Z. Angew. Math. Phys. (2024) 75:32

0044-2275/24/020001-22
published online February 10, 2024
https://doi.org/10.1007/s00033-023-02173-9

Zeitschrift für angewandte Mathematik und Physik ZAMP

Time-periodic traveling wave solutions of a reaction–diffusion Zika epidemic model with seasonality

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Abstract. In this paper, the full information about the existence and nonexistence of a time-periodic traveling wave solution of a reaction–diffusion Zika epidemic model with seasonality, which is non-monotonic, is investigated. More precisely, if the basic reproduction number, denoted by R_0, is larger than one, there exists a minimal wave speed $c^* > 0$ satisfying for each $c > c^*$, the system admits a nontrivial time-periodic traveling wave solution with wave speed c, and for $c < c^*$, there exist no nontrivial time-periodic traveling waves such that if $R_0 \leqslant 1$, the system admits no nontrivial time-periodic traveling waves.

Mathematics Subject Classification. 35R10, 35B40, 34K30, 58D25.

Keywords. Nontrivial time-periodic traveling wave solutions, Zika epidemic model, A reaction–diffusion system, Seasonality.

1. Introduction

In this paper, we focus on the following reaction–diffusion Zika epidemic model with seasonality

$$\begin{cases} \frac{\partial S_H(t,x)}{\partial t} = D_1\Delta S_H(t,x) - \beta_1(t)S_H(t,x)I_H(t,x) - \beta_2(t)S_H(t,x)I_V(t,x), \\ \frac{\partial I_H(t,x)}{\partial t} = d_1\Delta I_H(t,x) + \beta_1(t)S_H(t,x)I_H(t,x) + \beta_2(t)S_H(t,x)I_V(t,x) - r_1(t)I_H(t,x), \\ \frac{\partial S_V(t,x)}{\partial t} = D_2\Delta S_V(t,x) - \beta_3(t)S_V(t,x)I_H(t,x), \\ \frac{\partial I_V(t,x)}{\partial t} = d_2\Delta I_V(t,x) + \beta_3(t)S_V(t,x)I_H(t,x) - r_2(t)I_V(t,x), \end{cases} \quad t > 0, x \in \mathbb{R}, \tag{1.1}$$

where the total group of human can be divided into the susceptible group S_H and the infected group I_H. Similarly, the total group of mosquitoes can be separated into S_V-susceptible and I_V-infected. $D_i(i = 1, 2)$ and $d_i(i = 1, 2)$ are the diffusion rate of the susceptible individuals, the susceptible mosquitoes, the infectious individuals and the infectious mosquitoes, respectively. $\beta_1(t)$, $\beta_2(t)$, and $\beta_3(t)$ are the contact rates among the susceptible humans and the infected humans, the susceptible humans and the infected mosquitoes, and the infected humans and the susceptible mosquitoes, respectively. $r_1(t)$ and $r_2(t)$ are the removal rate of the infectious individuals and the infectious mosquitoes, respectively. Moreover, we make the following assumption:

(**A**) $D_i(i = 1, 2)$ and $d_i(i = 1, 2)$ are all positive constants. In addition, $\beta_i(t)(i = 1, 2, 3)$ and $r_i(t)(i = 1, 2)$ are Hölder continuous and positive nontrivial functions on $\mathbb{R}^+$ and periodic in time with the same period $T > 0$.

In the paper, we study the existence and non-existence of a time-periodic traveling wave solution of system (1.1). Namely, system (1.1) admits a nontrivial time-periodic traveling wave front with each wave speed $c > c^*$ if $R_0 > 1$. However, the system admits no nontrivial time-periodic traveling wave fronts with $0 < c < c^*$ and $R_0 > 1$ or $R_0 \leqslant 1$.

Model (1.1) describes the spatial transmission of Zika virus in human, which were first confirmed in Nigeria [25]. A first severe Zika outbreak has occurred in Island of Yap in 2007. After that, they have also experienced the subsequent outbreak of Zika, such as French Polynesia, South Pacific, New Caledonia, Easter Island, etc. [6]. In 2015, a large outbreak in Brazil was occurred and provided a large number of infected cases. Since then, it had spread freely to many other countries [35]. WHO called Zika a "Public Health Emergency of International Concern" in 2016 [42]. Up to now, there is still no effective drug used to treat Zika patients. In fact, Zika virus infection can be transmitted mainly by the bite of an infected Aedes species mosquito during the day and night. Then a mosquito can be infected with a virus when it bites an infected person during the period of time when the virus can be found in the person's blood, typically only through the first week of infection [5]. Similar to other viruses transmission through mosquito bites, such as dengue, fever, rash, headache and muscle pain are the most common symptoms of many infected people with Zika virus. However, unlike these infectious disease, Zika virus can be passed through sex [13]. In order to establish a theoretical framework for mathematical analysis of transmission of Zika virus, many ordinary differential epidemic models have been derived, see [5,8,13,15,19,26,28,32,34] and the cited reference therein.

Since the human individuals and the mosquitoes usually move randomly in the spatial space, it is reasonable to take to account the random walk of individuals, which can be described by a reaction–diffusion epidemic model. In the literature, there are many results studying the existence and non-existence of traveling wave solutions of some reaction–diffusion epidemic models, see Murray [27], Rass and Radcliffe [29], Ruan [30], Ruan and Wu [31], Ducrot et al. [9,10], Wang et al. [38,39], Li and Zou [21], Zhao et al. [51,52] and the references cited therein. Recently, Zhang and Zhao [49] studied traveling wave solutions for a nonlocal diffusive Zika transmission model with bilinear incidence. Zhao [54] firstly analyzed spreading speed of a reaction–diffusion Zika model with constant recruitment in terms of the basic reproduction number R_0 and the minimal wave speed c^*. On the basis of it, the full information about the existence and nonexistence of traveling wave solutions of the system is investigated.

It was reported that the transmission dynamics of infectious diseases can be significantly influenced by the seasonal change, see Bacaëra and Gomes [2], Buonomo [4], Eikenberry and Gumel [11], Grassly and Fraser [14], Hethcote [16], Hethcote and Levin [17] and Soper [33]. Thus, it is crucial to investigate the influence of the seasonal factor on the geographic transmission of infectious diseases. However, the study for traveling waves solutions of non-autonomous epidemic models is few. Wang et al. [40] studied the existence and nonexistence of a time-periodic traveling wave solution for a reaction–diffusion SIR epidemic model with the standard incidence rate and seasonality. After that, they [48] further investigated a traveling wave solution of a time-periodic reaction–diffusion SIR model with the bilinear incidence rate. Compared with the above system in [40], the infection group of such system, denoted by $I(t,x)$, is unbounded. Zhao et al. [53] took into account the asymptotic speed of spread and traveling wave solutions for a time-periodic reaction–diffusion SIR epidemic model with periodic recruitment and standard incidence rate determined by the basic reproduction number R_0 and the minimal wave speed c^*. Wang et al. [36] analyzed the existence and non-existence of a time-periodic traveling wave solution of a generalization of the classical Kermack–McKendrick model with seasonality and nonlocal delayed transmission derived by mobility of individuals during latent period of the infectious disease. Yang and Lin [47] established the speed of asymptotic spreading and minimal wave speed of traveling wave solutions for a time-periodic and diffusive DS-I-A epidemic model. Ambrosio et al. [1] studied the existence of generalized traveling waves for a time-dependent reaction–diffusion SIR epidemic model with the bilinear incidence rate on $\mathbb{R}^2$. Huang et al. [18] established the spreading speeds and periodic traveling waves for a class of time-periodic and partially degenerate reaction–diffusion systems with monotone and non-monotone nonlinearities. For other related results on the periodic traveling waves for time-periodic and spatially continuous non-monotone epidemic model, we refer to the literature [7,44,46]. Recently, Wu [43] analyzed the spreading speed and periodic traveling waves for a time-periodic epidemic model in discrete media, which is the lack of comparison principle and compactness of solution operators.

