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Global classical solutions to an indirect chemotaxis-consumption model with signal-dependent degenerate diffusion and logistic source

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Abstract. This paper deals with the following indirect chemotaxis-consumption model with signal-dependent degenerate diffusion and logistic source

$$\begin{cases} u_t = \Delta\left(uv^{\alpha}\right) + au - bu^{l}, & x \in \Omega, t > 0, \\ v_t = \Delta v - vw, & x \in \Omega, t > 0, \\ w_t = -\delta w + u, & x \in \Omega, t > 0, \end{cases}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$. Here, the parameters $a > 0$, $b > 0$, $\alpha \geq 1$, $\delta > 0$ and $l \geq 2$. For all suitably regular initial data, if one of the following cases holds:

(i) $l > 2$;

(ii) $l = 2, n \leq 3$;

(iii) $l = 2, n \geq 4$, and b is sufficiently large, then the corresponding initial boundary value problem possesses a global classical solution.

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1. Introduction

In 1971, Keller and Segel [16] proposed the following well-known Keller–Segel model

$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot (u\nabla v) + f(u), & x \in \Omega, t > 0, \\ v_t = \Delta v - uv, & x \in \Omega, t > 0, \end{cases} \tag{1.1}$$

where $u = u(x,t)$ represents the density of the bacteria, $v = v(x,t)$ denotes the oxygen concentration, $\chi \in \mathbb{R}$ represents the chemotactic sensitivity coefficient. When the logistic source vanishes (i.e., $f(u) = 0$), Tao [32] showed that (1.1) has global bounded classical solution under the conditions $\|v_0\|_{L^\infty(\Omega)} \leq \frac{1}{6(n+1)\chi}$. Tao and Winkler [34] proved that (1.1) admits at least one global weak solution in a three-dimensional domain which becomes smooth after some waiting time. The large time behavior of (1.1) has also been studied by Zhang and Li [57]. When $f(u) = ku - \mu u^2$, $k \in \mathbb{R}, \mu > 0$, Lankeit and Wang [18] found that (1.1) has global bounded classical solutions for sufficiently large μ and weak solutions exist for any $\mu > 0$. Furthermore, researchers have studied a modified version of system (1.1) (see [46–49]).

In view of (1.1), it is important to note that the utilization of the chemotaxis signal by cells may be more intricate in real-world scenarios. The signal could originate from external substances, be indirectly

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generated, or even consist of multiple signals generated through diverse mechanisms ([35]). In particular, a chemotaxis system with indirect signal consumption has been considered in [4]:

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v) + f(u), & x\in\Omega, t>0,\\ v_t = \Delta v - vw, & x\in\Omega, t>0,\\ w_t = -\delta w + u, & x\in\Omega, t>0, \end{cases} \tag{1.2}$$

where $\delta > 0$ is a constant. When $f(u) = 0$, Fuest [4] proved that either $n \le 2$ or $n \ge 3$ with $\|v_0\|_{L^\infty(\Omega)} \le \frac{1}{3n}$, (1.2) possesses global bounded classical solutions which converges to a spatially constant equilibrium in the large time. When $f(u) = \mu u(1-u)$, $\mu > 0$, if μ is suitably large, global existence of classical solutions has been established in Li et al. [26]. In addition, numerous findings pertain to the qualitative analysis of indirect signal mechanisms (see [8,12,21]).

It is widely recognized that chemotaxis systems with signal-dependent motility have garnered significant attention in the recent literature [11,27]. Firstly, let us introduce the following Keller–Segel-production models with signal-dependent motility

$$\begin{cases} u_t = \Delta(\gamma(v)u) + f(u), & x\in\Omega, t>0,\\ v_t = \Delta v - v + u, & x\in\Omega, t>0. \end{cases} \tag{1.3}$$

In the case of the absence of the logistic source (i.e., $f(u) = 0$), under the assumption that $\gamma(s)$ has a positive lower and upper bounds (i.e., $k_\gamma \le \gamma(s) \le K_\gamma$ for all $s \ge 0$, where $k_\gamma, K_\gamma > 0$), Tao and Winkler [38] showed that (1.3) possesses global bounded classical solutions in two dimensions and global weak solutions in high dimensions. In particular, such weak solution will eventually become smooth in three-dimensional settings. For the particular case $\gamma(s) = c_0/s^k (c_0, k > 0)$, the existence of global classical solutions has been studied in [56] if c_0 is small enough. If the motility function $\gamma(s) = s^{-\alpha}$ with $\alpha > 0$, global existence of classical solutions was shown in [1,6,10,14,43]. Moreover, global weak solutions in lower dimensions ($n \le 3$) were obtained in [3]. If the motility function $\gamma(s) = e^{-s}$ for all $s \ge 0$, certain critical mass phenomenon of (1.3) in the two-dimensional case has been detected in [7,15]. For another results on (1.3), we refer to [2,9,50].

When $f(u) = \mu u(1-u)$, $\mu > 0$ and $\gamma(s)$ satisfies $\gamma(s) > 0$, $\gamma'(s) < 0$ and $\lim\limits_{s\to+\infty}\frac{\gamma'(s)}{\gamma(s)}$ exists, Jin et al. [13] obtained the global classical solution of (1.3) in two-dimensional settings. Moreover, if $\mu > \frac{K_0}{16}$ with $K_0 = \max\limits_{0<v<\infty}\frac{|\gamma'(s)|^2}{\gamma(s)}$, the asymptotic stability was established. The similar result was proved in the higher dimensions in [25,40]. When $f(u) = \rho u - \mu u^l$, $\rho\in\mathbb{R}, \mu > 0$, global classical solutions was showed in [29,30] if $l > \max\left\{\frac{n+2}{2}, 2\right\}$. There are some other results on (1.3), see [5,31].

Whereas if the signal is degraded, rather than produced, by the cells, the chemotaxis-consumption with signal-dependent motility has also been considered

$$\begin{cases} u_t = \Delta(\gamma(v)u) + f(u), & x\in\Omega, t>0,\\ v_t = \Delta v - uv, & x\in\Omega, t>0. \end{cases} \tag{1.4}$$

In the case of vanishing logistic source (i.e., $f(u) = 0$), if $\gamma \in C^3([0,+\infty))$ is positive on $[0,+\infty)$, by constructing a weighted integral function, Li and Zhao [20] found that (1.4) has global bounded classical solutions if $\|v_0\|_{L^\infty(\Omega)}$ is sufficiently small. Li and Winkler [23] showed that (1.4) possesses global classical bounded solutions without the smallness assumption of v_0 when $n \le 2$ and global weak solutions when $n \ge 3$, such weak solutions become eventually smooth in the three-dimensional setting. If $\gamma \in C^0([0,+\infty))$ is positive on $[0,+\infty)$, (1.4) admits global very weak solutions for all $n \ge 1$ [24]. If the motility function $\gamma(s) = s^{-\alpha}$, $\alpha > 0$, Tao and Winkler [39] obtained that there exists a very weak-strong solution; under the additional restrictions that $2 \le n \le 5$ and $\alpha > \frac{n-2}{6-n}$, (1.4) has global weak solutions. If the motility function $\gamma(s) = s^\alpha, \alpha > 0$ for all $s \ge 0$, there are some another results in [52–55].

