Dynamics analysis of a reaction-diffusion malaria model accounting for asymptomatic carriers

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Abstract. A significant proportion of malaria infections in humans exhibit no symptoms, but it is a reservoir for maintaining malaria transmission. A time periodic reaction-diffusion model for malaria spread is introduced in this paper, incorporating spatial heterogeneity, incubation periods, symptomatic and asymptomatic carriers. This paper introduces the concept of the basic reproduction number $\mathcal{R}_0$, which is defined as the spectral radius of the next generation operator, and we present some preliminary results by elementary analysis. The threshold dynamic behavior analysis shows that when $\mathcal{R}_0 < 1$, the disease is extinct, and when $\mathcal{R}_0 > 1$, the disease is persistent. We investigate the case of constant system parameters, focusing on the global asymptotic stability of the disease-free steady state when $\mathcal{R}_0 = 1$. In the numerical simulation section, we validate the theoretical results obtained, and then use elasticity analysis methods to explore the influence of parameters on the output solution. In addition, sensitivity analysis of the basic reproduction number under homogeneous conditions indicates direction of controlling malaria transmission. And several control measures are evaluated in the following steps.

1. Introduction

Malaria is one of the most deadly and sophisticated parasitic diseases in underdeveloped countries, especially in Sub-Saharan Africa, causing high incidence rate and mortality, inducing a monstrous economic, social, and health burden [1], for which nearly half of the world population is at risk. There are currently approximately 219 million cases worldwide, with nearly 3.3 billion people exposed to contact the disease [2]. It stems from plasmodium parasite of protozoa spread in human after being effectively bitten by infected adult female *Anopheles* mosquitoes. *P. vivax*, *P. falciparum*, *P. malariae*, *P. ovale* and *P. knowlesi* are five types of malaria parasites that can infect humans.

The employment of mathematical models in the study of malaria transmission enables a deeper understanding of disease prevalence in communities and the exploration of how various factors, including migration, influence the evolution of the epidemic. Concerning this subject, researchers have developed rich mathematical models from the basic model of [3] and [4] to the spatiotemporal dynamic transmission of diseases. Epidemiologists widely use mathematical models as tools for predicting the prevalence of epidemics and infectious diseases, as well as guiding current study on malaria eradication [5]. Reaction-diffusion equations are usually used to comprehend the affect of the movement of humans and mosquitoes on diseases transmission taking into account the spatial structure [6,7].

In recent years, malaria has become increasingly prominent due to the unexpected impact of climate change or global warming on the incidence rate of malaria. The rise and undulate of temperature can affect the vectors and life process of parasites. As is well known, mosquito reproduction is subjected to temperature - a temperature vary from 12 to 31°C decreases the number of days needed for reproduction from 65 to 7.3 days [8]. The spore formation of the parasites in mosquito is finished within 55 days at 16°C and reduced to 7 days at 28°C [9]. Climate change will shift from low latitude areas to areas where the population has not yet developed immunity to malaria, thereby affecting the epidemic patterns of the disease [10,11]. The heightened interest in the relationship between

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global warming and malaria transmission has underscored the significance and relevance of modelling the impact of environmental factors on malaria spread. [11,12]. Consequently, we need to consider these factor in the model.

The presence or absence of immunity constitutes a key determinant in the manifestation of clinical symptoms among malaria patients [13]. In high transmission regions of malaria, individuals maintain their susceptibility to malaria infection, potentially leading to the acquisition of immunity against the disease. The improvement of immunity reduces the parasite density of individuals and alleviates the severity of symptoms [14]. The clinical manifestations of malaria vary from severe and complex to mild and uncomplicated to asymptomatic [15]. According to reports, asymptomatic malaria infections have occurred in multiple high and medium transmission areas such as Kenya and Nigeria [16], which can be defined as individuals who have no recent history of symptoms and/or signs of malaria carrying parasites that can transmit the disease [17]. The number of asymptomatic carriers in a specific population within a specific time interval may affect the dynamics of disease transmission. Interestingly, simulation studies to date have shown that targeting asymptomatic infected individuals can reduce the spread of malaria [17,18]. Therefore, it is believed that identifying and treating asymptomatic populations is an important path forward. By considering both symptomatic and asymptomatic cases of malaria, our aim is to gain insights into the influence of asymptomatic malaria individuals on the disease dynamics. A correct understanding of the impact of asymptomatic individuals on transmission dynamics will comprehensively describe the complex interplay between identified female *Anopheles* mosquitoes, intermediaries human, and pathogens (*Plasmodium* parasite). We hope that this qualitative analysis will fill the current gap in knowledge about asymptomatic malaria and help develop strategies that will further develop malaria control and eradication efforts. Understanding the contribution of asymptomatic carriers to the transmission of malaria among humans is crucial for the elimination of the disease.

The organizational structure of the remaining parts of this paper is as follows. A time-periodic reaction-diffusion model includes asymptomatic carriers, incubation periods and spatial heterogeneity is developed in Section 2. Section 3 brings some preliminary results. In Section 4, by applying to the theory of the next generation operator a threshold value, namely basic reproduction number $\mathcal{R}_0$ is introduced. In Section 5, we analyze the threshold dynamic behavior: if $\mathcal{R}_0 < 1$, then the disease becomes extinct, and if $\mathcal{R}_0 > 1$, then the disease persists. Just as importantly, numerical simulations are conducted in the next section to explain the main theoretical results, sensitivity analysis, and evaluate control measures. Finally, a summary concludes the paper.

2. Model description

Due to the fact that malaria is transmitted among humans by mosquitos, two populations of mosquitoes and humans are considered here, and in a bounded domain Ω which has smooth boundary $\partial\Omega$. Describe $N_h(t,x)$ and $N_v(t,x)$ as the total population size of the humans and mosquitoes, respectively, at time t and location x. We further decompose the human population into susceptible, exposed, symptomatic, asymptomatic and recovered compartments, and are expressed by S_h, E_h, I_h, J and R, respectively. Thus, $N_h = S_h + E_h + I_h + J + R$. In addition, the recovered compartment dose not contain asymptomatic carriers and there is no associated transmission probability, as it does not cause mosquito infection. Although asymptomatic carriers may not develop clinical disease, their blood may still contain gametocytes with low levels wich could transmit the infection to mosquitoes [19]. The occurrence of asymptomatic infection is often due to the individual's partial immunity to malaria caused by repeated exposures, so here we introduce a quantity c, $0 \leq c \leq 1$, which indicates a factor in reducing the infectivity of asymptomatic infections. It deems that the vector population consists only of female *Anopheles* mosquitoes, which are categorized into susceptible (S_v), exposed (E_v), and infectious compartments (I_v). The life span of mosquitoes is short, once they get infected they will not recover and can leave the infected class only through death. Accordingly, the total mosquito population is determined by $N_v = S_v + E_v + I_v$. To address the impact of seasonal variation, we suppose that for some $\omega > 0$, the coefficients $\Lambda_v(t,x)$, $\mu_v(t,x)$, $\alpha_v(t,x)$, $\beta_1(t,x)$ and $\beta_2(t,x)$ are all ω-periodic with respect to t. As the human population is not significantly affected by seasonal temperature, the coefficients only related to mosquitoes are periodic. Furthermore, the dispersal pattern is an unbiased random walk, where a single walker randomly walks on a solid line with a fixed step [20]. Especially, according to [21], S_h, E_h,
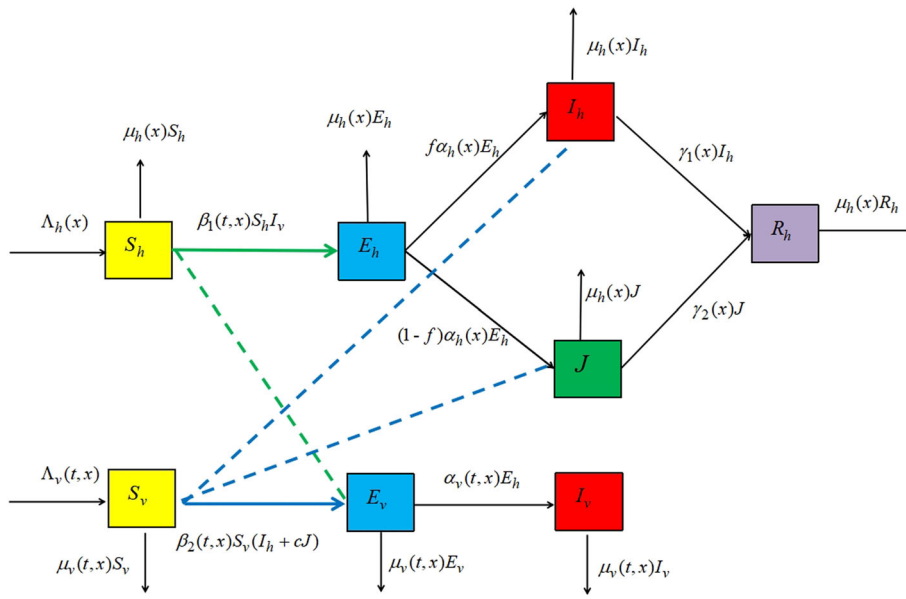
FIG. 1. A diagrammatic depiction of malaria transmission

I_h, J and R_h have the same coefficients represented by $D_h > 0$, while S_v, E_v and I_v own the same coefficient denoted by $D_v > 0$.

Figure 1 is employed to elaborate on how humans and mosquitoes transition between various compartments. Therefore, for $t > 0$, our model subjects to the ensuing reaction-diffusion system,

$$
\begin{cases}
\dfrac{\partial S_h}{\partial t} = D_h \Delta S_h + \Lambda_h(x) - \beta_1(t,x) S_h I_v - \mu_h(x) S_h, & x \in \Omega, \\[2mm]
\dfrac{\partial E_h}{\partial t} = D_h \Delta E_h + \beta_1(t,x) S_h I_v - (\mu_h(x) + \alpha_h(x)) E_h, & x \in \Omega, \\[2mm]
\dfrac{\partial I_h}{\partial t} = D_h \Delta I_h(t,x) + f\alpha_h(x) E_h - (\mu_h(x) + \gamma_1(x)) I_h, & x \in \Omega, \\[2mm]
\dfrac{\partial J}{\partial t} = D_h \Delta J + (1-f)\alpha_h(x) E_h - (\mu_h(x) + \gamma_2(x)) J, & x \in \Omega, \\[2mm]
\dfrac{\partial R_h}{\partial t} = D_h \Delta R_h + \gamma_1(x) I_h + \gamma_2(x) J - \mu_h(x) R_h, & x \in \Omega, \\[2mm]
\dfrac{\partial S_v}{\partial t} = D_v \Delta S_v + \Lambda_v(t,x) - \beta_2(t,x) S_v(I_h + cJ) - \mu_v(t,x) S_v, & x \in \Omega, \\[2mm]
\dfrac{\partial E_v}{\partial t} = D_v \Delta E_v + \beta_2(t,x) S_v(I_h + cJ) - (\mu_v(t,x) + \alpha_v(t,x)) E_v, & x \in \Omega, \\[2mm]
\dfrac{\partial I_v}{\partial t} = D_v \Delta I_v + \alpha_v(t,x) E_v - \mu_v(t,x) I_v, & x \in \Omega, \\[2mm]
\dfrac{\partial S_h}{\partial \nu} = \dfrac{\partial E_h}{\partial \nu} = \dfrac{\partial I_h}{\partial \nu} = \dfrac{\partial J}{\partial \nu} = \dfrac{\partial R_h}{\partial \nu} = \dfrac{\partial S_v}{\partial \nu} = \dfrac{\partial E_v}{\partial \nu} = \dfrac{\partial I_v}{\partial \nu} = 0, & x \in \partial\Omega,
\end{cases}
\tag{2.1}
$$

where $\frac{\partial}{\partial \nu}$ denotes the differentiation along the outward normal ν to $\partial\Omega$, Δ is a Laplacian operator. The last equation of system (2.3) is boundary condition which means that the individuals do not move across the boundary $\partial\Omega$. About initial data,

$$
A(0, x) = A^0(x) \geq 0, \quad x \in \Omega,
\tag{2.2}
$$

TABLE 1. *The meanings of all parameters in model (2.1)*

Symbol	Meanings
D_h	The diffusion rate of humans
D_v	The diffusion rate of mosquitoes
Λ_h	The recruitment rate of the humans
Λ_v	The recruitment rate of the adult female mosquitoes
β_1	The rate of transmission from virus-infected mosquitoes to vulnerable humans
β_2	The transmission rate from infected humans to susceptible mosquitoes
μ_h	The natural death rate of humans
μ_v	The natural death rate of mosquitoes
$\frac{1}{\alpha_h}$	The mean latent period in humans
$\frac{1}{\alpha_v}$	The mean latent period in mosquitoes
f	The proportion develops symptomatic
$1 - f$	The proportion develops asymptomatic
c	The factor that reduce the infectivity of asymptomatic humans
γ_1	The recovery rate of symptomatic humans
γ_2	The recovery rate of asymptomatic humans

where $A = S_h, E_h, I_h, J, R_h, S_v, E_v, I_v$. All position-dependent parameters are strictly positive, continuous and uniformly bounded functions on Ω.

The other equations in system (2.1) are not coupled with R_h, then it is enough to explore the next system,

$$
\begin{cases}
\dfrac{\partial S_h}{\partial t} = D_h \Delta S_h + \Lambda_h(x) - \beta_1(t,x) S_h I_v - \mu_h(x) S_h, & x \in \Omega, \\[2mm]
\dfrac{\partial E_h}{\partial t} = D_h \Delta E_h + \beta_1(t,x) S_h I_v - (\mu_h(x) + \alpha_h(x)) E_h, & x \in \Omega, \\[2mm]
\dfrac{\partial I_h}{\partial t} = D_h \Delta I_h + f \alpha_h(x) E_h - (\mu_h(x) + \gamma_1(x)) I_h, & x \in \Omega, \\[2mm]
\dfrac{\partial J}{\partial t} = D_h \Delta J + (1-f) \alpha_h(x) E_h - (\mu_h(x) + \gamma_2(x)) J, & x \in \Omega, \\[2mm]
\dfrac{\partial S_v}{\partial t} = D_v \Delta S_v + \Lambda_v(t,x) - \beta_2(t,x) S_v(I_h + cJ) - \mu_v(t,x) S_v, & x \in \Omega, \\[2mm]
\dfrac{\partial E_v}{\partial t} = D_v \Delta E_v + \beta_2(t,x) S_v(I_h + cJ) - (\mu_v(t,x) + \alpha_v(t,x)) E_v, & x \in \Omega, \\[2mm]
\dfrac{\partial I_v}{\partial t} = D_v \Delta I_v + \alpha_v(t,x) E_v - \mu_v(t,x) I_v, & x \in \Omega, \\[2mm]
\dfrac{\partial S_h}{\partial \nu} = \dfrac{\partial E_h}{\partial \nu} = \dfrac{\partial I_h}{\partial \nu} = \dfrac{\partial J}{\partial \nu} = \dfrac{\partial S_v}{\partial \nu} = \dfrac{\partial E_v}{\partial \nu} = \dfrac{\partial I_v}{\partial \nu} = 0, & x \in \partial\Omega,
\end{cases}
\tag{2.3}
$$

for $t > 0$. The biological significance of parameters is described in Table 1.

3. Preliminary results

Define $\mathbb{X} := C\left(\overline{\Omega}, \mathbb{R}^7\right)$ as the Banach space consisting of continuous functions mapping $\overline{\Omega}$ to $\mathbb{R}^7$. The space is equipped with the supremum norm, denoted by $\|\cdot\|_{\mathbb{X}}$, and $\mathbb{X}^+ := C\left(\overline{\Omega}, \mathbb{R}^7_+\right)$. Set $\mathbb{Y} := C\left(\overline{\Omega}, \mathbb{R}\right)$ and $\mathbb{Y}^+ := C\left(\overline{\Omega}, \mathbb{R}_+\right)$. Suppose that $T_i(t,s)(i = 1, \ldots, 6) : \mathbb{Y} \to \mathbb{Y}$, are evolution operators intimately related to

$$\frac{\partial u_1}{\partial t} = D_h \Delta u_1 - \mu_h(x) u_1,$$

$$\frac{\partial u_2}{\partial t} = D_h \Delta u_2 - (\mu_h(x) + \alpha_h(x)) u_2,$$

$$\frac{\partial u_3}{\partial t} = D_h \Delta u_3 - (\mu_h(x) + \gamma_1(x)) u_3,$$

$$\frac{\partial u_4}{\partial t} = D_h \Delta u_4 - (\mu_h(x) + \gamma_2(x)) u_4,$$

$$\frac{\partial u_5}{\partial t} = D_v \Delta u_5 - \mu_v(t,x) u_5,$$

$$\frac{\partial u_6}{\partial t} = D_v \Delta u_6 - (\mu_v(t,x) + \alpha_v(t,x)) u_6,$$

depending on the Neumann boundary condition, respectively. To clarify that $T_j(t,s) = T_j(t-s)$, then $T_j(t+\omega, s+\omega) = T_j(t,s)$ $(j = 1,2,3,4)$ with $t \geq s$ for $(t,s) \in (0,\infty) \times (0,\infty)$. Given that $\mu_v(t,\cdot)$ and $\alpha_v(t,\cdot)$ is ω-periodic in t, Lemma 6.1 in [22] signifies that for $(t,s) \in \mathbb{R}^2$ with $t \geq s$, $T_5(t+\omega, s+\omega) = T_5(t,s)$ and $T_6(t+\omega, s+\omega) = T_6(t,s)$. Additionally, based on [23, Corollary 7.2.3], T_i $(i = 1, \ldots, 6)$ is compact and strongly positive. Then for $(t,s) \in \mathbb{R}^2$ with $t \geq s$, $T(t,s) = \text{diag}(T_1(t,s), T_2(t,s), T_3(t,s), T_4(t,s), T_5(t,s), T_6(t,s), T_5(t,s)) : \mathbb{X} \to \mathbb{X}$ is an evolution operator.

For $t > 0$, $x \in \overline{\Omega}$ and $\phi \in \mathbb{X}^+$, denote a family operator $\{\Phi(t)\}_{t>0}$ on $\mathbb{X}^+$ by $\Phi(t)(\phi)(x) = u(t,x;\phi)$. According to the proof of Lemma 2.1 in [24], it is straightforward to get that $\{\Phi_t\}_{t\geq 0}$ is an ω-periodic semiflow on $\mathbb{X}^+$, which means that $\Phi(t)$ is point dissipative. Furthermore, Theorem 2.1.8 in [25] reveals that $\Phi(t)$ is compact.

Denote $F = (F_1, F_2, F_3, F_4, F_5, F_6, F_7) : [0, +\infty) \times \mathbb{X} \to \mathbb{X}$ by

$$
\begin{aligned}
F_1(t,\phi) &:= \Lambda_h(\cdot) - \beta_1(t,\cdot)\phi_1(\cdot)\phi_7(\cdot), \\
F_2(t,\phi) &:= \beta_1(t,\cdot)\phi_1(\cdot)\phi_7(\cdot), \\
F_3(t,\phi) &:= f\alpha_h(\cdot)\phi_2(\cdot), \\
F_4(t,\phi) &:= (1-f)\alpha_h(\cdot)\phi_2(\cdot), \\
F_5(t,\phi) &:= \Lambda_v(t,\cdot) - \beta_2(t,\cdot)\phi_5(\cdot)(\phi_3(\cdot) + c\phi_4(\cdot)), \\
F_6(t,\phi) &:= \beta_2(t,x)\phi_5(\cdot)(\phi_3(\cdot) + c\phi_4(\cdot)), \\
F_7(t,\phi) &:= \alpha_v(t,x)\phi_6(\cdot),
\end{aligned}
\tag{3.1}
$$

for $\phi = (\phi_1, \cdots, \phi_7) \in \mathbb{X}^+$, $t \geq 0$ and $x \in \overline{\Omega}$. Let $A(t) = \text{diag}(A_1, A_2, A_3, A_4, A_5(t), A_6(t), A_5(t))$, then $T(t)$ is a simgroup generated by the operaator A defined on $D(A) = (D(A_1) \times D(A_2) \times D(A_3) \times D(A_4) \times D(A_5) \times D(A_6(t)) \times D(A_7(t)))$. Then (2.3) can be rewritten as

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = A(t)u + F, & t > 0, \ x \in \Omega, \\
u(0,x) = \phi(x), & x \in \Omega,
\end{cases}
\tag{3.2}
$$

where $u = (S_h, E_h, I_h, J, S_v, E_v, I_v)$. Here, $A_i(i = 1, \ldots, 4)$ is decided by

$$
\begin{cases}
D(A_i) = \left\{ \phi \in C^2\left(\overline{\Omega}\right) : \dfrac{\partial \phi}{\partial n} = 0, \text{ on } \partial\Omega \right\}, \\
A_i\phi(x) = D_h\Delta\phi(x) - m_i(x)\phi(x), & x \in \Omega, \quad i = 1, \ldots, 4,
\end{cases}
\tag{3.3}
$$

where $m_1(x) = \mu_h(x)$, $m_2(x) = \mu_h(x) + \alpha_h(x)$, $m_3(x) = \mu_h(x) + \gamma_1(x)$ and $m_4(x) = \mu_h(x) + \gamma_2(x)$. $A_j(t)(j = 5, 6)$ is defined by

$$
\begin{cases}
D(A_j(t)) = \left\{ \phi \in C^2\left(\overline{\Omega}\right) : \dfrac{\partial \phi}{\partial n} = 0, \text{ on } \partial\Omega \right\}, \\
A_j(t)\phi(x) = D_v\Delta\phi(x) - p_j(t,x)\phi(x), & x \in \Omega, \quad j = 5, 6,
\end{cases}
\tag{3.4}
$$

where $p_5 = \mu_v$, $p_6 = \mu_v + \alpha_v$.