We mention that the major difficulty to study (1.1) is that it lacks the classical comparison principle. Thus, the theory on the traveling wave solutions for monotone semiflows, see [12,22,23,41] and the cited references therein, doesn't directly work for system (1.1). In addition, a reaction–diffusion epidemic model describing Zika virus spreading is more complex. Thus, except for [49,54], there seem no results on a time-periodic traveling wave solution for such a reaction–diffusion Zika epidemic model with seasonality.

The rest of this paper is organized as follows. In Sect. 2, the basic reproduction number R_0 and the minimal wave speed c^* of the system are defined. On the basis of it, the full information with the existence and non-existence of a time-periodic traveling wave solution of system (1.1) is established for $(t,x) \in \mathbb{R}^2$ in Sects. 3 and 4.

2. Preliminary

The aim of the preliminary is to find the basic reproduction number and the minimal wave speed of system (1.1), denoted by R_0 and c^*, which is related with the existence and non-existence of a time-periodic traveling wave solution for the system. Firstly, let $\mathbb{C}_T$ be the Banach space of all T-periodic continuous functions from $\mathbb{R}$ to $\mathbb{R}^2$, which is endowed with the usual supremum norm. Its positive cone $\mathbb{C}_T^+$ consists of all functions in $\mathbb{C}_T$ with both nonnegative components.

Secondly, consider the following ODE system

$$\begin{cases} \frac{\mathrm{d}\tilde{S}_H}{\mathrm{d}t} = -\beta_1(t)\tilde{S}_H(t)\tilde{I}_H(t) - \beta_2(t)\tilde{S}_H(t)\tilde{I}_V(t), \ t>0, \\ \frac{\mathrm{d}\tilde{I}_H}{\mathrm{d}t} = \beta_1(t)\tilde{S}_H(t)\tilde{I}_H(t) + \beta_2(t)\tilde{S}_H(t)\tilde{I}_V(t) - r_1(t)\tilde{I}_H(t), \ t>0, \\ \frac{\mathrm{d}\tilde{S}_V}{\mathrm{d}t} = -\beta_3(t)\tilde{S}_V(t)\tilde{I}_H(t), \ t>0, \\ \frac{\mathrm{d}\tilde{I}_V}{\mathrm{d}t} = \beta_3(t)\tilde{S}_V(t)\tilde{I}_H(t) - r_2(t)\tilde{I}_V(t), \ t>0. \end{cases} \tag{2.1}$$

It is clear that $(S_H^0, 0, 0, S_V^0, 0)$ is always an equilibrium of (2.1), denoted by E_0, which is called the disease-free equilibrium of (2.1). Let

$$\mathcal{F}(t) := \begin{pmatrix} \beta_1(t)S_H^0 & \beta_2(t)S_H^0 \\ \beta_3(t)S_V^0 & 0 \end{pmatrix} \text{ and } \mathcal{V}(t) := \begin{pmatrix} r_1(t) & 0 \\ 0 & r_2(t) \end{pmatrix}.$$

There is an evolution operator $U(t,s)$ for $t \geqslant s$ such that the following linear T-periodic system

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\mathcal{V}(t)y.$$

Precisely speaking, for each $s \in \mathbb{R}$, the 2×2 matrix $U(t,s)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t,s) = -\mathcal{V}(t)U(t,s), \ \forall t \geqslant s, \ U(s,s) = I,$$

where I is the 2×2 identify matrix. Define a linear operator $\mathcal{L} : \mathbb{C}_T \to \mathbb{C}_T$ by

$$(\mathcal{L}v)(t) = \int_0^\infty U(t,t-s)\mathcal{F}(t-s)v(t-s)\mathrm{d}s, \ \forall t \in \mathbb{R}, \ v \in \mathbb{C}_T.$$

According to [37], $\mathcal{L}$ is called by the next infection operator and define the basic reproduction number of system (2.1) by $R_0 := r(\mathcal{L})$, where $r(\mathcal{L})$ is the spectral radius of $\mathcal{L}$.

Linearizing the second equation and the forth equation of system (1.1) at the disease-free equilibrium E_0 yields

$$\begin{cases} \partial_t I_H(t,x) = d_1\Delta I_H(t,x) + S_H^0\beta_1(t)I_H(t,x) + S_H^0\beta_2(t)I_V(t,x) - r_1(t)I_H(t,x), \ t>0, \ x \in \mathbb{R}, \\ \partial_t I_V(t,x) = d_2\Delta I_V(t,x) + S_V^0\beta_3(t)I_H(t,x) - r_2(t)I_V(t,x), \ t>0, \ x \in \mathbb{R}. \end{cases} \tag{2.2}$$

Letting $\binom{I_H}{I_V}(t,x) = e^{\mu x}\binom{\eta_H(t)}{\eta_V(t)}$ and then plugging it into equation (2.2), we obtain the characteristic equations as below

$$\begin{cases}\eta_H'(t) = d_1\mu^2\eta_H(t) + S_H^0\beta_1(t)\eta_H(t) + S_H^0\beta_2(t)\eta_V(t) - r_1(t)\eta_H(t), \ \forall t > 0,\\ \eta_V'(t) = d_2\mu^2\eta_V(t) + S_V^0\beta_3(t)\eta_H(t) - r_2(t)\eta_V(t), \ \forall t > 0.\end{cases} \tag{2.3}$$

Denote the solution map of system (2.3) by $(\eta_H, \eta_V)_t(\tilde\eta_{H0}, \tilde\eta_{V0}) := (\eta_H, \eta_V)(t; \tilde\eta_{H0}, \tilde\eta_{V0})$, where $(\eta_H, \eta_V)(t; \tilde\eta_{H0}, \tilde\eta_{V0})$ is the solution of system (2.3) with initial value $(\tilde\eta_{H0}, \tilde\eta_{V0}) \in \mathbb{R}_+^2$. Assume that $r(\mu)$ denotes the spectral radius of the Poincaré map $B_c := (\eta_H, \eta_V)_T$ with system (2.3). By using the similar arguments as those in [45], (η_H^*, η_V^*) is a eigenvalue vector of B_c associated with the corresponding principal eigenvalue $r(\mu)$. Furthermore, according to [37] with $R_0 > 1$, one has $r_0 := r(0) > 1$, indicating that $r(\mu) > r_0 > 1$. Define $\lambda(\mu) := \frac{\ln r(\mu)}{T}$ and $\Phi(\mu) := \frac{\lambda(\mu)}{\mu}$, $\forall \mu \in (0, \infty)$. It then follows from Lemma 3.8 in [23] that there exist $\mu^*, c^* \in (0, +\infty)$ such that

$$c^* = \Phi(\mu^*) = \inf_{\mu>0} \Phi(\mu). \tag{2.4}$$

Choose a small enough constant $\epsilon > 0$, which is determined later. Then consider the following system

$$\begin{cases}\frac{\partial I_H(t,x)}{\partial t} = d_1\Delta I_H(t,x) + S_H^0(1-\epsilon)\big(\beta_1(t)I_H(t,x) + \beta_2(t)I_V(t,x)\big) - r_1(t)I_H(t,x),\\ \frac{\partial I_V(t,x)}{\partial t} = d_2\Delta I_V(t,x) + S_V^0(1-\epsilon)\beta_3(t)I_H(t,x) - r_2(t)I_V(t,x),\end{cases}$$

