infty(\mathbb{R}^+, [0,1])$ such that $\eta = 1$ if $0 \le t \le \frac{\delta}{2}$ and $\eta = 0$ if $t \ge \delta$. We define

$$\Psi_{\varepsilon,y}(x) := \eta(|\varepsilon x - y|)\omega\left(\frac{\varepsilon x - y}{\varepsilon}\right)\exp\left(i\tau_y\left(\frac{\varepsilon x - y}{\varepsilon}\right)\right),$$

for each $y \in M$, where $\tau_y := \sum_1^N A_i(y)x_i$. Let $t_\varepsilon > 0$ be the unique positive number such that

$$\max_{t\ge 0} J_\varepsilon(t\Psi_{\varepsilon,y}) = J_\varepsilon(t_\varepsilon\Psi_{\varepsilon,y}).$$

By noticing that $t_\varepsilon\Psi_{\varepsilon,y} \in \mathcal{N}_\varepsilon$, we consider the function $\Phi_\varepsilon := M \to \mathcal{N}_\varepsilon$ defined by setting

$$\Phi_\varepsilon := t_\varepsilon\Psi_{\varepsilon,y}.$$

By construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in M$.

Lemma 5.1. *The limit*

$$\lim_{\varepsilon\to 0^+} J_\varepsilon(\Phi_\varepsilon(y)) = c_{V_0}.$$

holds uniformly in $y \in M$.

Proof. Arguing by contradiction, we deduce that there exist $\delta_0 > 0, \{y_n\} \subset M$ and $\varepsilon_n \to 0^+$ satisfying

$$|J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y)) - c_{V_0}| \ge \delta_0. \tag{5.1}$$

For simplicity, we write Φ_n, Ψ_n and t_n for $\Phi_{\varepsilon_n}, \Psi_{\varepsilon_n,y_n}$ and t_{ε_n}, respectively.

Arguing as in [13, Lemma 3.2], we see that

$$\|\Psi_n\|^2_{\varepsilon_n} \to \|u\|_{V_0} \text{ as } n \to \infty. \tag{5.2}$$

On the other hand, since $\langle J'_{\varepsilon_n}(t_n\Psi_n), t_n\Psi_n\rangle = 0$, by the change of variables $z = (\varepsilon_n x - y_n)/\varepsilon_n$, we obtain

$$\begin{aligned}\|\Psi_n\|^2_{\varepsilon_n} &= \frac{1}{2^*_\mu}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * G(\varepsilon_n x, |t_n\Psi_n|^2)\right) g(\varepsilon_n x, |t_n\Psi_n|^2)|\Psi_n|^2 \mathrm{d}x \\ &= \frac{1}{2^*_\mu}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * G(\varepsilon_n z + y_n, |t_n\Psi_n|^2)\right) g(\varepsilon_n z + y_n, t_n^2\eta^2(|\varepsilon_n z|)\omega^2(z)\eta^2(|\varepsilon_n z|)\omega^2(z)\mathrm{d}z.\end{aligned} \tag{5.3}$$

If $z \in B_{\delta/\varepsilon_n}(0)$, then $\varepsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta \subset \Lambda$. Hence, $g(x,s) = f(s)$ for any $x \in \Lambda$ and $\Psi(s) = 0$ for $s \ge \delta$, the last equality leads to

$$\|\Psi_n\|^2_{\varepsilon_n} \ge \frac{F(|t_n\alpha|^2)f(|t_n\alpha|^2)}{2^*_\mu\cdot|\alpha|^2}\int_{B_{\delta/2}(0)}\int_{B_{\delta/2}(0)}\frac{|\omega(y)|^2|\omega(z)|^2}{|y-z|^\mu}\mathrm{d}x\mathrm{d}y,$$

where $\alpha = \min\{\omega(z) : |z| \le \delta/2\}$.

If $t_n \to \infty$, by (f_3) we deduce that $\|\Psi_n\|^2_{\varepsilon_n} \to \infty$ which contradicts (5.2). Therefore, up to a subsequence, we may assume that $t_n \to t_0 \ge 0$.