Model (2.3) can be construed as

$$\begin{cases} u(t,\phi) = T(t,0)\phi(x) + \displaystyle\int_0^t T(t,s)F(s,u(s))\mathrm{d}s, \quad t > 0, \\ \\ u_0 = \phi. \end{cases} \tag{3.5}$$

For each $\phi \in \mathbb{X}^+$, according to [27, Theorem 1.1 and Remark 1.1], it can be inferred that (2.3) allows a single mild solution meeting $u_0 = \phi$ and $u(t,\phi) \in \mathbb{X}^+$ for any t on its maximum existence interval $[0, \sigma_\phi)$. According to the analytically of $T(t,s)$, $t > s$ and $(t,s) \in (0,\infty) \times (0,\infty)$, $u(t,x,\phi)$ is a classical solution.

Theorem 3.1. *For all $\phi \in \mathbb{X}^+$, system (2.3) has a single solution $u(t,;\phi) \in \mathbb{X}^+$ on $[0,\infty)$ with $u_0 = \phi$. Moreover, system (2.3) yields an ω-periodic semiflow $\Phi(t) = u(t,\cdot)$, that is, $\Phi(t)(\phi)(x) = u(t,x;\phi)$, for $(t,x) \in (0,+\infty) \times \Omega$, additionally, $\Phi := \Phi(\omega)$ admits a global compact attractor in $\mathbb{X}^+$.*

Proof. According to the comparison principle, it is easy to know that on $[0,\sigma_\phi)$, $S_h(t,\cdot;\phi)$ is bounded. There is some positive integer $l_1 = l_1(\phi) > 0$ fulfilling $S_h(t,x;\phi) \leq M_1$, for $t > l_1\omega$ and $x \in \overline{\Omega}$. Let $(S_h(t,x), E_h(t,x), I_h(t,x), J(t,x), S_v(t,x), E_v(t,x), I_v(t,x)) := (S_h(t,\phi)(x), E_h(t,\phi)(x), I_h(t,\phi)(x), J(t,\phi)(x), S_v(t,\phi)(x), E_v(t,\phi)(x), I_v(t,\phi)(x))$, $t \geq 0$, $x \in \Omega$ and

$$\begin{gathered} \bar{S}_h(t) = \int_\Omega S_h \mathrm{d}x, \quad \bar{E}_h(t) = \int_\Omega E_h \mathrm{d}x, \quad \bar{I}_h(t) = \int_\Omega I_h \mathrm{d}x, \quad \bar{J}(t) = \int_\Omega J \mathrm{d}x, \\ \bar{S}_v(t) = \int_\Omega S_v \mathrm{d}x, \quad \bar{E}_v(t) = \int_\Omega E_v \mathrm{d}x, \quad \bar{I}_v(t) = \int_\Omega I_v \mathrm{d}x. \end{gathered} \tag{3.6}$$

Denote $\hat{f} = \max\limits_{t \in [0,\omega], x \in \overline{\Omega}} f(t,x)$, $\underline{f} = \min\limits_{t \in [0,\omega], x \in \overline{\Omega}} f(t,x)$, $\hat{g} = \max\limits_{x \in \overline{\Omega}} g(x)$, and $\tilde{g} = \min\limits_{x \in \overline{\Omega}} g(x)$ where $f = \Lambda_v(t,x)$, $\mu_v(t,x), \alpha_v(t,x)$ and $g = \Lambda_h(x), \mu_h(x), \alpha_h(x), \gamma_1(x), \gamma_2(x)$.

Integrating of the S_h equation of (2.3), gets

$$\frac{\mathrm{d}\bar{S}_h(t)}{\mathrm{d}t} \leq \hat{\Lambda}_h|\Omega| - \int_\Omega \beta_1(t,x)S_h I_v \mathrm{d}x.$$

That is,

$$\int_\Omega \beta_1(t,x)S_h I_v \mathrm{d}x \leq \hat{\Lambda}_h|\Omega| - \frac{\mathrm{d}\bar{S}_h(t)}{\mathrm{d}t}, \quad t > 0.$$

Application of the Green's formula to the integrated form of the E_h equation in (2.3) yields

$$\begin{aligned} \frac{\mathrm{d}\bar{E}_h(t)}{\mathrm{d}t} &\leq \int_\Omega \beta_1(t,x)S_h I_v \mathrm{d}x - (\tilde{\mu}_h + \tilde{\alpha}_h)\int_\Omega E_h \mathrm{d}x, \\ &\leq \hat{\Lambda}_h|\Omega| - \frac{\mathrm{d}\bar{S}_h(t)}{\mathrm{d}t} - (\tilde{\mu}_h + \tilde{\alpha}_h)\bar{E}_h(t), \ t > 0. \end{aligned}$$

For $t > l_1\omega$, we get

$$\frac{\mathrm{d}[\bar{S}_h(t) + \bar{E}_h(t)]}{\mathrm{d}t} \leq -(\tilde{\mu}_h + \tilde{\alpha}_h)[\bar{E}_h(t) + \bar{S}_h(t)] + \hat{\Lambda}_h|\Omega| + M_1|\Omega|(1 + \tilde{\mu}_h + \tilde{\alpha}_h),$$

which gives rise to

$$\bar{S}_h(t) + \bar{E}_h(t) \leq \frac{\hat{\Lambda}_h|\Omega| + M_1|\Omega|(1 + \tilde{\mu}_h + \tilde{\alpha}_h)}{\tilde{\mu}_h + \tilde{\alpha}_h}, \ t \geq l_2\omega,$$

where $l_2 > l_1$ is some integer. Thereupon,

$$\bar{E}_h(t) = \|E_h(t, \cdot)\|_{L^1(\Omega)} \leq \frac{\hat{\Lambda}_h|\Omega| + M_1|\Omega|(1 + \tilde{\mu}_h + \tilde{\alpha}_h)}{\tilde{\mu}_h + \tilde{\alpha}_h} + 1.$$

Integrating the I_h and J equations in (2.3), respectively, and applying Green's formula obtains

$$\frac{\mathrm{d}\bar{I}_h(t)}{\mathrm{d}t} = f\int_{\Omega} \alpha_h(x)E_h\mathrm{d}x - \int_{\Omega} (\mu_h(x) + \gamma_1(x))I_h\mathrm{d}x$$

$$\leq f\hat{\alpha}_h \left[\frac{\hat{\Lambda}_h|\Omega| + M_1|\Omega|(1 + \tilde{\mu}_h + \tilde{\alpha}_h)}{\tilde{\mu}_h + \tilde{\alpha}_h} + 1\right]|\Omega| - (\tilde{\mu}_h + \tilde{\gamma}_1)\bar{I}_h(t),$$

$$\frac{\mathrm{d}\bar{J}(t)}{\mathrm{d}t} = (1 - f)\int_{\Omega} \alpha_h(x)E_h\mathrm{d}x - \int_{\Omega} (\mu_h(x) + \gamma_2(x))J\mathrm{d}x$$

$$\leq (1 - f)\hat{\alpha}_h \left[\frac{\hat{\Lambda}_h|\Omega| + M_1|\Omega|(1 + \tilde{\mu}_h + \tilde{\alpha}_h)}{\tilde{\mu}_h + \tilde{\alpha}_h} + 1\right]|\Omega| - (\tilde{\mu}_h + \tilde{\gamma}_2)\bar{J}(t),$$

Thus,

$$\frac{\mathrm{d}\left(\bar{I}_h(t) + \bar{J}(t)\right)}{\mathrm{d}t} \leq \hat{\alpha}_h \left[\frac{\hat{\Lambda}_h|\Omega| + M_1|\Omega|(1 + \tilde{\mu}_h + \tilde{\alpha}_h)}{\tilde{\mu}_h + \tilde{\alpha}_h} + 1\right]|\Omega| - (\tilde{\mu}_h + \tilde{\gamma}_1)\bar{I}_h(t) - (\tilde{\mu}_h + \tilde{\gamma}_2)\bar{J}(t)$$

$$\leq M_2 - m[\bar{I}_h(t) + \bar{J}(t)],$$

where $M_2 = \hat{\alpha}_h \left[\frac{\hat{\Lambda}_h|\Omega| + M_1|\Omega|(1+\tilde{\mu}_h+\tilde{\alpha}_h)}{\tilde{\mu}_h + \tilde{\alpha}_h} + 1\right]|\Omega|$ and $m = \min\{\tilde{\mu}_h + \tilde{\gamma}_1, \tilde{\mu}_h + \tilde{\gamma}_2\}$. Then $\bar{I}_h(t) + \bar{J}(t) \leq \frac{M_2}{m}$ for $t \geq l_3\omega$ ($l_3 \geq l_2$).

Based on comparison principle, on $[0, \sigma_\phi)$, $S_v(t, \cdot; \phi)$ is bounded, then there is some positive integer $l_4 = l_4(\phi)$ meeting $S_v(t, \cdot; \phi) \leq M_3$ for $t \geq l_4\omega$ and $x \in \overline{\Omega}$.

Integrate the S_v equation in system (2.3) produces

$$\frac{\mathrm{d}\bar{S}_v(t)}{\mathrm{d}t} \leq \hat{\Lambda}_v|\Omega| - \int_{\Omega} \beta_2(t, x)S_v(I_h + cJ)\mathrm{d}x.$$

That is

$$\int_{\Omega} \beta_2(t, x)S_v(I_h + cJ)\mathrm{d}x \leq \hat{\Lambda}_v|\Omega| - \frac{\mathrm{d}\bar{S}_v(t)}{\mathrm{d}t}.$$

Integrating of the sixth equation of model (2.3) and applying Green's formula acquires

$$\frac{\mathrm{d}\bar{E}_v(t)}{\mathrm{d}t} \leq \int_{\Omega} \beta(t, x)S_v(I_h + cJ)\mathrm{d}x - \left(\underline{\mu}_v + \underline{\alpha}_v\right)\overline{E}_v(t)$$

$$\leq \hat{\Lambda}_v|\Omega| - \frac{\mathrm{d}\bar{S}_v(t)}{\mathrm{d}t} - \left(\underline{\mu}_v + \underline{\alpha}_v\right)\bar{E}_v(t), \ t > 0.$$

For $t > l_4\omega$,

$$\frac{\mathrm{d}[\bar{S}_v(t) + \bar{E}_v(t)]}{\mathrm{d}t} \leq \hat{\Lambda}_v|\Omega| - \left(\underline{\mu}_v + \underline{\alpha}_v\right)[\bar{S}_v(t) + \bar{E}_v(t)] + M_3|\Omega|\left(1 + \underline{\mu}_v + \underline{\alpha}_v\right),$$

which leads to

$$\bar{S}_v(t) + \bar{E}_v(t) \leq \frac{\hat{\Lambda}_v|\Omega| + M_3|\Omega|\left(1 + \underline{\mu}_v + \underline{\alpha}_v\right)}{\underline{\mu}_v + \underline{\alpha}_v}, \ t \geq l_5\omega,$$

where $l_5 > l_4$ is positive integer. Consequently,

$$\bar{E}_v(t) = \|E_v(t,\cdot)\|_{L^1(\Omega)} \leq \frac{\hat{\Lambda}_v|\Omega| + M_3|\Omega| \left(1 + \underline{\mu_v} + \underline{\alpha_v}\right)}{\underline{\mu_v} + \underline{\alpha_v}} + 1, \ t \geq l_5\omega.$$

Integrating the seventh equation of model (2.3)

$$\frac{\mathrm{d}I_v(t)}{\mathrm{d}t} \leq \int_\Omega \alpha_v(t,x)E_v\mathrm{d}x - \underline{\mu_v}\bar{I}_v(t)$$

$$\leq \hat{\alpha}_v \left[\frac{\hat{\Lambda}_v|\Omega| + M_3|\Omega| \left(1 + \underline{\mu_v} + \underline{\alpha_v}\right)}{\underline{\mu_v} + \underline{\alpha_v}} + 1\right]|\Omega| - \underline{\mu_v}\bar{I}_v(t).$$

Thus,

$$\bar{I}_v(t) \leq \frac{\hat{\alpha}_v \left[\frac{\hat{\Lambda}_v|\Omega| + M_3|\Omega|\left(1 + \underline{\mu_v} + \underline{\alpha_v}\right)}{\underline{\mu_v} + \underline{\alpha_v}} + 1\right]|\Omega|}{\underline{\mu_v}} := M_4, \ \text{for } t \geq l_5\omega \ (l_5 > l_4 > l_3 > l_2 > l_1).$$

By Lemma 3.1 in [28], there exist constants K_1 and K_2 independent ϕ satisfying

$$\|E_h\|_{L^\infty(\Omega)} + \|I_h\|_{L^\infty(\Omega)} + \|J\|_{L^\infty(\Omega)} \leq K_1, \ t \geq l_3\omega,$$

and

$$\|E_v\|_{L^\infty(\Omega)} + \|I_v\|_{L^\infty(\Omega)} \leq K_2, \ t \geq l_5\omega.$$

Therefore, S_h, E_h, I_h, J, S_v, E_v and I_v are uniformly bounded. Hence, $\sigma_\phi = \infty$ for $\phi \in \mathbb{X}^+$. $\qquad\square$

Describe a operator family $\{\Phi(t)\}_{t>0}$ by $\Phi(t)(\phi)(x) = u(t,x,\phi)$ for $\phi \in \mathbb{X}^+$, and $(t,x) \in \mathbb{R} \times \overline{\Omega}$. Then $\{\Phi_t\}_{t\geq 0}$ is an ω-periodic semiflow according to the proof of Lemma 2.1 in [24]. In addition, $\Phi(t)$ is point dissipative and Theorem 2.1.8 in [25] implies that $\Phi(t)$ is compact. As a consequence, $\Phi = \Phi(t)$ owns a global compact attractor [26, Theorem 2.9].

4. Basic reproduction number

The primary aim of this study is to explore the threshold dynamics of model (2.3). Basic reproduction number $\mathcal{R}_0$ is among the most critical concepts in the study of infectious diseases, which will perform as the threshold for disease extinction and persistence. It is usually construed as the mean number of secondary infections that occur when a type of infected individual is introduced into a utterly susceptible population during the full infection period [29]. Here, according to the theory developed in [30,32–34], the $\mathcal{R}_0$ is introduced.

Set $\mathbb{E} := C\left(\overline{\Omega}, \mathbb{R}^5\right)$, $\mathbb{E}^+ := C\left(\overline{\Omega}, \mathbb{R}_+^5\right)$ and $C_\omega(\mathbb{R}, \mathbb{E})$ to be the Banach space consisting of wholly ω-periodic and continuous functions from $\mathbb{R}$ to $\mathbb{E}$, and for $\psi \in C_\omega(\mathbb{R}, \mathbb{E})$, $\|\psi\|_{C_\omega(\mathbb{R},\mathbb{E})} := \max_{\theta \in [0,\omega]} \|\psi(\theta)\|_{\mathbb{E}}$. Then use [32] to acquire $\mathcal{R}_0$ for model (2.3). Setting $E_h = I_h = J = E_v = I_v = 0$ in system (2.3), one gets

$$\begin{cases} \dfrac{\partial S_h}{\partial t} = D_h \Delta S_h + \Lambda_h(x) - \mu_h(x)S_h, \\ \dfrac{\partial S_v}{\partial t} = D_v \Delta S_v + \Lambda_v(t,x) - \mu_v(t,x)S_v, \\ \dfrac{\partial S_h}{\partial \nu} = \dfrac{\partial S_v}{\partial \nu} = 0. \end{cases} \tag{4.1}$$

Lemma 2.1 in [24] suggests that model (4.1) has a positive globally attractive ω-periodic solution $\left(\tilde{S}_h(\cdot), \tilde{S}_v(t,\cdot)\right)$ on $C\left(\overline{\Omega}, \mathbb{R}_+^2\right)$. Linearizing model (2.3) at $\left(\tilde{S}_h(\cdot), 0, 0, 0, \tilde{S}_v(t,\cdot), 0, 0\right)$ and regarding infection compartments, for

$t > 0$, one obtains

$$
\begin{cases}
\dfrac{\partial v_1}{\partial t} = D_h \Delta v_1 + \beta_1(t,x)\tilde{S}_h(x)v_5 - (\mu_h(x) + \alpha_h(x))v_1, & x \in \Omega, \\[2mm]
\dfrac{\partial v_2}{\partial t} = D_h \Delta v_2 + f\alpha_h(x)v_1 - (\mu_h(x) + \gamma_1(x))v_2, & x \in \Omega, \\[2mm]
\dfrac{\partial v_3}{\partial t} = D_h \Delta v_3 + (1-f)\alpha_h(x)v_1 - (\mu_h(x) + \gamma_2(x))v_3, & x \in \Omega, \\[2mm]
\dfrac{\partial v_4}{\partial t} = D_v \Delta v_4(t,x) + \beta_2(t,x)\tilde{S}_v(v_2 + cv_3) - (\mu_v(t,x) + \alpha_v(t,x))v_4, & x \in \Omega, \\[2mm]
\dfrac{\partial v_5}{\partial t} = D_v \Delta v_5 + \alpha_v(t,x)v_4 - \mu_v(t,x)v_5, & x \in \Omega, \\[2mm]
\dfrac{\partial v_1}{\partial \nu} = \dfrac{\partial v_2}{\partial \nu} = \dfrac{\partial v_3}{\partial \nu} = \dfrac{\partial v_4}{\partial \nu} = \dfrac{\partial v_5}{\partial \nu} = 0, & x \in \partial\Omega.
\end{cases}
\tag{4.2}
$$

Define $\mathcal{F}(t) : \mathbb{E} \to \mathbb{E}$ by

$$
\mathcal{F}(t)\begin{pmatrix} \varphi_1(\cdot) \\ \varphi_2(\cdot) \\ \varphi_3(\cdot) \\ \varphi_4(\cdot) \\ \varphi_5(\cdot) \end{pmatrix} = \begin{pmatrix} \beta_1(t,\cdot)\tilde{S}_h(\cdot)\varphi_5(\cdot) \\ f\alpha_h(\cdot)\varphi_1(\cdot) \\ (1-f)\alpha_h(\cdot)\varphi_1(\cdot) \\ \beta_2(t,\cdot)\tilde{S}_v(t,\cdot)(\varphi_2(\cdot) + c\varphi_3(\cdot)) \\ \alpha_v(t,\cdot)\varphi_4(\cdot) \end{pmatrix},
$$

for $(\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5) \in \mathbb{E}$, $t \in \mathbb{R}$, $-V(t)v = \mathcal{D}\Delta v - W(t)v$, with $\mathcal{D} = diag(D_h, D_h, D_h, D_v, D_v)$ and

$$
-[W(t)](\cdot) = \begin{pmatrix} -(\mu_h(\cdot) + \alpha_h(\cdot)) & 0 & 0 & 0 & 0 \\ 0 & -(\mu_h(\cdot) + \gamma_1(\cdot)) & 0 & 0 & 0 \\ 0 & 0 & -(\mu_h(\cdot) + \gamma_2(\cdot)) & 0 & 0 \\ 0 & 0 & 0 & -(\mu_v(t,\cdot) + \alpha_v(t,\cdot)) & 0 \\ 0 & 0 & 0 & 0 & -\mu_v(t,\cdot) \end{pmatrix}.
$$

Allow $\Psi(t,s) = \mathrm{diag}(T_2(t,s), T_3(t,s), T_4(t,s), T_6(t,s), T_5(t,s))$, $t \geq s$ to be the evolution operators intimately related to the subsequent mechanism

$$
\frac{\mathrm{d}v}{\mathrm{d}t} = -V(t)v.
$$

Then [35, Theorem 3.12] reveals that $-V(t)$ is resolvent positive.

