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Well-posedness of a nonlinear Hilfer fractional derivative model for the Antarctic circumpolar current

H. M. Srivastava, Kanika Dhawan, Ramesh Kumar Vats and Ankit Kumar Nain

Abstract. This article explores the Hilfer fractional derivative within the context of fractional differential equations and investigates a mathematical model formulated as a three-point boundary value problem (BVP). The primary focus is on the application of these models to analyze the jet flow of the Antarctic Circumpolar Current. The study establishes the existence of stream functions using Schaefer's fixed point theorem under the assumption of the continuity of the vorticity function Φ. Furthermore, the article delves into the existence and uniqueness results of the stream functions by employing the Banach fixed point theorem. This analysis is conducted under the condition that the vorticity function Φ is Lipschitz continuous with respect to the stream function. Additionally, the stability of the stream functions of the BVP is explored through Ulam–Hyers and generalized Ulam–Hyers stability analyses. In contrast to the foundational results presented for the three-point BVP, the article includes illustrative examples aimed at validating the findings.

Mathematics Subject Classification. Primary 26A33, 47B01, 47H10; Secondary 33B15, 34K20.

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1. Introduction

The Antarctic Circumpolar Current (ACC) is a significant oceanic current that follows a clockwise direction (as observed from the south pole) while moving from west to east around Antarctica, commonly referred to as the west-wind drift. Its crucial role in the global climate lies in facilitating water exchange between the Atlantic, Indian, and Pacific oceans. A comprehensive understanding of ACC transport during the Last Glacial Maximum is essential for accurately assessing ACC dynamics and past global climate changes (see [17,23]). Unlike a continuous flow, the ACC comprises thin jets spanning 40 to 50 kilometers in width, with average speeds exceeding 1 meter per second. The absence of land barriers in the latitude region of the Drake Passage results in the ACC being one of the world's most potent gyres. It exhibits a mean transit of approximately 134 ± 13 Sv through the Drake Passage, surpassing the combined transport of all the world's rivers by more than 100 times (see [3]).

Constantin and Johnson [6] introduced an approach that employed Euler's equation in conjunction with the integration of the equation of mass conservation and associated boundary conditions. This methodology resulted in a solution capable of accurately depicting the fundamental characteristics of gyres on the Earth's surface, regardless of their size. Subsequent to their work, Hus and Martin [16] delved into an exploration of solutions and the function pertaining to the ACC. Building upon these contributions, Marynets [20] reexamined the governing equation from Constantin and Johnson's study [6], transforming it into a second-order two-point boundary value problem (BVP) suitable for analyzing ocean flows devoid of azimuthal variations

$$\begin{cases} \phi''(\mathsf{s}) &= \mathsf{p}(\mathsf{s})\Phi\big(\mathsf{s},\phi(\mathsf{s})\big)+\mathsf{q}(\mathsf{s}), \mathsf{s}\in\mathcal{J}=[\mathfrak{e},\mathfrak{f}],\\ \phi(\mathfrak{e}) &= 0,\ \ \phi(\mathfrak{f})=0, \end{cases} \tag{1.1}$$

where $\Phi:[\mathfrak{e},\mathfrak{f}]\times\mathbb{R}\to\mathbb{R}$ denotes the given continuous nonlinear oceanic vorticity functions, $\mathfrak{e}\geqq 0$ and $\mathfrak{f}>\mathfrak{e}$. Also, $\mathsf{p},\mathsf{q}:[\mathfrak{e},\mathfrak{f}]\to\mathbb{R}$ are continuous functions given by

$$\mathsf{p}(\mathsf{s})=\frac{e^{\mathsf{s}}}{(1+e^{\mathsf{s}})^2}\quad\text{and}\quad\mathsf{q}(\mathsf{s})=-\frac{2\omega e^{\mathsf{s}}(e^{\mathsf{s}}-1)}{(1+e^{\mathsf{s}})^3},$$

and the Coriolis parameter is denoted by the dimensionless symbol $\omega>0$. The author employs an approach based on the topological transversality theorem to prove the existence of solutions for a class of oceanic vorticities. Chu and Marynets [5] further extended the work of [20] and studied the same second-order two-point BVP. The authors of [5] considered the semi-linear case and established their result by using the topological degree theory and weighted eigenvalues. Also, for the sub-linear and super-linear cases, the existence results were given by using FPTs. Wang et al. [33] studied nonlocal formulations for modeling ACC by using unknown functions to represent horizontal flow components without taking vertical motion into account.

$$\begin{cases} \phi''(\mathsf{s})=\mathsf{p}(\mathsf{s})\Phi\big(\mathsf{s},\phi(\mathsf{s})\big)+\mathsf{q}(\mathsf{s}),\quad \mathsf{s}\in\mathcal{J}=[\mathfrak{e},\mathfrak{f}],\\ \phi(\mathfrak{e})=\sum\limits_{i=1}^{m-2}\mathfrak{c}_i\phi(\varkappa_i)\ \text{ and }\ \phi(\mathfrak{f})=\sum\limits_{i=1}^{m-2}\mathfrak{d}_i\phi(\varkappa_i), \end{cases} \tag{1.2}$$

where $\varkappa_i (i=1,2,3,\ \cdots,m-2)$ satisfies the condition:

$$\mathfrak{e}<\varkappa_1<\varkappa_2<\cdots<\varkappa_{m-2}<\mathfrak{f},$$

$\mathfrak{c}_i$ and $\mathfrak{d}_i$ satisfy the condition:

$$\sum_{i=1}^{m-2}\mathfrak{c}_i=\sum_{i=1}^{m-2}\mathfrak{d}_i=1.$$

Using topological degree, the zero-exponent theory, and the fixed point method, the authors prove the existence of positive solutions to nonlocal BVP with nonlinear vorticity. The recent works on the gyre models for the modeling of the ACC can be seen in [7,36]. Also, the application of the fixed point theory can be seen in understanding the mathematical framework of the ACC.

For a second-order ordinary differential equation that describes the ACC's flow, Yang et al. [35] investigated the two-point BVP. In order to establish that the bounded solutions exist, Yang et al. [35] applied fixed point theory. For some class of oceanic vorticities, they found that the solutions are unique and radially symmetric. The model of the ACC's second-order elliptic equation with Dirichlet boundary was studied by Zhang et al. [37]. Nonlinear ocean vorticity with sub-critical growth and super-quadratic velocities leads to the presence of infinitely many solutions for the nonlinear elliptic equations using the truncation function and perturbation method.

While studying a particular phenomenon, Hilfer [14] introduced a new approach for the definition of fractional derivative, which is known as the Hilfer fractional derivative (HFD). It is a unification and generalization of the Riemann–Liouville (R-L) and the Liouville–Caputo (L-C) fractional derivative. In fact, HFD can be viewed as an interpolation between R-L and L-C fractional derivatives. Hilfer's derivative is more general and flexible, making it suitable for a wide range of applications, particularly where an interpolation between two types of fractional derivatives is beneficial. The Hadamard derivative is more specialized and often used in contexts where logarithmic behavior is prominent. It may be remarked in passing that, as already observed in the survey-cum-expository review articles by Srivastava (see, [26] and [27]), the so-called Caputo fractional derivative should be called the Liouville–Caputo fractional derivatives, thereby giving credits to Liouville who considered such fractional derivatives many decades earlier in 1832.

The HFD is used in the modeling of several theoretical fractional-order simulations in applied problems such as those involving dielectric relaxation in glass forming materials [15], a thermally sensitive resistor problem [31], and so on (see, for example, [1,2]). In the year 2012, Furati et al. [13] published the first article in which he proposed the qualitative analysis concerning the existence and uniqueness of the solution for the initial value problem (IVP) involving the HFD. For further theoretical developments on the HFD, see [9,10,14,22,29,32]. Some general families of fractional derivatives were introduced and applied in the recent works (see [26,28,30]).

The stability theory is also an important topic in several fields of mathematics and its widespread applications. It studies the behavior of solutions of differential equations and the trajectories of dynamical systems by using small perturbations. This type of stability analysis is more suitable to a dynamical system and quite appropriate in applications when it is not possible to find an exact solution (see [4]).

Positive solutions for the ACC's nonlinear model have been studied by Fečkan et al. [11], who also provided a qualitative examination of the model in terms of the UH stability. In the year 2022, Dhawan et al. [8] worked on the qualitative analysis for coupled Hilfer fractional differential equation (HFDE) with nonlocal conditions. For the purpose of modeling the jet flow of the ACC, Wang et al. [34] looked into the existence and uniqueness results where the vorticity function satisfies either a Lipschitz condition or is continuous. The presence of solutions for nonlinear elliptic equations representing the steady flow of the ACC was investigated by Fečkan et al. [12].

From the above discussion, we can observe that the qualitative analysis of the ACC models is quite significant in order to observe ocean currents. Also, the HFD has numerous applications. To the best of our knowledge, till now there is no such work on the qualitative analysis of the ACC model which involves the HFD. Therefore, motivated by the works discussed above, we consider the following ACC model involving HFD:

$$\begin{cases} D_{\mathfrak{e}+}^{\varrho,\varphi}\phi(\mathsf{s}) = \mathsf{p}(\mathsf{s})\Phi\big(\mathsf{s},\phi(\mathsf{s})\big) + \mathsf{q}(\mathsf{s}), & \mathsf{s}\in\mathscr{J}=[\mathfrak{e},\mathfrak{f}], \\ \phi(\mathfrak{e}) = 0 \;\text{ and }\; \phi(\mathfrak{f}) = {}^{\mathrm{RL}}I_{\mathfrak{e}+}^{1-\gamma}\phi(\eta), & \mathfrak{e}<\eta<\mathfrak{f}, \end{cases} \tag{1.3}$$

where $D_{\mathfrak{e}+}^{\varrho,\varphi}$ represents the HFD of order ϱ $(1<\varrho\leqq 2)$ and type φ $(0\leqq\varphi\leqq 1)$ and ${}^{\mathrm{RL}}I_{\mathfrak{e}+}^{1-\gamma}$ represent the RL fractional derivative of order $1-\gamma$, $\gamma=\varrho+2\varphi-\varrho\varphi$, $(1<\gamma\leq 2)$. Also, $\Phi:[\mathfrak{e},\mathfrak{f}]\times\mathbb{R}\to\mathbb{R}$ denotes the given continuous nonlinear oceanic vorticity functions, $\mathfrak{e}\geqq 0$ and $\mathfrak{f}>\mathfrak{e}$.