Since g has critical growth and $t_n\Psi_n \in \mathcal{N}_\varepsilon$, it follows that $t_0 > 0$. Thereby, taking the limit in (5.3), we obtain

$$\int_{\mathbb{R}^N}(|\nabla\omega|^2 + V_0\omega^2)\mathrm{d}x = \frac{1}{2^*_\mu}\int_{\mathbb{R}^N}\left(\frac{1}{|x|^\mu} * G(\varepsilon_n x, |t_0\omega|^2)\right) g(\varepsilon_n x, |t_0\omega|^2)|\omega|^2\mathrm{d}x,$$

which implies that $t_0\omega \in \mathcal{N}_{V_0}$. Since $\omega \in \mathcal{N}_{V_0}$, we obtain $t_0 = 1$. Using the Lebesgue dominated convergence theorem, we get

$$\int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon_n x, |t_n\Psi_n|^2)\right) G(\varepsilon_n y, |t_n\Psi_n|^2)\mathrm{d}x = \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon_n x, |\omega|^2)\right) G(\varepsilon_n y, |\omega|^2)\mathrm{d}x.$$

Hence,

$$\lim_{n\to\infty} J_{\varepsilon_n}(\Phi_n(y_n)) = I_0(\omega) = c_{V_0},$$

which is a contradiction with (5.1) and the proof is complete. □

Now we define the barycenter map.

Let $\rho > 0$ be such that $M_\delta \subset B_\rho$ and $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ be defined by

$$\Upsilon(x) := \begin{cases} x & \text{if } |x| < \rho, \\ \rho x/|x| & \text{if } |x| \geq \rho. \end{cases}$$

The barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ is defined by

$$\beta_\varepsilon(u) := \frac{1}{\|u\|_2^2} \int_{\mathbb{R}^N} \Upsilon(\varepsilon x)|u|^2 \mathrm{d}x.$$

Arguing as Lemma 4.3 in [8], it is easy to see that the function β_ε verifies the following limit:

Lemma 5.2.

$$\lim_{\varepsilon\to 0^+} \beta_\varepsilon\left(\Phi_\varepsilon(y)\right) = y$$

holds uniformly in $y \in M$.

We are next to establish the following useful compactness result.

Lemma 5.3. *Let $\varepsilon_n \to 0^+$ and $(u_n) \subset \mathcal{N}_{\varepsilon_n}$ be such that $J_{\varepsilon_n}(u_n) \to c_{V_0}$. Then, there exists a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $v_n := |u_n|(\cdot + \tilde{y}_n)$ has a convergent subsequence in $H^1(\mathbb{R}^N, \mathbb{R})$. Moreover, up to a subsequence, $y_n := \varepsilon_n \tilde{y}_n \to y \in M$ as $n \to \infty$.*

Proof. Since $\langle J'_{\varepsilon_n}(u_n), u_n\rangle = 0$ and $J_{\varepsilon_n}(u_n) \to c_{V_0}$, arguing as in the proof of Lemma 3.4, we can prove that there exists $C > 0$ such that $\|u_n\|_{\varepsilon_n} \leq C$ for all $n \in \mathbb{N}$. We claim that there exist a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ and constants $R, \beta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} |u_n|^2 \geqslant \beta. \tag{5.4}$$

Thereby, for some subsequence,

$$v_n := |u_n|(\cdot + \tilde{y}_n) \to v \neq 0 \ \text{ weakly in } \ H^1\left(\mathbb{R}^N, \mathbb{R}\right).$$

Let $t_n > 0$ be such that $\tilde{v}_n := t_n v_n \in \mathcal{N}_{V_0}$. By the diamagnetic inequality (2.1), we have

$$c_{V_0} \leq J_0\left(\tilde{v}_n\right) \leq \max_{t\geq 0} J_{\varepsilon_n}\left(tv_n\right) = J_{\varepsilon_n}\left(u_n\right) = c_{V_0} + o_n(1).$$

Hence, $J_0(\tilde{v}_n) \to c_{V_0}$ as $n \to +\infty$.