Recall that the exponential growth bound of $\Psi(t,s)$ is defined by

$$
\bar{\omega}(\Psi) = \inf\left\{ \tilde{\omega} : \exists L \geq 1 \text{ such that } \|\Psi(t+s,s)\| \leq Le^{\tilde{\omega}t}, \ \forall s \in \mathbb{R}, \ t \geq 0 \right\}.
$$

Proposition A.2 in [35] shows that

$$
\bar{\omega}(\Psi) = \frac{\ln r(\Psi(\omega,0))}{\omega} = \frac{\ln r(\Psi(\omega + \triangle t, \triangle t))}{\omega}, \ \triangle t \in [0, \omega].
$$

According to [36, Lemma 14.2] and Krein-Rutman theorem, one has

$$
0 < r(\Psi(\omega,0)) = \max\{r(T_2(\omega,0)), r(T_3(\omega,0)), r(T_4(\omega,0)), r(T_5(\omega,0)), r(T_6(\omega,0))\} < 1,
$$

and $r(\Psi(\omega,0))$ is the spectral radius of $\Psi(\omega,0)$. Let $s = 0$ in [35, Proposition 5.6], one brings $\bar{\omega}(\Psi) < 0$. Clarify that $\Psi(t,s)$ is a positive operator, as meaning that for $t \geq s$, $\Psi(t,s)\mathbb{E}^+ \subset \mathbb{E}^+$. Thus $\mathcal{F}(t)$ and $W(t)$ satisfying (i) for any $t > 0$, $\mathcal{F}(t)$ maps $\mathbb{E}^+$ into $\mathbb{E}^+$; (ii) $-W(t)$ is cooperative.

In order to introduce $\mathcal{R}_0$ for system (2.3), keeping both human and mosquito populations are near the disease-free ω-periodic solution $\left(\tilde{S}_h(x), 0, 0, 0, \tilde{S}_v(t,x), 0, 0\right)$. Suppose that $\bar{v} \in C(\mathbb{R}, \mathbb{E})$ and $\bar{v}(t,x) = \bar{v}(t)(x)$ is the initial distribution of infectious humans and mosquitoes introduced at $t \in \mathbb{R}$, $x \in \overline{\Omega}$. Note that for $s \geq 0$,

$\mathcal{F}(t-s)\bar{v}(t-s,x)$ represents the density distribution of lately infected individuals at $t-s$ ($s<t$) and location x. Subsequently,

$$\int_0^\infty \Psi(t,t-s)\mathcal{F}(t-s)\bar{v}(t-s,\cdot)\mathrm{d}s$$

represents the distribution of cumulative infected individuals, which is created by whole infected individuals at prior to time t. Define

$$\mathcal{L}\bar{v}t(t) := \int_0^\infty \Psi(t,t-s)\mathcal{F}(t-s)\bar{v}(t-s,\cdot)\mathrm{d}s,\ \forall t \in \mathbb{R},\ \bar{v} \in C_\omega(\mathbb{R},\mathbb{E}).$$

Subsequently $\mathcal{L}$ is a continuous and positive operator, which maps the distribution of initial infection $\bar{v}(t)$ to the whole infected distribution developed during among the infectious periodic. Encouraged by the doctrine of the next generation operators in [32,35], $\mathcal{R}_0$ for model (2.3) is described as the spectral radius of $\mathcal{L}$,

$$\mathcal{R}_0 = r(\mathcal{L}).$$

For all $\varphi \in \mathbb{E}$, $P(t)$ is the solution map of (4.2) on $\mathbb{E}$, namely, $P(t)(\varphi) = v_t(\varphi)$, $t \geq 0$, with $v_t(\varphi)(x) = (v_1(t,x;\varphi), v_2(t,x;\varphi), v_3(t,x;\varphi), v_4(t,x;\varphi), v_5(t,x;\varphi))$ and $v(t,x;\varphi)$ is a single solution of (4.2) and $v_0(x) = \varphi(x)$, for all $x \in \overline{\Omega}$. Hence, $P := P(\omega)$ is the Poincaré map intimately related to (4.2). $r(P)$ is set to be the spectral radius of P. Similar to [7, Section 3], one has $v(t,x,\varphi) \gg 0$. According to [25, Theorem 2.1.8], for $t > 0$, $v(t,x,\varphi)$ is compact on $\mathbb{E}$. Thereby, P^n is compact and strongly positive. Based on [30, Lemma 3.1], $r(P)$ is a simple eigenvalue of P which is intimately related to a positive eigenfunction $\varphi \in Int(\mathbb{E}^+)$.

In order to characterize $\mathcal{R}_0$, we consider the following linear ω-periodic equation

$$\begin{cases} -\mathcal{D}\Delta\varphi + W(t)\varphi = \hat{\mu}\mathcal{F}(t), & x \in \Omega, \\ \dfrac{\partial\varphi_i}{\partial\nu} = 0, \quad i = 1,\cdots,5, & x \in \partial\Omega. \end{cases} \tag{4.3}$$

Based on [30,31], we can obtain the following lemma.

Lemma 4.1. *Problem (4.3) admits a unique principal eigenvalue $\mu^* > 0$, associated with a strictly positive eigenvector $(\varphi_1^*, \varphi_2^*, \varphi_3^*, \varphi_4^*, \varphi_5^*)$, then $\mathcal{R}_0 = \frac{1}{\mu^*}$.*

According to [32, Theorem 2.1], one gets the subsequent result.

Lemma 4.2. *The sign of $\mathcal{R}_0 - 1$ and $r(P) - 1$ are the same.*

5. Threshold dynamics

Next analysis the strictly positive of the solution for system (2.3).

Lemma 5.1. *Allow $u(t,x,\phi)$ to be the solution of (2.3) with $u_0 = \phi \in \mathbb{X}^+$. In the event that there exists $t_0 \geq 0$ in such a way that $E_h(t_0,\cdot;\phi) \not\equiv 0$, $I_h(t_0,\cdot;\phi) \not\equiv 0$, $J(t_0,\cdot;\phi) \not\equiv 0$, $E_v(t_0,\cdot;\phi) \not\equiv 0$ and $I_v(t_0,\cdot;\phi) \not\equiv 0$, then the solution of (2.3) meets*

$$E_h > 0,\ I_h > 0,\ J > 0,\ E_v > 0,\ I_v > 0,\ t \geq t_0,\ x \in \overline{\Omega}.$$

Besides, for all initial data $\phi \in \mathbb{X}^+$, $\forall t \geq 0$, $x \in \overline{\Omega}$, one has $S_h > 0$, $S_v > 0$, and

$$\liminf_{t\to\infty} S_h \geq \tilde{\varepsilon},\ \liminf_{t\to\infty} S_v \geq \tilde{\varepsilon},\ uniformly\ for\ x \in \overline{\Omega},$$

where $\tilde{\varepsilon} > 0$ is ϕ-independent constant.

Proof. We know that for $t > 0$, E_h, I_h, J, E_v and I_v meet

$$\begin{cases} \dfrac{\partial E_h}{\partial t} \geq D_h \Delta E_h - (\mu_h(x) + \alpha_h(x))E_h, & x \in \Omega, \\[2mm] \dfrac{\partial I_h}{\partial t} \geq D_h \Delta I_h - (\mu_h(x) + \gamma_1(x))I_h, & x \in \Omega, \\[2mm] \dfrac{\partial J}{\partial t} \geq D_h \Delta J - (\mu_h(x) + \gamma_2(x))J, & x \in \Omega, \\[2mm] \dfrac{\partial E_v}{\partial t} \geq D_v \Delta E_v - (\mu_v(t,x) + \alpha_v(t,x))E_v, & x \in \Omega, \\[2mm] \dfrac{\partial I_v}{\partial t} \geq D_v \Delta I_v - \mu_v(t,x)I_v, & x \in \Omega, \\[2mm] \dfrac{\partial E_h}{\partial \nu} = \dfrac{\partial I_h}{\partial \nu} = \dfrac{\partial E_v}{\partial \nu} = \dfrac{\partial I_v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

Suppose there exists $t_0 \geq 0$, in such a way that $E_h(t_0, \cdot; \phi) \not\equiv 0$, $I_h(t_0, \cdot; \phi) \not\equiv 0$, $J(t_0, \cdot; \phi) \not\equiv 0$, $E_v(t_0, \cdot; \phi) \not\equiv 0$ and $I_v(t_0, \cdot; \phi) \not\equiv 0$, then according to parabolic maximum principle, one has that $E_h(t, \cdot; \phi) > 0$, $I_h(t, \cdot; \phi) > 0$, $J(t, \cdot; \phi) > 0$, $E_v(t, \cdot; \phi) > 0$ and $I_v(t, \cdot; \phi) > 0$, for $t > t_0$. Set $\hat{S}_h(t, \cdot; \phi)$ to be the solution of

$$\begin{cases} \dfrac{\partial \hat{S}_h}{\partial t} = D_h \Delta \hat{S}_h + \Lambda_h(x) - \beta_1(t,x)\hat{S}_h M_4 - \mu_h(x)\hat{S}_v, & t > 0, \, x \in \Omega, \\[2mm] \dfrac{\partial \hat{S}_h}{\partial \nu} = 0, & t > 0, \, x \in \partial\Omega, \\[2mm] \hat{S}_h(0, x) = \phi_1(x), & x \in \Omega. \end{cases} \tag{5.1}$$

In virtue of the comparison principle, one concludes that $S_h \geq \hat{S}_h$ for $x \in \overline{\Omega}$. Additionally, system (5.1) has a single positive global attractive ω-periodic solution $\hat{S}_h^*(t, x)$ based on [24, Lemma 2.1]. Then

$$\liminf_{t \to \infty} S_h \geq \varepsilon_1 := \min_{t \in [0,\omega], x \in \overline{\Omega}} \hat{S}_h^* \text{ uniformly for } x \in \overline{\Omega}.$$

Denote $\hat{S}_v$ to be the solution of

$$\begin{cases} \dfrac{\partial \hat{S}_v}{\partial t} = D_v \Delta \hat{S}_v + \Lambda_v(t,x) - \beta_2(t,x)\hat{S}_v \dfrac{M_2}{m} - \mu_v(x)\hat{S}_v, & x \in \Omega, \\[2mm] \dfrac{\partial \hat{S}_v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{5.2}$$

for $t > 0$, with $\hat{S}_v(0, x) = \phi_5(x)$. Similarly, $S_v \geq \hat{S}_v$ for $x \in \overline{\Omega}$. Indisputably, system (5.2) has a single positive global attractive ω-periodic solution $\hat{S}_v^*(t, \cdot)$. Then

$$\liminf_{t \to \infty} S_v \geq \varepsilon_2 := \min_{t \in [0,\omega], x \in \overline{\Omega}} \hat{S}_v^*, \text{ uniformly for } x \in \overline{\Omega}.$$

Choosing $\tilde{\varepsilon} = \min\{\varepsilon_1, \varepsilon_2\}$. The proof is complete. $\qquad\square$

Lemma 5.2. *Define* $\mu = \frac{\ln r(P)}{\omega}$. *There exists a ω-periodic function $v^*(t, x)$, which is positive, in such a way that $e^{\mu t}v^*(t, x)$ is a solution of (4.2).*

The proof of Lemma 5.2 is given in Appendix A.

Theorem 5.1. *Denote $u(t, x, \phi)$ to be the solution of (2.3) with $u_0 = \phi \in \mathbb{X}^+$. The following statements are true.*

(i) If $\mathcal{R}_0 < 1$, then the disease-free ω-periodic solution $\left(\tilde{S}_h(x), 0, 0, 0, \tilde{S}_v(t, x), 0, 0\right)$ is globally attractive.

(ii) If $\mathcal{R}_0 > 1$, then model (2.3) admits no less than one positive ω-periodic solution, and there exists an $\eta > 0$ such that for $\phi \in \mathbb{X}^+$ with $\phi_2(\cdot) \not\equiv 0$, $\phi_3(\cdot) \not\equiv 0$, $\phi_4(\cdot) \not\equiv 0$, $\phi_6(\cdot) \not\equiv 0$ and $\phi_7(\cdot) \not\equiv 0$, we have

$$\liminf_{t\to\infty} S_h \geq \eta, \ \liminf_{t\to\infty} E_h \geq \eta, \ \liminf_{t\to\infty} I_h \geq \eta, \ \liminf_{t\to\infty} J \geq \eta, \quad \liminf_{t\to\infty} S_v \geq \eta, \ \liminf_{t\to\infty} E_v \geq \eta, \ \liminf_{t\to\infty} I_v \geq \eta,$$

uniformly for $x \in \overline{\Omega}$.

Proof. (i) If $\mathcal{R}_0 < 1$, from Lemma 4.2, we know that $r(P) < 1$, and thereupon $\mu = \frac{\ln r(P)}{\omega} < 0$. For $t > 0$, view the following equations with $\epsilon > 0$,

$$
\begin{cases}
\dfrac{\partial v_1^\epsilon}{\partial t} = D_h \Delta v_1^\epsilon + \beta_1(t,x)\left(\tilde{S}_h(x) + \epsilon\right) v_5^\epsilon - (\mu_h(x) + \alpha_h(x))v_1^\epsilon, & x \in \Omega, \\[2mm]
\dfrac{\partial v_2^\epsilon}{\partial t} = D_h \Delta v_2^\epsilon + f\alpha_h(x)v_1^\epsilon - (\mu_h(x) + \gamma_1(x))v_2^\epsilon, & x \in \Omega, \\[2mm]
\dfrac{\partial v_3^\epsilon}{\partial t} = D_h \Delta v_3^\epsilon + (1-f)\alpha_h(x)v_1^\epsilon - (\mu_h(x) + \gamma_2(x))v_3^\epsilon, & x \in \Omega, \\[2mm]
\dfrac{\partial v_4^\epsilon}{\partial t} = D_v \Delta v_4^\epsilon + \beta_2(t,x)\left(\tilde{S}_v^\epsilon + \epsilon\right) v_2^\epsilon - (\mu_v(t,x) + \alpha_v(t,x))v_4^\epsilon, & x \in \Omega, \\[2mm]
\dfrac{\partial v_5^\epsilon}{\partial t} = D_v \Delta v_5^\epsilon + \alpha_v(t,x)v_4^\epsilon - \mu_v(t,x)v_5^\epsilon, & x \in \Omega, \\[2mm]
\dfrac{\partial v_1^\epsilon}{\partial \nu} = \dfrac{\partial v_2^\epsilon}{\partial \nu} = \dfrac{\partial v_3^\epsilon}{\partial \nu} = \dfrac{\partial v_4^\epsilon}{\partial \nu} = \dfrac{\partial v_5^\epsilon}{\partial \nu} = 0, & x \in \partial\Omega.
\end{cases}
\tag{5.3}
$$

For $\varphi \in \mathbb{E}$, we assume that $v^\epsilon = (v_1^\epsilon, v_2^\epsilon, v_3^\epsilon, v_4^\epsilon)$ is the unique solution of (5.3) with $v_0^\epsilon(\varphi)(t,x) = v^\epsilon(t,x;\varphi) = (v_1^\epsilon(t,x;\varphi), v_2^\epsilon(t,x;\varphi), v_3^\epsilon(t,x;\varphi), v_4^\epsilon(t,x;\varphi))$, $t \geq 0$, $x \in \overline{\Omega}$. Let $P_\epsilon := \mathbb{E} \to \mathbb{E}$ be the Poincaré map of (5.3), i.e., $P_\epsilon(\varphi) = v_\omega^\epsilon(\varphi)$, $\varphi \in \mathbb{E}$. Allow $r(P_\epsilon)$ to represent the spectral radius of P_ϵ. Given that $\lim_{\epsilon\to 0} r(P_\epsilon) = r(P) < 1$, one can choose sufficiently small $\epsilon > 0$ in such a way that $r(P_\epsilon) < 1$. On the basis of Lemma 5.2, there can be a positive ω-periodic function $v_\epsilon^*(t,x)$, then $v^\epsilon = e^{\mu t}v_\epsilon^*$ is a solution of model (5.3), where $\mu_\epsilon = \frac{\ln r(P_\epsilon)}{\omega} < 0$. For given $\epsilon > 0$, applying the comparison principle, there is some adequately large integer $n_1 > 0$ in such manner as to

$$S_h \leq \tilde{S}_h(x) + \epsilon, \ S_v \leq \tilde{S}_v(t,x) + \epsilon, \ t \geq n_1\omega, \ x \in \overline{\Omega}.$$

Then, for $t \geq n_1\omega$,

$$
\begin{cases}
\dfrac{\partial E_h}{\partial t} \leq D_h \Delta E_h + \beta_1(t,x)\left(\tilde{S}_h(x) + \epsilon\right) I_v - (\mu_h(x) + \alpha_h(x))E_h, & x \in \Omega, \\[2mm]
\dfrac{\partial I_h}{\partial t} \leq D_h \Delta I_h + f\alpha_h(x)E_h - (\mu_h(x) + \gamma_1(x))I_h, & x \in \Omega, \\[2mm]
\dfrac{\partial J}{\partial t} \leq D_h \Delta J + (1-f)\alpha_h(x)E_h - (\mu_h(x) + \gamma_2(x))J, & x \in \Omega, \\[2mm]
\dfrac{\partial E_v}{\partial t} \leq D_v \Delta E_v + \beta_2(t,x)\left(\tilde{S}_v + \epsilon\right) I_h - (\mu_v(t,x) + \alpha_v(t,x))E_v, & x \in \Omega, \\[2mm]
\dfrac{\partial I_v}{\partial t} \leq D_v \Delta I_v + \alpha_v(t,x)I_v - \mu_v(t,x)I_v, & x \in \Omega, \\[2mm]
\dfrac{\partial E_h}{\partial \nu} = \dfrac{\partial I_h}{\partial \nu} = \dfrac{\partial J}{\partial \nu} = \dfrac{\partial E_v}{\partial \nu} = \dfrac{\partial I_v}{\partial \nu} = 0, & x \in \partial\Omega.
\end{cases}
\tag{5.4}
$$

Using (5.3), (5.4) and the comparison theorem, there is $\alpha_1 > 0$ in such a manner that $(E_h, I_h, J, E_v, I_v) \leq \alpha_1 e^{\mu_\epsilon t}v_\epsilon^*(t,\cdot)$, $t \geq n_1\omega$, and

$$\lim_{t\to\infty}(E_h, I_h, J, E_v, I_v) = (0,0,0,0,0) \text{ uniformly for } x \in \overline{\Omega}.$$

Then the equations of S_h and S_v are asymptomatic to (4.1). By the internally chain transitive sets [37, Section 2.1], we obtain that $\lim\limits_{t\to\infty} \left[(S_h(t,x), S_v(t,x)) - \left(\tilde{S}_h(x), \tilde{S}_v(t,x) \right) \right] = 0$ uniformly for $x \in \overline{\Omega}$, where $\left(\tilde{S}(x), \tilde{S}_v(t,x) \right)$ is the globally attractive solution of (4.1).

(ii) For $\mathcal{R}_0 > 1$, one achieves $r(P) > 1$ and $\mu = \frac{\ln r(P)}{\omega} > 0$.