When $f(u) = \mu u(1-u)$, the global classical solutions in two dimensions for any $\mu > 0$ and in the higher dimensions for large $\mu > 0$ were established in [22]. When $f(u) = au - bu^l$ $(a, b > 0)$, Wang [41] obtained that (1.4) possesses global bounded classical solutions and then (1.4) admits at least one global weak solution ($n \geq 3$), which becomes smooth after some waiting time. If $\gamma \in C^1([0,\infty)) \cap C^3((0,\infty))$ is positive on $(0,+\infty)$ with $\alpha \geq 1$, then global classical solutions can be established in [42]. On the other hand, if γ has rather mild regularities, then (1.4) admits at least one global weak solution in case $\alpha > 0$. In addition, then the above weak solutions become eventually smooth if $\alpha > 1$. Some scholars also consider the system (1.4) in other situations, readers can refer to [19,28].

Motivated by the above works, in this paper, we consider the following system

$$\begin{cases} u_t = \Delta\left(uv^\alpha\right) + au - bu^l, & x \in \Omega, \quad t > 0, \\ v_t = \Delta v - vw, & x \in \Omega, \quad t > 0, \\ w_t = -\delta w + u, & x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \quad t > 0, \\ u\left(x,0\right) = u_0\left(x\right), v\left(x,0\right) = v_0\left(x\right), w\left(x,0\right) = w_0\left(x\right), & x \in \Omega, \end{cases} \tag{1.5}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$, where $a > 0, b > 0, \delta > 0, l \geq 2, \alpha \geq 1$. The initial data

$$\begin{cases} u_0 \in C^0(\Omega) & \text{is nonnegative in } \Omega, \\ v_0 \in W^{1,\infty}(\Omega) & \text{is positive in } \Omega \text{ and} \\ w_0 \in W^{1,\infty}(\Omega) & \text{is nonnegative in } \Omega. \end{cases} \tag{1.6}$$

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary. Suppose that $a > 0, b > 0$, $\delta > 0$ and $\alpha \geq 1$, and that the initial data (u_0, v_0, w_0) satisfy (1.6). If one of the following cases holds:*

(i) $l > 2$;
(ii) $l = 2, n \leq 3$;
(iii) $l = 2, n \geq 4$, *and* $b > (\frac{n-2}{2})^{\frac{n+2}{n}} (n+\sqrt{n})^{\frac{4}{n}} (n-1)^{-\frac{2}{n}} \alpha^{\frac{2(n+2)}{n}} \|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(n+2)-2}{n}} + \frac{2^{2n+5}(n+\sqrt{n}+1)^n (n-2+\sqrt{n})^{\frac{n+2}{2}}}{(n+2)(n-1)^{\frac{n}{2}} \delta^{\frac{n+2}{2}}} \|v_0\|_{L^\infty(\Omega)}$, *then one can find nonnegative functions*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ v \in \bigcap_{\theta > n} C^0([0,\infty); W^{1,\theta}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ w \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{0,1}(\overline{\Omega} \times (0,\infty)), \end{cases}$$

such that (u, v, w) solves the problem (1.5) in the classical sense.

This paper is arranged as follows. In Sect. 2, we will get some preliminary inequalities and some basic lemmas. Some estimates of the solution and the proof of Theorem 1.1 are shown in Sect. 3.

2. Preliminaries

In this section, based on the well-established parabolic theory in [13,33], we can obtain the local-in-time existence result of a classical solution of (1.5).

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary, and let a, b, δ, α be some positive constants. If the initial data fulfill (1.6), then there exist a triple (u, v, w) of nonnegative functions*

$$\begin{cases} u \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})), \\ v \in \bigcap_{\theta > n} C^0([0,T_{\max}); W^{1,\theta}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\max})), \\ w \in C^0(\overline{\Omega} \times [0,T_{\max})) \cap C^{0,1}(\overline{\Omega} \times (0,T_{\max})), \end{cases}$$

which solves (1.5) in the classical sense. Moreover, if $T_{\max} < \infty$, we have

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

The following lemma can be easily obtained.

Lemma 2.2. *Let* (1.6) *hold and* $l > 1$. *Then, there exists* $C > 0$ *such that*

$$\|u(\cdot,t)\|_{L^1(\Omega)} \le C \quad \text{for all } t \in (0, T_{\max}), \tag{2.1}$$

$$0 \le v \le \|v_0\|_{L^\infty(\Omega)} \quad \text{in } \Omega \times (0, T_{\max}) \tag{2.2}$$

and

$$\int_t^{t+\tau} \int_\Omega u^l \le C \quad \text{for all } t \in (0, T_{\max} - \tau), \tag{2.3}$$

where $\tau := \min\left\{1, \frac{1}{2}T_{\max}\right\}$.

Proof. Integrating the first Eq. in (1.5), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u = a\int_\Omega u - b\int_\Omega u^l \le a\int_\Omega u - b|\Omega|^{1-l}\left(\int_\Omega u\right)^l \tag{2.4}$$

for all $t \in (0, T_{\max})$. Then, an ODE comparison argument implies (2.1). Consequently, an integration of (2.4) shows (2.3). From the nonnegativity v, w and the maximum principle, we derive (2.2). □

In order to prove our main results later, we quote a basic property of parabolic Eq. in [17, Lemma 1.2] (see also [21]).

Lemma 2.3. *Let* $T \in (0, +\infty)$. *Suppose that* $z_0 \in W^{1,\infty}(\Omega)$, *and that* $z \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ *is the solution of*

$$\begin{cases} z_t = \Delta z - zg, & x \in \Omega, t \in (0,T), \\ \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0,T), \\ z(x,0) = z_0(x), & x \in \Omega, \end{cases}$$

where $g \in C^0(\overline{\Omega} \times (0,T))$ *satisfies* $g \in L^\infty((0,T); L^p(\Omega))$ *with* $p > 0$. *Then for each*

$$r \in \begin{cases} \left[1, \frac{np}{n-p}\right) & \text{if } p \le n, \\ [1, \infty] & \text{if } p > n, \end{cases}$$

there exists a constant $C > 0$ *such that*

$$\|z(\cdot,t)\|_{W^{1,r}(\Omega)} \le C \quad \text{for all} \quad t \in (0,T).$$

The following auxiliary statement on a boundedness property will be used in the time-independent estimates (see [37, Lemma 3.2]).

Lemma 2.4. *Let* $T > 0$, $t_0 \in (0,T)$, $a > 0$, $b > 0$. *Suppose that* $y : [0,T) \to [0,\infty)$ *is absolutely continuous and that*

$$y'(t) + ay(t) \le h(t) \quad \text{for a.e.} \quad t \in (0,T),$$

where h *is a nonnegative function in* $L^l_{loc}([0,T))$ *satisfying*

$$\int_t^{t+t_0} h(s)\mathrm{d}s \le b \quad t \in [0, T-t_0).$$

Then, we have

$$y(t) \le \max\left\{y(0) + b, \frac{b}{at_0} + 2b\right\} \quad \text{forall} \quad t \in (0,T).$$

Next, we shall collect two lemmas that will be frequently used later.