Since the sequences $\{v_n\}$ and $\{\tilde{v}_n\}$ are bounded in $H^1\left(\mathbb{R}^N, \mathbb{R}\right)$ and $v_n \nrightarrow 0$ in $H^1\left(\mathbb{R}^N, \mathbb{R}\right)$, we deduce (t_n) is also bounded and up to a subsequence, we may assume that $t_n \to t_0 \geq 0$.

If $t_0 = 0$, in view of the boundedness of v_n in H_0, we have $\tilde{v}_n := t_n v_n \to 0$ in H_0. Hence, $I_0(\tilde{v}_n) \to 0$, which contradicts $c_{V_0} > 0$. Thus, up to a subsequence, we may assume that $\tilde{v}_n \rightharpoonup \tilde{v} := t_0 v \neq 0$ in H_0, and, by Lemma 4.5, we can deduce that $\tilde{v}_n \to \tilde{v}$ in H_0, which gives $v_n \to v$ in H_0.

Now we show the final part, namely that $\{y_n\}$ has a subsequence such hat $y_n \to y \in M$. Assume by contradiction that $\{y_n\}$ is not bounded and so, up to a subsequence, $|y_n| \to +\infty$ as $n \to +\infty$. Choose $R > 0$ such that $\Lambda \subset B_R(0)$. Then for n large enough, we have $|y_n| > 2R$, and for any $x \in B_{R/\varepsilon_n}(0)$,

$$|\varepsilon_n x + y_n| \geq |y_n| - \varepsilon_n |x| > R.$$

Since $u_n \in \mathcal{N}_{\varepsilon_n}$, using (V_1) and the diamagnetic inequality (2.2), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + V_0 |v_n|^2\right) \mathrm{d}x \\
&\leq \int_{\mathbb{R}^N} \left(|\nabla_\varepsilon v_n|^2 + V(\varepsilon_n x + y_n)|v_n|^2\right) \mathrm{d}x \\
&= \frac{1}{2_\mu^*} \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon_n x + y_n, |v_n|^2)\right) g(\varepsilon_n x + y_n, |v_n|^2)\, |v_n|^2 \, \mathrm{d}x \\
&\leq \frac{1}{2_\mu^*} \int_{B_{R/\varepsilon_n}(0)} \left(\frac{1}{|x|^\mu} * G(\varepsilon_n x + y_n, |v_n|^2)\right) \tilde{f}(|v_n|^2)|v_n|^2 \mathrm{d}x \\
&\quad + \frac{1}{2_\mu^*} \int_{B^c_{R/\varepsilon_n}(0)} \left(\frac{1}{|x|^\mu} * G(\varepsilon_n x + y_n, |v_n|^2)\right) f(|v_n|^2)|v_n|^2 \mathrm{d}x.
\end{aligned} \tag{5.5}$$

Since $v_n \to v$ in H_0, $v_n \in \mathcal{B}$ and $\tilde{f}(t) \leq V_0/K$, we obtain

$$\min\left\{1, \frac{V_0}{2 \cdot 2_\mu^*}\right\} \int_{\mathbb{R}^N} \left(|\nabla v_n|^2 + |v_n|^2\right) \mathrm{d}x = o_n(1),$$

that is to say $v_n \to 0$ in H_0, which contradicts $v \neq 0$.

Therefore, we may assume that $y_n \to y_0 \in \mathbb{R}^N$. Assume by contradiction that $y_0 \notin \bar{\Lambda}$. Then, there exists $r > 0$ such that for every n large enough, we have that $|y_n - y_0| < r$ and $B_{2r}(y_0) \subset \bar{\Lambda}^c$. Then, if $x \in B_{r/\varepsilon_n}(0)$, we have that $|\varepsilon_n x + y_n - y_0| < 2r$ so that $\varepsilon_n x + y_n \in \bar{\Lambda}^c$, and so, arguing as before, we achieve a contradiction. Thus, $y_0 \in \bar{\Lambda}$.