In our present investigation, we have used a variety of different approaches to obtain the primary conclusions, which differ from the methodologies used in [19,21]. The primary purpose of this study is to obtain the existence result for the Hilfer fractional differential equation (HFDE) with three-point boundary conditions by using the Schaefer FPT. Also, for the uniqueness of the solution to the investigated problem, the Banach fixed point theorem is used. In addition to the existence and uniqueness result, the *a priori* estimate of the solution is also obtained. Furthermore, by adding to the qualitative properties, the UH stability analysis is also proposed for the HFDE with three-point boundary conditions (1.3) following the methodology given by Rus [25].

The proposed work is structured as follows. In Sect. 2, a few preliminaries of fractional calculus and some basic lemmas are provided, which are required to analyze the mathematical model (1.3). Section 3 provides a few sufficient conditions for the analytical properties by using Schaefer's FPT and the Banach contraction principle and also obtains an *a priori* bound of the solution. In Sect. 4, we determine Ulam's stability of the solution of the FDE. Validation of our findings is provided by a number of appropriate illustrative examples in Sect. 5. Finally, in Sect 6, the conclusion of our present study has been given.

2. Preliminaries

Throughout this article, $\mathsf{E} = C([\mathfrak{e},\mathfrak{f}],\mathbb{R})$ is a Banach space of all continuous functions defined on $[\mathfrak{e},\mathfrak{f}] \to \mathbb{R}$, which is endowed with the norm given as follows:

$$\|\phi\|_{\mathsf{E}} = \sup\{|\phi(\mathsf{s})| : \mathsf{s} \in [\mathfrak{e},\mathfrak{f}]\}.$$

Definition 1. (see [18]and [24]) The R-L fractional integral operator ${}^{\mathrm{RL}}I^{\varrho}_{\mathfrak{e}+}$, for a function $\phi : [\mathfrak{e},\mathfrak{f}] \to \mathbb{R}$, is given by

$${}^{\mathrm{RL}}I^{\varrho}_{\mathfrak{e}+}\phi(\mathsf{s}) = \frac{1}{\Gamma(\varrho)} \int\limits_{\mathfrak{e}}^{\mathsf{s}} (\mathsf{s}-\tau)^{\varrho-1}\phi(\tau)\, d\tau,$$

provided that the integral exists, where Γ denotes the familiar gamma function.

Remark 1. (see [18]) **If $\mathfrak{p} > -1$ and $n > 0$, then**

$${}^{\mathrm{RL}}I^{n}_{0+}(x^{\mathfrak{p}}) = \frac{\Gamma(\mathfrak{p}+1)}{\Gamma(n+\mathfrak{p}+1)}\, x^{n+\mathfrak{p}}.$$

Definition 2. (see [18]and [24]) The R-L fractional derivative operator ${}^{\mathrm{RL}}D^{\varrho}_{\mathfrak{e}+}$ for a function ϕ, is defined as follows:

$${}^{\mathrm{RL}}D^{\varrho}_{\mathfrak{e}+}\phi(\mathsf{s}) = \frac{1}{\Gamma(n-\varrho)} \frac{d^n}{d\mathsf{s}^n} \int\limits_{\mathfrak{e}}^{\mathsf{s}} \frac{\phi(\tau)}{(\mathsf{s}-\tau)^{\varrho+1-n}}\, d\tau.$$

Definition 3. (see [18], [24]and [27]) The L-C fractional derivative operator ${}^{\mathrm{LC}}D^{\varrho}_{\mathfrak{e}+}$ of order ϱ, for a function $\phi : [\mathfrak{e},\mathfrak{f}] \to \mathbb{R}$, is defined by

$${}^{\mathrm{LC}}D^{\varrho}_{\mathfrak{e}+}\phi(\mathsf{s}) = \frac{1}{\Gamma(n-\varrho)} \int\limits_{\mathfrak{e}}^{\mathsf{s}} (\mathsf{s}-\tau)^{n-\varrho-1}\phi^{(n)}(\tau)\, d\tau, \quad n = [\varrho]+1,$$

provided that the integral exists, where $[\varrho]$ denotes the integer part of the real number ϱ and $\phi^{(n)}$ denotes the derivative of the integer order n.

Definition 4. (see [14];see also [26] and [29])

For $\phi \in \mathsf{E}$, the HFD of order ϱ and type φ is of the form given by

$$D^{\varrho,\varphi}_{\mathfrak{e}+}\phi(\mathsf{s}) = \left({}^{\mathrm{RL}}I^{\varphi(n-\varrho)}_{\mathfrak{e}+} \frac{d^n}{d\mathsf{s}^n}\big({}^{\mathrm{RL}}I^{(1-\varphi)(n-\varrho)}_{\mathfrak{e}+}\phi\big)\right)(\mathsf{s}), \quad v = \varrho + n\varphi - \varrho\varphi, \tag{2.1}$$

such that the expression on the right-hand side exists, where $\varrho \in \mathbb{R}$, $n-1 < \varrho \leqq n$, $n \in \mathbb{N}$, and $\varphi \in \mathbb{R}$ $(0 \leqq \varphi \leqq 1)$.

Remark 2. **If $\varphi = 0$, $n-1 < \varrho \leqq n$, then the HFD corresponds to the R-L fractional derivative and we have**

$$D^{\varrho,0}_{\mathfrak{e}+}\phi(\mathsf{s}) = \frac{d^n}{d\mathsf{s}^n}\left({}^{\mathrm{RL}}I^{n-\varrho}_{\mathfrak{e}+}\phi\right)(\mathsf{s}) = {}^{\mathrm{RL}}D^{\varrho}_{\mathfrak{e}+}\phi(\mathsf{s}).$$

Remark 3. **If $\varphi = 1$, $n-1 < \varrho \leqq n$, then the HFD corresponds to the L-C fractional derivative and we thus find that**

$$D^{\varrho,1}_{\mathfrak{e}+}\phi(\mathsf{s}) = {}^{\mathrm{RL}}I^{n-\varrho}_{\mathfrak{e}+}\frac{d^n}{d\mathsf{s}^n}\big(\phi(\mathsf{s})\big) = {}^{\mathrm{LC}}D^{\varrho}_{\mathfrak{e}+}\phi(\mathsf{s}).$$

Lemma 1. *[18] If $\phi \in C^n[\mathfrak{e},\mathfrak{f}]$ and $n-1 < \varrho \leqq n$, then* *(i)* ${}^{\mathrm{RL}}I^{\varrho}_{\mathfrak{e}+}\,{}^{\mathrm{RL}}D^{\varrho}_{\mathfrak{e}+}\phi(\mathsf{s}) = \phi(\mathsf{s}) - \sum\limits_{k=1}^{n} C_k(\mathsf{s}-\mathfrak{e})^{\varrho-k}$.

(ii) ${}^{\mathrm{RL}}D^{\varrho}_{\mathfrak{e}+}\,{}^{\mathrm{RL}}I^{\varrho}_{\mathfrak{e}+}\phi(\mathsf{s}) = \phi(\mathsf{s})$.

The following lemma addresses a linear version of the three-point BVP (1.3).

Lemma 2. *Let $\mathfrak{P} \in C([\mathfrak{e},\mathfrak{f}],\mathbb{R})$ be a given function. Then, the solution of the following three-point BVP:*

$$\begin{cases} D^{\varrho,\varphi}_{\mathfrak{e}+}\phi(\mathsf{s}) = \mathfrak{P}(\mathsf{s}), & \mathsf{s} \in \mathcal{J} = [\mathfrak{e},\mathfrak{f}], \\ \phi(\mathfrak{e}) = 0 \ \ \text{and} \ \ \phi(\mathfrak{f}) = {}^{\mathrm{RL}}I^{1-\gamma}_{\mathfrak{e}+}\phi(\eta), & \mathfrak{e} < \eta < \mathfrak{f} \end{cases} \tag{2.2}$$

is given by

$$\begin{aligned} \phi(\mathsf{s}) = {} & \frac{(\mathsf{s}-\mathfrak{e})^{\gamma-1}}{\mathfrak{A}}\left(\frac{1}{\Gamma(\varrho\varphi-2\varphi+1)}\int_{\mathfrak{e}}^{\eta}(\eta-\mathsf{w})^{\varrho\varphi-2\varphi}\mathfrak{P}(\mathsf{w})d\mathsf{w} - \frac{1}{\Gamma(\varrho)}\int_{\mathfrak{e}}^{\mathfrak{f}}(\mathfrak{f}-\mathsf{w})^{\varrho-1}\mathfrak{P}(\mathsf{w})\,d\mathsf{w}\right) \\ & + \frac{1}{\Gamma(\varrho)}\int_{\mathfrak{e}}^{\mathsf{s}}(\mathsf{s}-\mathsf{w})^{\varrho-1}\mathfrak{P}(\mathsf{w})\,d\mathsf{w}, \end{aligned} \tag{2.3}$$

where

$$\mathfrak{A} = (\mathfrak{f}-\mathfrak{e})^{\gamma-1} - \Gamma(\gamma) \quad \text{and} \quad \gamma = \varrho + 2\varphi - \varrho\varphi.$$