Lemma 2.5. ([51, Lemma 3.4]) *Let $q \geq 2$ and $\psi \in C^2(\overline{\Omega})$ be positive fulfilling $\frac{\partial \psi}{\partial \nu} = 0$ on $\partial\Omega$. Then, we obtain*

$$\int_\Omega \psi^{-q-1}|\nabla\psi|^{q+2} \leq (q+\sqrt{n})^2 \int_\Omega \psi^{-q+3}|\nabla\psi|^{q-2}\left|D^2 \ln\psi\right|^2$$

and

$$\int_\Omega \psi^{-q+1}|\nabla\psi|^{q-2}\left|D^2\psi\right|^2 \leq (q+\sqrt{n}+1)^2 \int_\Omega \psi^{-q+3}|\nabla\psi|^{q-2}\left|D^2 \ln\psi\right|^2.$$

Lemma 2.6. ([51, Lemma 3.5]) *Let $q \geq 2$ and $\eta > 0$. There is $C = C(q, \eta) > 0$ such that every positive $\psi \in C^2(\overline{\Omega})$ with $\frac{\partial \psi}{\partial \nu} = 0$ on $\partial\Omega$ satisfies*

$$\int_{\partial\Omega} \psi^{-q+1}|\nabla\psi|^{q-2}\frac{\partial|\nabla\psi|^2}{\partial\nu} \leq \eta\int_\Omega \psi^{-q-1}|\nabla\psi|^{q+2} + \eta\int_\Omega \psi^{-q+1}|\nabla\psi|^{q-2}\left|D^2\psi\right|^2 + C\int_\Omega \psi.$$

Combining (2.3) and Lemma 2.4, we can derive the boundedness of $\int_\Omega w^l(\cdot,t)$ with $l > 1$.

Lemma 2.7. *If (1.6) and $l > 1$ hold, then there exists $C > 0$ such that*

$$\int_\Omega w^l(\cdot,t) \leq C \quad \text{for all} \quad t \in (0, T_{\max}). \tag{2.5}$$

Proof. We test the w-equation of (1.5) by w^{l-1} and integrate to obtain

$$\begin{aligned} \frac{1}{l}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega w^l &= -\delta\int_\Omega w^l + \int_\Omega uw^{l-1} \\ &\leq -\delta\int_\Omega w^l + \frac{\delta}{2}\int_\Omega w^l + \frac{l-1}{l}\left(\frac{\delta l}{2}\right)^{\frac{1}{1-l}}\int_\Omega u^l \end{aligned} \tag{2.6}$$

for all $t \in (0, T_{\max})$. Combining (2.3) and Lemma 2.4, we obtain (2.5) □

By a transformation $z(x,t) = -\ln\frac{v(x,t)}{\|v_0\|_{L^\infty(\Omega)}}$, we can construct a time-dependent pointwise lower bound for v. The ideas come from [42,45].

Lemma 2.8. *Let (1.6) hold and $n \geq 1$. For some $p > \frac{n}{2}$, assume that there exists $C_1 > 0$ satisfying*

$$\|w(\cdot,t)\|_{L^p(\Omega)} \leq C_1 \quad \text{for all} \quad t \in (0, T_{\max}). \tag{2.7}$$

Then, given any $T \in (0, T_{\max})$ there exists $C_2(T) > 0$ such that

$$v(x,t) \geq C_2(T) \quad \text{for all} \quad t \in (0, T). \tag{2.8}$$

Proof. Let $z(x,t) = -\ln\frac{v(x,t)}{\|v_0\|_{L^\infty(\Omega)}}$. Then, using the second equation of (1.5), we derive

$$\begin{cases} z_t = \Delta z - |\nabla z|^2 + w, & x \in \Omega, t \in (0, T_{\max}), \\ \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, t \in (0, T_{\max}), \\ z(x,0) = z_0(x) = -\ln\frac{v_0(x)}{\|v_0\|_{L^\infty(\Omega)}}, & x \in \Omega. \end{cases} \tag{2.9}$$

On the basis of the variation-of-constants formula of (2.9), using the nonnegativity of w and a comparison principle, we have

$$\begin{aligned} z(\cdot,t) &= e^{\Delta t} z_0 - \int_0^t e^{(t-s)\Delta} |\nabla z|^2 + \int_0^t e^{(t-s)\Delta} w(\cdot,s)\mathrm{d}s \\ &\le e^{\Delta t} z_0 + \int_0^t e^{(t-s)\Delta} w(\cdot,s)\mathrm{d}s \end{aligned}$$

for all $t \in (0, T_{\max})$. Then by virtue of (2.7) and the smoothing properties of Neumann heat semigroup $(e^{\Delta t})_{t\ge 0}$ on Ω ([44, Lemma 1.3]), we infer the existences of $c_1 > 0$ and $c_2 > 0$ such that

$$\begin{aligned} \|z(\cdot,t)\|_{L^\infty(\Omega)} &\le \left\|e^{t\Delta} z_0\right\|_{L^\infty(\Omega)} + \int_0^t \left\|e^{(t-s)\Delta} w(\cdot,s)\right\|_{L^\infty(\Omega)} \mathrm{d}s \\ &\le \|z_0\|_{L^\infty(\Omega)} + c_1 \int_0^t \left\{1 + (t-s)^{-\frac{n}{2p}}\right\} \|w(\cdot,s)\|_{L^p(\Omega)}\mathrm{d}s \\ &\le \|z_0\|_{L^\infty(\Omega)} + c_2 \int_0^t \left(1 + \sigma^{-\frac{n}{2p}}\right) \mathrm{d}\sigma \end{aligned} \tag{2.10}$$

for all $t \in (0, T_{\max})$. Hence, thanks to $p > \frac{n}{2}$, for any $T \in (0, T_{\max})$, (2.10) in conjunction with (2.11) implies that one can find some $c_3(T,p) > 0$ fulfilling

$$\|z(\cdot,t)\|_{L^\infty(\Omega)} \le c_3(T,p) \quad \text{for all } t \in (0,T). \tag{2.11}$$

Then according to the definition of $z(\cdot,t)$, we can readily obtain (2.8). □

With the lower bound of v at hand, we can show local boundedness criterion of solutions of (1.5).

Lemma 2.9. *Let* (1.6) *hold. For all $n \ge 1$, assume that there exist $C > 0$ and $q \ge 1$ with $q > \frac{n}{2}$ such that*

$$\|u(\cdot,t)\|_{L^q(\Omega)} \le C \quad \text{for all } t \in (0, T_{\max}). \tag{2.12}$$

Then for all $T \in (0, T_{\max})$, one can find $C(T) > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \le C(T) \quad \text{for all } t \in (0,T). \tag{2.13}$$

Proof. We may test the third equation in (1.5) by qw^{q-1} and use Young's inequality to find some $c_1 > 0$ such that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega w^q &= -q\delta \int_\Omega w^q + q \int_\Omega u w^{q-1} \\ &\le -\frac{q\delta}{2} \int_\Omega w^q + c_1 \int_\Omega u^q \end{aligned} \tag{2.14}$$

for all $t \in (0, T_{\max})$. Using (2.12) and (2.14), we can show that there exists a constant $c_2 > 0$ fulfilling

$$\int_\Omega w^q \le c_2 \quad \text{for all } t \in (0, T_{\max}). \tag{2.15}$$

Because $q > \frac{n}{2}$, we have $\frac{nq}{(n-q)_+} > n$. Thus, we can pick $\theta > \max\left\{1, \frac{n}{2}\right\}$ such that $\frac{nq}{(n-q)_+} > 2\theta > n$. An application of (2.15) and Lemma 2.3 implies that there exists $c_3 > 0$ such that $\|\nabla v(\cdot,t)\|_{L^{2\theta}(\Omega)} \le c_3$ for all $t \in (0, T_{\max})$. Moreover, (2.15) in conjunction with Lemma 2.8 shows that for any $T \in (0, T_{\max})$, we can find $c_4(T) > 0$ fulfilling $v \geq c_4(T)$ in $\Omega \times (0, T)$.