Denote

$$M_0 := \{\phi \in \mathbb{X}^+ : \phi_2(\cdot) \not\equiv 0 \text{ and } \phi_3(\cdot) \not\equiv 0 \text{ and } \phi_4(\cdot) \not\equiv 0 \text{ and } \phi_6(\cdot) \not\equiv 0 \text{ and } \phi_7(\cdot) \not\equiv 0\}$$

and

$$\partial M_0 := \mathbb{X}^+ \backslash M_0 = \left\{ \phi \in \mathbb{X}^+ : \phi_2(\cdot) \equiv 0 \text{ or } \phi_3(\cdot) \equiv 0 \text{ or } \phi_4(\cdot) \equiv 0 \text{ or } \phi_6(\cdot) \equiv 0 \text{ or } \phi_7(\cdot) \equiv 0 \right\}.$$

To highlight the fact that for $\phi \in M_0$, $t > 0$, $x \in \overline{\Omega}$, Lemma 5.1 reveals that

$$E_h > 0, \ I_h > 0, \ E_v > 0, \ I_v > 0.$$

It follows that $\Phi^n(M_0) \subset M_0$, $n \in \mathbb{N}$. According to Theorem 3.1, Φ allows a global attractor in $\mathbb{X}^+$. Denote

$$M_\partial := \{\phi \in \partial M_0, \Phi^n(\phi) \in \partial M_0, n \in \mathbb{N}\},$$

meanwhile $\omega(\phi)$ is the omega limit set of the orbit $\gamma^+(\phi) = \{\Phi^n(\phi) : \forall n \in \mathbb{N}\}$. Set $\mathcal{M} = \left\{ \left(\tilde{S}_h(\cdot), 0, 0, 0, \tilde{S}_v(t,\cdot), 0, 0 \right) \right\}$. The fact that $\mathcal{M}$ cannot develop a cycle for $\Phi(\omega)$ in M_0 is shown in the next claim. **Claim 1** For any $\tilde{\phi} \in M_\partial$, the omega limit set $\omega(\tilde{\phi}) = \mathcal{M}$. For $\tilde{\phi} \in M_\partial$, $\Phi^n\left(\tilde{\phi}\right) \in \partial M_0$, $n \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$, either $E_h\left(n\omega, \cdot; \tilde{\phi}\right) \equiv 0$ or $I_h\left(n\omega, \cdot; \tilde{\phi}\right) \equiv 0$ or $J\left(n\omega, \cdot; \tilde{\phi}\right) \equiv 0$ or $E_v\left(n\omega, \cdot; \tilde{\phi}\right) \equiv 0$ or $I_v\left(n\omega, \cdot; \tilde{\phi}\right) \equiv 0$. Accordingly, for each $t \geq 0$, $E_h\left(t, \cdot; \tilde{\phi}\right) \equiv 0$ or $I_h\left(t, \cdot; \tilde{\phi}\right) \equiv 0$ or $J\left(t, \cdot; \tilde{\phi}\right) \equiv 0$ or $E_v\left(t, \cdot; \tilde{\phi}\right) \equiv 0$ or $I_v\left(t, \cdot; \tilde{\phi}\right) \equiv 0$. Conversely, it contradicts with Lemma 5.1. If $E_h\left(t, \cdot; \tilde{\phi}\right) \equiv 0$, the third and forth equations in system (2.3) satisfy

$$\begin{cases} \dfrac{\partial I_h}{\partial t} \leq D_h \Delta I_h - (\tilde{\mu}_h + \tilde{\gamma}_1)\, I_h, \\[2mm] \dfrac{\partial J}{\partial t} \leq D_h \Delta J - (\tilde{\mu}_h + \tilde{\gamma}_2)\, J. \end{cases}$$

By the comparison principle, one has $\lim\limits_{t\to\infty} \left(I_h\left(t, x; \tilde{\phi}\right), J\left(t, x; \tilde{\phi}\right) \right) = (0,0)$ uniformly for $x \in \overline{\Omega}$. From the E_v equation of model (2.3), it is easy to check that $\lim\limits_{t\to\infty} E_v\left(t, x; \tilde{\phi}\right) = 0$ uniformly for $x \in \overline{\Omega}$, and then $\lim\limits_{t\to\infty} I_v\left(t, x; \tilde{\phi}\right) = 0$. Furthermore, S_h and S_v equations satisfy an nonautonomous system which is asymptomatic to the periodic system (4.1). Furthermore, it can be demonstrated through application of the internally chain transitive sets method, as presented in [37], that $\lim\limits_{t\to\infty} \left(S_h\left(t, \cdot; \tilde{\phi}\right), S_v\left(t, \cdot; \tilde{\phi}\right) \right) - \left(\tilde{S}_h(\cdot), \tilde{S}_v(t,\cdot) \right) = 0$. If $E_h\left(t_1, \cdot; \tilde{\phi}\right) \not\equiv 0$, for some $t_1 \geq 0$. It follows from Lemma 5.1 that $E_h\left(t_1, \cdot; \tilde{\phi}\right) > 0$ for $t \geq t_1$. Thus, $I_h\left(t, \cdot; \tilde{\phi}\right) \equiv 0$ or $J\left(t, \cdot; \tilde{\phi}\right) \equiv 0$ or $E_v\left(t, \cdot; \tilde{\phi}\right) \equiv 0$ or $I_v(t, \cdot; \tilde{\phi}) \equiv 0$, $t \geq t_1$. For the case of $I_h\left(t, \cdot; \tilde{\phi}\right) \equiv 0$, $t \geq t_1$, then $\lim\limits_{t\to\infty} E_h\left(t, \cdot; \tilde{\phi}\right) = 0$. In this case, it is easy to get $\lim\limits_{t\to\infty} J\left(t, \cdot; \tilde{\phi}\right) = 0$, $\lim\limits_{t\to\infty} E_v\left(t, \cdot; \tilde{\phi}\right) = 0$ and then $\lim\limits_{t\to\infty} I_v\left(t, \cdot; \tilde{\phi}\right) = 0$. Based on S_h and S_v equations, one obtains $\lim\limits_{t\to+\infty} \left(\left(S_h\left(t, \cdot; \tilde{\phi}\right), S_v\left(t, \cdot; \tilde{\phi}\right) \right) - \left(\tilde{S}_h(\cdot), \tilde{S}_v(t,\cdot) \right) \right) = 0$. The same way can also be used in other cases. Consequently, $\omega(\tilde{\phi}) = \mathcal{M}$ for any $\tilde{\phi} \in M_\partial$.

Take into account the next equations with $\delta > 0$ and $t > 0$,

$$
\begin{cases}
\dfrac{\partial v_1^\delta}{\partial t} = D_h \Delta v_1^\delta + \beta_1(t,x)\left(\tilde{S}_h(x) - \delta\right) v_4^\delta - (\mu_h(x) + \alpha_h(x))v_1^\delta, & x \in \Omega, \\[2mm]
\dfrac{\partial v_2^\delta}{\partial t} = D_h \Delta v_2^\delta + f\alpha_h(x)v_1^\delta - (\mu_h(x) + \gamma_1(x))v_2^\delta, & x \in \Omega, \\[2mm]
\dfrac{\partial v_3^\delta}{\partial t} = D_h \Delta v_3^\delta + (1-f)\alpha_h(x)v_1^\delta - (\mu_h(x) + \gamma_2(x))v_2^\delta, & x \in \Omega, \\[2mm]
\dfrac{\partial v_4^\delta}{\partial t} = D_v \Delta v_4^\delta + \beta_2(t,x)\left(\tilde{S}_v - \delta\right) v_2^\delta - (\mu_v(t,x) + \alpha_v(t,x))v_4^\delta, & x \in \Omega, \\[2mm]
\dfrac{\partial v_5^\delta}{\partial t} = D_v \Delta v_5^\delta + \alpha_v(t,x)v_4^\delta - \mu_v(t,x)v_5^\delta, & x \in \Omega, \\[2mm]
\dfrac{\partial v_1^\delta}{\partial \nu} = \dfrac{\partial v_2^\delta}{\partial \nu} = \dfrac{\partial v_3^\delta}{\partial \nu} = \dfrac{\partial v_4^\delta}{\partial n} = \dfrac{\partial v_5^\delta}{\partial \nu} = 0, & x \in \partial\Omega.
\end{cases}
\tag{5.5}
$$

For $\tilde{\psi} \in \mathbb{E}$, denote $v^\delta = \left(v_1^\delta, v_2^\delta, v_3^\delta, v_4^\delta\right)$ to be the solution of (5.5) with $v_0^\delta\left(\tilde{\psi}\right)(t,x) = \tilde{\psi}(x)$. Let $P_\delta := P_\delta(\omega)$ is the Poincaré map of (5.5) in $\mathbb{E}$, i.e., $P_\delta\left(\tilde{\psi}\right) = v_\omega^\delta\left(\tilde{\psi}\right), \forall \tilde{\psi} \in \mathbb{E}$, and $r(P_\delta)$ be the spectral radius of P_δ. Due to $\lim_{\delta \to 0} r(P_\delta) = r(P) > 1$, select $\delta > 0$ to be sufficiently small, so that

$$
\delta < \min\left\{ \min_{x \in \overline{\Omega}} \tilde{S}_h(x), \min_{t \in [0,\omega], x \in \overline{\Omega}} \tilde{S}_v(t,x) \right\}, \text{ and } r(P) > 1.
$$

For the above fixed $\delta > 0$, by the continuous dependence of solutions on initial value, there exists $\bar{\delta} > 0$ such that for all $\tilde{\psi}$ with $\left\| \tilde{\psi} - \mathcal{M} \right\| \le \bar{\delta}$, one has $\|\Phi(t)\tilde{\psi} - \Phi(t)\mathcal{M}\| < \delta$ for $t \in [0, \omega]$. We now prove the following claim.

Claim 2 For every $\phi \in M_0$, there holds $\lim_{n \to \infty} \|\Phi^n \phi - \mathcal{M}\| \ge \bar{\delta}$.

Using contradictory proof, assume that for some $\phi_0 \in M_0$, there is $\lim_{n \to \infty} \|\Phi^n \phi_0 - \mathcal{M}\| < \bar{\delta}$. Given $n_2 \ge 1$, for $n \ge n_2$, one has $\|\Phi^n(\phi_0) - \mathcal{M}\| < \bar{\delta}$. For $t \ge n_2\omega$, letting $t = n\omega + t'$ with $n = [t/\omega]$ and $t' \in [0, \omega)$, we obtain that

$$
\|\Phi(t)\phi_0 - \Phi(t)\mathcal{M}\| = \|\Phi(t')(\Phi^n(\phi_0)) - \Phi(t')\mathcal{M}\|.
\tag{5.6}
$$

Based on (5.6) and Lemma 5.1,

$$
\tilde{S}_h(x) - \delta < S_h < \tilde{S}_h(x) + \delta, \ 0 < E_h < \delta, \ 0 < I_h < \delta, \ 0 < J < \delta,
$$
$$
\tilde{S}_v(t,x) - \delta < S_v < \tilde{S}_v(t,x) + \delta, \ 0 < E_v < \delta, \ 0 < I_v < \delta,
$$

for $t \ge n_2\omega$, $x \in \overline{\Omega}$. As a consequence, when $t \ge n_2\omega$, $E_h(t,x;\phi_0)$, $I_h(t,x;\phi_0)$, $J(t,x;\phi_0)$, $E_v(t,x;\phi_0)$ and $I_v(t,x;\phi_0)$ meet

$$
\begin{cases}
\dfrac{\partial E_h}{\partial t} \ge D_h \Delta E_h + \beta_1(t,x)\left(\tilde{S}_h(x) - \delta\right) I_v - (\mu_h(x) + \alpha_h(x))E_h, & x \in \Omega, \\[2mm]
\dfrac{\partial I_h}{\partial t} \ge D_h \Delta I_h + f\alpha_h(x)E_h - (\mu_h(x) + \gamma_1(x))I_h, & x \in \Omega, \\[2mm]
\dfrac{\partial J}{\partial t} \ge D_h \Delta J + (1-f)\alpha_h(x)E_h - (\mu_h(x) + \gamma_2(x))J, & x \in \Omega, \\[2mm]
\dfrac{\partial E_v}{\partial t} \ge D_v \Delta E_v + \beta_2(t,x)\left(\tilde{S}_v(t,x) - \delta\right) I_h - (\mu_v(t,x) + \alpha_v(t,x))E_v, & x \in \Omega, \\[2mm]
\dfrac{\partial I_v}{\partial t} \ge D_v \Delta I_v + \alpha_v(t,x)I_v - \mu_v(t,x)I_v, & x \in \Omega, \\[2mm]
\dfrac{\partial E_h}{\partial \nu} = \dfrac{\partial I_h}{\partial \nu} = \dfrac{\partial J}{\partial \nu} = \dfrac{\partial E_v}{\partial \nu} = \dfrac{\partial I_v}{\partial \nu} = 0, & x \in \partial\Omega.
\end{cases}
\tag{5.7}
$$

Given that $u(t, x; \phi_0) \gg 0$ for $(t, x) \in (0, +\infty) \times \overline{\Omega}$, there exists $\alpha_2 > 0$ in such a way that

$$(E_h, I_h, J, E_v, I_v) \geq \alpha_2 e^{\mu_\delta t} v_\delta^*, \ t \geq n_2 \omega, \ x \in \overline{\Omega},$$

where $v_\delta^*(t, x)$ is a positive ω-periodic function in a manner that $e^{\mu_\delta t} v_\delta^*(t, x)$ is a solution of system (5.5), and $\mu_\delta = \frac{\ln r(P_\delta)}{\omega}$. Since $\mu_\delta > 0$, it yields that $E_h(t, \cdot; \phi_0) \to \infty$, $I_h(t, \cdot; \phi_0) \to \infty$, $J(t, \cdot; \phi_0) \to \infty$, $E_v(t, \cdot; \phi_0) \to \infty$ and $I_v(t, \cdot; \phi_0) \to \infty$ as $t \to \infty$, which is a contradiction. Thence, $\mathcal{M}$ is an isolated invariant set and $W^s(\mathcal{M}) \cap M_0 = \emptyset$, $W^s(\mathcal{M})$ is the stable set. Pursuant to [26, Theorem 3.7], Φ allows a global attractor A_0 in M_0. Based on [37, Theorem 1.3.1 and Remark 1.3.1], one has that Φ is uniformly persistent about $(M_0, \partial M_0)$. That is to say, there exists $\bar{\eta} > 0$, in such a way that

$$\liminf_{n \to \infty} \mathrm{d}(\Phi^n(\phi), \partial M_0) \geq \bar{\eta}, \ \phi \in M_0. \tag{5.8}$$

Since $A_0 = \Phi A_0$, we find that $\phi_i(\cdot) > 0$, $i = 2, 3, 4, 6, 7$ for $\phi \in A_0$. Denote $B_0 := \bigcup_{t \in [0,\omega]} \Phi(t) A_0$. Subsequently, $B_0 \subset M_0$ and $\lim_{t \to \infty} \mathrm{d}(\Phi(t), B_0) = 0, \forall \phi \in M_0$. Define $p : \mathbb{X}^+ \to \mathbb{R}_+$ as a continuous function,

$$p(\phi) := \min \left\{ \min_{x \in \overline{\Omega}} \phi_2(x), \min_{x \in \overline{\Omega}} \phi_3(x), \min_{x \in \overline{\Omega}} \phi_4(x), \min_{x \in \overline{\Omega}} \phi_6(x), \min_{x \in \overline{\Omega}} \phi_7(x) \right\}, \ \phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7) \in \mathbb{X}^+.$$

In view of B_0 is compact subset of M_0, it follows that $\inf_{\phi \in B_0} p(\phi) = \min_{\phi \in B_0} p(\phi) > 0$. Consequently, there is an $\eta^* > 0$ that

$$\liminf p(\Phi(t)\phi) = \liminf_{t \to \infty} \min \left(\min_{x \in \overline{\Omega}} E_h(t, x; \phi), \min_{x \in \overline{\Omega}} I_h(t, x; \phi), \min_{x \in \overline{\Omega}} J(t, x; \phi), \min_{x \in \overline{\Omega}} E_v(t, x; \phi), \min_{x \in \overline{\Omega}} I_v(t, x; \phi) \right)$$
$$\geq \eta^*, \ \forall \phi \in M_0.$$

Additionally, according to Theorem 5.1, there is a constant $\hat{\eta} \in (0, \eta^*)$ such that

$$\liminf_{t \to \infty} \min_{x \in \overline{\Omega}} u \geq \hat{\eta}, \quad \phi \in M_0, \ u = (S_h, E_h, I_h, J, S_v, E_v, I_v).$$

Based on [20, Theorem 4.6] and [26, Theorem 4.5], model (2.3) has at least one positive ω-periodic solution. We complete the proof. $\qquad \square$

6. Global asymptotic stability analysis

We presume that β_1, β_2, μ_v and α_v are independent of time t. Following [38, 39], we cogitate the global asymptotic stability of the disease-free steady state in the critical case of $\mathcal{R}_0 = 1$ for the next system at $t > 0$,

$$\begin{cases} \dfrac{\partial S_h}{\partial t} = D_h \Delta S_h + \Lambda_h(x) - \beta_1(x) S_h I_v - \mu_h(x) S_h, & x \in \Omega, \\[2mm] \dfrac{\partial E_h}{\partial t} = D_h \Delta E_h + \beta_1(t, x) S_h I_v - (\mu_h(x) + \alpha_h(x)) E_h, & x \in \Omega, \\[2mm] \dfrac{\partial I_h}{\partial t} = D_h \Delta I_h + f \alpha_h(x) E_h - (\mu_h(x) + \gamma_1(x)) I_h, & x \in \Omega, \\[2mm] \dfrac{\partial J}{\partial t} = D_h \Delta J + (1 - f) \alpha_h(x) E_h - (\mu_h(x) + \gamma_2(x)) J, & x \in \Omega, \\[2mm] \dfrac{\partial S_v}{\partial t} = D_v \Delta S_v + \Lambda_v(x) - \beta_2(x) S_v (I_h + cJ) - \mu_v(x) S_v, & x \in \Omega, \\[2mm] \dfrac{\partial E_v}{\partial t} = D_v \Delta E_v + \beta_2(x) S_v (I_h + cJ) - (\mu_v(x) + \alpha_v(x)) E_v, & x \in \Omega, \\[2mm] \dfrac{\partial I_v}{\partial t} = D_v \Delta I_v + \alpha_v(x) E_v - \mu_v(x) I_v, & x \in \Omega, \\[2mm] \dfrac{\partial S_h}{\partial \nu} = \dfrac{\partial E_h}{\partial \nu} = \dfrac{\partial I_h}{\partial \nu} = \dfrac{\partial J}{\partial \nu} = \dfrac{\partial S_v}{\partial \nu} = \dfrac{\partial E_v}{\partial \nu} = \dfrac{\partial I_v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{6.1}$$

For simplicity, we still use the previous symbols.

Theorem 6.1. *If $\mathcal{R}_0 = 1$, then the disease-free steady state $E^0 = \left(\tilde{S}_h(x), 0, 0, 0, \tilde{S}_v(x), 0, 0 \right)$ of system (6.1) is globally asymptotical stable.*

Proof. We first explore that $\left(\tilde{S}_h(x), 0, 0, 0, \tilde{S}_v(x), 0, 0 \right)$ is locally asymptotically stable. Assume $\delta_1 > 0$ and let $u_0 = \left(S_h^0, E_h^0, I_h^0, J^0, S_v^0, E_h^0, I_v^0 \right)$ with $\left\| u_0 - \left(\tilde{S}_h(x), 0, 0, 0, \tilde{S}_v(x), 0, 0 \right) \right\| \le \delta_1$. Denote

$$r_1(t, x) = \frac{S_h(t, x)}{\tilde{S}_h(x)} - 1, \ r_2(t, x) = \frac{S_v(t, x)}{\tilde{S}_v(x)} - 1, \text{ and } z(t) = \max_{x \in \overline{\Omega}} \{ r_1(t, x), r_2(t, x), 0 \},$$

In view of

$$\begin{cases} D_h \Delta \tilde{S}_h(x) + \Lambda_h(x) - \mu_h(x) \tilde{S}_h(x) = 0, \\ D_v \Delta \tilde{S}_v(x) + \Lambda_v(x) - \mu_v(x) \tilde{S}_v(x) = 0, \end{cases}$$

one has

$$\begin{cases} \dfrac{\partial r_1(t, x)}{\partial t} - D_h \Delta r_1(t, x) - 2 D_h \dfrac{\nabla \tilde{S}_h(x) \nabla r_1(t, x)}{\tilde{S}_h(x)} + \dfrac{\Lambda_h(x)}{\tilde{S}_h(x)} r_1(t, x) = -\dfrac{\beta_1(x) S_h I_v}{\tilde{S}_h(x)}, \\ \dfrac{\partial r_2(t, x)}{\partial t} - D_v \Delta r_2(t, x) - 2 D_v \dfrac{\nabla \tilde{S}_v(x) \nabla r_2(t, x)}{\tilde{S}_v(x)} + \dfrac{\Lambda_v(x)}{\tilde{S}_v(x)} r_2(t, x) = -\dfrac{\beta_2(x) S_v (I_h + cJ)}{\tilde{S}_v(x)}. \end{cases}$$

Set $\hat{T}_1(t)$ and $\hat{T}_2(t)$ to be designated as the semigroups produced by the generator

$$D_h \Delta + 2 D_h \frac{\nabla \tilde{S}_h(x) \nabla}{\tilde{S}_h(x)} - \frac{\Lambda_h(x)}{\tilde{S}_h(x)},$$

and

$$D_v \Delta + 2 D_v \frac{\nabla \tilde{S}_v(x) \nabla}{\tilde{S}_v(x)} - \frac{\Lambda_v(x)}{\tilde{S}_v(x)},$$

subject to Nuemann boundary condition, respectively.

Choose $\delta_2 > 0$ such that $\| \hat{T}_1(t), \hat{T}_2(t) \| \le M_5 e^{-\delta_2 t}$ for some $M_5 > 0$. Then

$$\begin{cases} r_1(t, \cdot) = \hat{T}_1(t) r_1^0 - \displaystyle\int_0^\infty \hat{T}_1(t - s) \frac{\beta_1(\cdot) S_h I_v}{\tilde{S}_h(\cdot)} \mathrm{d}s, \\ r_2(t, \cdot) = \hat{T}_2(t) r_2^0 - \displaystyle\int_0^\infty \hat{T}_2(t - s) \frac{\beta_2(\cdot) S_v (I_h + cJ)}{\tilde{S}_v(\cdot)} \mathrm{d}s, \end{cases}$$

where $r_1^0 = \frac{S_h^0}{\tilde{S}_h(x)} - 1$ and $r_2^0 = \frac{S_v^0}{\tilde{S}_v(x)} - 1$.