Proof. In order to solve the three-point BVP (2.2), we first apply the R-L integral of order ϱ on both sides of the given equation (2.2). We thus find that

$${}^{\mathrm{RL}}I^{\varrho}_{\mathfrak{e}+}D^{\varrho,\varphi}_{\mathfrak{e}+}\phi(\mathsf{s}) = {}^{\mathrm{RL}}I^{\varrho}_{\mathfrak{e}+}\mathfrak{P}(\mathsf{s}).$$

Since $2-\gamma = (2-\varrho)(1-\varphi)$ and $1 < \gamma \leqq 2$, by using Definition 4, we have

$${}^{\mathrm{RL}}I^{\varrho}_{\mathfrak{e}+}\,{}^{\mathrm{RL}}I^{\varphi(2-\varrho)}_{\mathfrak{e}+}D^2\left({}^{\mathrm{RL}}I^{(1-\varphi)(2-\varrho)}_{\mathfrak{e}+}\phi\right)(\mathsf{s}) = {}^{\mathrm{RL}}I^{\varrho}_{\mathfrak{e}+}\mathfrak{P}(\mathsf{s})$$

$${}^{\mathrm{RL}}I^{\gamma}_{\mathfrak{e}+}D^2\,{}^{\mathrm{RL}}I^{2-\gamma}_{\mathfrak{e}+} = {}^{\mathrm{RL}}I^{\varrho}_{\mathfrak{e}+}\mathfrak{P}(\mathsf{s}). \tag{2.4}$$

Now, applying Lemma 1 in the equation (2.4), we have

$$\begin{aligned} \phi(\mathsf{s}) - \sum_{k=1}^{2} C_k(\mathsf{s}-\mathfrak{e})^{\gamma-k} &= {}^{\mathrm{RL}}I^{\varrho}_{\mathfrak{e}+}\mathfrak{P}(\mathsf{s}), \\ \phi(\mathsf{s}) &= C_1(\mathsf{s}-\mathfrak{e})^{\gamma-1} + C_2(\mathsf{s}-\mathfrak{e})^{\gamma-2} + {}^{\mathrm{RL}}I^{\varrho}_{\mathfrak{e}+}\mathfrak{P}(\mathsf{s}). \end{aligned} \tag{2.5}$$

Using the boundary condition $\phi(\mathfrak{e}) = 0$ implies that $C_2 = 0$, so we get

$$\phi(\mathsf{s}) = C_1(\mathsf{s}-\mathfrak{e})^{\gamma-1} + {}^{\mathrm{RL}}I^{\varrho}_{\mathfrak{e}+}\mathfrak{P}(\mathsf{s}).$$

Again, using the another boundary condition $\phi(\mathfrak{f}) = {}^{\mathrm{RL}}I^{1-\gamma}_{\mathfrak{e}+}\phi(\eta)$, we have

$$\phi(\mathfrak{f}) = C_1(\mathfrak{f}-\mathfrak{e})^{\gamma-1} + {}^{\mathrm{RL}}I^{\varrho}_{\mathfrak{e}+}\mathfrak{P}(\mathfrak{f}).$$

We also have

$$\begin{aligned} {}^{\mathrm{RL}}I^{1-\gamma}_{\mathfrak{e}+}\phi(\eta) &= C_1\,{}^{\mathrm{RL}}I^{1-\gamma}_{\mathfrak{e}+}(\eta-\mathfrak{e})^{\gamma-1} + {}^{\mathrm{RL}}I^{1-\gamma}_{\mathfrak{e}+}\,{}^{\mathrm{RL}}I^{\varrho}_{\mathfrak{e}+}\mathfrak{P}(\eta) \\ &= C_1\Gamma(\gamma) + {}^{\mathrm{RL}}I^{\varrho\varphi-2\varphi+1}_{\mathfrak{e}+}\mathfrak{P}(\eta). \end{aligned}$$

On comparing the above equations, we obtain the value of C_1 as follows:

$$\phi(\mathfrak{f}) = {}^{\mathrm{RL}}I^{1-\gamma}_{\mathfrak{e}+}\phi(\eta)$$

$$C_1 = \frac{1}{\mathfrak{A}}\left({}^{\mathrm{RL}}I_{\mathfrak{e}+}^{\varrho\varphi-2\varphi+1}\mathfrak{P}(\eta) - {}^{\mathrm{RL}}I_{\mathfrak{e}+}^{\varrho}\mathfrak{P}(\mathfrak{f})\right).$$

Substituting the value of C_1 and C_2 in the equation (2.5), the desired result is obtained. □

3. Existence and uniqueness results

This section discusses the suggested three-point BVP and the fundamental conclusions proving its existence and uniqueness of the solution of (1.3). With the help of linear growth condition imposed on the nonlinear vorticity function, the result for the existence of the solution of the three-point BVP was established and the result for the uniqueness of the solution for three-point BVP was investigated using the Lipschitz condition. For this, operator $\mathbb{F} : C([\mathfrak{e},\mathfrak{f}],\mathbb{R}) \to C([\mathfrak{e},\mathfrak{f}],\mathbb{R})$ is defined as follows:

$$\begin{aligned}\mathbb{F}\big(\phi(\mathsf{s})\big) = \frac{(\mathsf{s}-\mathfrak{e})^{\gamma-1}}{\mathfrak{A}}\Bigg(&\frac{1}{\Gamma(\varrho\varphi-2\varphi+1)}\int_{\mathfrak{e}}^{\eta}(\eta-\mathsf{w})^{\varrho\varphi-2\varphi}[\mathsf{p}(\mathsf{w})\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)+\mathsf{q}(\mathsf{w})]\,d\mathsf{w}\\ &-\frac{1}{\Gamma(\varrho)}\int_{\mathfrak{e}}^{\mathfrak{f}}(\mathfrak{f}-\mathsf{w})^{\varrho-1}[\mathsf{p}(\mathsf{w})\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)+\mathsf{q}(\mathsf{w})]d\mathsf{w}\Bigg)\\ &+\frac{1}{\Gamma(\varrho)}\int_{\mathfrak{e}}^{\mathsf{s}}(\mathsf{s}-\mathsf{w})^{\varrho-1}[\mathsf{p}(\mathsf{w})\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)+\mathsf{q}(\mathsf{w})]\,d\mathsf{w},\end{aligned} \tag{3.1}$$

Therefore, three-point BVP is transformed into a fixed point problem $\mathsf{s} = \mathbb{F}(\mathsf{s})$. Observe that the fixed point of the operator $\mathbb{F}$ is the solution of the three-point BVP. We now present the necessary assumptions for proving the existence and uniqueness results:

(H1) For $\phi, \bar{\phi} \in \mathbb{R}$, there exist constants $\mathfrak{L}$, such that

$$|\Phi(\mathsf{s},\phi) - \Phi(\mathsf{s},\bar{\phi})| \leqq \mathfrak{L}|\phi-\bar{\phi}| \quad \forall\, \mathsf{s} \in [\mathfrak{e},\mathfrak{f}].$$

(H2) $\mathsf{p},\mathsf{q} : [\mathfrak{e},\mathfrak{f}] \to \mathbb{R}$ be continuous and there exists $\mathfrak{m}_1, \mathfrak{m}_2 > 0$ to be a constant such that

$$|\mathsf{p}(\mathsf{s})| < \mathfrak{m}_1 \text{ and } |\mathsf{q}(\mathsf{s})| < \mathfrak{m}_2.$$

(H3) There exist two real-valued functions $p_1, p_2 \in \mathbb{R}$ such that

$$|\Phi(\mathsf{s},\phi)| \leqq p_1 + p_2|\phi|$$

for $\mathsf{s} \in \mathcal{J}$ and $\phi \in \mathbb{R}$.

(H4) There exists $\phi_1 \in C([\mathfrak{e},\mathfrak{f}],\mathbb{R}^+)$ to be an increasing function and $K^* > 0$ such that

$${}^{\mathrm{RL}}I_{\mathfrak{e}+}^{\varrho}\phi_1(\mathsf{s}) \leqq K^*\phi_1(\mathsf{s}) \quad \text{for each } \mathsf{s} \in [\mathfrak{e},\mathfrak{f}]$$

Theorem 1. *Suppose that the hypotheses* (H2) *and* (H3) *are satisfied along with the condition* $1-\Lambda_2 > 0$. *Then, the three-point BVP* (1.3) *has a solution where*

$$\Lambda_2 = \frac{\mathfrak{m}_1 p_2(\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}(\mathfrak{f}-\mathfrak{e})^{\varrho-1}}{\Gamma(2-2\varphi+\varrho\varphi)} - \frac{\mathfrak{m}_1 p_2(\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)}\left(\frac{(\mathfrak{f}-\mathfrak{e})^{\varrho-1}}{|\mathfrak{A}|d_2}+1\right).$$

Proof. We prove that the solution of the three-point BVP (1.3) exists using Schaefer's FPT. For this first, we prove that $\mathbb{F}$ is continuous. Let $\{\phi_n\}$ be a sequence of functions that converges to ϕ.