On the same way, plugging $\binom{I_H^\epsilon}{I_V^\epsilon}(t,x) = e^{\mu x}\binom{\eta_H^\epsilon(t)}{\eta_V^\epsilon(t)}$ into the above equations causes to

$$\begin{cases}(\eta_H^\epsilon)'(t) = d_1\mu^2\eta_H^\epsilon(t) + S_H^0(1-\epsilon)\left[\beta_1(t)\eta_H^\epsilon(t) + \beta_2(t)\eta_V^\epsilon(t)\right] - r_1(t)\eta_H^\epsilon(t), \ \forall t > 0,\\ (\eta_V^\epsilon)'(t) = d_2\mu^2\eta_V^\epsilon(t) + S_V^0(1-\epsilon)\beta_3(t)\eta_H^\epsilon(t) - r_2(t)\eta_V^\epsilon(t), \ \forall t > 0,\end{cases} \tag{2.5}$$

Similarly, define the spectral radius of the Poincaré map with system (2.5) by $r^\epsilon(\mu)$. Due to $R_0 > 1$, there exists a $\epsilon_0 > 0$ small enough such that for any $\epsilon \in (0, \epsilon_0)$, one has $r_0^\epsilon := r^\epsilon(0) > 1$, indicating that $r^\epsilon(\mu) > r_0^\epsilon > 1$. Let $\lambda^\epsilon(\mu) := \frac{\ln r^\epsilon(\mu)}{T}$ and $\Phi^\epsilon(\mu) := \frac{\lambda^\epsilon(\mu)}{\mu}$, $\forall \mu \in (0, \infty)$. Then there exist $\mu_\epsilon^*, c_\epsilon^* \in (0, +\infty)$ such that $c_\epsilon^* = \Phi^\epsilon(\mu_\epsilon^*) = \inf_{\mu>0} \Phi^\epsilon(\mu)$ and

$$c_\epsilon^* = \inf_{\mu>0} \frac{\ln r^\epsilon(\mu)}{T\mu} \leqslant \frac{\ln r^\epsilon(\mu^*)}{T\mu^*} < \frac{\ln r(\mu^*)}{T\mu^*} = c^*$$

by using (2.4) and (2.5). In addition, it is obvious that $\lim_{\epsilon\to 0^+} c_\epsilon^* = c^*$.

3. Existence of periodic traveling wave solutions

In the section, we establish the existence of the time-periodic traveling wave solutions of model (1.1). We firstly define a time T-periodic traveling wave solution for system (1.1), namely, it is a special solution with the form as follows

$$\begin{aligned}&S_H(t,x) = u_1(t, x+ct) := u_1(t,z), \ I_H(t,x) = v_1(t, x+ct) := v_1(t,z),\\ &S_V(t,x) = u_2(t, x+ct) := u_2(t,z), \ I_V(t,x) = v_2(t, x+ct) := v_2(t,z), \ \forall (t,z) \in \mathbb{R}\times\mathbb{R},\\ &\text{and } u_i(t,z) = u_i(t+T, z), \ v_i(t,z) = v_i(t+T,z), \ \forall (t,z) \in \mathbb{R}\times\mathbb{R}, \ i = 1,2.\end{aligned} \tag{3.1}$$

In addition, it can satisfy the following epidemic model

$$\begin{cases}\partial_t u_1(t,z) = D_1\partial_{zz}u_1(t,z) - c\partial_z u_1(t,z) - u_1(t,z)(\beta_1(t)v_1(t,z) + \beta_2(t)v_2(t,z)),\\ \partial_t v_1(t,z) = d_1\partial_{zz}v_1(t,z) - c\partial_z v_1(t,z) + u_1(t,z)(\beta_1(t)v_1(t,z) + \beta_2(t)v_2(t,z)) - r_1(t)v_1(t,z),\\ \partial_t u_2(t,z) = D_2\partial_{zz}u_2(t,z) - c\partial_z u_2(t,z) - \beta_3(t)u_2(t,z)v_1(t,z),\\ \partial_t v_2(t,z) = d_2\partial_{zz}v_2(t,z) - c\partial_z v_2(t,z) + \beta_3(t)u_2(t,z)v_1(t,z) - r_2(t)v_2(t,z)\end{cases} \tag{3.2}$$

posed for $\forall(t,z)\in\mathbb{R}\times\mathbb{R}$. We intend to find a nonnegative solution $(u_1(t,z),u_2(t,z),v_1(t,z),v_2(t,z))$ of system (3.2) so that the following boundary conditions

$$\begin{aligned} &u_1(t,-\infty)=S_H^0,\ u_2(t,-\infty)=S_V^0,\ v_1(t,-\infty)=v_2(t,-\infty)=0,\\ &u_1(t,+\infty)=S_H^\infty,\ u_2(t,+\infty)=S_V^\infty,\ v_1(t,+\infty)=v_2(t,+\infty)=0 \end{aligned} \tag{3.3}$$

uniformly $t\in\mathbb{R}$, where $S_H^0>S_H^\infty$ and $S_V^0>S_V^\infty$, S_H^∞ and S_V^∞ are determined later.

Linearizing the second and the last equation of system (3.2) causes to

$$\begin{cases}\partial_t\bar v_1(t,z)=d_1\partial_{zz}\bar v_1(t,z)-c\partial_z\bar v_1(t,z)+S_H^0\beta_1(t)\bar v_1(t,z)+S_H^0\beta_2(t)\bar v_2(t,z)-r_1(t)\bar v_1(t,z),\\ \partial_t\bar v_2(t,z)=d_2\partial_{zz}\bar v_2(t,z)-c\partial_z\bar v_2(t,z)+S_V^0\beta_3(t)\bar v_1(t,z)-r_2(t)\bar v_2(t,z).\end{cases} \tag{3.4}$$

Letting $\binom{\bar v_1}{\bar v_2}(t,z)=e^{\mu z}\binom{\mathcal{J}_1(t)}{\mathcal{J}_2(t))}$ and then plugging it into (3.4), we get the characteristic equations as below

$$\begin{cases}\frac{\mathrm{d}\mathcal{J}_1}{\mathrm{d}t}(t)-d_1\mu^2\mathcal{J}_1(t)-S_H^0\big(\beta_1(t)\mathcal{J}_1(t)+\beta_2(t)\mathcal{J}_2(t)\big)+r_1(t)\mathcal{J}_1(t)=-c\mu\mathcal{J}_1(t),\\ \frac{\mathrm{d}\mathcal{J}_2}{\mathrm{d}t}(t)-d_2\mu^2\mathcal{J}_2(t)-S_V^0\beta_3(t)\mathcal{J}_1(t)+r_2(t)\mathcal{J}_2(t)=-c\mu\mathcal{J}_2(t).\end{cases} \tag{3.5}$$