To prove that $V(y_0) = V_0$, we suppose by contradiction that $V(y_0) > V_0$. Using Fatou's lemma, the change of variable $z = x + \tilde{y}_n$ and $\max_{t \geq 0} J_{\varepsilon_n}(tu_n) = J_{\varepsilon_n}(u_n)$, we obtain

$$\begin{aligned}
c_{V_0} = I_0(\tilde{v}) &< \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla \tilde{v}|^2 + V(y_0)|\tilde{v}|^2\right) \mathrm{d}x - \frac{1}{2 \cdot 2_\mu^*} \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, |\tilde{v}|^2)\right) G(\varepsilon x, |\tilde{v}|^2) \mathrm{d}x \\
&\leq \liminf_n \left(\frac{1}{2} \int_{\mathbb{R}^N} (|\nabla \tilde{v}_n|^2 + V(\varepsilon_n x + y_n)|\tilde{v}_n|^2) \mathrm{d}x \right.\\
&\qquad \left. - \frac{1}{2 \cdot 2_\mu^*} \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon_n x, |\tilde{v}_n|^2)\right) G(\varepsilon_n x, |\tilde{v}_n|^2) \mathrm{d}x\right) \\
&= \liminf_n \left(\frac{t_n^2}{2} \int_{\mathbb{R}^N} (|\nabla |u_n||^2 + V(\varepsilon_n z)|u_n|^2) \mathrm{d}x \right.\\
&\qquad \left. - \frac{1}{2 \cdot 2_\mu^*} \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon_n x, |t_n u_n|^2)\right) G(\varepsilon_n x, |t_n u_n|^2) \mathrm{d}x\right) \\
&\leq \liminf_n J_{\varepsilon_n}(t_n u_n) \leq \liminf_n J_{\varepsilon_n}(u_n) = c_{V_0},
\end{aligned}$$

which is impossible and the proof is complete. $\square$

Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be any positive function satisfying $h(\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Consider the following subset of the Nehari manifold

$$\tilde{\mathcal{N}}_\varepsilon = \{u \in \mathcal{N}_\varepsilon : J_\varepsilon(u) \le c_{V_0} + h(\varepsilon)\}.$$

Fixed $y \in M$, we conclude from Lemma 5.1 that $|J_\varepsilon(\Phi_\varepsilon(y)) - c_{V_0}| \to 0$ as $\varepsilon \to 0^+$. Thus, $\Phi_\varepsilon(y) \in \widetilde{\mathcal{N}}_\varepsilon$ and $\tilde{\mathcal{N}}_\varepsilon \neq \emptyset$ for $\varepsilon > 0$ small enough.

Arguing as Lemma 4.5 in [8], the next statement is to involve in the relationship between $\widetilde{\mathcal{N}}_\varepsilon$ and the barycenter map.

Lemma 5.4. *For any $\delta > 0$, we have*

$$\lim_{\varepsilon\to 0^+} \sup_{u\in\tilde{\mathcal{N}}_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u), M_\delta) = 0.$$

Next we prove our multiplicity result by presenting a relation between the topology of M and the number of solutions of the modified problem (3.1).

Theorem 5.5. *For any $\delta > 0$ such that $M_\delta \subset \Lambda$, there exists $\tilde{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$, problem (3.1) has at least $\operatorname{cat}_{M_\delta} M$ nontrivial solutions.*