Denote $S = \min \left\{ \min_{x \in \overline{\Omega}} \tilde{S}_h(x), \min_{x \in \overline{\Omega}} \tilde{S}_h(x) \right\}$, then

$$z(t) = \max_{x \in \overline{\Omega}} \left\{ \hat{T}_1(t) r_1^0 - \int_0^\infty \hat{T}_1(t - s) \frac{\beta_1(\cdot) S_h I_v}{\tilde{S}_h(\cdot)} \mathrm{d}s, \hat{T}_2(t) r_2^0 - \int_0^\infty \hat{T}_2(t - s) \frac{\beta_2(\cdot) S_h (I_h + cJ)}{\tilde{S}_h(\cdot)} \mathrm{d}s, 0 \right\}$$

$$\le \max_{x \in \overline{\Omega}} \left\{ \hat{T}_1(t) r_1^0, \hat{T}_2(t) r_2^0, 0 \right\} \le \max \left\{ \left\| \hat{T}_1(t) r_1^0 \right\|, \left\| \hat{T}_2(t) r_2^0 \right\| \right\}$$

$$\le \frac{\delta_1 M_5 e^{-\delta_2 t}}{S}.$$

Note that for $t > 0$, (E_h, I_h, J, E_v, I_v) satisfies

$$
\begin{cases}
\dfrac{\partial E_h}{\partial t} = D_h \Delta E_h + \beta_1(x)\tilde{S}_h(x)I_v - (\mu_h(x) + \alpha_h(x))E_h + \beta_1(x)\tilde{S}_h(x)\left(\dfrac{S_h}{\tilde{S}_h(x)} - 1\right)I_v, & x \in \Omega, \\[2mm]
\dfrac{\partial I_h}{\partial t} = D_h \Delta I_h + f\alpha_h(x)E_h - (\mu_h(x) + \gamma_1(x))I_h, & x \in \Omega, \\[2mm]
\dfrac{\partial J}{\partial t} = D_h \Delta J + (1-f)\alpha_h(x)E_h - (\mu_h(x) + \gamma_2(x))J, & x \in \Omega, \\[2mm]
\dfrac{\partial E_v}{\partial t} = D_v \Delta E_v + \beta_2(x)\tilde{S}_v(x)(I_h + cJ) - (\mu_v(x) + \alpha_v(x))E_v + \beta_2(x)\tilde{S}_v(x)\left(\dfrac{S_v}{\tilde{S}_v(x)} - 1\right)(I_h + cJ), & x \in \Omega, \\[2mm]
\dfrac{\partial I_v}{\partial t} = D_v \Delta I_v + \alpha_v(x)E_v - \mu_v(x)I_v, & x \in \Omega, \\[2mm]
\dfrac{\partial E_h}{\partial \nu} = \dfrac{\partial I_h}{\partial \nu} = \dfrac{\partial J}{\partial \nu} = \dfrac{\partial E_v}{\partial \nu} = \dfrac{\partial I_v}{\partial \nu} = 0, & x \in \partial\Omega.
\end{cases}
$$

Assuming $\tilde{T}(t) = \left(\tilde{T}_1(t), \tilde{T}_2(t), \tilde{T}_3(t), \tilde{T}_4(t), \tilde{T}_5(t)\right)$ represents the semigroup of the system,

$$
\begin{cases}
\dfrac{\partial E_h}{\partial t} = D_h \Delta E_h + \beta_1(x)\tilde{S}_h(x)I_v - (\mu_h(x) + \alpha_h(x))E_h, \\[2mm]
\dfrac{\partial I_h}{\partial t} = D_h \Delta I_h + f\alpha_h(x)E_h - (\mu_h(x) + \gamma_1(x))I_h, \\[2mm]
\dfrac{\partial J}{\partial t} = D_h \Delta J + (1-f)\alpha_h(x)E_h - (\mu_h(x) + \gamma_2(x))J, \\[2mm]
\dfrac{\partial E_v}{\partial t} = D_v \Delta E_v + \beta_2(x)\tilde{S}_v(x)(I_h + cJ) - (\mu_v(x) + \alpha_v(x))E_v, \\[2mm]
\dfrac{\partial I_v}{\partial t} = D_v \Delta I_v + \alpha_v(x)E_v - \mu_v(x)I_v.
\end{cases}
$$

Then, one has

$$
\begin{pmatrix} E_h(t,\cdot) \\ I_h(t,\cdot) \\ J(t,\cdot) \\ E_v(t,\cdot) \\ I_v(t,\cdot) \end{pmatrix} = \tilde{T}(t)\begin{pmatrix} E_h^0(\cdot) \\ I_h^0(\cdot) \\ J^0(\cdot) \\ E_v^0(\cdot) \\ I_v^0(\cdot) \end{pmatrix} + \int_0^\infty \tilde{T}(t-s)\begin{pmatrix} \beta_1(\cdot)\tilde{S}_h(\cdot)\left(\frac{S_h(s,\cdot)}{\tilde{S}_h(\cdot)} - 1\right)I_v(s,\cdot) \\ 0 \\ 0 \\ \beta_2(\cdot)\tilde{S}_v(\cdot)\left(\frac{S_v(s,\cdot)}{\tilde{S}_v(\cdot)} - 1\right)(I_h(s,\cdot) + cJ(s,\cdot)) \\ 0 \end{pmatrix}\mathrm{d}s.
$$

Since $\mathcal{R}_0 = 1$, due to Proposition 4.15 in [40], there is $M_5 > 0$, in such a way that $\|\tilde{T}(t)\| \le M_5$ for $t \ge 0$. Using $z(s) \le \frac{\delta_1 M_5 e^{-\delta_2 t}}{S}$, one deduces that

$$
\begin{aligned}
\max &\left\{\|E_h(t,\cdot)\|, \|I_h(t,\cdot)\|, \|J(t,\cdot)\|, \|E_v(t,\cdot)\|, \|I_v(t,\cdot)\|\right\} \\
&\le M_5 \max\{\|E_h^0(\cdot)\|, \|I_h^0(\cdot)\|, \|J^0(\cdot)\|, \|E_v^0(\cdot)\|, \|I_v^0(\cdot)\|\} \\
&\quad + M_5 \max\left\{\|\beta_1\|\,\|\tilde{S}_h\|\int_0^\infty z(s)\|I_v(s)\|, \|\beta_2\|\,\|\tilde{S}_v\|\int_0^\infty z(s)\|(I_h(s) + cJ(s))\|\mathrm{d}s\right\} \\
&\le M_5\delta_1 + \max\left\{M_6\delta_1\int_0^\infty e^{-\delta_2 s}\|I_v(s)\|, M_7\delta_1\int_0^\infty e^{-\delta_2 s}\|(I_h(s) + cJ(s))\|\right\},
\end{aligned}
$$

where $M_6 = \frac{M_5^2 \|\beta_1\| \, \|\tilde{S}_h\|}{S}$ and $M_7 = \frac{M_5^2 \|\beta_2\| \, \|\tilde{S}_v\|}{S}$. Applying Gronwall's inequality yields

$$\max\left\{\|E_h(t,\cdot)\|, \|I_h(t,\cdot)\|, \|J(t,\cdot)\|, \|E_v(t,\cdot)\|, \|I_v(t,\cdot)\|\right\} \le \max\left\{M_5\delta_1 e^{\int_0^\infty M_6\delta_1 e^{-\delta_2 s}\mathrm{d}s}, 2M_5\delta_1 e^{\int_0^\infty M_7\delta_1 e^{-\delta_2 s}\mathrm{d}s}\right\}$$

$$\le M_8,$$

where $M_8 = \max\left\{M_5\delta_1 \frac{\delta_1 M_6}{\delta_2}, 2M_5\delta_1 \frac{\delta_1 M_7}{\delta_2}\right\}$. Thus,

$$\begin{cases} \dfrac{\partial S_h}{\partial t} - D_h\Delta S_h > \Lambda_h(x) - \mu_h(x)S_h - \beta_1(x)S_h M_8, \\[2mm] \dfrac{\partial S_v}{\partial t} - D_v\Delta S_v > \Lambda_v(x) - \mu_v(x)S_v - \beta_2(x)S_v M_8. \end{cases}$$

For $t > 0$, let $(\hat{v}_1, \hat{v}_2)$ be a solution of the system,

$$\begin{cases} \dfrac{\partial \hat{v}_1}{\partial t} = D_h\Delta \hat{v}_1 + \Lambda_h(x) - \mu_h(x)\hat{v}_1 - \beta_1(x)\hat{v}_1 M_8, & x \in \Omega, \\[2mm] \dfrac{\partial \hat{v}_2}{\partial t} = D_v\Delta \hat{v}_2 + \Lambda_v(x) - \mu_v(x)\hat{v}_2 - \beta_2(x)\hat{v}_2 M_8, & x \in \Omega, \\[2mm] \dfrac{\partial \hat{v}_2}{\partial \nu} = \dfrac{\partial \hat{v}_2}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{6.2}$$

with $\hat{v}_1(0,x) = S_h^0$, $\hat{v}_2(0,x) = S_v^0$, for $x \in \overline{\Omega}$. By the comparison principle, one has $(S_h(t,x), S_v(t,x)) \ge (\hat{v}_1(t,x), \hat{v}_2(t,x))$, with $t \ge 0$ and $x \in \overline{\Omega}$. Let $\left(S_h^{\delta_1}(x), S_v^{\delta_1}(x)\right)$ be the positive steady state of system (6.2) and $w_1(t,x) = \hat{v}_1(t,x) - S_h^{\delta_1}(x)$, $w_2(t,x) = \hat{v}_2(t,x) - S_v^{\delta_1}(x)$. Then for $t > 0$, $(w_1(t,x), w_2(t,x))$ satisfies

$$\begin{cases} \dfrac{\partial w_1}{\partial t} = D_h\Delta w_1 - (\mu_h(x) + \beta_1(x)M_8)w_1, & x \in \Omega, \\[2mm] \dfrac{\partial w_2}{\partial t} = D_v\Delta w_2 - (\mu_v(x) - \beta_2(x)M_8)w_2, & x \in \Omega, \\[2mm] \dfrac{\partial w_1}{\partial \nu} = \dfrac{\partial w_2}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{6.3}$$

with $w_1(0,x) = S_h^0 - S_h^{\delta_1}(x)$, $w_2(0,x) = S_v^0 - S_v^{\delta_1}(x)$. Let $P_1(t)$, $P_2(t)$ be the semigroups generated by $D_h\Delta - \mu_h(x)$ and $D_v\Delta - \mu_v(x)$ with Neumann boundary condition, respectively. Set $P(t) = (P_1(t), P_2(t))$. It is possible to select M_5 in such a manner that $\|P(t)\| \le M_5 e^{\alpha_3 t}$, provided that M_5 is sufficiently large. By (6.3), we have

$$\begin{cases} w_1(t,\cdot) = P_1(t)\left(S_h^0(\cdot) - S_h^{\delta_1}(\cdot)\right) - \displaystyle\int_0^\infty P_1(t-s)\beta_1(\cdot)M_8 w_1(s,\cdot)\mathrm{d}s, \\[4mm] w_2(t,\cdot) = P_2(t)\left(S_v^0(\cdot) - S_v^{\delta_1}(\cdot)\right) - \displaystyle\int_0^\infty P_2(t-s)\beta_2(\cdot)M_8 w_2(s,\cdot)\mathrm{d}s. \end{cases}$$

Hence,

$$\begin{cases} \|w_1(t,\cdot)\| \le M_5 e^{\alpha_3 t}\left\|S_h^0(\cdot) - S_h^{\delta_1}(\cdot)\right\| - \displaystyle\int_0^\infty M_8 M_5 e^{\alpha_3(t-s)}\|\beta_1(\cdot)\| \, \|w_1(s,\cdot)\|\mathrm{d}s, \\[4mm] \|w_2(t,\cdot)\| = M_5 e^{\alpha_3 t}\left\|S_v^0(\cdot) - S_v^{\delta_1}(\cdot)\right\| - \displaystyle\int_0^\infty M_8 M_5 e^{\alpha_3(t-s)}\|\beta_2(\cdot)\| \, \|w_2(s,\cdot)\|\mathrm{d}s. \end{cases}$$

Let $K_3 = M_8 M_5 \|\beta_1(\cdot)\|$ and $K_4 = M_8 M_5 \|\beta_2(\cdot)\|$. Then

$$\begin{cases} \left\| \hat{v}_1(t,\cdot) - S_h^{\delta_1}(\cdot) \right\| = \|w_1(t,\cdot)\| \le M_5 \left\| S_h^0(\cdot) - S_h^{\delta_1}(\cdot) \right\| e^{\alpha_3 t + K_3 t}, \\ \left\| \hat{v}_2(t,\cdot) - S_v^{\delta_1}(\cdot) \right\| = \|w_2(t,\cdot)\| \le M_5 \left\| S_v^0(\cdot) - S_v^{\delta_1}(\cdot) \right\| e^{\alpha_3 t + K_4 t}. \end{cases}$$

Selecting $\delta_4 > 0$ small enough to satisfy $\max\{K_3, K_4\} < -\frac{\alpha_3 t}{2}$, one has

$$\begin{cases} \left\| \hat{v}_1(t,\cdot) - S_h^{\delta_1}(\cdot) \right\| \le M_5 \left\| S_h^0(\cdot) - S_h^{\delta_1}(\cdot) \right\| e^{\frac{\alpha_3 t}{2}}, \\ \left\| \hat{v}_2(t,\cdot) - S_v^{\delta_1}(\cdot) \right\| \le M_5 \left\| S_v^0(\cdot) - S_v^{\delta_1}(\cdot) \right\| e^{\frac{\alpha_3 t}{2}}. \end{cases} \tag{6.4}$$

Now by (6.4), one has

$$\begin{aligned} S_h(t,\cdot) - \tilde{S}_h(\cdot) \ge & \hat{v}_1(t,\cdot) - \tilde{S}_h(\cdot) = \hat{v}_1(t,\cdot) - S_h^{\delta_1}(\cdot) + S_h^{\delta_1}(\cdot) - \tilde{S}_h(\cdot) \\ \ge & - M_5 \left\| S_h^0(\cdot) - S_h^{\delta_1}(\cdot) \right\| e^{\frac{\alpha_3 t}{2}} + S_h^{\delta_1}(\cdot) - \tilde{S}_h(\cdot) \\ \ge & - M_5 \left(\left\| S_h^0(\cdot) - \tilde{S}_h(\cdot) \right\| + \left\| \tilde{S}_h(\cdot) - S_h^{\delta_1}(\cdot) \right\| \right) - \left\| S_h^{\delta_1}(\cdot) - \tilde{S}_h(\cdot) \right\| \\ \ge & - M_5 \delta_1 - (M_5 + 1) \left\| S_h^{\delta_1}(\cdot) - \tilde{S}_h(\cdot) \right\|, \\ S_v(t,\cdot) - \tilde{S}_v(\cdot) \ge & \hat{v}_2(t,\cdot) - \tilde{S}_v(\cdot) = \hat{v}_2(t,\cdot) - S_v^{\delta_1}(\cdot) + S_v^{\delta_1}(\cdot) - \tilde{S}_v(\cdot) \\ \ge & - M_5 \left\| S_v^0(\cdot) - S_v^{\delta_1}(\cdot) \right\| e^{\frac{\alpha_3 t}{2}} + S_v^{\epsilon}(\cdot) - \tilde{S}_v(\cdot) \\ \ge & - M_5 \left(\left\| S_v^0(\cdot) - \tilde{S}_v(\cdot) \right\| + \left\| \tilde{S}_v(\cdot) - S_v^{\delta_1}(\cdot) \right\| \right) - \left\| S_v^{\delta_1}(\cdot) - \tilde{S}_v(\cdot) \right\| \\ \ge & - M_5 \delta_1 - (M_5 + 1) \left\| S_v^{\delta_1}(\cdot) - \tilde{S}_v(\cdot) \right\|. \end{aligned}$$

Noticing that $z(s) \le \frac{\delta_1 M_5 e^{-\delta_2 t}}{S} \le \frac{\delta_1 M_5}{S}$, we get that

$$\begin{aligned} S_h(t,\cdot) - \tilde{S}_h(\cdot) = & \tilde{S}_h(\cdot) \left(\frac{S_h(t,\cdot)}{\tilde{S}_h(\cdot)} - 1 \right) \le \delta_1 M_5 \frac{\|\tilde{S}_h(\cdot)\|}{S}, \\ S_v(t,\cdot) - \tilde{S}_v(\cdot) = & \tilde{S}_v(\cdot) \left(\frac{S_v(t,\cdot)}{\tilde{S}_v(\cdot)} - 1 \right) \le \delta_1 M_5 \frac{\|\tilde{S}_v(\cdot)\|}{S}, \end{aligned}$$

and

$$\left\| S_h(t,\cdot) - \tilde{S}_h(\cdot) \right\| \le \max \left\{ M_5 \delta_1 + (M_5 + 1) \left\| S_h^{\delta_1}(\cdot) - \tilde{S}_h(\cdot) \right\|, \delta_1 M_5 \frac{\|\tilde{S}_h(\cdot)\|}{S} \right\},$$

$$\| S_v(t,\cdot) - \tilde{S}_v(\cdot) \| \le \max \left\{ M_5 \delta_1 + (M_5 + 1) \left\| S_v^{\delta_1}(\cdot) - \tilde{S}_v(\cdot) \right\|, \delta_1 M_5 \frac{\|\tilde{S}_v(\cdot)\|}{S} \right\}.$$

In view of that $\lim_{\delta_1 \to 0} \left(S_h^{\delta_1}(x), S_v^{\delta_1}(x) \right) = \left(\tilde{S}_h(x), \tilde{S}_v(x) \right)$, we can choose δ_1 sufficiently small, for $t > 0$, one derives that

$$\left\| S_h(t,\cdot) - \tilde{S}_h(\cdot) \right\| \le \delta_1, \quad \|E_h(t,\cdot)\| \le \delta_1, \quad \|I_h(t,\cdot)\| \le \delta_1, \quad \|J(t,\cdot)\| \le \delta_1,$$

$$\left\| S_v(t,\cdot) - \tilde{S}_v(\cdot) \right\| \le \delta_1, \quad \|E_v(t,\cdot)\| \le \delta_1, \quad \|I_v(t,\cdot)\| \le \delta_1,$$

proving the local stability of $E_0 = \left\{ \tilde{S}_h(x), 0, 0, 0, \tilde{S}_v(x), 0, 0 \right\}$.

Next, we prove the global attractivity of $\left\{\tilde{S}_h(x), 0, 0, 0, \tilde{S}_v(x), 0, 0\right\}$. On view of Theorem 3.1, $\Phi(t)$ has a global attractor $\mathcal{A}$. Define

$$\partial X_1 = \left\{(S_h^\varepsilon, E_h^\varepsilon, I_h^\varepsilon, J^\varepsilon, S_v^\varepsilon, E_v^\varepsilon, I_v^\varepsilon) \in \mathbb{X}^+ : E_h^\varepsilon = I_h^\varepsilon = J^\varepsilon = E_v^\varepsilon = I_v^\varepsilon = 0\right\}.$$

Claim 1. For $u_0 = (S_h^0, E_h^0, I_h^0, J^0, S_v^0, E_v^0, I_v^0) \in \mathcal{A}$, the omega limit set $\omega(u_0) \in \partial X_1$.