Next, we show that system (3.5) generates a positive time-periodic solution with the period $T>0$, still denoted by $(\mathcal{J}_1,\mathcal{J}_2)$. Firstly, consider the following system

$$\begin{cases}\frac{\mathrm{d}\tilde\eta_1}{\mathrm{d}t}(t)=(d_1\mu^2-c\mu)\tilde\eta_1(t)+S_H^0(\beta_1(t)\tilde\eta_1(t)+\beta_2(t)\tilde\eta_2(t))-r_1(t)\tilde\eta_1(t),\\ \frac{\mathrm{d}\tilde\eta_2}{\mathrm{d}t}(t)=(d_2\mu^2-c\mu)\tilde\eta_2(t)+S_V^0\beta_3(t)\tilde\eta_1(t)-r_2(t)\tilde\eta_2(t).\end{cases} \tag{3.6}$$

Define the solution semiflow of system (3.6) by $(\tilde\eta_1,\tilde\eta_2)_t(\tilde\eta_{10},\tilde\eta_{20}):=(\tilde\eta_1,\tilde\eta_2)(t;\tilde\eta_{10},\tilde\eta_{20})$, where $(\tilde\eta_1,\tilde\eta_2)(t;\tilde\eta_{10},\tilde\eta_{20})$ is the solution of system (3.6) with initial value $(\tilde\eta_{10},\tilde\eta_{20})\in\mathbb{R}_+^2$. In addition, denote the Poincaré map of system (3.6) by $\mathcal{P}_c:=(\tilde\eta_1,\tilde\eta_2)_T$. It further follows that

$$\mathcal{P}_c(\kappa_1,\kappa_2)=(\tilde\eta_1,\tilde\eta_2)_T(\kappa_1,\kappa_2)=(\tilde\eta_1,\tilde\eta_2)(T;\kappa_1,\kappa_2)=e^{-c\mu T}(\eta_H,\eta_V)(T;\kappa_1,\kappa_2),$$

where (κ_1,κ_2) is the initial value of system (3.6) and $(\eta_H,\eta_V)(t;\kappa_1,\kappa_2)$ is the solution of system (2.3) with initial value $(\kappa_1,\kappa_2)\in\mathbb{R}_+^2$. Consequently, one has

$$\mathcal{P}_c(\eta_H^*,\eta_V^*)=e^{-c\mu T}(B_c(\eta_H^*,\eta_V^*))=e^{-c\mu T}r(\mu)(\eta_H^*,\eta_V^*),$$

where (η_H^*,η_V^*) is a eigenvalue vector of the Poincaré map B_c of system (3.6) with the principal eigenvalue $r(\mu)$. Obviously, if $\mu=\frac{\lambda(\mu)}{c}$, then (η_H^*,η_V^*) is a fixed point of the Poincaré map $\mathcal{P}_c$, where $\lambda(\mu)$ has been defined in (2.3). Consequently, $(\tilde\eta_1,\tilde\eta_2)_t:=(\tilde\eta_1,\tilde\eta_2)(t;\eta_H^*,\eta_V^*)$ is a positive time-periodic solution of system (3.6) with $c\mu=\lambda(\mu)$.

According to [23, Theorem 3.8], we obtain that if $R_0>1$, for each $c>c^*$, there exist $\mu_1(c)$ and $\mu_2(c)$ such that $0<\mu_1(c)<\mu_2(c)<\infty$, $\Phi(\mu_1)=c$ and $\Phi(\mu)<c,\ \mu\in(\mu_1,\mu_2)$. Let $\epsilon_2\in(0,\mu_2-\mu_1)$, which is determined later, $\mu_{\epsilon_2}=\mu_1+\epsilon_2$, $\lambda(\mu_{\epsilon_2}):=\frac{\ln\rho(\mu_{\epsilon_2})}{T}$, $\Phi(\mu_{\epsilon_2}):=\inf_{\mu_{\epsilon_2}>0}\frac{\lambda(\mu_{\epsilon_2})}{\mu_{\epsilon_2}}$ and $c^*<c_{\epsilon_2}:=\Phi(\mu_{\epsilon_2})<c$, where $\rho(\mu_{\epsilon_2})$ is the spectral radius of the Poincaré map of the system as follows

$$\begin{cases}\frac{\mathrm{d}\mathcal{P}_1}{\mathrm{d}t}(t)-d_1\mu_{\epsilon_2}^2\mathcal{P}_1(t)-S_H^0\big(\beta_1(t)\mathcal{P}_1(t)+\beta_2(t)\mathcal{P}_2(t)\big)+r_1(t)\mathcal{P}_1(t)=-c_{\epsilon_2}\mu_{\epsilon_2}\mathcal{P}_1(t),\\ \frac{\mathrm{d}\mathcal{P}_2}{\mathrm{d}t}(t)-d_2\mu_{\epsilon_2}^2\mathcal{P}_2(t)-S_V^0\beta_3(t)\mathcal{P}_1(t)+r_2(t)\mathcal{P}_2(t)=-c_{\epsilon_2}\mu_{\epsilon_2}\mathcal{P}_2.\end{cases} \tag{3.7}$$

On the same way, system (3.7) generates a positive time-periodic solution with the period $T>0$, denoted by $(\mathcal{P}_1(t),\mathcal{P}_2(t))$.

Based on the above arguments, we can obtain the following lemmas.

Lemma 3.1. *The vector function* $\binom{v_1^+}{v_2^+}(t,z):=\binom{\mathcal{J}_1(t)}{\mathcal{J}_2(t)}e^{\mu_1 z}$ *satisfies the following equations*

$$\begin{cases}\partial_t v_1^+(t,z)=d_1\partial_{zz}v_1^+(t,z)-c\partial_z v_1^+(t,z)+S_H^0\beta_1(t)v_1^+(t,z)+S_H^0\beta_2(t)v_2^+(t,z)-r_1(t)v_1^+(t,z),\\ \partial_t v_2^+(t,z)=d_2\partial_{zz}v_2^+(t,z)-c\partial_z v_2^+(t,z)+S_V^0\beta_3(t)v_1^+(t,z)-r_2(t)v_2^+(t,z).\end{cases}$$

Lemma 3.2. *Assume that* ϵ_1 *is sufficiently small with* $0 < \epsilon_1 < \min\{\mu_1, \frac{c}{D_i}\}(i = 1, 2)$ *and* $\mathcal{M} := \frac{1}{\epsilon_1}$ *is large enough. Then the functions* $u_1^-(t, z) := \max\{S_H^0(1 - \mathcal{M}e^{\epsilon_1 z}), 0\}$ *and* $u_2^-(t, z) := \max\{S_V^0(1 - \mathcal{M}e^{\epsilon_1 z}), 0\}$ *satisfy*

$$\begin{cases} \partial_t u_1^- - D_1\partial_{zz}u_1^- + c\partial_z u_1^- \leqslant -\beta_1(t)u_1^- v_1^+ - \beta_2(t)u_1^- v_2^+, \\ \partial_t u_2^- - D_2\partial_{zz}u_2^- + c\partial_z u_2^- \leqslant -\beta_3(t)u_2^- v_1^+, \end{cases} \quad \forall z \neq z_1 := -\epsilon_1^{-1}\ln\mathcal{M}. \tag{3.8}$$

Proof. Here, we show that u_1^- satisfies (3.8). If $z > -\epsilon_1^{-1}\ln\mathcal{M}$, then $u_1^-(t, z) = 0$, and thus, the first equation of (3.8) is valid.