$$|(\mathbb{F}\phi_n)(\mathsf{s}) - (\mathbb{F}\phi)(\mathsf{s})| \leqq \frac{(\mathsf{s}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\Bigg[\frac{1}{\Gamma(\varrho\varphi-2\varphi+1)}\int_{\mathfrak{e}}^{\eta}(\eta-\mathsf{w})^{\varrho\varphi-2\varphi}\Big(|\mathsf{p}(\mathsf{w})|\cdot\big|\Phi_n\big(\mathsf{w},\phi(\mathsf{w})\big)$$

$$
\begin{aligned}
&- \Phi\big(\mathsf{w}, \phi(\mathsf{w})\big)\big|\Big) \, d\mathsf{w} + \frac{1}{\Gamma(\varrho)} \int\limits_{\mathfrak{e}}^{\mathfrak{f}} (\mathfrak{f}-\mathsf{w})^{\varrho-1} \big[|\mathsf{p}(\mathsf{w})| \\
&\quad \cdot \big|\Phi_n\big(\mathsf{w}, \phi(\mathsf{w})\big) - \Phi\big(\mathsf{w}, \phi(\mathsf{w})\big)\big|\big] \, d\mathsf{w}\Big] \\
&+ \frac{1}{\Gamma(\varrho)} \int\limits_{\mathfrak{e}}^{\mathsf{s}} (\mathsf{s}-\mathsf{w})^{\varrho-1} \big[|\mathsf{p}(\mathsf{w})| \cdot \big|\Phi_n\big(\mathsf{w}, \phi(\mathsf{w})\big) - \Phi\big(\mathsf{w}, \phi(\mathsf{w})\big)\big|\big] \, d\mathsf{w} \\
\leqq\ & \frac{(\mathsf{s}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}| d_2} \bigg(\frac{1}{\Gamma(\varrho\varphi - 2\varphi + 1)} \int\limits_{\mathfrak{e}}^{\eta} (\eta-\mathsf{w})^{\varrho\varphi-2\varphi} \big[\mathfrak{m}_1 |\Phi_n\big(\mathsf{w}, \phi(\mathsf{w})\big) \\
&- \Phi\big(\mathsf{w}, \phi(\mathsf{w})\big)\big|\big] \, d\mathsf{w} + \frac{1}{\Gamma(\varrho)} \int\limits_{\mathfrak{e}}^{\mathfrak{f}} (\mathfrak{f}-\mathsf{w})^{\varrho-1} \\
&\quad \cdot \big[\mathfrak{m}_1 |\Phi_n(\mathsf{w}, \phi\big(\mathsf{w})\big) - \Phi\big(\mathsf{w}, \phi(\mathsf{w})\big)\big|\big] \, d\mathsf{w}\bigg) \\
&+ \frac{1}{\Gamma(\varrho)} \int\limits_{\mathfrak{e}}^{\mathsf{s}} (\mathsf{s}-\mathsf{w})^{\varrho-1} \big[\mathfrak{m}_1 \big|\Phi_n\big(\mathsf{w}, \phi(\mathsf{w})\big) - \Phi\big(\mathsf{w}, \phi(\mathsf{w})\big)\big|\big] \, d\mathsf{w} \\
\leqq\ & \frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}| d_2} \ \frac{\mathfrak{m}_1 \ [p_2(|\phi_n| - |\phi|)] \ (\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi - 2\varphi + 2)} \\
&+ \frac{\mathfrak{m}_1 \ [p_2(|\phi_n| - |\phi|)] \ (\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)} \left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}| d_2} + 1 \right) \\
\leqq\ & \frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}| d_2} \ \frac{\mathfrak{m}_1 \ p_2(\|\phi_n - \phi\|_{\mathsf{E}}) \ (\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi - 2\varphi + 2)} \\
&+ \frac{\mathfrak{m}_1 \ p_2(\|\phi_n - \phi\|_{\mathsf{E}}) \ (\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)} \left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}| d_2} + 1 \right).
\end{aligned}
$$

Since the function $\Phi\big(\mathsf{s}, \phi(\mathsf{s})\big)$ is continuous, we have $\|(\mathbb{F}\phi_n)(\mathsf{s}) - (\mathbb{F}\phi)(\mathsf{s})\|_{\mathsf{E}} \to 0$ as $n \to \infty$. Hence, the function $\mathbb{F}$ is continuous.

We next prove that $\mathbb{F}$ maps bounded sets in $C([\mathfrak{e}, \mathfrak{f}], \mathbb{R})$ into bounded sets. Indeed, for any $\mathfrak{r} > 0$, we define $\mathcal{B}_{\mathfrak{r}} = \{\phi \in \mathsf{E} : \|\phi\|_{\mathsf{E}} \leqq \mathfrak{r}\}$.

$$
\begin{aligned}
\big|\mathbb{F}\big(\phi(\mathsf{s})\big)\big| \leqq\ & \frac{(\mathsf{s}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}| d_2} \bigg(\frac{1}{\Gamma(\varrho\varphi - 2\varphi + 1)} \int\limits_{\mathfrak{e}}^{\eta} (\eta-\mathsf{w})^{\varrho\varphi-2\varphi} \big[|\mathsf{p}(\mathsf{w})| \cdot \big|\Phi\big(\mathsf{w}, \phi(\mathsf{w})\big)\big| + |\mathsf{q}(\mathsf{w})|\big] \, d\mathsf{w} \\
&+ \frac{1}{\Gamma(\varrho)} \int\limits_{\mathfrak{e}}^{\mathfrak{f}} (\mathfrak{f}-\mathsf{w})^{\varrho-1} \big[|\mathsf{p}(\mathsf{w})| \cdot \big|\Phi\big(\mathsf{w}, \phi(\mathsf{w})\big)\big| + |\mathsf{q}(\mathsf{w})|\big] \, d\mathsf{w}\bigg) \\
&+ \frac{1}{\Gamma(\varrho)} \int\limits_{\mathfrak{e}}^{\mathsf{s}} (\mathsf{s}-\mathsf{w})^{\varrho-1} \big[|\mathsf{p}(\mathsf{w})| \cdot |\Phi\big(\mathsf{w}, \phi(\mathsf{w})\big)\big| + |\mathsf{q}(\mathsf{w})|\big] \, d\mathsf{w} \\
\leqq\ & \frac{(\mathsf{s}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}| d_2} \bigg(\frac{1}{\Gamma(\varrho\varphi - 2\varphi + 1)} \int\limits_{\mathfrak{e}}^{\eta} (\eta-\mathsf{w})^{\varrho\varphi-2\varphi} [\mathfrak{m}_1(p_1 + p_2|\phi(\mathsf{s})|) + \mathfrak{m}_2] dw
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\varrho)}\int\limits_{\mathfrak{e}}^{\mathfrak{f}}(\mathfrak{f}-\mathsf{w})^{\varrho-1}[\mathfrak{m}_1(p_1+p_2|\phi(\mathsf{s})|)+\mathfrak{m}_2]d\mathsf{w}\Bigg)\\
&+\frac{1}{\Gamma(\varrho)}\int\limits_{\mathfrak{e}}^{\mathsf{s}}(\mathsf{s}-\mathsf{w})^{\varrho-1}[\mathfrak{m}_1(p_1+p_2|\phi(\mathsf{s})|)+\mathfrak{m}_2]d\mathsf{w}\\
\leqq\; &\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\frac{[\mathfrak{m}_1(p_1+p_2\|\phi\|_{\mathsf{E}})+\mathfrak{m}_2](\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi-2\varphi+2)}\\
&+\frac{[\mathfrak{m}_1(p_1+p_2\|\phi\|_{\mathsf{E}})+\mathfrak{m}_2](\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)}\left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}+1\right),
\end{aligned}\tag{3.2}
$$

which shows that $\mathbb{F}$ is uniformly bounded. Moreover, we prove that $\mathbb{F}(\mathcal{B}_{\mathfrak{r}})$ is equi-continuous. Let $0 \leqq \mathsf{s}_1 < \mathsf{s}_2 \leqq 1$ and $\phi \in \mathcal{B}_{\mathfrak{r}}$. Then, we have

$$
\begin{aligned}
|(\mathbb{F}\phi)(\mathsf{s}_2)-(\mathbb{F}\phi)(\mathsf{s}_1)| = &\frac{(\mathsf{s}_2-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\Bigg(\frac{1}{\Gamma(\varrho\varphi-2\varphi+1)}\int\limits_{\mathfrak{e}}^{\eta}(\eta-\mathsf{w})^{\varrho\varphi-2\varphi}\big[|\mathsf{p}(\mathsf{w})|\cdot\big|\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)\big|\\
&+|\mathsf{q}(\mathsf{w})|\big]\,d\mathsf{w}+\frac{1}{\Gamma(\varrho)}\int\limits_{\mathfrak{e}}^{\mathfrak{f}}(\mathfrak{f}-\mathsf{w})^{\varrho-1}\big[|\mathsf{p}(\mathsf{w})|\cdot\big|\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)\big|\\
&+|\mathsf{q}(\mathsf{w})|\big]\,d\mathsf{w}\Bigg)+\frac{1}{\Gamma(\varrho)}\int\limits_{\mathfrak{e}}^{\mathsf{s}_2}(\mathsf{s}-\mathsf{w})^{\varrho-1}\big[|\mathsf{p}(\mathsf{w})|\cdot\big|\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)\big|+|\mathsf{q}(\mathsf{w})|\big]\,d\mathsf{w}\\
&+\frac{(\mathsf{s}_1-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\Bigg(\frac{1}{\Gamma(\varrho\varphi-2\varphi+1)}\int\limits_{\mathfrak{e}}^{\eta}(\eta-\mathsf{w})^{\varrho\varphi-2\varphi}\big[|\mathsf{p}(\mathsf{w})|\cdot\big|\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)\big|\\
&+|\mathsf{q}(\mathsf{w})|\big]\,d\mathsf{w}+\frac{1}{\Gamma(\varrho)}\int\limits_{\mathfrak{e}}^{\mathfrak{f}}(\mathfrak{f}-\mathsf{w})^{\varrho-1}\big[|\mathsf{p}(\mathsf{w})|\cdot\big|\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)\big|+|\mathsf{q}(\mathsf{w})|\big]d\mathsf{w}\Bigg)\\
&+\frac{1}{\Gamma(\varrho)}\int\limits_{\mathfrak{e}}^{\mathsf{s}_1}(\mathsf{s}-\mathsf{w})^{\varrho-1}\big[|\mathsf{p}(\mathsf{w})|\cdot\big|\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)\big|+|\mathsf{q}(\mathsf{w})|\big]\,d\mathsf{w}\\
\leqq\; &\frac{(\mathsf{s}_2-\mathfrak{e})^{\gamma-1}-(\mathsf{s}_1-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\Bigg(\frac{1}{\Gamma(\varrho\varphi-2\varphi+1)}\int\limits_{\mathfrak{e}}^{\eta}(\eta-\mathsf{w})^{\varrho\varphi-2\varphi}\\
&\cdot\big[|\mathsf{p}(\mathsf{w})|\cdot\big|\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)\big|+|\mathsf{q}(\mathsf{w})|\big]\,d\mathsf{w}\\
&-\frac{1}{\Gamma(\varrho)}\int\limits_{\mathfrak{e}}^{\mathfrak{f}}(\mathfrak{f}-\mathsf{w})^{\varrho-1}\big[|\mathsf{p}(\mathsf{w})|\cdot\big|\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)\big|\\
&+|\mathsf{q}(\mathsf{w})|\big]\,d\mathsf{w}\Bigg)+\frac{1}{\Gamma(\varrho)}\int\limits_{\mathsf{s}_1}^{\mathsf{s}_2}(\mathsf{s}-\mathsf{w})^{\varrho-1}\big[|\mathsf{p}(\mathsf{w})|\cdot|\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)\big|+|\mathsf{q}(\mathsf{w})|\big]\,d\mathsf{w}\\
\leqq\; &\frac{(\mathsf{s}_2-\mathfrak{e})^{\gamma-1}-(\mathsf{s}_1-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\Bigg(\frac{[\mathfrak{m}_1(p_1+p_2\|\phi\|_{\mathsf{E}})+\mathfrak{m}_2]}{\Gamma(\varrho\varphi-2\varphi+2)}(\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}
\end{aligned}
$$