We know that $S_h^0(\cdot) \leq \tilde{S}_h(\cdot)$ and $S_v^0(\cdot) \leq \tilde{S}_v(\cdot)$. If $E_h^\varepsilon = I_h^\varepsilon = J^\varepsilon = E_v^\varepsilon = I_v^\varepsilon = 0$, the claim easily follows from the fact that ∂X_1 is invariant for $\Phi(t)$. Assuming that either $E_h^0 \neq 0$ or $I_h^0 \neq 0$ or $J^0 \neq 0$ or $S_v^0 \neq 0$ or $E_v^0 \neq 0$ or $I_v^0 \neq 0$, then one has $E_h(t,x) > 0$, $I_h(t,x) > 0$, $E_v(t,x) > 0$, $I_v(t,x) > 0$ and $J(t,x) > 0$ for $t > 0$ and $x \in \overline{\Omega}$. Then for $t > 0$, S_v and S_h satisfy

$$\begin{cases} \dfrac{\partial S_h}{\partial t} < D_h \Delta S_h + \Lambda_h(x) - \mu_h(x) S_h, & x \in \Omega, \\ \dfrac{\partial S_v}{\partial t} < D_v \Delta S_v + \Lambda_v(x) - \mu_v(t,x) S_v, & x \in \Omega, \\ \dfrac{\partial S_h}{\partial \nu} = \dfrac{\partial S_v}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases}$$

with $S_h(0,x) \leq \tilde{S}_h(x)$, $S_v(0,x) \leq \tilde{S}_v(x)$ for $x \in \overline{\Omega}$. Apply the comparison principle, $S_h < \tilde{S}_h(x)$ and $S_v < \tilde{S}_v(x)$ for $t > 0$ and $x \in \overline{\Omega}$. According to [39], we introduce

$$c(t; u_0) := \inf\{\tilde{c} \in \mathbb{R} : E_h(t,\cdot) \leq \tilde{c}\phi_2, \ I_h(t,\cdot) \leq \tilde{c}\phi_3, \ J(t,\cdot) \leq \tilde{c}\phi_4, \ E_v(t,\cdot) \leq \tilde{c}\phi_5, \ I_v(t,\cdot) \leq \tilde{c}\phi_6\}.$$

Then for $t > 0$, $c(t; u_0) > 0$. We conclude that $c(t; u_0)$ is strictly decreasing. Give $t_2 > 0$ and set $E_h^\varepsilon(t,\cdot) = c(t_2; u_0)\phi_2$, $I_h^\varepsilon(t,\cdot) = c(t_2; u_0)\phi_3$, $J^\varepsilon(t,\cdot) = c(t_2; u_0)\phi_4$, $E_v^\varepsilon(t,\cdot) = c(t_2; u_0)\phi_5$, $I_v^\varepsilon(t,\cdot) = c(t_2; u_0)\phi_6$ for $t \geq t_2$. It follows from $S_h < \tilde{S}_h(\cdot)$ and $S_v < \tilde{S}_v(\cdot)$ that

$$\begin{cases} \dfrac{\partial E_h^\varepsilon}{\partial t} = D_h \Delta E_h^\varepsilon + \beta_1(t,x) S_h I_v^\varepsilon - (\mu_h(x) + \alpha_h(x)) E_h^\varepsilon, \\ \dfrac{\partial I_h^\varepsilon}{\partial t} = D_h \Delta I_h^\varepsilon + f\alpha_h(x) E_h^\varepsilon - (\mu_h(x) + \gamma_1(x)) I_h^\varepsilon, \\ \dfrac{\partial J^\varepsilon}{\partial t} = D_h \Delta J + (1-f)\alpha_h(x) E_h^\varepsilon - (\mu_h(x) + \gamma_2(x)) J^\varepsilon, \\ \dfrac{\partial E_v^\varepsilon}{\partial t} = D_v \Delta E_v^\varepsilon + \beta_2(x) S_v (I_h^\varepsilon + cJ^\varepsilon) - (\mu_v(x) + \alpha_v(x)) E_v^\varepsilon, \\ \dfrac{\partial I_v^\varepsilon}{\partial t} = D_v \Delta I_v^\varepsilon + \alpha_v(x) E_v^\varepsilon - \mu_v(x) I_v^\varepsilon(t,x), \\ E_h^\varepsilon(t_2,x) \geq E_h(t_2,x), \ I_h^\varepsilon(t_2,x) \geq I_h(t_2,x), \ J^\varepsilon(t_2,x) \geq J(t_2,x), \ E_v^\varepsilon(t_2,x) \geq E_v(t_2,x), \ I_v^\varepsilon(t_2,x) \geq I_v(t_2,x). \end{cases}$$
$$(6.5)$$

Therefore, $(E_h^\varepsilon, I_h^\varepsilon, J^\varepsilon, E_v^\varepsilon, I_v^\varepsilon) \geq (E_h, I_h, J, E_v, I_v)$ for $t > t_2$ and $x \in \overline{\Omega}$. For system (6.5), we see that $c(t_2; u_0)\phi_2(x) = E_h^\varepsilon > E_h$ for $(t,x) \in (t_2, +\infty) \times \overline{\Omega}$. Similarly, $c(t_2; u_0)\phi_3(x) = I_h^\varepsilon > I_h$, $c(t_2; u_0)\phi_4(x) = J^\varepsilon > J$, $c(t_2; u_0)\phi_6(x) = E_v^\varepsilon > E_v$, $c(t_2; u_0)\phi_7(x) = I_v^\varepsilon > I_v$, for $t > t_2$ and $x \in \overline{\Omega}$. Due to t_2 is arbitrary, $c(t; u_0)$ is strictly decreasing. Let $c_* = \lim\limits_{t\to\infty} c(t; u_0)$. Then $c_* = 0$. Actually let $Z = (Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7) \in \omega(u_0)$. There is $\{t_n\}$ with $t_n \to +\infty$ as $t \to +\infty$ such that $\Phi(t_n)u_0 \to Z$. Then for $t \geq 0$, one gets that $c(t; Z) = c_*$ due to $\lim\limits_{t\to+\infty} \Phi(t + t_n)u_0 = \Phi(t)\lim\limits_{t\to+\infty} \Phi(t_n)u_0 = \Phi(t)Z$. If $Z_2 \neq 0$, $Z_3 \neq 0$, $Z_4 \neq 0$, $Z_5 \neq 0$, $Z_6 \neq 0$ and $Z_7 \neq 0$, based on the above viewpoint, it can be concluded that $c(t; Z)$ is strictly decreasing, which yields a contradiction to $c(t; Z) = c_*$. Consequently, $Z_2 = Z_3 = Z_4 = Z_6 = Z_6 = 0$.

Claim 2. $\mathcal{A} = \left\{\left(\tilde{S}_h(x), 0, 0, 0, \tilde{S}_v(x), 0, 0\right)\right\}$.

Since $\mathcal{A}$ is globally attractive in ∂X_1, then $\left\{\left(\tilde{S}_h(x), 0, 0, 0, \tilde{S}_v(x), 0, 0\right)\right\}$ is the only compact invariant subset of system (6.1). From the invariance of $\omega(u_0)$ and $u_0 \in \partial X_1$, one has $\omega(u_0) = \left\{\left(\tilde{S}_h(x), 0, 0, 0, \tilde{S}_v(x), 0, 0\right)\right\}$.

Since the global attractor $\mathcal{A}$ is compact invariant in $\mathbb{X}^+$, $\left(\tilde{S}_h(x), 0, 0, 0, \tilde{S}_v(x), 0, 0\right)$ is stable, and by [38, Lemma 3.11], one has $\mathcal{A} = \left\{\left(\tilde{S}_h(x), 0, 0, 0, \tilde{S}_v(x), 0, 0\right)\right\}$.

The globally asymptotical stability of $\left(\tilde{S}_h(x), 0, 0, 0, \tilde{S}_v(x), 0, 0\right)$ is immediately obtained based on the global attractivity and local stability. $\qquad\square$

7. Numerical simulation

This part uses numerical simulations to clarify the analytical results and to show how to gain some cognisance of epidemiology.

7.1. Long term behavior

The one-dimensional domain $(0, \pi)$ is generally employed to simulate the long-time dynamics as suggested by [6,7,21]. We refer to system (2.3) as a model for the spread of malaria in Maputo Province, Mozambique. Fix periodic $\omega = 12$ months. b means biting rate on humans, $\frac{b}{N_h}$ is the per human biting rate. Denote $\tilde{\beta}_1$ as the role of mosquito biting behavior in the probability of acquiring malaria from infectious humans, and $\tilde{\beta}_2$ is expressed as the transmission probability of the infectivity of mosquito bites in transmitting malaria from human to mosquito. As a consequence, we can express $\beta_1 = \frac{b}{N_h}\tilde{\beta}_1$ and $\beta_2 = \frac{b}{N_h}\tilde{\beta}_2$, which represent the spread probability of the disease in mosquitoes and humans. [41] ascertained the interval of critical parameters about the temporal and spatial patterns of malaria. Since the climate in Maputo is conducive to the spread of malaria, [42] explored the seasonality impacts on the spread of malaria, containing seasonal forced biting rate $b(t)$, mosquitoes mortality rate $\mu_v(t)$ and recruitment rate $\Lambda_v(t)$, where

$$
\begin{aligned}
b(t) = &\ 6.983 - 1.993\cos(\pi t/6) - 1.459\sin(\pi t/6) - 0.128\cos(\pi t/2) - 0.04095\cos(2\pi t/3) \\
&+ 0.0005486\cos(5\pi t/6) - 0.4247\cos(\pi t/3) + 0.05452\sin(2\pi t/3) - 0.0709\sin(\pi t/2) \\
&- 0.007642\sin(\pi t/3) - 0.06235\sin(5\pi t/6)\ \text{Month}^{-1},
\end{aligned}
\tag{7.1}
$$

$$
\begin{aligned}
\mu_v(t) = &\ 3.086 + 0.0007665\cos(2\pi t/3) + 0.007133\cos(\pi t/2) + 0.01942\cos(\pi t/3) + 0.04788\cos(\pi t/6) \\
&+ 0.01135\sin(\pi t/2) + 0.02655\sin(\pi t/6) + 0.003198\sin(5\pi t/6) + 0.01819\sin(\pi t/3) \\
&+ 0.005687\sin(2\pi t/3) - 0.001459\cos(5\pi t/6)\ \text{Month}^{-1},
\end{aligned}
\tag{7.2}
$$

and

$$
\Lambda_v(t) = \hat{k} \times b(t)\ (\text{km}^2\text{Month})^{-1},\ \text{with}\ \hat{k} = 53.13 \times 5.
\tag{7.3}
$$

The description of the parameters can be found in Table 2.

We choose $\tilde{\beta}_1 = 0.17$, $\tilde{\beta}_2 = 0.15$. Let $\gamma_1 = a_1 \cdot (1.05 - \cos(2x))\ \text{Month}^{-1}$ and $\gamma_2 = a_2 \cdot (1.05 - \cos(2x))\ \text{Month}^{-1}$, where $a_1 = 0.055$, $a_2 = 0.051$. It can be observed that individuals residing in urban areas, especially those nearer to the city center, have access to superior medical treatment due to a higher concentration of doctors, hospitals, and state-of-the-art medical equipment. Consequently, the recovery rate of patients is expected to be higher in urban areas compared to rural areas. The other parameters remain consistent with those in Table 2. For these given parameters, numerical calculations can yield $\mathcal{R}_0 = 4.3652 > 1$, indicating that the disease is persistent. In this case, the long-term behavior of system (2.3) is illustrated in Figure 2, with initial data

TABLE 2. *The value or range of parameters*

Parameter	Value(Range)	Dimension	References
N_h	53	$(\text{km}^2)^{-1}$	[21]
μ_h	0.00157	Month^{-1}	[42]
Λ_h	0.0984	$(\text{km}^2\text{Month})^{-1}$	[42]
γ_1	(0.04256,0.5168)	Month^{-1}	[21]
γ_2	(0.04256,0.5168)	Month^{-1}	[21]
α_h	(0.067,0.20)	Month^{-1}	[41]
α_v	(0.029,0.33)	Month^{-1}	[41]
$\tilde{\beta}_1$	(0.01,0.27)	Dimensionless	[41]
$\tilde{\beta}_2$	(0.072,0.64)	Dimensionless	[41]
D_h	0.1	$\text{km}^2\text{Month}^{-1}$	[21]
D_v	0.0125	$\text{km}^2\text{Month}^{-1}$	[21]
c	(0,1)	Dimensionless	See text
b	(7.1)	Month^{-1}	[42]
μ_v	(7.2)	Month^{-1}	[42]
Λ_v	(7.3)	$(\text{km}^2\text{Month})^{-1}$	[42]

(a)

(b)

(c)

(d)

(e)
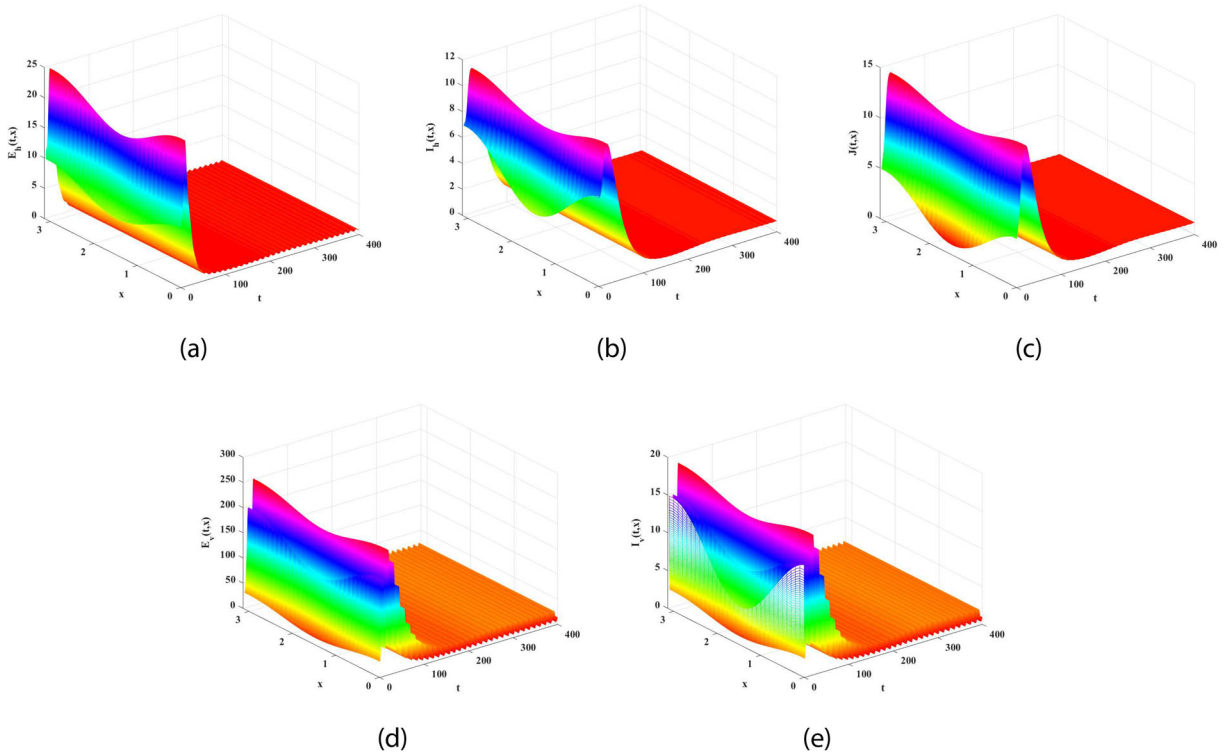
FIG. 2. The evolution of infection compartments when $\mathcal{R}_0 > 1$. (a), (b) and (c) are infected human compartments. (d) and (e) are infected mosquitoes compartments
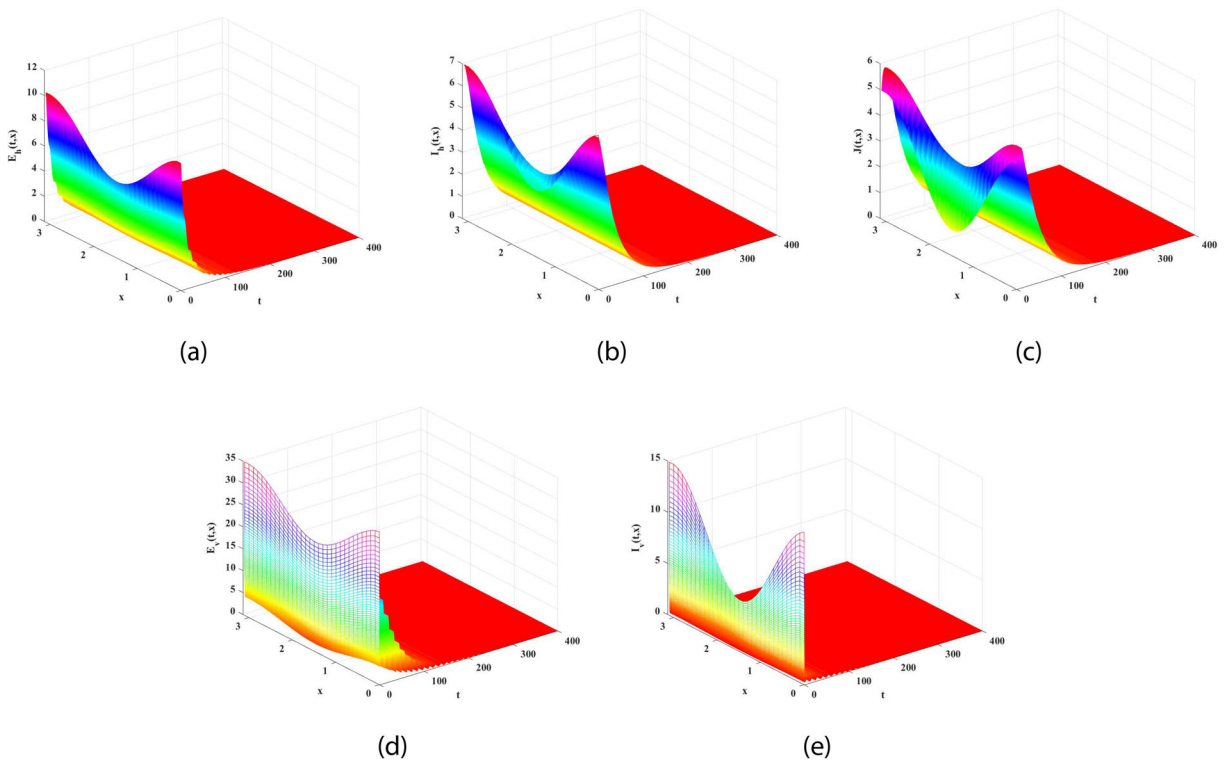
FIG. 3. The evolution of infection compartments when $\mathcal{R}_0 < 1$. (a), (b) and (c) are infected human compartments. (d) and (e) are infected mosquitoes compartments

$$u(0,x) = \begin{pmatrix} 34 - 2\cos 2x \\ 8 - 2\cos 2x \\ 5 - 2\cos 2x \\ 3 - 2\cos 2x \\ 300 - 5\cos 2x \\ 30 - 5\cos 2x \\ 10 - 5\cos 2x \end{pmatrix} , \ x \in [0, \pi].$$

This is consistent with the Theorem 5.1(ii).

By disinfection and sterilization mosquito breeding sites and take advantage of insecticide treated mosquito nets, the bite rate is reduced to $0.5b$, and the death rate of mosquito is increased to $1.5\mu_v$. People pay more attention to the spread of diseases and invest more and more medical resources. Therefore, it is assumed that the recovery rate of infected persons will increase to $1.1\gamma_1$ and $1.1\gamma_2$, then $\mathcal{R}_0 = 0.5164 < 1$. Figure 3 describes that the infectious hosts and vectors go to 0, i.e., the disease will be eradicated.

7.2. Sensitivity analysis

Sensitivity analysis can be applied to quantify the impact of uncertainty on model input parameters and subsequently impact on model outputs [43,44]. The purpose of sensitivity analysis is to quantify this relationship by using the ubiquitous derivative of the output as a function of the input and to accurately quantify the ratio of output

perturbation relative to the input perturbation. This section analyzes the sensitivity of the output solution and the basic reproduction number.

7.2.1. Sensitivity analysis to output solutions. For our model, this involves under the 13×5 sensitive coefficient matrix given by

$$H_1 = \frac{\mathrm{d}\tilde{Q}}{\mathrm{d}\tilde{P}},$$

for convenience, let

$$\tilde{Q} = (E_h, I_h, J, E_v, I_v), \quad \tilde{P} = (\Lambda_h, \mu_h, N_h, \alpha_h, f, c, \gamma_1, \gamma_2, \beta_1, \beta_2, \Lambda_v, \mu_v, \alpha_v).$$

Since our model parameters are assessed in distinct units and of different orders of magnitude, which leads to a challenge to interpret the sensitivity results. First, we focus on elasticity analysis [45]. The field of elasticity analysis is concerned with studying the proportional response of a model's output to proportional changes in its input parameters, as opposed to additive changes. The elasticity coefficient matrix is given by

$$H_2 = \frac{\mathrm{d}\tilde{Q}}{\mathrm{d}\tilde{P}} \cdot \frac{\tilde{P}}{\tilde{Q}}.$$

Especially, elasticity is dimensionless due to it is proportional sensitivities and is scaled. Therefore, the elasticity between all model parameters can be directly compared.