If $z < -\epsilon_1^{-1}\ln\mathcal{M}$, then $u_1^-(t, z) = S_H^0(1 - \mathcal{M}e^{\epsilon_1 z})$. In addition, it is needed only to prove that

$$\mathcal{M}\epsilon_1 e^{\epsilon_1 z}(c - d_1\epsilon_1) \geqslant S_H^0\beta_1(t)\mathcal{J}_1(t)e^{\mu_1 z}(1 - \mathcal{M}e^{\epsilon_1 z}) + S_H^0\beta_2(t)\mathcal{J}_2(t)e^{\mu_1 z}(1 - \mathcal{M}e^{\epsilon_1 z}).$$

Therefore, it is sufficient to verify

$$\mathcal{M}\epsilon_1(c - d_1\epsilon_1) \geqslant S_H^0(\beta_1(t)\mathcal{J}_1(t) + \beta_2(t)\mathcal{J}_2(t))e^{(\mu_1-\epsilon_1)z} = S_H^0(\beta_1(t)\mathcal{J}_1(t) + \beta_2(t)\mathcal{J}_2(t))\mathcal{M}^{-\epsilon_1^{-1}(\mu_1-\epsilon_1)}, \quad i = 1, 2.$$

It is obvious that the above conclusion holds provided that $\mathcal{M} := \frac{1}{\epsilon_1}$ is sufficiently large. In addition, $u_2^-(t, z)$ is discussed similarly and thus we omit it. The proof is completed. □

Lemma 3.3. *Suppose that* ϵ_2 *with* $\epsilon_2 < \min\{\epsilon_1, \mu_2 - \mu_1\}$ *is sufficiently small and* $\mathcal{K}$ *is large enough such that*

$$\mathcal{K} > \max_{[0,T]}\left\{\frac{MS_H^0(\beta_1(t)\mathcal{J}_1(t) + \beta_2(t)\mathcal{J}_2(t))}{(c - c_{\epsilon_2})\mu_{\epsilon_2}\mathcal{P}_1(t)}, \frac{MS_V^0\beta_3(t)\mathcal{J}_2(t)}{(c - c_{\epsilon_2})\mu_{\epsilon_2}\mathcal{P}_2(t)}\right\}, \tag{3.9}$$

where c_{ϵ_2}*,* μ_{ϵ_2} *and* $\mathcal{P}_i(t)$ *have been defined in* (3.7) *and* $\mathcal{J}_i(t)(i = 1, 2)$ *has been defined in* (3.5)*. Then the function* $v_i^-(t, z) := \max\{(\mathcal{J}_i(t)e^{\mu_1 z} - \mathcal{K}e^{\mu_{\epsilon_2} z}\mathcal{P}_i(t)), 0\}(i = 1, 2)$ *satisfies*

$$\begin{cases} \partial_t v_1^- - d_1\partial_{zz}v_1^- + c\partial_z v_1^- \leqslant -r_1(t)v_1^- + u_1^-(\beta_1(t)v_1^- + \beta_2(t)v_2^-), \\ \partial_t v_2^- - d_2\partial_{zz}v_2^- + c\partial_z v_2^- \leqslant -r_2(t)v_2^- + \beta_3(t)u_2^- v_1^- \end{cases}$$

for any $z \neq z_2, z_3$*,* $z_2(t) := (\epsilon_2)^{-1}\ln\frac{\mathcal{J}_1(t)}{\mathcal{K}\mathcal{P}_1(t)}$*,* $z_3(t) := (\epsilon_2)^{-1}\ln\frac{\mathcal{J}_2(t)}{\mathcal{K}\mathcal{P}_2(t)}$ *and* $z_2, z_3 < z_1$*.*

Proof. There may be the two following cases $z_3(t) \leq z_2(t)$ and $z_2(t) < z_3(t)$ for some $t \in \mathbb{R}$. Next we show $z_3(t) \leq z_2(t)$ for some $t \in \mathbb{R}$ and then we omit the condition of $z_2(t) > z_3(t)$ for some $t \in \mathbb{R}$.
If $z > z_2(t)$, then $v_i^- = 0$ for $i = 1, 2$.
If $z_3(t) < z < z_2(t) < z_1$ for some $t \in \mathbb{R}$, then $u_1^-(t, z) = S_H^0(1 - \mathcal{M}e^{\epsilon_1 z})$, $v_1^-(t, z) = \mathcal{J}_1(t)e^{\mu_1 z} - \mathcal{K}e^{\mu_{\epsilon_2} z}\mathcal{P}_1(t)$ and $v_2^-(t, z) = 0$. Due to (3.9), one can get

$$\begin{aligned}
&d_1\partial_{zz}v_1^- - c\partial_z v_1^- - \partial_t v_1^- - r_1(t)v_1^- + \beta_1(t)v_1^- u_1^- \\
&= d_1\left[\mu_1^2\mathcal{J}_1(t)e^{\mu_1 z} - \mu_{\epsilon_2}^2\mathcal{K}e^{\mu_{\epsilon_2} z}\mathcal{P}_1(t)\right] - c\left[\mu_1\mathcal{J}_1(t)e^{\mu_1 z} - \mu_{\epsilon_2}\mathcal{K}e^{\mu_{\epsilon_2} z}\mathcal{P}_1(t)\right] \\
&\quad - \left[\mathcal{J}_1'(t)e^{\mu_1 z} - \mathcal{K}e^{\mu_{\epsilon_2} z}\mathcal{P}_1'(t)\right] - r_1(t)\left[\mathcal{J}_1(t)e^{\mu_1 z} - \mathcal{K}e^{\mu_{\epsilon_2} z}\mathcal{P}_1(t)\right] \\
&\quad + S_H^0\beta_1(t)(1 - \mathcal{M}e^{\epsilon_1 z})\left[\mathcal{J}_1(t)e^{\mu_1 z} - \mathcal{K}e^{\mu_{\epsilon_2} z}\mathcal{P}_1(t)\right] \\
&= \left\{-\mathcal{J}_1'(t) + d_1\mu_1^2\mathcal{J}_1(t) - c\mu_1\mathcal{J}_1(t) + \beta_1(t)S_H^0\mathcal{J}_1(t) - r_1(t)\mathcal{J}_1(t)\right\}e^{\mu_1 z} \\
&\quad - \mathcal{K}e^{\mu_{\epsilon_2} z}\left\{-\mathcal{P}_1'(t) + d_1\mu_{\epsilon_2}^2\mathcal{P}_1(t) - c_{\epsilon_2}\mu_{\epsilon_2}\mathcal{P}_1(t) - r_1(t)\mathcal{P}_1(t) + \beta_1(t)S_H^0\mathcal{P}_1(t)\right\} \\
&\quad + (c - c_{\epsilon_2})\mu_{\epsilon_2}\mathcal{P}_1(t)\mathcal{K}e^{\mu_{\epsilon_2} z} - MS_H^0\beta_1(t)e^{\epsilon_1 z}(\mathcal{J}_1(t)e^{\mu_1 z} - \mathcal{K}e^{\mu_{\epsilon_2} z}\mathcal{P}_1(t)) \\
&\geqslant e^{\mu_{\epsilon_2} z}\left\{(c - c_{\epsilon_2})\mu_{\epsilon_2}\mathcal{P}_1(t)\mathcal{K} - MS_H^0\beta_1(t)\mathcal{J}_1(t)\right\} \geqslant 0.
\end{aligned}$$