$$+\frac{[\mathfrak{m}_1(p_1+p_2\|\phi\|_{\mathsf{E}})+\mathfrak{m}_2]}{\Gamma(\varrho+1)}(\mathfrak{f}-\mathfrak{e})^{\varrho\varphi-2\varphi+1}\Bigg)$$
$$+\frac{[\mathfrak{m}_1(p_1+p_2\|\phi\|_{\mathsf{E}})+\mathfrak{m}_2]}{\Gamma(\varrho+1)}(\mathsf{s}_2-\mathsf{s}_1)^{\varrho}.$$

We thus obtain

$$\|(\mathbb{F}\phi)(\mathsf{s}_2)-(\mathbb{F}\phi)(\mathsf{s}_1)\|_{\mathsf{E}}\to 0 \text{ as } \mathsf{s}_2\to\mathsf{s}_1.$$

Since $\mathbb{F}(\mathcal{B}_{\mathfrak{r}})$ is equi-continuous, from the Arzelá–Ascoli theorem, the function $\mathbb{F}$ is completely continuous.

Lastly, we prove that the set $\Theta=\{\phi\in C([\mathfrak{e},\mathfrak{f}],\mathbb{R}):\phi=\theta\mathbb{F}(\phi)\text{ for some }\theta\in(0,1)\}$ is bounded.
Let $\phi\in\Theta$. Then, $\phi(\mathsf{s})=\theta\mathbb{F}(\phi)(\mathsf{s})$ for some $0<\theta<1$. Therefore, for each $\mathsf{s}\in[\mathfrak{e},\mathfrak{f}]$, we have

$$\begin{aligned}
|\phi(\mathsf{s})| &= |\theta\mathbb{F}(\phi)(\mathsf{s})|\\
&\leqq \frac{(\mathsf{s}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\Bigg(\frac{1}{\Gamma(\varrho\varphi-2\varphi+1)}\int_{\mathfrak{e}}^{\eta}(\eta-\mathsf{w})^{\varrho\varphi-2\varphi}\left[|\mathsf{p}(\mathsf{w})|\cdot\left|\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)\right|+|\mathsf{q}(\mathsf{w})|\right]d\mathsf{w}\\
&\quad+\frac{1}{\Gamma(\varrho)}\int_{\mathfrak{e}}^{\mathfrak{f}}(\mathfrak{f}-\mathsf{w})^{\varrho-1}\left[|\mathsf{p}(\mathsf{w})|\cdot\left|\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)\right|+|\mathsf{q}(\mathsf{w})|\right]d\mathsf{w}\Bigg)\\
&\quad+\frac{\theta}{\Gamma(\varrho)}\int_{\mathfrak{e}}^{\mathsf{s}}(\mathsf{s}-\mathsf{w})^{\varrho-1}\left[|\mathsf{p}(\mathsf{w})|\cdot\left|\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)\right|+|\mathsf{q}(\mathsf{w})|\right]d\mathsf{w}\\
&\leqq \frac{(\mathsf{s}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\Bigg(\frac{1}{\Gamma(\varrho\varphi-2\varphi+1)}\int_{\mathfrak{e}}^{\eta}(\eta-\mathsf{w})^{\varrho\varphi-2\varphi}[\mathfrak{m}_1(p_1+p_2|\phi(\mathsf{s})|)+\mathfrak{m}_2]d\mathsf{w}\\
&\quad+\frac{1}{\Gamma(\varrho)}\int_{\mathfrak{e}}^{\mathfrak{f}}(\mathfrak{f}-\mathsf{w})^{\varrho-1}[\mathfrak{m}_1(p_1+p_2|\phi(\mathsf{s})|)+\mathfrak{m}_2]d\mathsf{w}\Bigg)\\
&\quad+\frac{1}{\Gamma(\varrho)}\int_{\mathfrak{e}}^{\mathsf{s}}(\mathsf{s}-\mathsf{w})^{\varrho-1}[\mathfrak{m}_1(p_1+p_2|\phi(\mathsf{s})|)+\mathfrak{m}_2]d\mathsf{w}\\
&\leqq \frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\frac{[\mathfrak{m}_1(p_1+p_2\|\phi\|_{\mathsf{E}})+\mathfrak{m}_2](\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi-2\varphi+2)}\\
&\quad+\frac{[\mathfrak{m}_1(p_1+p_2\|\phi\|_{\mathsf{E}})+\mathfrak{m}_2](\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)}\left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}+1\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|\phi\|_{\mathsf{E}} &\leqq \frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\frac{[\mathfrak{m}_1(p_1+p_2\|\phi\|_{\mathsf{E}})+\mathfrak{m}_2](\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi-2\varphi+2)}\\
&\quad+\frac{[\mathfrak{m}_1(p_1+p_2\|\phi\|_{\mathsf{E}})+\mathfrak{m}_2](\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)}\left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}+1\right)\\
\|\phi\|_{\mathsf{E}}&\left(1-\frac{\mathfrak{m}_1\ p_2\ (\mathfrak{f}-\mathfrak{e})^{\gamma-1}(\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi-2\varphi+2)}-\frac{\mathfrak{m}_1\ p_2\ (\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)}\left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}+1\right)\right)\\
&\leqq \frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\frac{[\mathfrak{m}_1p_1+\mathfrak{m}_2](\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi-2\varphi+2)}+\frac{[\mathfrak{m}_1p_1++\mathfrak{m}_2](\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)}\left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}+1\right)
\end{aligned}$$

$$\|\phi\|_{\mathsf{E}} \leqq \frac{\Lambda_1}{1-\Lambda_2},$$

where

$$\Lambda_1 = \frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2} \frac{[\mathfrak{m}_1 p_1 + \mathfrak{m}_2](\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi-2\varphi+2)} + \frac{[\mathfrak{m}_1 p_1 + \mathfrak{m}_2](\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)} \left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2} + 1 \right)$$

and

$$\Lambda_2 = \frac{\mathfrak{m}_1 \ p_2 \ (\mathfrak{f}-\mathfrak{e})^{\gamma-1}(\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi-2\varphi+2)} - \frac{\mathfrak{m}_1 \ p_2 \ (\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)} \left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2} + 1 \right).$$

Hence, clearly, the set Θ is bounded. Therefore, by referring to Schaefer's FPT, $\mathbb{F}$ has a fixed point which is a solution of (1.3). □

Theorem 2. *If the conditions* (H1) *and* (H2) *are satisfied along with the condition* $\mathfrak{N} < 1$, *then the BVP* (1.3) *has a unique solution on* $[\mathfrak{e}, \mathfrak{f}]$, *where*

$$\mathfrak{N} = \frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2} \frac{\mathfrak{m}_1 \mathfrak{L}(\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi-2\varphi+2)} + \frac{\mathfrak{m}_1 \mathfrak{L}(\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)} \left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2} + 1 \right).$$