Proof. For any given $\delta > 0$ such that $M_\delta \subset \Lambda$, by Lemmas 5.1, 5.2, and 5.3, we employ an argument as in [15] to deduce the existence of $\tilde{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, the following diagram

$$M \xrightarrow{\Phi_\varepsilon} \tilde{\mathcal{N}}_\varepsilon \xrightarrow{\beta_\varepsilon} M_\delta$$

is well defined and $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopically equivalent to the embedding $\iota : M \to M_\delta$. This fact implies that $\operatorname{cat}_{\widetilde{\mathcal{N}}_\varepsilon}(\widetilde{\mathcal{N}}_\varepsilon) \ge \operatorname{cat}_{M_\delta}(M)$ due to [15, Lemma 2.2]. It follows from Lemma 3.9 and standard Ljusternik–Schnirelmann theory that J_ε possesses at least $\operatorname{cat}_{\tilde{\mathcal{N}}_\varepsilon}(\tilde{\mathcal{N}}_\varepsilon)$ critical points on $\mathcal{N}_\varepsilon$. Consequently, equation (3.1) possess at least $\operatorname{cat}_{M_\delta}(M)$ critical points. $\square$

6. Proof of Theorem (1.2)

In this section, we prove our main result. The idea is to show that the solutions u_ε obtained in Theorem 5.5 satisfy

$$|u_\varepsilon(x)| \le a \quad \text{for } x \in \Lambda_\varepsilon^c$$

for ε small enough. Arguing as in [4, Lemma 4.1], we have the following important result.

Lemma 6.1. *Let $\varepsilon_n \to 0^+$ and $u_n \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ be a solution of (3.1). Then, $J_{\varepsilon_n}(u_n) \to c_{V_0}$ and $|u_n| \in L^\infty(\mathbb{R}^N)$. Moreover, for any given $\gamma > 0$, there exists $R > 0$ and $n_0 \in \mathbb{N}$ such that*

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n)^c)} < \gamma \quad \text{for all } n \ge n_0, \tag{6.1}$$

where $\tilde{y}_n$ is given by Lemma 5.3.

Now, we are ready to give a proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\delta > 0$ be such that $M_\delta \subset \Lambda$. We first claim that there exists $\tilde{\varepsilon}_\delta > 0$ such that for any $0 < \varepsilon < \tilde{\varepsilon}_\delta$ and any solution $u \in \widetilde{\mathcal{N}}_\varepsilon$ of the problem (3.1), it holds

$$\|u\|_{L^\infty(\mathbb{R}^N\setminus\Lambda_\varepsilon)} < a. \tag{6.2}$$

To prove the above claim, we argue by contradiction. Assume that there are two sequences $\varepsilon_n \to 0^+$and $u_n \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ verifying $J'_{\varepsilon_n}(u_n) = 0$ and

$$\|u_n\|_{L^\infty(\mathbb{R}^N\setminus\Lambda_{\varepsilon_n})} \ge a. \tag{6.3}$$

As in Lemma 5.1, we have that $J_{\varepsilon_n}(u_n) \to c_{V_0}$. Therefore, by Lemma 5.3, there exists obtain a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $\varepsilon_n \tilde{y}_n \to y_0 \in M$. If we take $r > 0$ such that $B_r(y_0) \subset B_{2r}(y_0) \subset \Lambda$ we have that

$$B_{r/\varepsilon_n}(y_0/\varepsilon_n) = \frac{1}{\varepsilon_n} B_r(y_0) \subset \Lambda_{\varepsilon_n}.$$

Moreover, for any $z \in B_{r/\varepsilon_n}(\tilde{y}_n)$, it holds

$$\left| z - \frac{y_0}{\varepsilon_n} \right| \leqslant |z - \tilde{y}_n| + \left| \tilde{y}_n - \frac{y_0}{\varepsilon_n} \right| < \frac{1}{\varepsilon_n}(r + o_n(1)) < \frac{2r}{\varepsilon_n}$$

for n large. For these values of n, we have that $B_{r/\varepsilon_n}(\tilde{y}_n) \subset \Lambda_{\varepsilon_n}$, that is, $\mathbb{R}^N \backslash \Lambda_{\varepsilon_n} \subset \mathbb{R}^N \backslash B_{r/\varepsilon_n}(\tilde{y}_n)$. On the other hand, it follows from Lemma 6.1 with $\gamma = a$ that, for any $n \geq n_0$ such that $r/\varepsilon_n > R$, it holds

$$\|u_n\|_{L^\infty(\mathbb{R}^N \backslash \Lambda_{\varepsilon_n})} \leqslant \|u_n\|_{L^\infty\left(\mathbb{R}^N \backslash B_{r/\varepsilon_n}(\tilde{y}_n)\right)} \leqslant \|u_n\|_{L^\infty(\mathbb{R}^N \backslash B_R(\tilde{y}_n))} < a,$$

which contradicts with (6.3). So the claim is correct.