If $z < z_2(t), z_3(t) < z_1$, then $u_1^-(t,z) = S_H^0(1-\mathcal{M}e^{\epsilon_1 z})$, $v_1^-(t,z) = \mathcal{J}_1(t)e^{\mu_1 z} - \mathcal{K}e^{\mu_{\epsilon_2} z}\mathcal{P}_1(t)$ and $v_2^-(t,z) = \mathcal{J}_2(t)e^{\mu_1 z} - \mathcal{K}e^{\mu_{\epsilon_2} z}\mathcal{P}_2(t)$. Furthermore, we need to verify that

$$\begin{aligned}
& d_1\partial_{zz}v_1^- - c\partial_z v_1^- - \partial_t v_1^- - r_1(t)v_1^- + (\beta_1(t)v_1^- + \beta_2(t)v_2^-)u_1^- \\
&= d_1\big[\mu_1^2\mathcal{J}_1(t)e^{\mu_1 z} - \mu_{\epsilon_2}^2\mathcal{K}e^{\mu_{\epsilon_2} z}\mathcal{P}_1(t)\big] - c\big[\mu_1\mathcal{J}_1(t)e^{\mu_1 z} - \mu_{\epsilon_2}\mathcal{K}e^{\mu_{\epsilon_2} z}\mathcal{P}_1(t)\big] \\
&\quad - \big[\mathcal{J}_1'(t)e^{\mu_1 z} - \mathcal{K}e^{\mu_{\epsilon_2} z}\mathcal{P}_1'(t)\big] - r_1(t)\big[\mathcal{J}_1(t)e^{\mu_1 z} - \mathcal{K}e^{\mu_{\epsilon_2} z}\mathcal{P}_1(t)\big] \\
&\quad + \beta_1(t)S_H^0(1-\mathcal{M}e^{\epsilon_1 z})\big[\mathcal{J}_1(t)e^{\mu_1 z} - \mathcal{K}e^{\mu_{\epsilon_2} z}\mathcal{P}_1(t)\big] + \beta_2(t)S_H^0(1-\mathcal{M}e^{\epsilon_1 z})\big[\mathcal{J}_2(t)e^{\mu_1 z} - \mathcal{K}e^{\mu_{\epsilon_2} z}\mathcal{P}_2(t)\big] \\
&= \big\{-\mathcal{J}_1'(t) + d_1\mu_1^2\mathcal{J}_1(t) - c\mu_1\mathcal{J}_1(t) + \beta_1 S_H^0\mathcal{J}_1(t) + \beta_2 S_H^0\mathcal{J}_2(t) - r_1(t)\mathcal{J}_1(t)\big\}e^{\mu_1 z} \\
&\quad - \mathcal{K}e^{\mu_{\epsilon_2} z}\big\{-\mathcal{P}_1'(t) + d_1\mu_{\epsilon_2}^2\mathcal{P}_1(t) - c_{\epsilon_2}\mu_{\epsilon_2}\mathcal{P}_1(t) + \beta_1(t)S_H^0\mathcal{P}_1(t) + \beta_2(t)S_H^0\mathcal{P}_2(t) - r_1(t)\mathcal{P}_1(t)\big\} \\
&\quad + (c-c_{\epsilon_2})\mu_{\epsilon_2}\mathcal{P}_1(t)\mathcal{K}e^{\mu_{\epsilon_2} z} - MS_H^0 e^{\epsilon_1 z}\big(\beta_1(t)v_1^- + \beta_2(t)v_2^-\big) \\
&\geqslant e^{\mu_{\epsilon_2} z}\big\{(c-c_{\epsilon_2})\mu_{\epsilon_2}\mathcal{P}_1(t)\mathcal{K} - MS_H^0\big(\beta_1(t)\mathcal{J}_1(t) + \beta_2(t)\mathcal{J}_2(t)\big)\big\} \geqslant 0.
\end{aligned}$$

According to (3.9), the above inequality holds true. In addition, v_2^- is proved similarly. It completes the proof. □

Let $N > -\min\{z_2, z_3\}$ and $C_N := C(\mathbb{R}\times[-N,N], \mathbb{R}^4)$. Define a convex cone $\mathcal{D}_N$ by

$$\mathcal{D}_N = \left\{ (\bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2) \in C_N \;\middle|\; \begin{array}{l} \bar{u}_i(t,z) = \bar{u}_i(t+T,z),\ \forall (t,z)\in\mathbb{R}\times[-N,N], \\ \bar{v}_i(t,z) = \bar{v}_i(t+T,z),\ \forall (t,z)\in\mathbb{R}\times[-N,N], \\ u_i^-(t,z) \leqslant \bar{u}_i(t,z) \leqslant S_H^0(S_V^0),\ \forall (t,z)\in\mathbb{R}\times[-N,N], \\ v_i^-(t,z) \leqslant \bar{v}_i(t,z) \leqslant v_i^+(t,z),\ \forall (t,z)\in\mathbb{R}\times[-N,N], \\ \bar{u}_i(t,\pm N) = u_i^-(t,\pm N),\ \forall t\in\mathbb{R}, \\ \bar{v}_i(t,\pm N) = v_i^-(t,\pm N),\ \forall t\in\mathbb{R}, i=1,2, \end{array} \right\}.$$

For any given $(\bar{u}_1, \bar{u}_2, \bar{v}_1, \bar{v}_2) \in \mathcal{D}_N$, consider the following initial value problem:

$$\begin{cases} \partial_t\bar{u}_1 - \mathcal{B}_1\bar{u}_1 = p_1[\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2],\ t>0,\ z\in[-N,N], \\ \partial_t\bar{u}_2 - \mathcal{B}_2\bar{u}_2 = p_2[\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2],\ t>0,\ z\in[-N,N], \\ \partial_t\bar{v}_1 - \mathcal{T}_1\bar{v}_1 = q_1[\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2],\ t>0,\ z\in[-N,N], \\ \partial_t\bar{v}_2 - \mathcal{T}_2\bar{v}_2 = q_2[\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2],\ t>0,\ z\in[-N,N], \\ \bar{u}_i(0,z) = \bar{u}_{i0}(z),\ \bar{v}_i(0,z) = \bar{v}_{i0}(z),\ z\in[-N,N],\ \bar{u}_{i0},\bar{v}_{i0}\in C([-N,N]), \\ \bar{u}_i(t,\pm N) = \bar{G}_{\bar{u}_i}(t,\pm N),\ \bar{v}_i(t,\pm N) = \bar{G}_{\bar{v}_i}(t,\pm N),\ \forall t>0, \end{cases} \tag{3.10}$$

where

$$\begin{aligned}
&\mathcal{B}_i\bar{u}_i = D_i\partial_{zz}\bar{u}_i - c\partial_z\bar{u}_i - \alpha_i\bar{u}_i,\ \mathcal{T}_i\bar{v}_i = d_i\partial_{zz}\bar{v}_i - c\partial_z\bar{v}_i - \chi_i\bar{v}_i,\ i=1,2, \\
&p_1[\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2] := \alpha_1\bar{u}_1 - \big(\beta_1\bar{v}_1(t,z) + \beta_2\bar{v}_2(t,z)\big)\bar{u}_1(t,z), \\
&p_2[\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2] := \alpha_2\bar{u}_2 - \beta_3\bar{v}_1(t,z)\bar{u}_2(t,z), \\
&q_1[\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2] := \chi_1\bar{v}_1 + \big(\beta_1\bar{v}_1(t,z) + \beta_2\bar{v}_2(t,z)\big)\bar{u}_1(t,z) - r_1(t)\bar{v}_1, \\
&q_2[\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2] := \chi_2\bar{v}_2 + \beta_3\bar{v}_1(t,z)\bar{u}_2(t,z) - r_2(t)\bar{v}_2, \\
&\alpha_1 > \max_{t\in[0,T]}\{(\beta_1(t)\mathcal{J}_1(t) + \beta_2(t)\mathcal{J}_2(t))e^{\mu_1 N}\},\ \alpha_2 > \max_{t\in[0,T]}\beta_3(t)\mathcal{J}_1(t)e^{\mu_1 N},\ \chi_i > \max_{t\in[0,T]} r_i(t),\ i=1,2
\end{aligned}$$