In this study, we employ a basic finite differences approach to estimate the derivative and computationally solve the elasticity coefficient matrix. More precisely, we utilize the following formula to numerically calculate the elasticity coefficients

$$(H_2)_{i,j}(t,x;\tilde{P}) = \frac{\partial \tilde{Q}_i(t,x;\tilde{P})}{\partial \tilde{P}_j} \frac{\tilde{P}_j}{\tilde{Q}_i(t,x;\tilde{P})} \approx \left(\frac{\tilde{Q}_i(t,x;\tilde{P} + \triangle\tilde{P}_j) - \tilde{Q}_i(t,x;\tilde{P})}{\triangle\tilde{P}_j} \right) \frac{\tilde{P}_j}{\tilde{Q}_i(t,x;\mathcal{P})},$$

for $i = 1, 2, 3, 4, 5$ and $j = 1, \cdots, 13$ and $\triangle\tilde{P}_j > 0$. According to [46,47], $\frac{\partial \tilde{Q}_i(t,x;\tilde{P})}{\partial \tilde{P}_j} \frac{\tilde{P}_j}{\tilde{Q}_i(t,x;\tilde{P})}$ is called normalized sensitivity index. The aforementioned metric quantifies the ratio of the change in the output to a small proportionate variation in the input.

Figure 4 presents the full elasticity coefficient at the $t = 350$ and $x = 3$. The presented data illustrates in a lucid manner that collective equilibrium concentrations are primarily influenced by the parameter μ_v, while the parameter Λ_v exhibits a secondary level of sensitivity. And through observation, it finds that the influence of these parameters on the infected compartment is relatively large, while the impact on the susceptible compartment is almost negligible.

The following evaluates the effect of evolving model parameters over time on outbreak size, sensitivity analysis is performed by Latin Hypercube Sampling and partial rank correlation coefficient (PRCC) method [48]. To assess whether the significance of a parameter appears at a certain time interval during the progression of the model dynamics, PRCC indices are calculated at multiple time points and plotted against time. Figure 5 shows the significance of the effect of parameters on outputs E_h, I_h, J, E_v and I_v. This figure shows the absolute value of the PRCC to make it easier to compare their relative magnitudes. Looking at Figure 5, the mathematical model is a dynamic system, PRCC can depend on time, and the relative importance of parameters can also be contingent on time. For E_h, shown in Figure 5 (a), for example, for $t_1 \leq t \leq t_2$, the effect of parameters on the solution is $\beta_1 > N_h > \Lambda_v > f > \beta_2 > \gamma_1 > \gamma_2 > \mu_v > \alpha_h$, while when $t_4 \leq t \leq t_5$, the order is $\beta_2 > \gamma_1 > \mu_v > f > \Lambda_v > \beta_1 > \Lambda_h > \alpha_v > \gamma_2 > c > N_h > \alpha_h > \mu_h$. In addition, it is also found that in this time interval, when t is less than about 0.8, the influence of β_1 might be the greatest on the solution, and when $t > t_3$, γ is the greatest. For I_h, shown in Figure 5 (b), when $t_1 < t < t_2$, the the effect of parameters on the solution is $\gamma_1 > f > \Lambda_v > N_h > \gamma_2 > \mu_h > \Lambda_h > \beta_1 > \alpha_h > c >$ others, while when $t_4 < t < t_5$, the order is $\alpha_h > \gamma_2(\gamma_1) > \beta_1 > f > \beta_2 > \Lambda_v > \Lambda_h > \mu_v > c(\mu_h) > N_h > \alpha_v$. Furthermore, during this time interval, if $t < t_3$, the γ_1 might have the greatest influence on I_h. For J (see Figure (5) (c)), during $t_1 < t < t_2$, the influence of parameters on the output solution is in the order of $\alpha_h > \gamma_2 > \beta_2 > \mu_v > \alpha_v > \gamma_1 > f > c >$
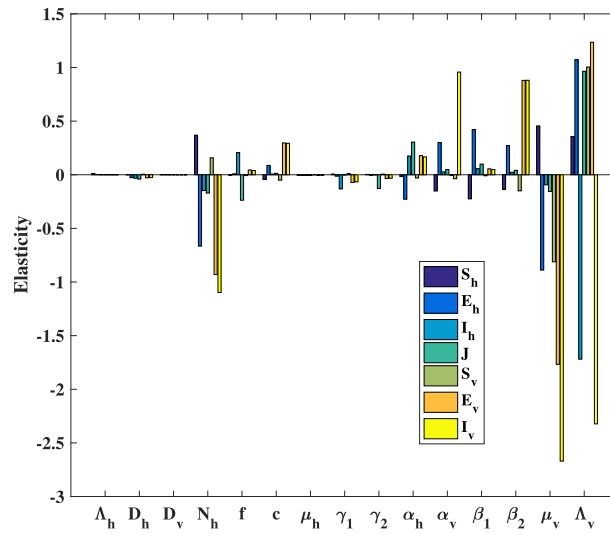
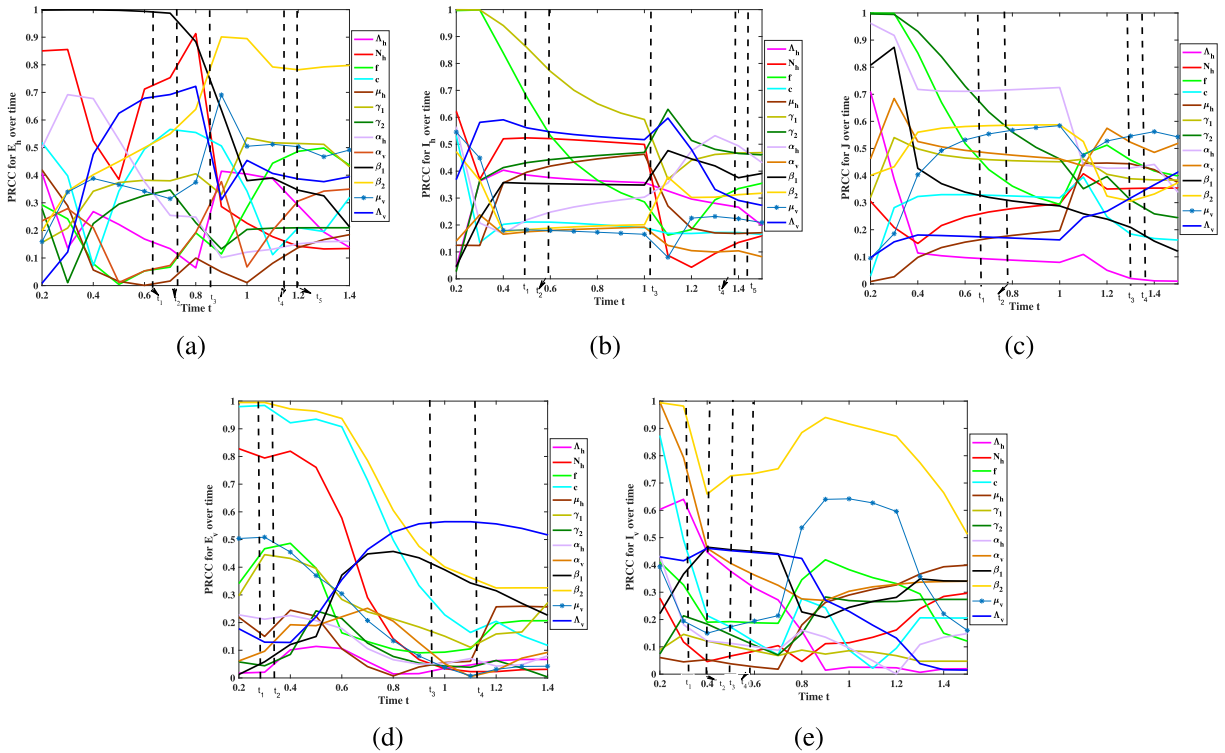FIG. 4. The normalized sensitivity indexes

(a)

(b)

(c)

(d)

(e)

FIG. 5. Plot of the absolute value of PRCC over time. The PRCC indexes are calculated with respect to the infectious humans and mosquitoes

$\beta_1 > N_h > \Lambda_v > \mu_h > \Lambda_h$, and when $t_3 < t < t_4$, the order is $\mu_v > \alpha_v > f > \mu_h > \alpha_h > \gamma_1 > N_h > \Lambda_v > \beta_2 > \gamma_2 > \beta_1 > c > \Lambda_h$. For E_v (see Figure 5 (d)), when $t_1 < t < t_2$, the order of influence of parameters on output solution is $\beta_2 > c > N_h > \mu_v > f > \gamma_1 > \alpha_h > \mu_h > \Lambda_v > \alpha_v > \beta_1 > \gamma_2 > \Lambda_h$, then it is

$\Lambda_v > \beta_2 > \beta_1 > c > \gamma_1 > \alpha_v > f > \mu_v >$ others when $t_3 < t < t_4$. It is easy to see that when $t < t_3$, β_2 has the greatest impact on E_v, while when $t > t_3$, Λ_v has the greatest impact. Finally, for I_v, when $t_1 < t < t_2$, the order of parameter influence is $\beta_2 > \alpha_v > \Lambda_h > \Lambda_v > \beta_1 > c > f > \gamma_2 > \mu_v > \alpha_h > \gamma_1 > N_h > \mu_h$, while $t_3 < t < t_4$, the order is $\beta_2 > \beta_1(\Lambda_v) > \alpha_v > \Lambda_h > f > \mu_v > c > \gamma_2 > \alpha_h > \gamma_1 > N_h > \mu_h$. During the entire time interval in Figure 5 (e), β_2 is the parameter that has the greatest impact on the output solution I_v.

7.2.2. Sensitivity analysis to basic reproduction number in a homogeneous case. When all coefficients are constants, we obtain a explicit formula of basic reproduction number, written by R_0. It is easy to get

$$R_0 = (R_1 + R_2) \cdot R_3,$$

where $R_1 = \frac{f\alpha_h}{\mu_h+\alpha_h} \cdot \frac{\beta_1}{\mu_h+\gamma_1} \cdot \frac{\Lambda_h}{\mu_h}$, $R_2 = \frac{(1-f)\alpha_h}{\mu_h+\alpha_h} \cdot \frac{c\beta_1}{\mu_h+\gamma_2} \cdot \frac{\Lambda_h}{\mu_h}$ and $R_3 = \frac{\alpha_v}{\mu_v+\alpha_v} \cdot \frac{\beta_2}{\mu_v} \cdot \frac{\Lambda_v}{\mu_v}$. In addition, R_1, R_2 and R_3 have their own biological meanings: R_1 implies the impact of one symptomatic infected human on the susceptible mosquitoes, where $\frac{f\alpha_h}{\mu_h+\alpha_h}$ is the proportion of exposed humans developing into symptomatic compartment, $\frac{1}{\mu_h+\gamma_1}$ and β_1 represent the period of infection and infection rate of symptomatic humans, respectively; R_2 denotes the effect of one asymptomatic infected human on the susceptible mosquitoes, where $\frac{(1-f)\alpha_h}{\mu_h+\alpha_h}$ is the proportion of exposed humans developing into asymptomatic compartment, $\frac{1}{\mu_h+\gamma_2}$ and $c\beta_1$ represent the period of infection and infection rate of asymptomatic humans, respectively; R_3 denotes the effect of one infected mosquito on susceptible humans, where $\frac{\alpha_v}{\mu_v+\alpha_v}$ is the proportion of exposed mosquitoes developing into infectious compartment, $\frac{1}{\mu_v}$ and β_2 represent the period of infection and infection rate of symptomatic humans, respectively.

As we all know, the basic reproduction number is one of the most important concepts in epidemiology. More meaningfully, it explains the threshold behavior of many infectious illness models. Therefore, it is necessary to analyze the affect of parameters on the basic reproduction number. Obviously, there are 14 parameters in system (2.3), however, it can be known from the expression of R_0 that D_v and D_h have no influence on R_0, so here we analyze the influence of the remaining eight parameters.

To analyze the effects of the parameter values on R_0 by Latin Hypercube Sampling and PRCC method [48]. PRCC scatter plots in Figure 6 of parameters Λ_h, f, c, μ_h, γ_1, γ_2, α_h, α_v, β_1, β_2, μ_v and Λ_v (all eight parameters are changed concomitantly, sample size $N = 8000$). The abscissa stands for a uniform distribution of all input parameters with the minimum and maximum values. The ordinate denotes R_0. From Figures 6 and 7, one can investigate the dependence of R_0 on parameters to get more information. For instance, numeric plots indicate that R_0 is monotonically decreasing with respect to μ_h, μ_v, γ_1 and γ_2, whereas R_0 is a monotonically increasing function of Λ_h, Λ_v, f, c, α_h, α_v, β_1 and β_2, respectively. According to the biological significance of the parameters, the means of control like applying insecticide treated mosquito nets and spraying insecticides can reduce the bite rate of mosquitoes, increase death rate and then reduce R_0. The reduction of bite rate can also decrease the recruitment rate of mosquitoes, thus reducing R_0. Moreover, increasing medical resources can improve the recovery rate of host and then reduce the basic reproduction number. For parameters μ_h and Λ_h related to host, increasing the death rate and reducing the recruitment rate can reduce the R_0, but this is not desirable. Therefore, we ignored the related strategies when evaluating the control measures later. The result illustrated in Figure 7 suggests that, R_0 is more sensitive to μ_h, followed by f, c and β_1, which means that the presence of asymptomatic carriers cannot be ignored when exploring malaria transmission patterns and developing strategies to prevent and control malaria transmission.

Figure 8 provides more results on the analysis of basic reproduction number. The gray plane in this figure represents $R_0 = 1$, which means that the values of the two parameters are combined below the gray plane, there will be $R_0 < 1$ indicating that the disease is extinct, otherwise $R_0 > 1$ meaning that the disease is persistent.

7.3. Effectiveness of preventive control measures

In view of the influence of the above parameters on the spread of disease, we propose the following measures to prevent and control the spread of disease, and analyze the effects of these measures. Let us consider seven cases of the intervening measures at vectors and hosts.
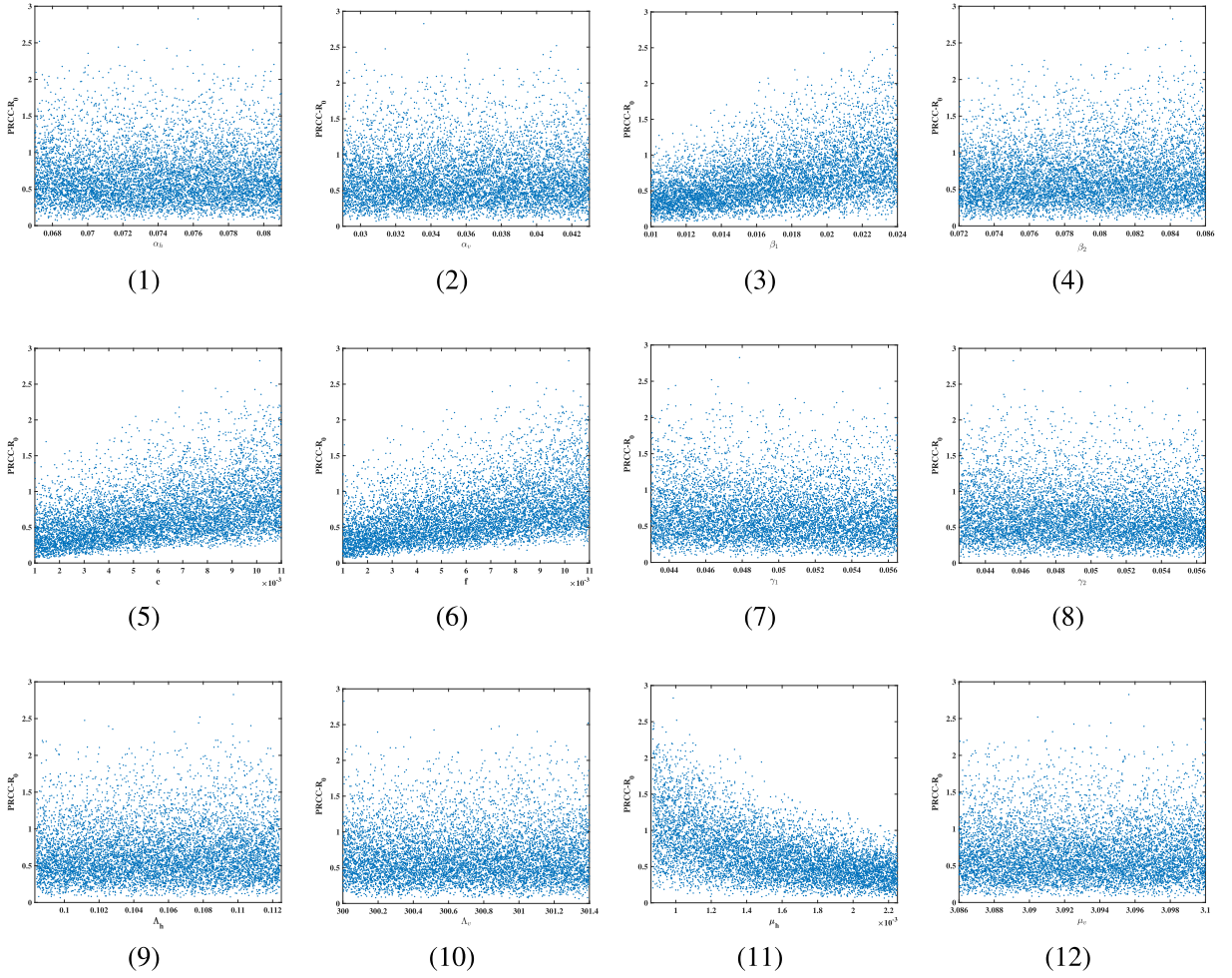
FIG. 6. PRCC for the basic reproduction number R_0

Baseline scenario (BS): The value of parameters in this case is consistent with that in Figure 2.

Strategy I: The contact of mosquitoes and humans can be reduced by using mosquito nets, mosquito repellent sprays, etc. Here has four steps:

Strategy I-I: Adjust $\beta_1 = 0.65 \times 0.17$.

Strategy I-II: Adjust $\beta_2 = 0.65 \times 0.44$.

Strategy I-III: Adjust $\beta_1 = 0.65 \times 0.17$ and $\beta_2 = 0.65 \times 0.44$.

The use of insecticides can increase mosquito mortality and improve the medical environment to increase the recovery rate of the host.

Strategy II-I: Adjust $\gamma_1 = 1.3 \times 0.04256$ Month^{-1}.

Strategy II-II: Adjust $\gamma_2 = 1.3 \times 0.045$ Month^{-1}.

Strategy II-III: Adjust $\mu_v = 1.3 \times \mu_v(t)$ Month^{-1}.

Strategy II-IV: Adjust $\gamma_1 = 1.3 \times 0.04256$ Month^{-1}, $\gamma_2 = 1.3 \times 0.045$ Month^{-1} and $\mu_v = 1.3 \times \mu_v(t)$ Month^{-1}.