For all $p > 1$, multiplying the first equation of (1.5) by u^{p-1} and using Young's inequality, we end up with

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^p + \frac{c_4^\alpha(T)p(p-1)}{2} \int_\Omega u^{p-2}|\nabla u|^2 + \int_\Omega u^p \leq \frac{\alpha^2 c_5(T)p(p-1)}{2} \int_\Omega u^p |\nabla v|^2 + (ap+1) \int_\Omega u^p \tag{2.16}$$

for all $t \in (0, T)$, where $c_5(T) := \max\left\{c_4^{\alpha-2}(T), \|v_0\|^{\alpha-2}_{L^\infty(\Omega)}\right\}$. Regarding for $\theta > \max\left\{1, \frac{n}{2}\right\}$, thus $\frac{2\theta}{\theta-1} < \frac{2n}{(n-2)_+}$, by an Ehrling-type inequality and (2.1), there exists a positive constant $c_6(p, T)$ such that

$$\begin{aligned} \frac{\alpha^2 c_5(T)p(p-1)}{2} \int_\Omega u^p |\nabla v|^2 + (ap+1) \int_\Omega u^p &\leq \frac{\alpha^2 c_5(T)p(p-1)}{2} \|u^{\frac{p}{2}}\|^2_{L^{\frac{2\theta}{\theta-1}}(\Omega)} \|\nabla v\|^2_{L^{2\theta}(\Omega)} \\ + (ap+1)\|u^{\frac{p}{2}}\|^2_{L^2(\Omega)} &\leq \frac{\alpha^2 c_3^2 c_5(T)p(p-1)}{2} \|u^{\frac{p}{2}}\|^2_{L^{\frac{2\theta}{\theta-1}}(\Omega)} \\ + (ap+1)\|u^{\frac{p}{2}}\|^2_{L^2(\Omega)} &\leq \frac{c_4^\alpha(T)p(p-1)}{2} \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_6(p, T) \end{aligned} \tag{2.17}$$

for all $t \in (0, T)$. Combining (2.16) and (2.17), and using an ODE argument, we infer that there exists $c_7(p, T) > 0$ satisfying $\|u(\cdot, t)\|_{L^p(\Omega)} \leq c_7(p, T)$ for all $t \in (0, T)$. This along with Lemma 2.3 implies that one can find some $c_8(T) > 0$ such that

$$\|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq c_8(T) \quad \text{for all } t \in (0, T). \tag{2.18}$$

Thus, according to [36, Lemma A.1], we can find a constant $c_9(T) > 0$ satisfying

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq c_9(T) \quad \text{for all} \quad t \in (0, T). \tag{2.19}$$

Applying the variation-of-constants formula for w, we have

$$w(\cdot, t) = e^{-\delta t} w_0 + \int_0^t e^{-\delta(t-s)} u(\cdot, s)\, \mathrm{d}s \quad \text{for all} \quad t \in (0, T),$$

which together with (2.19) implies that there exists $c_{10}(T) > 0$ such that

$$\|w(\cdot, t)\|_{L^\infty(\Omega)} \leq c_{10}(T) \quad \text{for all} \quad t \in (0, T). \tag{2.20}$$

Hence, collecting (2.18)–(2.20), we can derive the claimed conclusion (2.13). □

3. Proof of Theorem 1.1

In this section, our goal is to obtain global classical solutions of (1.5). To this end, we will establish certain integral inequalities of (1.5). The ideas used in this section are mainly taken from [18,42,51,52]. We start with the following integral of the type $\int_\Omega u^p$.

Lemma 3.1. *Let* (1.6) *hold. We have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u^p + \frac{p(p-1)}{2} \int_\Omega u^{p-2} v^\alpha |\nabla u|^2 \leq \frac{\alpha^2 p(p-1)}{2} \int_\Omega u^p v^{\alpha-2} |\nabla v|^2 + ap \int_\Omega u^p - bp \int_\Omega u^{p+l-1} \tag{3.1}$$

for all $t \in (0, T_{\max})$.

Proof. We test the u-equation of (1.5) by u^{p-1} and use Young's inequality to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p =& -p(p-1)\int_\Omega u^{p-2}v^\alpha|\nabla u|^2-\alpha p(p-1)\int_\Omega u^{p-1}v^{\alpha-1}\nabla u\cdot\nabla v\\ &+ap\int_\Omega u^p-bp\int_\Omega u^{p+l-1}\\ \leq& -\frac{p(p-1)}{2}\int_\Omega u^{p-2}v^\alpha|\nabla u|^2+\frac{\alpha^2p(p-1)}{2}\int_\Omega u^pv^{\alpha-2}|\nabla v|^2\\ &+ap\int_\Omega u^p-bp\int_\Omega u^{p+l-1}\end{aligned}$$

for all $t\in(0,T_{\max})$. Hence, we obtain (3.1). □

Next we establish a differential inequality of $\int_\Omega v^{-q+1}|\nabla v|^q$ for all $q\geq 2$. This idea comes from [42,51], which is the key to the proof of this paper.

Lemma 3.2. *If (1.6) holds, then for all $q\geq 2$, we obtain*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^{-q+1}|\nabla v|^q+q(q-1)\int_\Omega v^{-q+3}\qquad|\nabla v|^{q-2}|D^2\ln v|^2\leq\frac{q}{2}\int_{\partial\Omega}v^{-q+1}|\nabla v|^{q-2}\cdot\frac{\partial|\nabla v|^2}{\partial\nu}\\ &+q(q-2+\sqrt{n})\int_\Omega wv^{-q+2}|\nabla v|^{q-2}|D^2v|\end{aligned}\tag{3.2}$$

for all $t\in(0,T_{\max})$.

Proof. Integrating by parts in the second Eq. in (1.5) and using $2\nabla v\cdot\nabla\Delta v=\Delta|\nabla v|^2-2\left|D^2v\right|^2$, we can find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^{-q+1}|\nabla v|^q =& q\int_\Omega v^{-q+1}|\nabla v|^{q-2}\nabla v\cdot\nabla(\Delta v-vw)-(q-1)\int_\Omega v^{-q}|\nabla v|^q(\Delta v-vw)\\ =&\frac{q}{2}\int_\Omega v^{-q+1}|\nabla v|^{q-2}\Delta|\nabla v|^2-q\int_\Omega v^{-q+1}|\nabla v|^{q-2}|D^2v|^2\\ &-q\int_\Omega v^{-q+1}|\nabla v|^{q-2}\nabla v\cdot\nabla(vw)\\ &-(q-1)\int_\Omega v^{-q}|\nabla v|^q\Delta v+(q-1)\int_\Omega wv^{-q+1}|\nabla v|^q\\ =&q(q-1)\int_\Omega v^{-q}|\nabla v|^{q-2}\nabla v\cdot\nabla|\nabla v|^2-q\int_\Omega v^{-q+1}|\nabla v|^{q-2}|D^2v|^2\\ &-\frac{q(q-2)}{4}\int_\Omega v^{-q+1}|\nabla v|^{q-4}|\nabla|\nabla v|^2|^2-q(q-1)\int_\Omega v^{-q-1}|\nabla v|^{q+2}\\ &+\frac{q}{2}\int_{\partial\Omega}v^{-q+1}|\nabla v|^{q-2}\cdot\frac{\partial|\nabla v|^2}{\partial\nu}+\frac{q(q-2)}{2}\int_\Omega wv^{-q+2}|\nabla v|^{q-4}\nabla v\cdot\nabla|\nabla v|^2\end{aligned}$$

$$+ q \int_\Omega w v^{-q+2} |\nabla v|^{q-2} \Delta v - (q-1)^2 \int_\Omega w v^{-q+1} |\nabla v|^q \tag{3.3}$$