and

$$\bar{G}_{\bar{u}_i}(t,z) := \frac{1}{2}u_i^-(t,-N) - \frac{z}{2N}u_i^-(t,-N),\ \bar{G}_{\bar{v}_i}(t,z) := \frac{1}{2}v_i^-(t,-N) - \frac{z}{2N}v_i^-(t,-N)$$

for any $t\in[0,T]$ and $z\in[-N,N]$. It is easy to see that $\bar{G}_{\bar{u}_i}(t,\pm N) = u_i^-(t,\pm N)$ and $\bar{G}_{\bar{v}_i}(t,\pm N) = v_i^-(t,\pm N)$ for $t\in\mathbb{R}$ and $i=1,2$. Moreover, the functions $\bar{G}_{\bar{u}_i}$ and $\bar{G}_{\bar{v}_i}$ are T-periodic and belong to

$C^{1,2}(\mathbb{R}\times[-N,N])$. Set $\tilde{u}_i(t,z)=\bar{u}_i(t,z)-\bar{G}_{\bar{u}_i}(t,z)$, $\tilde{v}_i(t,z)=\bar{v}_i(t,z)-\bar{G}_{\bar{v}_i}(t,z)$, $\tilde{F}_{\bar{u}_i}=\mathcal{B}_i\bar{G}_{\bar{u}_i}(t,z)-\partial_t\bar{G}_{\bar{u}_i}(t,z)$ and $\tilde{F}_{\bar{v}_i}=\mathcal{T}_i\bar{G}_{\bar{v}_i}(t,z)-\partial_t\bar{G}_{\bar{v}_i}(t,z)$ for $i=1,2$. Then the problem (3.10) reduces to

$$\begin{cases}\partial_t\tilde{u}_1-\mathcal{B}_1\tilde{u}_1=p_1[\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2]+\tilde{F}_{\bar{u}_1}(t,z),\ t>0,\ z\in[-N,N],\\ \partial_t\tilde{u}_2-\mathcal{B}_2\tilde{u}_2=p_2[\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2]+\tilde{F}_{\bar{u}_2}(t,z),\ t>0,\ z\in[-N,N],\\ \partial_t\tilde{v}_1-\mathcal{T}_1\tilde{v}_1=q_1[\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2]+\tilde{F}_{\bar{v}_1}(t,z),\ t>0,\ z\in[-N,N],\\ \partial_t\tilde{v}_2-\mathcal{T}_2\tilde{v}_2=q_2[\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2]+\tilde{F}_{\bar{v}_2}(t,z),\ t>0,\ z\in[-N,N],\\ \tilde{u}_i(0,z)=\bar{u}_{i0}(z)-\bar{G}_{\bar{u}_i}(0,z),\ \tilde{v}_i(0,z)=\bar{v}_{i0}(z)-\bar{G}_{\bar{v}_i}(0,z),\ z\in[-N,N],\ \tilde{u}_{i0},\tilde{v}_{i0}\in C([-N,N]),\\ \tilde{u}_i(t,\pm N)=0,\ \tilde{v}_i(t,\pm N)=0,\ \forall t>0.\end{cases}\tag{3.11}$$

The realization of A_i in $C([-N,N])$ with the homogenous Dirichlet boundary condition can be defined by

$$D(A_i^0)=\left\{w\in\bigcap_{p\geqslant 1}W^{2,p}_{loc}((-N,N)):w,A_iw\in C([-N,N]),w|_{\pm N}=0\right\},$$
$$A_i^0w=\mathcal{B}_iw,\ A_j^0w=\mathcal{T}_iw,\ i=1,2,\ j=3,4.$$

In fact, $D(A_i)=\{u\in C^2([-N,N]),u|_{\pm N}=0\}$ (see, e.g., [24, Section 5.1.2]). Assume that $\{H_i(t)\}_{t\geqslant 0}$ is the strongly continuous analytic semigroup generated by $A_i^0:D(A_i^0)\subset C([-N,N])\to C([-N,N])$ for $i=1,2$ (see [24]). Note that

$$H_i(t)w(x)=e^{-\alpha_it}\int_{-N}^{N}\Gamma_i(t,x,y)w(y)\mathrm{d}y,\ i=1,2,\ w(x)\in C([-N,N])$$

and

$$H_j(t)w(x)=e^{-\chi_{j-2}t}\int_{-N}^{N}\Gamma_j(t,x,y)w(y)\mathrm{d}y,\ i=3,4,\ w(x)\in C([-N,N])$$

for $t>0$ and $x\in[-N,N]$, where $\Gamma_i(i=1,2)$ and $\Gamma_j(j=3,4)$ are the Green functions associated with $D_i\partial_{xx}-c\partial_x$ and $d_i\partial_{xx}-c\partial_x$ and Dirichlet boundary condition, respectively. Then system (3.11) can be treated as the following integral system

$$\begin{cases}\tilde{u}_i(t,z)=H_i(t)\tilde{u}_i(0)(z)+\int_0^t H_i(t-s)\big(p_i[\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2](s)+\tilde{F}_{\bar{u}_i}(s)\big)(z)\mathrm{d}s,\ i=1,2,\\ \tilde{v}_i(t,z)=H_{i+2}(t)\tilde{v}_i(0)(z)+\int_0^t H_{i+2}(t-s)\big(q_i[\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2](s)+\tilde{F}_{\bar{v}_i}(s)\big)(z)\mathrm{d}s,\ i=1,2,\end{cases}$$

where $t\geqslant 0$ and $z\in[-N,N]$, indicating that $(\bar{u}_1(t,z),\bar{u}_2(t,z),\bar{v}_1(t,z),\bar{v}_2(t,z))$ satisfies

$$\begin{cases}\bar{u}_i(t,z)=H_i(t)\tilde{u}_i(0)(z)+\int_0^t H_i(t-s)\big(p_i[\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2](s)+\tilde{F}_{\bar{u}_i}(s)\big)(z)\mathrm{d}s+\bar{G}_{\bar{u}_i}(t,z),\\ \bar{v}_i(t,z)=H_{i+2}(t)\tilde{v}_i(0)(z)+\int_0^t H_{i+2}(t-s)\big(q_i[\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2](s)+\tilde{F}_{\bar{v}_i}(s)\big)(z)\mathrm{d}s+\bar{G}_{\bar{v}_i}(t,z)\end{cases}\tag{3.12}$$

where $t\geqslant 0$, $z\in[-N,N]$ and $i=1,2$. A solution of (3.12) can be called as a mild solution of (3.11). Note that $p_i[\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2],q_i[\bar{u}_1,\bar{u}_2,\bar{v}_1,\bar{v}_2]\in C(\mathbb{R}\times[-N,N])$, then it follows from [24, Theorem 5.1.17] that the functions $\bar{u}_i$ and $\bar{v}_i(i=1,2)$ defined by (3.12) belong to $C([0,2T]\times[-N,N])\bigcap C^{\theta,2\theta}([\epsilon,2T]\times[-N,N])$ for every $\epsilon\in(0,2T)$ and $\theta\in(0,1)$. Define a set

$$\mathcal{D}_N^0=\left\{(u_{10},u_{20},v_{10},v_{20})\in C([-N,N],\mathbb{R}^4)\left|\begin{array}{l}u_i^-(0,z)\leqslant u_{i0}(z)\leqslant u_i^+(0,z),\ \forall z\in[-N,N],\\ v_i^-(0,z)\leqslant v_{i0}(z)\leqslant v_i^+(0,z),\ \forall z\in[-N,N],\\ u_{i0}(\pm N)=u_i^-(0,\pm N),\\ v_{i0}(\pm N)=v_i^-(0,\pm N),\end{array}\right.\right\}.$$

Obviously, $\mathcal{D}_N^0$ is a closed and convex set.