Strategy III: Adjust $\beta_1 = 0.65 \times 0.17$, $\beta_2 = 0.65 \times 0.44$, $\gamma_1 = 1.3 \times 0.04256$ Month^{-1}, $\gamma_2 = 1.3 \times 0.045$ Month^{-1} and $\mu_v = 1.3 \times \mu_v(t)$ Month^{-1}.
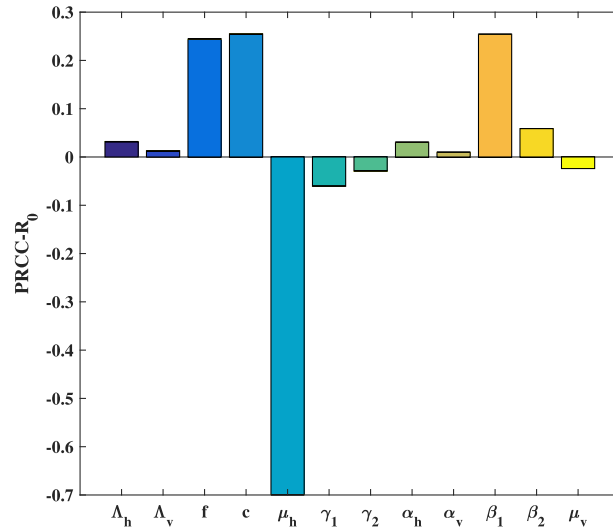
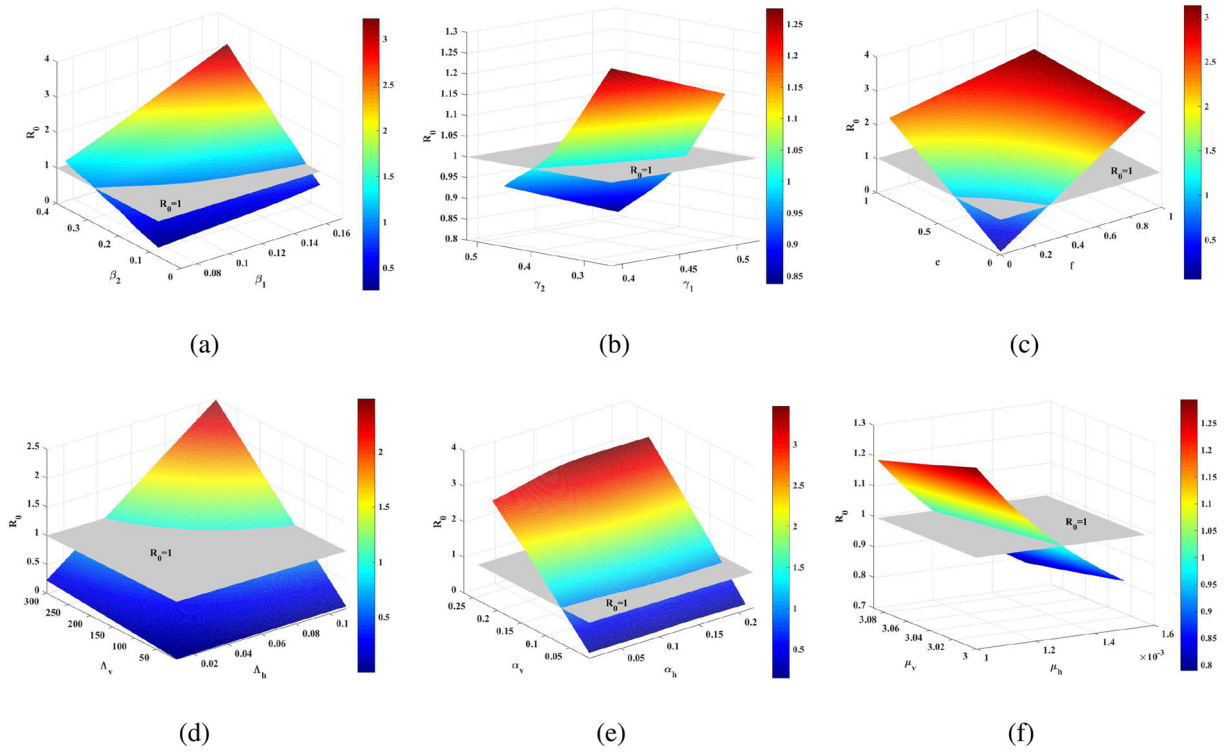FIG. 7. Sensitive analysis of the basic reproduction number R_0 via parameters

(a)   (b)   (c)

(d)   (e)   (f)

FIG. 8. Plots of R_0 as a function of parameters. The gray plane is $R_0 = 1$

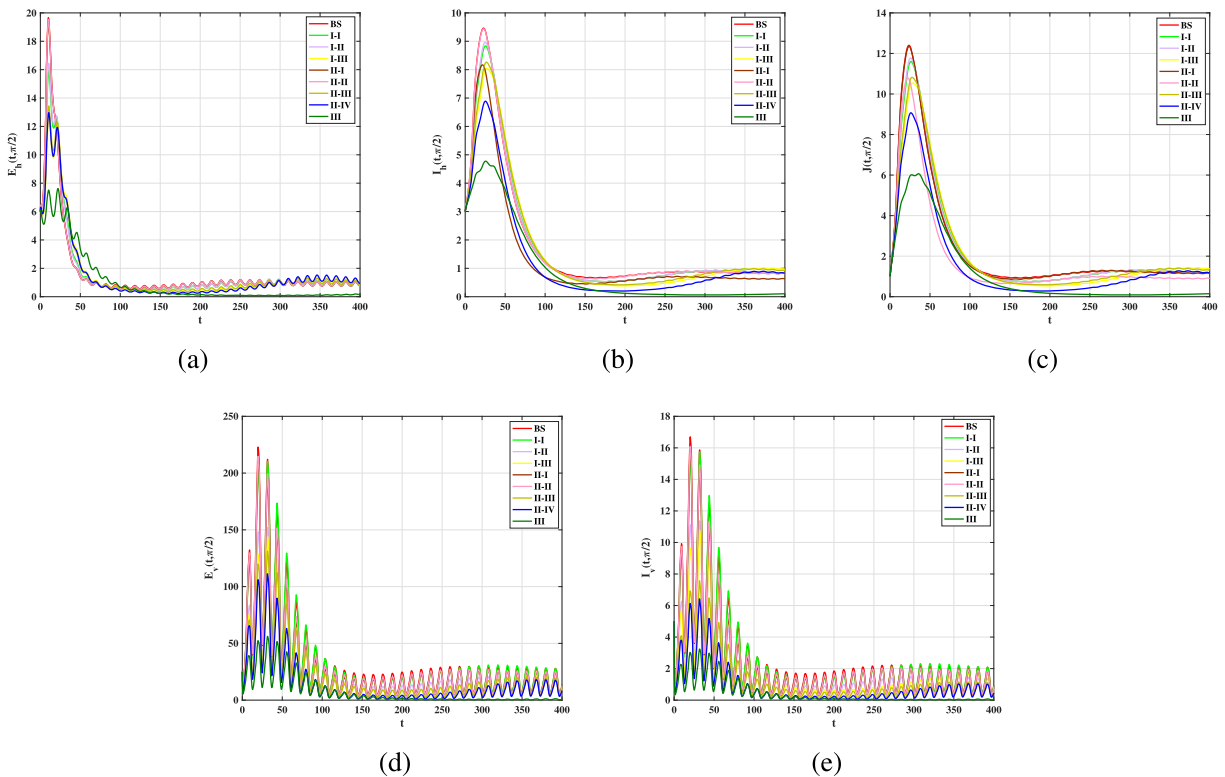(a)   (b)   (c)

(d)   (e)

FIG. 9. The impact of different intervening measures on malaria transmission

Figure 9 indicates that control measures can reduce the final size of infection mosquitoes and humans which helpful to lower the potential risk of the malaria transmission. Besides, we find that the use of both control measures delays the time to peak, which provides time for the department of disease control and prevention to take steps to control the disease when it emerges. However, the use of such control measures can also increase peaks, which in turn can lead to a shortage of medical resources such as hospital beds during peak disease. We point out that the comparison of strategies is not about which strategy is more effective (as they may be related to different costs), but more about which control measures can effectively control the spread of disease.

8. Discussion

This paper formulates a time periodic reaction-diffusion malaria model accounting for asymptomatic carriers. The genesis of this model is drawn from the following biological inquiries: (1) What is the impact of asymptomatic carriers on the transmission of malaria? (2) Are there regional variations in the role of mosquito and human propagation in the transmission of malaria? (3) What are the impacts of the seasonal changes in temperature on the malaria spread? For the model, our analyses include the stability of the infection-free ω-periodic solution, the existence and uniform persistence of positive ω-periodic solution. Assuming the parameter is not a time function, we can obtain the global asymptotic stability of the infection-free steady state in the critical case of $\mathcal{R}_0 = 1$. Aguilar and Gutierrez [49] considered asymptomatic models, but did not take into account the impact of human and mosquito spread on malaria transmission.

The numerical simulation part first verify the theoretical results of long-term dynamic behavior. Second, we analyze the impact of parameters in model (2.3) on disease transmission, which is divided into two parts. The impact

of parameters on the model state variables is analyzed using elasticity and PRCC indexes. When the mathematical model is a dynamic system, PRCC indexes depend on time, and the relative importance of parameters can also rely on time. In addition, this paper conducts sensitivity analysis on the basic reproduction number under special cases of spatial homogeneity. In this case, an explicit expression of the basic reproduction number can be obtained, elucidating the biological significance of each part, and it is found that R_0 is more sensitive to μ_h, followed by f, c and β_1, which means that the presence of asymptomatic carriers cannot be ignored when exploring malaria transmission patterns and developing strategies to prevent and control malaria transmission. The third part is the evaluation of control measures. It should be pointed out that the comparison of strategies is not about which strategies are more effective (as they may be related to different costs), but rather about which control measures can effectively control the spread of diseases. Based on the numerical results, it can be seen that, spraying insecticides, using mosquito nets and other means to disinfect vectors and reduce the contact between vector and host which can prevent or slow down the transmission of the malaria disease.

In our research, we've identified certain shortcomings that require improvement. In the course of theoretical analysis, the intricate nature of periodic has hindered a comprehensive discussion of all scenarios. Consequently, we have not yet addressed the global stability of the positive periodic solution in this model. In the realm of malaria transmission modeling, limited attention has been given to investigating the impact of infection age and spatial diffusion on disease transmission. In reality, the intensity of infectivity in malaria varies across different stages of infection. Referred to as the age of infection, this temporal factor significantly influences the number of secondary infections. Incorporating this crucial factor into the study of malaria transmission is essential. These issues hold significant importance in understanding the dynamics of malaria transmission and devising effective control measures. Therefore, in our future research, we aim to delve deeper into these aspects.

Data availibility Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations
Conflict of interest The authors declare that they have no competing interests.

Consent for publication All authors of this manuscript agree to publish in this journal.

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Appendices

Appendix A. Proof of Lemma 5.2

Proof. Denote $v = (v_1, v_2, v_3, v_4, v_5)$ to be the solution of (4.2) with $v_0(\bar{\varphi}) = \bar{\varphi}$. Let $v_1^* = e^{-\mu t} v_1$, $v_2^* = e^{-\mu t} v_2$, $v_3^* = e^{-\mu t} v_3$, $v_4^* = e^{-\mu t} v_4$, $v_5^* = e^{-\mu t} v_5$. Since $\bar{\varphi} \gg 0$, $v(t, x; \bar{\varphi}) \gg 0$ for $t \geq 0$, then

$$v^* = (v_1^*, v_2^*, v_3^*, v_4^*, v_5^*) \gg 0, \text{ for } x \in \overline{\Omega},$$

and v^* meets the next system with μ,

$$
\begin{cases}
\dfrac{\partial v_1^*}{\partial t} = D_h \Delta v_1^*(t, x) + \beta_1 \tilde{S}_h(x) v_5^* - (\mu_h(x) + \alpha_h(x) + \mu) v_1^*, & x \in \Omega, \\[2mm]
\dfrac{\partial v_2^*}{\partial t} = D_h \Delta v_2^*(t, x) + f \alpha_h(x) v_1^* - (\mu_h(x) + \gamma_1(x) + \mu) v_2^*, & x \in \Omega, \\[2mm]
\dfrac{\partial v_3^*}{\partial t} = D_h \Delta v_3^* + (1 - f) \alpha_h(x) v_1^* - (\mu_h(x) + \gamma_2(x) + \mu) v_3^*, & x \in \Omega, \\[2mm]
\dfrac{\partial v_4^*}{\partial t} = D_v \Delta v_4^* + \beta_2 \tilde{S}_v(t, x) (v_2^* + c v_3^*) - (\mu_v(t, x) + \alpha_v(t, x) + \mu) v_4^*, & x \in \Omega, \\[2mm]
\dfrac{\partial v_5^*}{\partial t} = D_v \Delta v_5^* + \alpha_v(t, x) v_4^* - (\mu_v(t, x) + \mu) v_5^*, & x \in \Omega.
\end{cases}
\tag{A.1}
$$

Thence, v^* is a solution of (A.1) with $\frac{\partial v_1^*}{\partial n} = \frac{\partial v_2^*}{\partial n} = \frac{\partial v_3^*}{\partial n} = \frac{\partial v_4^*}{\partial n} = \frac{\partial v_5^*}{\partial n} = 0$ on $(0, +\infty) \times \partial\Omega$ and $v^*(0, x) = (\bar{\varphi}_1(x), \bar{\varphi}_2(x), \bar{\varphi}_3(x), \bar{\varphi}_4(x), \bar{\varphi}_5(x))$ for $x \in \overline{\Omega}$. Then $v_i^*(\omega, x) = e^{-\mu\omega} r(P) \bar{\varphi}_i(x) v_i^*(0, x)$, $(i = 1, \ldots, 5)$. Thence

$$v_i^*(t, x) = v_i^*(t + \omega, x), \ t \geq 0, \ x \in \overline{\Omega}, \ i = 1, \ldots, 5.$$

Thereby, $e^{\mu t} v^*$ is a solution of (4.2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

References

[1] Gutierrez, J.B., Galinski, M.R., Cantrell, S., et al.: From within host dynamics to the epidemiology of infectious disease: scientific overview and challenges. Math. Biosci. **270**, 143–155 (2015)

[2] World Health Organization (2019), World Malaria Report, 2019

[3] Ross, R.: The Prevention of Malaria. John Murray, London (1911)

[4] Macdonald, G.: The analysis of infection rates in diseases in which super infection occurs. Trop. Dis. Bull. **47**, 907–915 (1950)

[5] Anderson, R.M., May, R.M.: Infectious Diseases of Humans: Dynamics and Control. Oxford University Press, London (1991)

[6] Bai, Z., Peng, R., Zhao, X.-Q.: A reaction-diffusion malaria model with seasonality and incubation period. J. Math. Biol. **77**, 201–228 (2018)

[7] Lou, Y., Zhao, X.-Q.: A reaction-diffusion malaria model with incubation period in the vector population. J. Math. Biol. **62**, 542–568 (2011)

[8] Li, J., Welch, R.M., Nair, U.S. et al.: Dynamic Malaria Models with Environmental Changes, Proceedings of the Thirty-Fourth Southeastern Symposium on System Theory Huntsville, AL, (2002) 396-400

[9] Martens, W.J.M., Niessen, L., Rotmans, J., et al.: Climate change and vector-borne disease: a global modelling perspective. Global Environ. Change **5**, 195–209 (1995)

[10] Hay, S.I., Cox, J., Rogers, D.J., et al.: Climate change and the resurgence of malaria in East African highlands. Nature **415**, 905–909 (2002)

[11] Gething, P.W., Smith, D.L., Patil, A.P., et al.: Climate change and the global malaria recession. Nature **465**, 342–346 (2010)

[12] Jetten, T.H., Martens, W.J.M., Takken, W.: Model simulations to estimate malaria risk under climate change. J. Med. Entomol. **33**, 361–371 (1996)

[13] Lindblade, K.A., Steinhardt, L., Samuels, A., et al.: The silent threat: asymptomatic parasitemia and malaria transmission. Expert Rev. Anti-Infect. Ther. **11**, 623–639 (2013)

[14] Crompton, P.D., Moebius, J., Portugal, S., et al.: Malaria immunity in man and mosquito: insights into unsolved mysteries of a deadly infectious disease. Annu. Rev. Immunol. **32**, 157–187 (2014)

[15] Babikera, H.A., Gadalla, A.A., Ranford-Cartwright, L.C.: The role of asymptomatic P. falciparum parasitaemia in the evolution of anti-malarial drug resistance in areas of seasonal transmission. Drug Resist. Updates **16**, 1–9 (2013)

[16] Eke, R.A., Chigbu, L.N., Nwachukwu, W.: High prevalence of asymptomatic Plasmodium infection in a suburb of Aba Town, Nigeria. Ann. Afr. Med. **5**, 42–45 (2006)

[17] Okell, L.C., Griffin, J.T., Kleinschmidt, I., et al.: The potential contribution of mass treatment to the control of Plasmodium falciparum malaria. PLoS One **6**, e20179 (2011)

[18] Griffin, J.T., Hollingsworth, T.D., Okell, L.C.: Reducing Plasmodium falciparum malaria transmission in Africa: a model-based evaluation of intervention strategies. PLoS Med. **7**, e1000324 (2010)

[19] Vinetz, J.M., Gilman, R.H.: Asymptomatic Plasmodium parasitemia and the ecology of malaria transmission. Am. J. Trop. Med. Hyg. **66**, 639–640 (2002)

[20] Cantrell, R.S., Cosner, C.: Spatial Ecology via Reaction-Diffusion Equations. In: Mathematical and Computational Biology. Wiley, West Sussex (2003)

[21] Wu, R., Zhao, X.-Q.: A reaction-diffusion model of vector-borne disease with periodic delays. J. Nonlinear Sci. **29**, 29–64 (2019)

[22] Daners, D., Medina, P.K.: Abstract Evolution Equations, Periodic Problems and Applications, Pitman research notes in mathematics series, vol. 279. Longman, Harlow (1992)

[23] Smith, H.L.: Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems. American Mathematical Society, Providence (1995)

[24] Zhang, L., Wang, Z.-C., Zhao, X.-Q.: Threshold dynamics of a time periodic reaction-diffusion epidemic model with latent periodic. J. Differ. Equ. **258**, 3011–3036 (2015)

[25] Wu, J.: Theory and Applications of Partial Functional Differential Equations. Springer, New York (1996)

[26] Magal, P., Zhao, X.-Q.: Global attractors and steady states for uniformly persistent dynamical systems. SIAM J. Math. Anal. **37**, 251–275 (2005)

[27] Martin, R.H., Smith, H.L.: Abstract functional differential equations and reaction-diffusion systems. Trans. Am. Math. Soc. **321**, 1–44 (1990)

[28] Peng, R., Zhao, X.-Q.: A reaction-diffusion SIS epidemic model in a time-periodic environment. Nonlinearity **25**, 1451–1471 (2012)

[29] Diekmann, O., Heesterbeek, J.A.P., Metz, J.A.J.: On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations. J. Math. Biol. **28**, 365–382 (1990)

[30] Liang, X., Zhang, L., Zhao, X.-Q.: Basic reproduction ratios for periodic abstract functional differential equations (with application to a spatial model for Lyme disease). J. Dyn. Diff. Equ. **31**, 1247–1278 (2019)

[31] Wang, J., Cui, R.: Analysis of a diffusive host-pathogen model with standard incidence and distinct dispersal rates. Adv. Nonlinear Anal. **10**, 922–951 (2021)

[32] Zhao, X.-Q.: Basic reproduction ratios for periodic compartmental models with time delay. J. Dyn. Differ. Equ. **29**, 67–82 (2017)

[33] Wang, B.-G., Xin, M.-Z., Huang, S., et al.: Basic reproduction ratios for almost periodic reaction-diffusion epidemic models. J. Differ. Equ. **352**, 189–220 (2023)

[34] Song, P., Lou, Y., Xiao, Y.: A spatial SEIRS reaction-diffusion model in heterogeneous environment. J. Differ. Equ. **267**, 5084–5114 (2019)

[35] Thieme, H.R.: Spectral bound and reproduction number for infinite-dimensional population structure and time heterogeneity. SIAM J. Appl. Math. **70**, 188–211 (2009)

[36] Hess, P.: Periodic-Parabolic Boundary Value Problem and Positivity, (Pitman reasearch notes in Mathematics vol 247) (Harlow: Longman Scientific and Technical), (1991)

[37] Zhao, X.-Q.: Dynamical Systems in Population Biology, 2nd edn. Springer, New York (2017)

[38] Wu, Y., Zou, X.: Dynamics and profiles of a diffusive host-pathogen system with distinct dispersal rates. J. Differ. Equ. **264**, 4989–5024 (2018)

[39] Cui, R., Lam, K.-Y., Lou, Y.: Dynamics and asymptotic profiles of steady states of an epidemic model in advective environments. J. Differ. Equ. **263**, 2343–2373 (2017)

[40] Webb, G.F.: Theory of Nonlinear Age-Dependent Population Dynamics. CRC Press (1985)

[41] Chitnis, N., Hyman, J.M., Cushing, J.M.: Determining important parameters in the spread of malaria through the sensitivity analysis of a mathematical model. Bull. Math. Biol. **70**, 1272–1296 (2008)

[42] Wang, X., Zhao, X.-Q.: A malaria transmission model with temperature-dependent incubation period. Bull. Math. Biol. **79**, 1155–1182 (2017)

[43] Cacuci, D.G.: Sensitivity theory for nonlinear systems. I. Nonlinear functional analysis approach. J. Math. Phys. **22**, 2794–2802 (1981)

[44] Levy, A.B.: Solution sensitivity from general principles. SIAM J. Control Optim. **40**, 1–38 (2001)

[45] Caswell, H.: Matrix Population Models, vol. 1. Sinauer, Sunderland (2000)

[46] Saltelli, A., Chan, K., Scott, E.: Sensitivity Analysis. Wiley Series in Probability and Statistics. Wiley, New York (2000)

[47] Arriola, L., Hyman, J.M.: Sensitivity Analysis for Uncertainty Quantification in Mathematical Models. In: Chowell, G., Hyman, J.M., Bettencourt, L.M.A., Castillo-Chavez, C. (eds.) Mathematical and Statistical Estimation Approaches in Epidemiology. Springer, Dordrecht (2009)

[48] Hoare, A., Regan, D.P., Wilson, D.G.: Sampling and sensitivity analyses tools (SaSAT) for computational modelling. Theor. Biol. Med. Model. **5**, 4 (2008)

[49] Aguilar, J.B., Gutierrez, J.B.: An epidemiological model of malaria accounting for asymptomatic carriers. Bull. Math. Biol. **82**, 42 (2020)

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