Proof. By applying the Banach FPT, we shall show that the function $\mathbb{F}$ has a unique fixed point. Let

$$\mathcal{M} = \sup_{\mathsf{s}\in[\mathfrak{e},\mathfrak{f}]} |\Phi(\mathsf{s},0)| < \infty$$

and suppose that $\phi, \psi \in \mathsf{E}$. Then, for $\mathsf{s} \in [\mathfrak{e}, \mathfrak{f}]$, we have

$$\begin{aligned}
|(\mathbb{F}\phi)(\mathsf{s}) - (\mathbb{F}\psi)(\mathsf{s})| \leqq{}& \frac{(\mathsf{s}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2} \left(\frac{1}{\Gamma(\varrho\varphi-2\varphi+1)} \int_{\mathfrak{e}}^{\eta} (\eta-\mathsf{w})^{\varrho\varphi-2\varphi} \big[|\mathsf{p}(\mathsf{w})| \cdot \big| \Phi\big(\mathsf{w}, \phi(\mathsf{w})\big) \right. \\
&- \Phi(\mathsf{w}, \psi(\mathsf{w}))\big|\big] \, d\mathsf{w} + \frac{1}{\Gamma(\varrho)} \int_{\mathfrak{e}}^{\mathfrak{f}} (\mathfrak{f}-\mathsf{w})^{\varrho-1} \big[|\mathsf{p}(\mathsf{w})| \\
&\left. \cdot \big| \Phi(\mathsf{w}, \phi(\mathsf{w})) - \Phi\big(\mathsf{w}, \psi(\mathsf{w})\big) \big| \big] \, d\mathsf{w} \right) \\
&+ \frac{1}{\Gamma(\varrho)} \int_{\mathfrak{e}}^{\mathsf{s}} (\mathsf{s}-\mathsf{w})^{\varrho-1} \big[|\mathsf{p}(\mathsf{w})| \cdot \big| \Phi\big(\mathsf{w}, \phi(\mathsf{w})\big) - \Phi\big(\mathsf{w}, \psi(\mathsf{w})\big) \big| \big] \, d\mathsf{w} \\
\leqq{}& \frac{(\mathsf{s}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2} \left(\frac{1}{\Gamma(\varrho\varphi-2\varphi+1)} \int_{\mathfrak{e}}^{\eta} (\eta-\mathsf{w})^{\varrho\varphi-2\varphi} \big[\mathfrak{m}_1 \big| \Phi\big(\mathsf{w}, \phi(\mathsf{w})\big) \right. \\
&- \Phi\big(\mathsf{w}, \psi(\mathsf{w})\big)\big|\big] \, d\mathsf{w} + \frac{1}{\Gamma(\varrho)} \int_{\mathfrak{e}}^{\mathfrak{f}} (\mathfrak{f}-\mathsf{w})^{\varrho-1} \\
&\left. \cdot \big[\mathfrak{m}_1 \big| \Phi\big(\mathsf{w}, \phi(\mathsf{w})\big) - \Phi\big(\mathsf{w}, \psi(\mathsf{w})\big) \big| \big] \, d\mathsf{w} \right) \\
&+ \frac{1}{\Gamma(\varrho)} \int_{\mathfrak{e}}^{\mathsf{s}} (\mathsf{s}-\mathsf{w})^{\varrho-1} \big[\mathfrak{m}_1 \big| \Phi\big(\mathsf{w}, \phi(\mathsf{w})\big) - \Phi\big(\mathsf{w}, \psi(\mathsf{w})\big) \big| \big] \, d\mathsf{w} \\
\leqq{}& \left[\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2} \frac{\mathfrak{m}_1 \mathfrak{L} \ (\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi-2\varphi+2)} + \frac{\mathfrak{m}_1 \mathfrak{L} \ (\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)} \right.
\end{aligned}$$

$$\cdot \left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2} + 1 \right) \Bigg] \|\phi - \psi\|_{\mathsf{E}},$$

which implies that

$$\|\mathbb{F}\phi - \mathbb{F}\psi\|_{\mathsf{E}} \leqq \mathfrak{N} \|\phi - \psi\|_{\mathsf{E}}.$$

Since $\mathfrak{N} < 1$, we see that $\mathbb{F}$ is a contraction. Therefore, the Banach FPT allows us to conclude that $\mathbb{F}$ has a fixed point that is the unique solution to the three-point BVP (1.3). □

The next result, which gives us the *a priori* bound of the solution, can be obtained readily at this point.

3.1. The *a priori* bound of the solution

In light of the assumptions made in Theorem 2, it is possible for us to derive an *a priori* bound on the solution $\phi(\mathsf{s})$ of (1.3). Let

$$\mathcal{B}_{\mathfrak{r}} = \{\phi \in \mathsf{E} : \|\phi\|_{\mathsf{E}} \leqq \mathfrak{r}\}$$

be the set of bounded continuous functions in $[\mathfrak{e}, \mathfrak{f}]$ with bound $\mathfrak{r} > 0$. We can estimate a value of $\mathfrak{r}$ as follows.

Assuming that the unique solution $\phi(\mathsf{s})$ is in $\mathcal{B}_{\mathfrak{r}}$, we have

$$\begin{aligned}
|\phi(\mathsf{s})| &= |\mathbb{F}(\phi)(\mathsf{s})| \\
&\leqq \frac{(\mathsf{s}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2} \Bigg(\frac{1}{\Gamma(\varrho\varphi - 2\varphi + 1)} \int_{\mathfrak{e}}^{\eta} (\eta - \mathsf{w})^{\varrho\varphi - 2\varphi} \left[|\mathsf{p}(\mathsf{w})| \cdot \left|\Phi\big(\mathsf{w}, \phi(\mathsf{w})\big)\right| + |\mathsf{q}(\mathsf{w})| \right] d\mathsf{w} \\
&\quad + \frac{1}{\Gamma(\varrho)} \int_{\mathfrak{e}}^{\mathfrak{f}} (\mathfrak{f} - \mathsf{w})^{\varrho-1} \left[|\mathsf{p}(\mathsf{w})| \cdot \left|\Phi\big(\mathsf{w}, \phi(\mathsf{w})\big)\right| + |\mathsf{q}(\mathsf{w})| \right] d\mathsf{w} \Bigg) \\
&\quad + \frac{1}{\Gamma(\varrho)} \int_{\mathfrak{e}}^{\mathsf{s}} (\mathsf{s} - \mathsf{w})^{\varrho-1} \left[|\mathsf{p}(\mathsf{w})| \cdot \left|\Phi\big(\mathsf{w}, \phi(\mathsf{w})\big)\right| + |\mathsf{q}(\mathsf{w})| \right] d\mathsf{w} \\
&\leqq \frac{(\mathsf{s}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2} \Bigg(\frac{1}{\Gamma(\varrho\varphi - 2\varphi + 1)} \int_{\mathfrak{e}}^{\eta} (\eta - \mathsf{w})^{\varrho\varphi - 2\varphi} \Big[|\mathsf{p}(\mathsf{w})| \cdot \left|\Phi\big(\mathsf{w}, \phi(\mathsf{w})\big) - \Phi(\mathsf{w}, 0)\right| \\
&\quad + |\Phi(\mathsf{w}, 0)| + |\mathsf{q}(\mathsf{w})| \Big] d\mathsf{w} \\
&+ \frac{1}{\Gamma(\varrho)} \int_{\mathfrak{e}}^{\mathfrak{f}} (\mathfrak{f} - \mathsf{w})^{\varrho-1} \left[|\mathsf{p}(\mathsf{w})| \cdot \left|\Phi\big(\mathsf{w}, \phi(\mathsf{w})\big) - \Phi(\mathsf{w}, 0)\right| + |\Phi(\mathsf{w}, 0)| + |\mathsf{q}(\mathsf{w})| \right] d\mathsf{w} \Bigg) \\
&\quad + \frac{1}{\Gamma(\varrho)} \int_{\mathfrak{e}}^{\mathsf{s}} (\mathsf{s} - \mathsf{w})^{\varrho-1} \left[|\mathsf{p}(\mathsf{w})| \cdot \left|\Phi\big(\mathsf{w}, \phi(\mathsf{w})\big) - \Phi(\mathsf{w}, 0)\right| + \left|\Phi(\mathsf{w}, 0)\right| + |\mathsf{q}(\mathsf{w})| \right] d\mathsf{w} \\
&\leqq \frac{(\mathsf{s}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2} \Bigg(\frac{1}{\Gamma(\varrho\varphi - 2\varphi + 1)} \int_{\mathfrak{e}}^{\eta} (\eta - \mathsf{w})^{\varrho\varphi - 2\varphi} [\mathfrak{m}_1(\mathfrak{L}|\phi(\mathsf{s})| + \mathcal{M}) + \mathfrak{m}_2] d\mathsf{w} \\
&\quad + \frac{1}{\Gamma(\varrho)} \int_{\mathfrak{e}}^{\mathfrak{f}} (\mathfrak{f} - \mathsf{w})^{\varrho-1} [\mathfrak{m}_1(\mathfrak{L}|\phi(\mathsf{s})| + \mathcal{M}) + \mathfrak{m}_2] d\mathsf{w} \Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\varrho)}\int_{\mathfrak{e}}^{\mathsf{s}}(\mathsf{s}-\mathsf{w})^{\varrho-1}[\mathfrak{m}_1(\mathfrak{L}|\phi(\mathsf{s})|+\mathcal{M})+\mathfrak{m}_2]d\mathsf{w}\\
&\leqq \frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\frac{[\mathfrak{m}_1(\mathfrak{L}|\|\phi\|_{\mathsf{E}}+\mathcal{M})+\mathfrak{m}_2](\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi-2\varphi+2)}\\
&\quad+\frac{[\mathfrak{m}_1(\mathfrak{L}|\|\phi\|_{\mathsf{E}}+\mathcal{M})+\mathfrak{m}_2](\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)}\left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}+1\right)\\
&\leqq \frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\frac{[\mathfrak{m}_1(\mathfrak{L}\mathfrak{r}+\mathcal{M})+\mathfrak{m}_2](\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi-2\varphi+2)}\\
&\quad+\frac{[\mathfrak{m}_1(\mathfrak{L}\mathfrak{r}+\mathcal{M})+\mathfrak{m}_2](\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)}\left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}+1\right)
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
\|\phi(\mathsf{s})\|_{\mathsf{E}} \leqq{}& \frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\frac{[\mathfrak{m}_1(\mathfrak{L}\mathfrak{r}+\mathcal{M})+\mathfrak{m}_2](\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi-2\varphi+2)}\\
&+\frac{[\mathfrak{m}_1(\mathfrak{L}\mathfrak{r}+\mathcal{M})+\mathfrak{m}_2](\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)}\left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}+1\right).
\end{aligned}
$$

We now let $\mathfrak{r}>0$ be a real number such that

$$
\mathfrak{r} \geqq \frac{\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\frac{[\mathfrak{m}_1\mathcal{M}+\mathfrak{m}_2](\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi-2\varphi+2)}+\frac{[\mathfrak{m}_1\mathcal{M}+\mathfrak{m}_2](\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)}\left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}+1\right)}{1-\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\frac{\mathfrak{m}_1\mathfrak{L}(\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi-2\varphi+2)}+\frac{\mathfrak{m}_1\mathfrak{L}(\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)}\left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}+1\right)}.
$$

This provides an *a priori* bound for $\phi(\mathsf{s})$.