for all $t \in (0, T_{\max})$. For the first four terms on the right of (3.3), we use the pointwise identity ([51, Lemma 3.2])

$$|D^2 \ln \varphi|^2 = -\frac{1}{\varphi^3} \nabla\varphi \cdot \nabla |\nabla \varphi|^2 + \frac{1}{\varphi^2} |D^2 \varphi|^2 + \frac{1}{\varphi^4} |\nabla \varphi|^4 \quad \text{for all positive } \varphi \in C^2(\overline{\Omega})$$

and $\nabla|\nabla v|^2 = 2D^2 v \cdot \nabla v$ to obtain

$$\begin{aligned} & q(q-1) \int_\Omega v^{-q} |\nabla v|^{q-2} \nabla v \cdot \nabla |\nabla v|^2 - q \int_\Omega v^{-q+1} |\nabla v|^{q-2} |D^2 v|^2 \\ & - \frac{q(q-2)}{4} \int_\Omega v^{-q+1} |\nabla v|^{q-4} |\nabla |\nabla v|^2|^2 - q(q-1) \int_\Omega v^{-q-1} |\nabla v|^{q+2} \\ = & - q(q-1) \int_\Omega v^{-q+3} |\nabla v|^{q-2} \left(-\frac{1}{v^3} \nabla v \cdot \nabla |\nabla v|^2 + \frac{1}{v^2} |D^2 v|^2 + \frac{1}{v^4} |\nabla v|^4 \right) \\ = & - q(q-1) \int_\Omega v^{-q+3} |\nabla v|^{q-2} |D^2 \ln v|^2 \end{aligned} \tag{3.4}$$

for all $t \in (0, T_{\max})$. The sixth and seventh summands on the right-hand side of (3.3) can be estimated as follows:

$$\begin{aligned} & \frac{q(q-2)}{2} \int_\Omega w v^{-q+2} |\nabla v|^{q-4} \nabla v \cdot \nabla |\nabla v|^2 + q \int_\Omega w v^{-q+2} |\nabla v|^{q-2} \Delta v \\ & \leq q \left(q - 2 + \sqrt{n} \right) \int_\Omega w v^{-q+2} |\nabla v|^{q-2} |D^2 v| \end{aligned} \tag{3.5}$$

for all $t \in (0, T_{\max})$, because of $|\Delta v| \leq \sqrt{n} \left| D^2 v \right|$. Inserting (3.4) and (3.5) into (3.3), we obtain (3.2). $\square$

In the following, we will drive a differential estimate of the type $\int_\Omega w^{p+1}$.

Lemma 3.3. *Let (1.6) hold. Then for all $p > 1$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega w^{p+1} + \frac{(p+1)\delta}{2} \int_\Omega w^{p+1} \leq \left(\frac{2}{\delta} \right)^p \int_\Omega u^{p+1} \quad \text{for all} \quad t \in (0, T_{\max}). \tag{3.6}$$

Proof. Testing the third equation in the model (1.5) by $(p+1) w^p$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega w^{p+1} + (p+1)\delta \int_\Omega w^{p+1} \leq (p+1) \int_\Omega w^p u \tag{3.7}$$

for all $t \in (0, T_{\max})$. Applying Young's inequality, we have

$$\begin{aligned}(p+1)\int_\Omega w^p u &\le \frac{(p+1)\delta}{2}\int_\Omega w^{p+1} + \left(\frac{2p}{(p+1)\delta}\right)^p \int_\Omega u^{p+1}\\ &\le \frac{(p+1)\delta}{2}\int_\Omega w^{p+1} + \left(\frac{2}{\delta}\right)^p \int_\Omega u^{p+1}\end{aligned} \tag{3.8}$$

for all $t \in (0, T_{\max})$. Plugging (3.8) into (3.7), we obtain (3.6). □

The following differential inequality of $\int_\Omega u^p + \int_\Omega v^{-q+1}|\nabla v|^q + \frac{8\kappa_1}{(q+2)\delta}\int_\Omega w^{\frac{q+2}{2}}$ with some $p > 1$, $q \ge 2$, $\kappa_1 > 0$ and $\delta > 0$ can be constructed.

Lemma 3.4. *Assume that (1.6) is valid, $\alpha \ge 1$ and $l > 1$. Then for all $p > 1$ and $q \ge 2$, there exists a constant $C > 0$ such that*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u^p + \int_\Omega v^{-q+1}|\nabla v|^q + \frac{8\kappa_1}{(q+2)\delta}\int_\Omega w^{\frac{q+2}{2}}\right) + \int_\Omega u^p + \int_\Omega v^{-q+1}|\nabla v|^q + \kappa_1\int_\Omega w^{\frac{q+2}{2}}\\ &\le \frac{p(p-1)}{2}\left(\frac{q(q-1)}{2p(p-1)(q+\sqrt{n})^2}\right)^{-\frac{2}{q}} \alpha^{\frac{2(q+2)}{q}} \|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(q+2)-2}{q}} \int_\Omega u^{\frac{p(q+2)}{q}}\\ &\quad + \frac{8\kappa_1}{(q+2)\delta}\left(\frac{2}{\delta}\right)^{\frac{q}{2}}\int_\Omega u^{\frac{q+2}{2}} - \frac{bp}{2}\int_\Omega u^{p+l-1} + C\end{aligned} \tag{3.9}$$

for all $t \in (0, T_{\max})$ with $\kappa_1 := \frac{8^{\frac{q}{2}} q(q+\sqrt{n}+1)^q (q-2+\sqrt{n})^{\frac{q+2}{2}}\|v_0\|_{L^\infty(\Omega)}}{(q-1)^{\frac{q}{2}}}$.

Proof. We only need to take a linear combination of (3.1) and (3.2) to get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u^p + \int_\Omega v^{-q+1}|\nabla v|^q\right) + \int_\Omega u^p + \int_\Omega v^{-q+1}|\nabla v|^q\\ &\quad + q(q-1)\int_\Omega v^{-q+3}|\nabla v|^{q-2}|D^2\ln v|^2\\ &\le \frac{\alpha^2 p(p-1)}{2}\int_\Omega u^p v^{\alpha-2}|\nabla v|^2 + (ap+1)\int_\Omega u^p - bp\int_\Omega u^{p+l-1}\\ &\quad + q(q-2+\sqrt{n})\int_\Omega w v^{-q+2}|\nabla v|^{q-2}|D^2 v|\\ &\quad + \frac{q}{2}\int_{\partial\Omega} v^{-q+1}|\nabla v|^{q-2}\cdot\frac{\partial|\nabla v|^2}{\partial\nu} + \int_\Omega v^{-q+1}|\nabla v|^q\end{aligned} \tag{3.10}$$

for all $t \in (0, T_{\max})$. We employ Young's inequality with any $\mu_1 > 0$ and Lemma 2.5 to estimate

$$\begin{aligned}\frac{\alpha^2 p(p-1)}{2}\int_\Omega u^p v^{\alpha-2}|\nabla v|^2 \le& \frac{p(p-1)}{2}\mu_1^{\frac{q+2}{2}}\int_\Omega v^{-q-1}|\nabla v|^{q+2}\\ &+\frac{p(p-1)}{2}\mu_1^{-\frac{q+2}{q}}\alpha^{\frac{2(q+2)}{q}}\int_\Omega u^{\frac{p(q+2)}{q}}v^{\frac{\alpha(q+2)-2}{q}}\\ \le& \frac{p(p-1)(q+\sqrt{n})^2}{2}\mu_1^{\frac{q+2}{2}}\int_\Omega v^{-q+3}|\nabla v|^{q-2}|D^2\ln v|^2\\ &+\frac{p(p-1)}{2}\mu_1^{-\frac{q+2}{q}}\alpha^{\frac{2(q+2)}{q}}\|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(q+2)-2}{q}}\int_\Omega u^{\frac{p(q+2)}{q}}\end{aligned} \tag{3.11}$$