Let $\hat{\varepsilon}_\delta > 0$ be given by Theorem 5.5 and set $\varepsilon_\delta := \min\{\hat{\varepsilon}_\delta, \tilde{\varepsilon}_\delta\}$. We will show the theorem holds for this choice of ε_δ. Let $0 < \varepsilon < \varepsilon_\delta$ be fixed. By Theorem 5.5, there is $\operatorname{cat}_{M_\delta}(M)$ nontrivial solutions of the problem (3.1). If $u \in H_\varepsilon$ is one of these solutions, we have that $u \in \widetilde{\mathcal{N}}_\varepsilon$. Then, by (6.2) $g(\varepsilon x, |u|^2) = f(|u|^2)$, u is a solution of the problem (3.1). An easy calculation shows that $\hat{u}(x) := u(x/\varepsilon)$ is a solution of the original problem (2.1). Then, problem (2.1) has at least $\operatorname{cat}_{M_\delta}(M)$ nontrivial solutions.

Take $\varepsilon_n \to 0^+$ and $\{u_n\}$ a sequence of solutions to (3.1). In order to study the behavior of the maximum points of $|u_n|$, we first notice that, by $(g4)$, there exists $\gamma > 0$ such that

$$g\left(\varepsilon x, s^2\right) s^2 \leqslant \frac{V_0}{K} s^2 \quad \text{for all } x \in \mathbb{R}^N, |s| \leqslant \gamma. \tag{6.4}$$

By Lemma 6.1, there are $R > 0$ and $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n)^c)} < \gamma. \tag{6.5}$$

Up to a subsequence, we can assume that

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n))} \geqslant \gamma. \tag{6.6}$$

Indeed, if this is not the case, we have $\|u_n\|_{L^\infty(\mathbb{R}^N)} < \gamma$, and therefore, it follows from $J'_{\varepsilon_n}(u_n) = 0$, (6.4) and the diamagnetic inequality (2.2) that

$$\begin{aligned}\int_{\mathbb{R}^N} (|\nabla |u_n||^2 + V_0|u_n|^2)\mathrm{d}x \leq \|u_n\|^2_{\varepsilon_n} \\ = \frac{1}{2^*_\mu} \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * G(\varepsilon x, |u_n|^2)\right) g_{\varepsilon_n}(x, |u_n|^2)|u_n|^2 \mathrm{d}x \\ \leq \frac{V_0}{2 \cdot 2^*_\mu} \int_{\mathbb{R}^N} |u_n|^2 \mathrm{d}x.\end{aligned}$$

The above expression implies that $\||u_n|\|_{H^1(\mathbb{R}^N,\mathbb{R})} = 0$, which is a contradiction. Thus, (6.6) holds. By (6.5) and (6.6), we can infer the maximum point $p_n \in \mathbb{R}^N$ of $|u_n|$ belongs to $B_R(\tilde{y}_n)$. Hence, $p_n = \tilde{y}_n + q_n$ for some $q_n \in B_R(0)$. Recalling that the associated solution of (2.1) is of the form $\hat{u}_n(x) = u_n(x/\varepsilon_n)$, we conclude that the maximum point η_n of $|\hat{u}_n|$ is $\eta_{\varepsilon_n} := \varepsilon_n \tilde{y}_n + \varepsilon_n q_n$. Since $(q_n) \subset B_R(0)$ is bounded and $\varepsilon_n \tilde{y}_n \to y_0 \in M$, we obtain

$$\lim_{n \to \infty} V(\eta_{\varepsilon_n}) = V(y_0) = V_0.$$

The proof is complete. $\square$

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