4. Stability results

In this section, we analyze the stability of the solution of HFDE by using the UH stability and the generalized UH stability. Implementing the methodology demonstrated in [25], we present the following definitions of the UH and the generalized UH stabilities conformable to the BVP (1.3).

Definition 5. The three-point BVP (1.3) is called UH stable whenever there exists a real constant $G_{\Phi}>0$ such that, for each $\delta>0$ and for each solution $\phi\in\mathsf{E}$ of the following inequality:

$$
|D^{\varrho,\varphi}_{\mathfrak{e}+}\phi(\mathsf{s})-\mathsf{p}(\mathsf{s})\Phi\big(\mathsf{s},\phi(\mathsf{s})\big)-\mathsf{q}(\mathsf{s})|\leqq\delta, \tag{4.1}
$$

there is a solution $\phi'\in\mathsf{E}$ of the system (1.3) such that

$$
|\phi(\mathsf{s})-\phi'(\mathsf{s})|\leqq G_{\Phi}\delta,\quad \mathsf{s}\in[\mathfrak{e},\mathfrak{f}].
$$

Definition 6. The system (1.3) is called a generalized UH stable whenever there exists a function $\Upsilon_{\Phi}\in C(\mathbb{R}^+,\ \mathbb{R}^+)$, and $\Upsilon_{\Phi}(0)=0$, such that, for each $\delta>0$ and for each solution $\phi\in\mathsf{E}$ of the following inequality:

$$
|D^{\varrho,\varphi}_{\mathfrak{e}+}\phi(\mathsf{s})-\mathsf{p}(\mathsf{s})\Phi\big(\mathsf{s},\phi(\mathsf{s})\big)-\mathsf{q}(\mathsf{s})|\leqq\delta, \tag{4.2}
$$

there is a solution $\phi'\in\mathsf{E}$ of the system (1.3) such that

$$
|\phi(\mathsf{s})-\phi'(\mathsf{s})|\leqq\Upsilon_{\Phi}(\delta),\quad \mathsf{s}\in[\mathfrak{e},\mathfrak{f}].
$$

Remark 4. **A function $\phi\in\mathsf{E}$ is the solution of the inequality (4.1) if and only if there exists a function $\mathfrak{g}\in\mathsf{E}$, which depends on ϕ, such that**

(i) $|\mathfrak{g}(\mathsf{s})|\leqq\delta$**.**

(ii) $D_{\mathfrak{e}+}^{\varrho;\varphi}\phi(\mathsf{s}) = \mathsf{p}(\mathsf{s})\Phi\big(\mathsf{s},\phi(\mathsf{s})\big) + \mathsf{q}(\mathsf{s}) + \mathfrak{g}(\mathsf{s})$.

Theorem 3. *If all of the assumptions of Theorem 2 are satisfied, then the three-point* BVP (1.3) *is UH stable.*

Proof. Let us consider $\delta > 0$ and $\phi(\mathsf{s}) \in \mathsf{E}$ such that

$$|D_{\mathfrak{e}+}^{\varrho;\varphi}\phi(\mathsf{s}) - \mathsf{p}(\mathsf{s})\Phi\big(\mathsf{s},\phi(\mathsf{s})\big) - \mathsf{q}(\mathsf{s})| \leqq \delta.$$

Then, owing to Remark (1), there is a continuous function $\mathfrak{g}$ such that

$$D_{\mathfrak{e}+}^{\varrho;\varphi}\phi(\mathsf{s}) = \mathsf{p}(\mathsf{s})\Phi\big(\mathsf{s},\phi(\mathsf{s})\big) + \mathsf{q}(\mathsf{s}) + \mathfrak{g}(\mathsf{s})$$

and

$$|\mathfrak{g}(\mathsf{s})| \leqq \delta$$

for all $\mathsf{s} \in [\mathfrak{e},\mathfrak{f}]$. We thus have

$$\begin{aligned}\phi(\mathsf{s}) = \frac{(\mathsf{s}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\Bigg(&\frac{1}{\Gamma(\varrho\varphi-2\varphi+1)}\int_{\mathfrak{e}}^{\eta}(\eta-\mathsf{w})^{\varrho\varphi-2\varphi}\big[\mathsf{p}(\mathsf{w})\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)+\mathsf{q}(\mathsf{w})+\mathfrak{g}(\mathsf{w})\big]\,d\mathsf{w}\\ &-\frac{1}{\Gamma(\varrho)}\int_{\mathfrak{e}}^{\mathfrak{f}}(\mathfrak{f}-\mathsf{w})^{\varrho-1}\big[\mathsf{p}(\mathsf{w})\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)+\mathsf{q}(\mathsf{w})+\mathfrak{g}(\mathsf{w})\big]\,d\mathsf{w}\Bigg)\\ &+\frac{1}{\Gamma(\varrho)}\int_{\mathfrak{e}}^{\mathsf{s}}(\mathsf{s}-\mathsf{w})^{\varrho-1}\big[\mathsf{p}(\mathsf{w})\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)+\mathsf{q}(\mathsf{w})+\mathfrak{g}(\mathsf{w})\big]\,d\mathsf{w},\end{aligned}$$

As we see from Theorem 2, there exists a unique solution $\phi' \in \mathsf{E}$ for the three-point BVP (1.3). Therefore, we get

$$\begin{aligned}|\phi(\mathsf{s})-\phi'(\mathsf{s})| \leqq \frac{(\mathsf{s}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\Bigg(&\frac{1}{\Gamma(\varrho\varphi-2\varphi+1)}\int_{\mathfrak{e}}^{\eta}(\eta-\mathsf{w})^{\varrho\varphi-2\varphi}\big[|\mathsf{p}(\mathsf{w})|\cdot\big|\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)-\Phi\big(\mathsf{w},\phi'(\mathsf{w})\big)\big|\\ &+|\mathfrak{g}(\mathsf{w})|\big]\,d\mathsf{w}-\frac{1}{\Gamma(\varrho)}\int_{\mathfrak{e}}^{\mathfrak{f}}(\mathfrak{f}-\mathsf{w})^{\varrho-1}\big[|\mathsf{p}(\mathsf{w})|\cdot\big|\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)-\Phi\big(\mathsf{w},\phi'(\mathsf{w})\big)\big|+|\mathfrak{g}(\mathsf{w})|\big]\,d\mathsf{w}\Bigg)\\ &+\frac{1}{\Gamma(\varrho)}\int_{\mathfrak{e}}^{\mathsf{s}}(\mathsf{s}-\mathsf{w})^{\varrho-1}\big[|\mathsf{p}(\mathsf{w})|\cdot\big|\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)-\Phi\big(\mathsf{w},\phi'(\mathsf{w})\big)\big|+|\mathfrak{g}(\mathsf{w})|\big]\,d\mathsf{w}\\ \leqq \frac{(\mathsf{s}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\Bigg(&\frac{1}{\Gamma(\varrho\varphi-2\varphi+1)}\int_{\mathfrak{e}}^{\eta}(\eta-\mathsf{w})^{\varrho\varphi-2\varphi}\big[\mathfrak{m}_1|\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)\\ &-\Phi\big(\mathsf{w},\phi'(\mathsf{w})\big)\big|+|\mathfrak{g}(\mathsf{w})|\big]\,d\mathsf{w}-\frac{1}{\Gamma(\varrho)}\int_{\mathfrak{e}}^{\mathfrak{f}}(\mathfrak{f}-\mathsf{w})^{\varrho-1}\big[\mathfrak{m}_1|\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)-\Phi\big(\mathsf{w},\phi'(\mathsf{w})\big)\big|\\ &+|\mathfrak{g}(\mathsf{w})|\big]\,d\mathsf{w}\Bigg)+\frac{1}{\Gamma(\varrho)}\int_{\mathfrak{e}}^{\mathsf{s}}(\mathsf{s}-\mathsf{w})^{\varrho-1}\big[\mathfrak{m}_1\big|\Phi\big(\mathsf{w},\phi(\mathsf{w})\big)-\Phi\big(\mathsf{w},\phi'(\mathsf{w})\big)\big|+|\mathfrak{g}(\mathsf{w})|\big]\,d\mathsf{w}\\ \leqq \Bigg[&\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\;\frac{[\mathfrak{m}_1\mathfrak{L}|\phi(\mathsf{s})-\phi'(\mathsf{s})|+\delta]\;(\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi-2\varphi+2)}\end{aligned}$$

$$+\frac{[\mathfrak{m}_1\mathfrak{L}|\phi(\mathsf{s})-\phi'(\mathsf{s})|+\delta]\ (\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)}\left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}+1\right)\Bigg].$$

We thus find that

$$\begin{aligned}(1-\mathfrak{N})\|\phi-\phi'\|_{\mathsf{E}} &\leqq \left[\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\ \frac{(\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi-2\varphi+2)}+\frac{(\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)}\left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}+1\right)\right]\delta;\\ \|\phi-\phi'\|_{\mathsf{E}} &\leqq \frac{\mathbb{M}}{1-\mathfrak{N}}\delta;\\ \|\phi-\phi'\|_{\mathsf{E}} &\leqq G_{\Phi}\delta,\end{aligned}\tag{4.3}$$

where

$$\mathbb{M}=(1-\mathfrak{N})\|\phi-\phi'\|_{\mathsf{E}}\leqq\left[\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\ \frac{(\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi-2\varphi+2)}+\frac{(\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)}\left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}+1\right)\right]$$

and

$$G_{\Phi}=\frac{\mathbb{M}}{1-\mathfrak{N}}.$$

Therefore, the system (1.3) is UH stable. □

Corollary 1. *Assume that $\Upsilon_{\Phi}:\mathbb{R}^+\to\mathbb{R}^+$ such that $\Upsilon_{\Phi}(\delta)=G_{\Phi}\delta$. Also, let Υ_{Φ} be a continuous function with $\Upsilon_{\Phi}(0)=0$. The equation (4.3), which is written as*

$$\|\phi-\phi'\|_{\mathsf{E}}\leqq\Upsilon_{\Phi}(\delta),$$

asserts that the system (1.3) is the generalized UH stable.