for all $t \in (0, T_{\max})$. We pick $\mu_1 = \left(\frac{q(q-1)}{2p(p-1)(q+\sqrt{n})^2}\right)^{\frac{2}{q+2}}$ and use (3.11) to see that

$$\begin{aligned}\frac{\alpha^2 p(p-1)}{2}\int_\Omega u^p v^{\alpha-2}|\nabla v|^2 \le& \frac{q(q-1)}{4}\int_\Omega v^{-q+3}|\nabla v|^{q-2}|D^2\ln v|^2\\ &+\frac{p(p-1)}{2}\left(\frac{q(q-1)}{2p(p-1)(q+\sqrt{n})^2}\right)^{-\frac{2}{q}}\alpha^{\frac{2(q+2)}{q}}\|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(q+2)-2}{q}}\int_\Omega u^{\frac{p(q+2)}{q}}\end{aligned} \tag{3.12}$$

for all $t \in (0, T_{\max})$. Applying Young's inequality with any $\mu_2 > 0$ and $\mu_3 > 0$ and Lemma 2.5 to the fourth term on the right hand of (3.10), we have

$$\begin{aligned}&q(q-2+\sqrt{n})\int_\Omega wv^{-q+2}|\nabla v|^{q-2}|D^2 v|\\ \le&\mu_2\int_\Omega v^{-q+1}|\nabla v|^{q-2}|D^2 v|^2+\mu_2^{-1}q^2(q-2+\sqrt{n})^2\int_\Omega w^2v^{-q+3}|\nabla v|^{q-2}\\ \le&\mu_2(q+\sqrt{n}+1)^2\int_\Omega v^{-q+3}|\nabla v|^{q-2}|D^2\ln v|^2+\mu_2^{-1}q^2(q-2+\sqrt{n})^2\mu_3^{\frac{q+2}{q-2}}\int_\Omega v^{-q-1}|\nabla v|^{q+2}\\ &+\mu_2^{-1}q^2(q-2+\sqrt{n})^2\mu_3^{-\frac{q+2}{4}}\int_\Omega w^{\frac{q+2}{2}}v\\ \le&(q+\sqrt{n}+1)^2\left\{\mu_2+\mu_2^{-1}q^2(q-2+\sqrt{n})^2\mu_3^{\frac{q+2}{q-2}}\right\}\int_\Omega v^{-q+3}|\nabla v|^{q-2}|D^2\ln v|^2\\ &+\mu_2^{-1}q^2(q-2+\sqrt{n})^2\mu_3^{-\frac{q+2}{4}}\|v_0\|_{L^\infty(\Omega)}\int_\Omega w^{\frac{q+2}{2}}\end{aligned} \tag{3.13}$$

for all $t \in (0, T_{\max})$. Letting $\mu_2 = \frac{q(q-1)}{8(q+\sqrt{n}+1)^2}, \mu_3 = \left(\frac{(q-1)^2}{64(q+\sqrt{n}+1)^4(q-2+\sqrt{n})^2}\right)^{\frac{q-2}{q+2}}$ and using (3.13), we know that

$$\begin{aligned}q(q-2+\sqrt{n})\int_\Omega wv^{-q+2}|\nabla v|^{q-2}|D^2 v| \le& \frac{q(q-1)}{4}\int_\Omega v^{-q+3}|\nabla v|^{q-2}|D^2\ln v|^2\\ &+\frac{8^{\frac{q}{2}}q(q+\sqrt{n}+1)^q(q-2+\sqrt{n})^{\frac{q+2}{2}}\|v_0\|_{L^\infty(\Omega)}}{(q-1)^{\frac{q}{2}}}\int_\Omega w^{\frac{q+2}{2}}\end{aligned} \tag{3.14}$$

for all $t \in (0, T_{\max})$. Inserting (3.12) and (3.14) into (3.10), we conclude that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u^p + \int_\Omega v^{-q+1}|\nabla v|^q\right) + \int_\Omega u^p + \int_\Omega v^{-q+1}|\nabla v|^q \\
&\quad + \frac{q(q-1)}{2}\int_\Omega v^{-q+3}|\nabla v|^{q-2}|D^2 \ln v|^2 \\
&\le \frac{p(p-1)}{2}\left(\frac{q(q-1)}{2p(p-1)(q+\sqrt{n})^2}\right)^{-\frac{2}{q}} \alpha^{\frac{2(q+2)}{q}} \|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(q+2)-2}{q}} \int_\Omega u^{\frac{p(q+2)}{q}} \\
&\quad + \frac{8^{\frac{q}{2}} q(q+\sqrt{n}+1)^q (q-2+\sqrt{n})^{\frac{q+2}{2}} \|v_0\|_{L^\infty(\Omega)}}{(q-1)^{\frac{q}{2}}} \int_\Omega w^{\frac{q+2}{2}} \\
&\quad + \frac{q}{2}\int_{\partial\Omega} v^{-q+1}|\nabla v|^{q-2}\cdot\frac{\partial|\nabla v|^2}{\partial\nu} + \int_\Omega v^{-q+1}|\nabla v|^q + (ap+1)\int_\Omega u^p - bp\int_\Omega u^{p+l-1}
\end{aligned} \tag{3.15}$$

for all $t \in (0, T_{\max})$. An application of (2.2), Lemmata 2.5, 2.6 and Young's inequality shows that for any $\eta > 0$, there exists some $c_1 > 0$ such that

$$\begin{aligned}
\frac{q}{2}\int_{\partial\Omega} v^{-q+1}|\nabla v|^{q-2}\cdot\frac{\partial|\nabla v|^2}{\partial\nu} &\le \eta\int_\Omega v^{-q-1}|\nabla v|^{q+2} + \eta\int_\Omega v^{-q+1}|\nabla v|^{q-2}|D^2 v|^2 + c_1\int_\Omega v \\
&\le 2(q+\sqrt{n}+1)^2\eta\int_\Omega v^{-q+3}|\nabla v|^{q-2}|D^2\ln v|^2 + c_1|\Omega|\|v_0\|_{L^\infty(\Omega)}
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
\int_\Omega v^{-q+1}|\nabla v|^q &\le \eta^{\frac{q+2}{q}}\int_\Omega v^{-q-1}|\nabla v|^{q+2} + \eta^{-\frac{q+2}{2}}\int_\Omega v \\
&\le (q+\sqrt{n})^2\eta^{\frac{q+2}{q}}\int_\Omega v^{-q+3}|\nabla v|^{q-2}|D^2\ln v|^2 + \eta^{-\frac{q+2}{2}}|\Omega|\|v_0\|_{L^\infty(\Omega)}
\end{aligned} \tag{3.17}$$

for all $t \in (0, T_{\max})$. Since $l > 1$, by Young's inequality, we can find $c_2 > 0$ such that

$$(ap+1)\int_\Omega u^p - bp\int_\Omega u^{p+l-1} \le -\frac{bp}{2}\int_\Omega u^{p+l-1} + c_2 \tag{3.18}$$