5. Applications and illustrative examples

In this section, we present two illustrative examples as applications in order to support our findings on the existence, uniqueness and stability of HFDE.

Example 1. Consider the following FDE involving the HFD:

$$\begin{cases} D_{0^+}^{\frac{3}{2},\frac{1}{2}}\phi(\mathsf{s})=\dfrac{e^{\mathsf{s}}}{(1+e^{\mathsf{s}})^2}\left(\dfrac{3+\phi(\mathsf{s})}{1+\phi(\mathsf{s})}\right)-\dfrac{2\omega e^{\mathsf{s}}(e^{\mathsf{s}}-1)}{(1+e^{\mathsf{s}})^3}, & \mathsf{s}\in\mathcal{J}_1=[0,1],\\ \phi(0)=0, & \\ \phi(1)={}^{\mathrm{RL}}I_{0^+}^{1-\gamma}\phi(\tfrac{1}{2}),\quad \mathfrak{e}<\eta<\mathfrak{f}. & \end{cases}\tag{5.1}$$

Comparing the problem (5.1) with the BVP (1.3), we have $\varrho=\frac{3}{2}$, $\varphi=\frac{1}{2}$, $\mathfrak{e}=0$, $\mathfrak{f}=1$, $\eta=\frac{1}{2}$, $\omega=2$.

We construct the function $\Phi:[0,1]\times\mathbb{R}\to\mathbb{R}$ given by

$$\Phi(\mathsf{s},\mathfrak{u})=\frac{3+\mathfrak{u}}{1+\mathfrak{u}},$$

so that, from the hypothesis (H3), we have

$$\begin{aligned}|\Phi(\mathsf{s},\mathfrak{u})| &= \left|\frac{3+\mathfrak{u}}{1+\mathfrak{u}}\right|\\ &\leqq (3+|\mathfrak{u}|).\end{aligned}$$

Therefore, we get the values of $p_1=3$ and $p_2=1$. Also, from the hypothesis (H2), we get

$$|\mathsf{p}(\mathsf{s})|=\left|\frac{e^{\mathsf{s}}}{(1+e^{\mathsf{s}})^2}\right|=0.1966=\mathfrak{m}_1$$

and

$$|\mathsf{q}(\mathsf{s})| = \left| -\frac{2\omega e^{\mathsf{s}}(e^{\mathsf{s}}-1)}{(1+e^{\mathsf{s}})^3} \right| = 0.3634 = \mathfrak{m}_2.$$

We also have the condition given by

$$\begin{aligned} 1-\Lambda_2 &= 1 - \frac{\mathfrak{m}_1\ p_2\ (\mathfrak{f}-\mathfrak{e})^{\gamma-1}(\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi-2\varphi+2)} - \frac{\mathfrak{m}_1\ p_2\ (\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)} \left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2} + 1 \right) \\ &= 1.0978 > 0, \end{aligned}$$

where $|\mathfrak{A}|d_2 = 1.9191$. Hence, all of the assumptions of Theorem 1 are satisfied. Therefore, the problem (5.1) has a solution on $[0,1]$.

Example 2. Consider FDE involving HFD

$$\begin{cases} D^{\frac{5}{4},\frac{3}{4}}_{\frac{1}{2}^+}\phi(\mathsf{s}) = \dfrac{e^{\mathsf{s}}}{(1+e^{\mathsf{s}})^2}[10+\sin\phi(\mathsf{s})] + \dfrac{2\omega e^{\mathsf{s}}(e^{\mathsf{s}}-1)}{(1+e^{\mathsf{s}})^3}, \quad \mathsf{s}\in\mathcal{J}_1 = [\frac{1}{2},5], \\ \phi(\frac{1}{2}) = 0, \\ \phi(5) = {}^{\mathrm{RL}}I^{1-\gamma}_{0^+}\phi(\frac{3}{2}), \quad \mathfrak{e}<\eta<\mathfrak{f}. \end{cases} \tag{5.2}$$

Comparing the problem (5.1) with the BVP (1.3), we have $\varrho = \frac{5}{4}$, $\varphi = \frac{3}{4}$, $\mathfrak{e} = \frac{1}{2}$, $\mathfrak{f} = 5$, $\eta = \frac{3}{2}$, $\omega = 5$.

We now construct the function $\Phi : [\frac{1}{2},5]\times\mathbb{R}\to\mathbb{R}$ given by

$$\Phi(\mathsf{s},\mathfrak{u}) = 10 + \sin\phi(\mathsf{s}).$$

Then, for any $\mathfrak{u}_i\in\mathbb{R}\ \ (i=1,2)$ and $\mathsf{s}\in\left[\frac{1}{2},5\right]$, we have

$$|\Phi(\mathsf{s},\mathfrak{u}_1) - \Phi(\mathsf{s},\mathfrak{u}_2)| \leqq |\sin(\mathfrak{u}_1) - \sin(\mathfrak{u}_2)|.$$

Here, $\mathfrak{L} = 1$. Also, from the hypothesis (H2), we have

$$|\mathsf{p}(\mathsf{s})| = \left| \frac{e^{\mathsf{s}}}{(1+e^{\mathsf{s}})^2} \right| = 0.0066 = \mathfrak{m}_1$$

and

$$|\mathsf{q}(\mathsf{s})| = \left| -\frac{2\omega e^{\mathsf{s}}(e^{\mathsf{s}}-1)}{(1+e^{\mathsf{s}})^3} \right| = 0.0656 = \mathfrak{m}_2.$$

We also have the following condition:

$$\begin{aligned} \mathfrak{N} &= \frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2}\ \frac{\mathfrak{m}_1\mathfrak{L}(\eta-\mathfrak{e})^{\varrho\varphi-2\varphi+1}}{\Gamma(\varrho\varphi-2\varphi+2)} + \frac{\mathfrak{m}_1\mathfrak{L}(\mathfrak{f}-\mathfrak{e})^{\varrho}}{\Gamma(\varrho+1)} \left(\frac{(\mathfrak{f}-\mathfrak{e})^{\gamma-1}}{|\mathfrak{A}|d_2} + 1 \right) \\ &\leqq 0.0745 < 1, \end{aligned}$$

where $|\mathfrak{A}|d_2 = 4.3289$. It can be seen that the assumptions of Theorem 2 are satisfied. Hence, the problem (5.2) has a unique solution and also the problem (5.2) is UH stable on $\mathcal{J}_2$.

6. Conclusion

The present research has discussed the ACC model involving the HFD of order $1 < \varrho \leqq 2$ with three-point integral boundary conditions. With the help of certain assumptions such as linear growth and the Lipschitz condition on the nonlinear function Φ defined in the BVP (1.3), we have successfully established sufficient conditions for the existence and uniqueness of the solution with the aid of Schaefer's FPT and Banach FPT, respectively. We have also acquired the *a priori* bound of the solution of the present problem. In order to enrich the literature further, we have investigated the stability analysis by using the UH and the generalized UH stabilities for the assumed BVP (1.3). The work presented in this paper is presumably new and we have used markedly different approaches for obtaining several results on the existence, uniqueness, and stability together with the *a priori* bound of the solution of ACC model.

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Author contributions KD and AKN conceived of the presented idea. KD and AKN developed the theory and performed the computations. RKV and HMS verified the analytical methods. HMS and RKV encouraged KD and AKN to investigate the stability aspect and supervised the findings of this work. All authors discussed the results and contributed to the final manuscript.

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H. M. Srivastava
Department of Mathematics and Statistics
University of Victoria
Victoria BCV8W 3R4
Canada
e-mail: harimsri@math.uvic.ca

H. M. Srivastava
Department of Medical Research, China Medical University Hospital
China Medical University
Taichung 40402
Taiwan, Republic of China

H. M. Srivastava
Center for Converging Humanities
Kyung Hee University
26 Kyungheedae-ro, Dongdaemun-gu
Seoul 02447
Republic of Korea

H. M. Srivastava
Department of Mathematics and Informatics
Azerbaijan University
71 Jeyhun Hajibeyli Street
AZ1007 Baku
Azerbaijan

H. M. Srivastava
Department of Applied Mathematics
Chung Yuan Christian University
Chung-LiTaoyuan City 320314
Taiwan

H. M. Srivastava
Section of Mathematics
International Telematic University Uninettuno
00186 Rome
Italy

Kanika Dhawan
Yogananda School of AI, Computer, and Data Science
Shoolini University
Bajhol, Solan Himachal Pradesh173229
India
e-mail: dhawankanika83@gmail.com

Ramesh Kumar Vats
Department of Mathematics and Scientific Computing
National Institute of Technology
Hamirpur Himachal Pradesh177005
India
e-mail: rkvats@nith.ac.in

Ankit Kumar Nain
Department of Mathematics and Scientific Computing
National Institute of Technology
Hamirpur Himachal Pradesh177005
India
e-mail: ankitnain744@gmail.com

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