for all $t \in (0, T_{\max})$. Choosing η appropriately small, plugging (3.16)–(3.18) into (3.15), we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u^p + \int_\Omega v^{-q+1}|\nabla v|^q\right) + \int_\Omega u^p + \int_\Omega v^{-q+1}|\nabla v|^q \\
&\le \frac{p(p-1)}{2}\left(\frac{q(q-1)}{2p(p-1)(q+\sqrt{n})^2}\right)^{-\frac{2}{q}} \alpha^{\frac{2(q+2)}{q}} \|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(q+2)-2}{q}} \int_\Omega u^{\frac{p(q+2)}{q}} \\
&\quad + \kappa_1\int_\Omega w^{\frac{q+2}{2}} - \frac{bp}{2}\int_\Omega u^{p+l-1} + C
\end{aligned} \tag{3.19}$$

for all $t \in (0, T_{\max})$ with $\kappa_1 := \frac{8^{\frac{q}{2}} q(q+\sqrt{n}+1)^q (q-2+\sqrt{n})^{\frac{q+2}{2}} \|v_0\|_{L^\infty(\Omega)}}{(q-1)^{\frac{q}{2}}}$. Letting $p := \frac{q}{2}$ in (3.6), and multiplying $\frac{8\kappa_1}{(q+2)\delta}$ in the both sides of (3.6), we have

$$\frac{8\kappa_1}{(q+2)\delta} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega w^{\frac{q+2}{2}} + 2\kappa_1 \int_\Omega w^{\frac{q+2}{2}} \leq \frac{8\kappa_1}{(q+2)\delta} \left(\frac{2}{\delta}\right)^{\frac{q}{2}} \int_\Omega u^{\frac{q+2}{2}} \tag{3.20}$$

for all $t \in (0, T_{\max})$. Substituting (3.20) into (3.19), we obtain (3.9). □

Our next plan is to deal with the first two integral terms on the right-hand side of (3.9).

Lemma 3.5. *Let $\alpha \geq 1$, $l > 1$, $p > 1$ and $q \geq 2$ be such that*

$$q > \frac{2p}{l-1}. \tag{3.21}$$

Then, there exists a constant $C > 0$ such that

$$\frac{p(p-1)}{2} \left(\frac{q(q-1)}{2p(p-1)(q+\sqrt{n})^2}\right)^{-\frac{2}{q}} \alpha^{\frac{2(q+2)}{q}} \|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(q+2)-2}{q}} \int_\Omega u^{\frac{p(q+2)}{q}} \leq \frac{bp}{4} \int_\Omega u^{p+l-1} + C \tag{3.22}$$

for all $t \in (0, T_{\max})$.

Proof. It follows from (3.21) that

$$p + l - 1 - \frac{p(q+2)}{q} = \frac{(l-1)q - 2p}{q} > 0.$$

Thus, we utilize Young's inequality to the first summand on the right hand of (3.9) to show the existence of $c > 0$ such that

$$\frac{p(p-1)}{2} \left(\frac{q(q-1)}{2p(p-1)(q+\sqrt{n})^2}\right)^{-\frac{2}{q}} \alpha^{\frac{2(q+2)}{q}} \|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(q+2)-2}{q}} \int_\Omega u^{\frac{p(q+2)}{q}} \leq \frac{bp}{4} \int_\Omega u^{p+l-1} + c$$

for all $t \in (0, T_{\max})$, which implies (3.22). □

Lemma 3.6. *Let $l > 1$, $p > 1$. Assume that $q \geq 2$ satisfies*

$$q < 2(p + l - 2). \tag{3.23}$$

Then, one can find a constant $C > 0$ such that

$$\frac{8\kappa_1}{(q+2)\delta} \left(\frac{2}{\delta}\right)^{\frac{q}{2}} \int_\Omega u^{\frac{q+2}{2}} \leq \frac{bp}{4} \int_\Omega u^{p+l-1} + C \tag{3.24}$$

for all $t \in (0, T_{\max})$, where κ_1 is given by (3.9).

Proof. Using (3.23), we obtain

$$p + l - 1 - \frac{q+2}{2} = p + l - 2 - \frac{q}{2} > 0.$$

Applying Young's inequality to the second term on the right of (3.9), one can find $c > 0$ satisfying

$$\frac{8\kappa_1}{(q+2)\delta} \left(\frac{2}{\delta}\right)^{\frac{q}{2}} \int_\Omega u^{\frac{q+2}{2}} \leq \frac{bp}{4} \int_\Omega u^{p+l-1} + c$$

for all $t \in (0, T_{\max})$, which immediately gives (3.24). □

In the case of $l > 2$, applying Lemma 3.4, we have the following result.

Lemma 3.7. *Let $l > 2$. Assume that (1.6) holds with some $\alpha \geq 1$. Then for all $p > 1$, there exists a constant $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^p(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}). \tag{3.25}$$

Proof. Since $l > 2$, we can readily obtain

$$2(p+l-2) - \frac{2p}{l-1} = \frac{2[(p+l-2)(l-1)-p]}{l-1} = \frac{2(l-2)(p+l-1)}{l-1} > 0.$$

Hence, for any $p > 1$, we can pick some $q \geq 2$ fulfilling

$$\frac{2p}{l-1} < q < 2(p+l-2). \tag{3.26}$$

It follows from Lemmata 3.4–3.6 and (3.26) that one can find $c_1 > 0$ such that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u^p + \int_\Omega v^{-q+1}|\nabla v|^q + \frac{8\kappa_1}{(q+2)\delta}\int_\Omega w^{\frac{q+2}{2}}\right)\\ &\quad + \min\left\{1, \frac{(q+2)\delta}{8}\right\}\left(\int_\Omega u^p + \int_\Omega v^{-q+1}|\nabla v|^q + \frac{8\kappa_1}{(q+2)\delta}\int_\Omega w^{\frac{q+2}{2}}\right) \leq c_1\end{aligned} \tag{3.27}$$

for all $t \in (0, T_{\max})$. Thus, a standard ODE comparison argument implies (3.25). □

In the case $l = 2$ and $n \geq 4$, we will derive L^p-bounds on u if b is sufficiently large.

Lemma 3.8. *Let $l = 2$, $n \geq 4$ and $p > 1$. Suppose that (1.6) holds, and that $b > 0$ satisfies*

$$b > \lambda_1(p,n)\alpha^{\frac{2(p+1)}{p}}\|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(p+1)-1}{p}} + \lambda_2(p,n,\delta)\|v_0\|_{L^\infty(\Omega)}, \tag{3.28}$$

where

$$\lambda_1(p,n) := (p-1)^{\frac{p+1}{p}}(2p+\sqrt{n})^{\frac{2}{p}}(2p-1)^{-\frac{1}{p}} \tag{3.29}$$

and

$$\lambda_2(p,n,\delta) := \frac{2^{4p+4}(2p+\sqrt{n}+1)^{2p}(2p-2+\sqrt{n})^{p+1}}{(p+1)(2p-1)^p\delta^{p+1}}. \tag{3.30}$$

Then, we infer the existence of $C > 0$ such that

$$\|u(\cdot,t)\|_{L^p(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max}). \tag{3.31}$$

Proof. Since $l = 2$, using (3.29) and (3.30), we apply Lemma 3.4 to $q := 2p$ to find

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u^p + \int_\Omega v^{-2p+1}|\nabla v|^{2p} + \frac{4\kappa_1}{(p+1)\delta}\int_\Omega w^{p+1}\right) + \int_\Omega u^p + \int_\Omega v^{-2p+1}|\nabla v|^{2p} + \kappa_1\int_\Omega w^{p+1}\\ &\leq \frac{p(p-1)}{2}\left(\frac{(2p-1)}{(p-1)(2p+\sqrt{n})^2}\right)^{-\frac{1}{p}}\alpha^{\frac{2(p+1)}{p}}\|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(p+1)-1}{p}}\int_\Omega u^{p+1}\\ &\quad + \frac{2^{4p+3}p(2p+\sqrt{n}+1)^{2p}(2p-2+\sqrt{n})^{p+1}\|v_0\|_{L^\infty(\Omega)}}{(p+1)(2p-1)^p\delta^{p+1}}\int_\Omega u^{p+1} - \frac{bp}{2}\int_\Omega u^{p+1} + C\\ &= -\frac{p}{2}\left\{b - \lambda_1(p,n)\alpha^{\frac{2(p+1)}{p}}\|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(p+1)-1}{p}} - \lambda_2(p,n)\|v_0\|_{L^\infty(\Omega)}\right\}\int_\Omega u^{p+1} + C\end{aligned} \tag{3.32}$$

for all $t \in (0, T_{\max})$. Using (3.28), we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u^p + \int_\Omega v^{-2p+1}|\nabla v|^{2p} + \frac{4\kappa_1}{(p+1)\delta}\int_\Omega w^{p+1}\right) \\ &\quad + \min\left\{1, \frac{(p+1)\delta}{4}\right\}\left(\int_\Omega u^p + \int_\Omega v^{-2p+1}|\nabla v|^2 p + \frac{4\kappa_1}{(p+1)\delta}\int_\Omega w^{p+1}\right) \le C \end{aligned} \tag{3.33}$$

for all $t \in (0, T_{\max})$. Therefore, by a standard ODE comparison argument, we have (3.31). □

Next, we only consider $n \in \{2, 3\}$ and $l = 2$, since the global bounded solution is ensured by (2.1) and Lemma 2.9 when $n = 1$.

Lemma 3.9. *Let $n \in \{2, 3\}$ and $l = 2$, and suppose that (1.6) holds. Then for all $T > 0$, one can find $C(T) > 0$ such that*

$$\|u(\cdot, t)\|_{L^2(\Omega)} \le C(T) \quad \text{for all } t \in (0, T_{\max}). \tag{3.34}$$

Proof. Letting $q = 2$ in (3.1), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2 + \int_\Omega v^\alpha|\nabla u|^2 + \int_\Omega u^2 \le \alpha^2\int_\Omega u^2 v^{\alpha-2}|\nabla v|^2 + (2a+1)\int_\Omega u^2 \tag{3.35}$$

for all $t \in (0, T_{\max})$. Due to $l = 2$, (2.5) in conjunction with Lemma 2.8 shows that for any $T > 0$, we can find $c_1(T) > 0$ fulfilling $v \ge c_1(T)$ in $\Omega \times (0, T)$. Therefore, (3.35) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2 + c_1^\alpha(T)\int_\Omega |\nabla u|^2 + \int_\Omega u^2 \le \alpha^2 c_2(T)\int_\Omega u^2|\nabla v|^2 + (2a+1)\int_\Omega u^2 \tag{3.36}$$

for all $t \in (0, T_{\max})$, where $c_2(T) := \max\left\{c_1^{\alpha-2}(T), \|v_0\|_{L^\infty(\Omega)}^{\alpha-2}\right\}$. Since $n \in \{2, 3\}$, $l = 2$ according to (2.5) and Lemma 2.3, we can find $c_3 > 0$ satisfying

$$\|\nabla v(\cdot, t)\|_{L^4(\Omega)} \le c_3 \quad \text{for all } t \in (0, T_{\max}). \tag{3.37}$$

Using $n \le 3$, (2.1), (3.37), Hölder's inequality and an Ehrling-type inequality, there exists a positive constant $c_4(T)$ such that

$$\begin{aligned} \alpha^2 c_2(T)\int_\Omega u^2|\nabla v|^2 + (2a+1)\int_\Omega u^2 &\le \alpha^2 c_2(T)\|u\|_{L^4(\Omega)}^2\|\nabla v\|_{L^4(\Omega)}^2 + (2a+1)\|u\|_{L^2(\Omega)}^2 \\ &\le \alpha^2 c_2(T)c_3^2\|u\|_{L^4(\Omega)}^2 + (2a+1)\|u\|_{L^2(\Omega)}^2 \le \frac{c_1^\alpha(T)}{2}\int_\Omega |\nabla u|^2 + c_4(T) \end{aligned} \tag{3.38}$$

for all $t \in (0, T_{\max})$. Plugging (3.38) into (3.36), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^2 + \frac{c_1^\alpha(T)}{2}\int_\Omega |\nabla u|^2 + \int_\Omega u^2 \le c_4(T) \quad \text{for all} \quad t \in (0, T_{\max}). \tag{3.39}$$

Thus, using an ODE argument, we complete the proof of (3.34). □

With the help of Lemma 2.9, the following result can be obtained.

Lemma 3.10. *Suppose that (1.6) and $\alpha \ge 1$ hold. For all $n \ge 1$, if one of the following cases holds:*

(i) $l > 2$;
(ii) $l = 2, n \le 3$;

(iii) $l = 2, n \geq 4$, *and* $b > \lambda_1(\frac{n}{2}, n)\alpha^{\frac{2(n+2)}{n}} \|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(n+2)-2}{n}} + \lambda_2(\frac{n}{2}, n, \delta)\|v_0\|_{L^\infty(\Omega)}$, *where* λ_1 *and* λ_2 *are as defined in* (3.29) *and* (3.30), *then for all* $T \in (0, T_{\max})$, *there exists* $C(T) > 0$ *fulfilling*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \leq C(T) \quad \textit{for all } t \in (0,T).$$

Proof. Let $p \geq 1$ such that $p > \frac{n}{2}$. In the case $l > 2$, Lemma 3.7 guarantees that $\|u(\cdot,t)\|_{L^p(\Omega)}$ is bounded for all $t \in (0, T_{\max})$. In the case $l = 2$, for any $b > 0$, in view of Lemma 3.9 and (2.1), we obtain that $\|u(\cdot,t)\|_{L^2(\Omega)}$ is bounded when $n \in \{2, 3\}$ and $\|u(\cdot,t)\|_{L^1(\Omega)}$ is bounded when $n = 1$. Whereas in the case $l = 2$ and $n \geq 4$, thanks to

$$b > \lambda_1\left(\frac{n}{2}, n\right)\alpha^{\frac{2(n+2)}{n}} \|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(n+2)-2}{n}} + \lambda_2\left(\frac{n}{2}, n, \delta\right)\|v_0\|_{L^\infty(\Omega)}$$

and the continuity of λ_1 and λ_2, there exists $p > \frac{n}{2}$ satisfying

$$b > \lambda_1(p, n)\alpha^{\frac{2(p+1)}{p}} \|v_0\|_{L^\infty(\Omega)}^{\frac{\alpha(p+1)-1}{p}} + \lambda_2(p, n, \delta)\|v_0\|_{L^\infty(\Omega)}.$$

Applying Lemma 3.8, we also see that $\|u(\cdot,t)\|_{L^p(\Omega)}$ is bounded for all $t \in (0, T_{\max})$. In conclusion, we utilize Lemma 2.9 to complete this proof. $\square$

Now, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Theorem 1.1 is a direct consequence of Lemmata 2.1 and 3.10. $\square$

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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