Lemma 3.4. *For any $U_0:=(u_{10},u_{20},v_{10},v_{20})\in\mathcal{D}_N^0$, let $(u_{1N}(t,z;U_0),u_{2N}(t,z;U_0),v_{1N}(t,z;U_0),v_{2N}(t,z;U_0))$ be the solutions of system (3.12) with the initial value U_0. Then*

$$u_i^-(t,z)\leqslant u_{iN}(t,z;U_0)\leqslant S_H^0(S_V^0),\ \ v_i^-(t,z)\leqslant v_{iN}(t,z;U_0)\leqslant v_i^+(t,z),\ \ i=1,2$$

for any $(t,z)\in[0,\infty)\times[-N,N]$.

Proof. The argumentations are essentially same as those in [53, Lemma 3.3] and [48, Lemma 2.4], so we omit them. ☐

For a given $U_0:=(u_{10},u_{20},v_{10},v_{20})\in\mathcal{D}_N^0$, define a map $F:\mathcal{D}_N^0\to C([-N,N],\mathbb{R}^4)$ by

$$F[u_{10},u_{20},v_{10},v_{20}](\cdot)=(u_{1N}(t,z;U_0),u_{2N}(t,z;U_0),v_{1N}(t,z;U_0),v_{2N}(t,z;U_0)),$$

where $(u_{1N}(t,z;U_0),u_{2N}(t,z;U_0),v_{1N}(t,z;U_0),v_{2N}(t,z;U_0))$ is the solution of system (3.12) with the initial value U_0. In view of Lemma 3.4 and the periodicity of u_i^-, v_i^- and v_i^+, we have $F[\mathcal{D}_N^0]\in\mathcal{D}_N^0$. Obviously, $\mathcal{D}_N^0$ is a complete metric space with a distance induced by the supreme norm. For any $U_0^1:=(u_{10}^1,u_{20}^1,v_{10}^1,v_{20}^1)$ and $U_0^2:=(u_{10}^2,u_{20}^2,v_{10}^2,v_{20}^2)\in\mathcal{D}_N^0$, (3.12) indicates

$$\begin{aligned}\|u_{iN}(T,z;U_0^1)-u_{iN}(T,z;U_0^2)\|=&\sup_{z\in[-N,N]}|e^{-\alpha_iT}\int_{-N}^{N}\Gamma_i(T,z,y)\big(U_0^1-U_0^2\big)\mathrm{d}y|\\ \leqslant& e^{-\alpha_iT}\|U_0^1-U_0^2\|_{C([-N,N])},\ i=1,2.\end{aligned}$$

On the same way,

$$\|v_{iN}(T,z;U_0^1)-v_{iN}(T,z;U_0^2)\|\leqslant e^{-\chi_iT}\|U_0^1-U_0^2\|_{C([-N,N])},\ i=1,2.$$

Since $e^{-\alpha_iT},e^{-\chi_iT}<1$ for $i=1,2$, one has that $F:\mathcal{D}_N^0\to\mathcal{D}_N^0$ is a contraction map. As a consequence, the Banach fixed point theorem implies that F admits a unique fixed point $U_0^*:=(u_{10}^*,u_{20}^*,v_{10}^*,v_{20}^*)\in\mathcal{D}_N^0$. Let $(u_{1N}^*(t,z),u_{2N}^*(t,z),v_{1N}^*(t,z),v_{2N}^*(t,z))=(u_{1N}(t,z;U_0^*),u_{2N}(t,z;U_0^*),v_{1N}(t,z;U_0^*),v_{2N}(t,z;U_0^*))$ for $t\in(0,+\infty)$ and $z\in[-N,N]$, where $(u_{1N}(t,z;U_0^*),u_{2N}(t,z;U_0^*),v_{1N}(t,z;U_0^*),v_{2N}(t,z;U_0^*))$ is the solution of system (3.10) with the initial value U_0^*. Furthermore, using the similar arguments to these in [53], one has $(u_{1N}^*(t,z),u_{2N}^*(t,z),v_{1N}^*(t,z),v_{2N}^*(t,z))=(u_{1N}^*(t+T,z),u_{2N}^*(t+T,z),v_{1N}^*(t+T,z),v_{2N}^*(t+T,z))$ for all $t\in[0,\infty)$ and $z\in[-N,N]$. According to Lemma 3.4, we can get $(u_{1N}^*(t,z),u_{2N}^*(t,z),v_{1N}^*(t,z),v_{2N}^*(t,z))\in\mathcal{D}_N$. Then $(u_{1N}^*(t,z),u_{2N}^*(t,z),v_{1N}^*(t,z),v_{2N}^*(t,z))$ satisfies

$$\begin{cases}u_{iN}^*(t)=H_i(t-s)(u_{iN}^*(s)-\bar{G}_{\bar{u}_i}(s))+\int_s^t H_i(t-m)\big(f_i[u_{1N}^*,u_{2N}^*,v_{1N}^*,v_{2N}^*](m)+\tilde{F}_{\bar{u}_i}(m)\big)dm+\bar{G}_{\bar{u}_i}(t),\\ v_{iN}^*(t)=H_{i+2}(t-s)(v_{iN}^*(s)-\bar{G}_{\bar{v}_i}(s))+\int_s^t H_{i+2}(t-m)\big(g_i[u_{1N}^*,u_{2N}^*,v_{1N}^*,v_{2N}^*](m)+\tilde{F}_{\bar{v}_i}(m)\big)dm\\ \quad+\bar{G}_{\bar{v}_i}(t)\end{cases}\tag{3.13}$$

for any $t\geqslant s$ and $i=1,2$. On the basis of the above discussion, we obtain the theorem as follows.

Theorem 3.5. *For any given $(u_{1N},u_{2N},v_{1N},v_{2N})\in\mathcal{D}_N$, there exists a unique solution $(u_{1N}^*,u_{2N}^*,v_{1N}^*,v_{2N}^*)\in\mathcal{D}_N$ satisfying (3.13).*

By virtue of Theorem 3.5, we can define an operator $\mathcal{R}:\mathcal{D}_N\to\mathcal{D}_N$ by $\mathcal{R}(u_{1N},u_{2N},v_{1N},v_{2N})=(u_{1N}^*,u_{2N}^*,v_{1N}^*,v_{2N}^*)$. In what follows, by using the similar arguments to those in [53, Lemma 3.5] and [48, Lemma 2.6], we present the complete continuity of the operator $\mathcal{R}$ without proof.

Lemma 3.6. *The operator $\mathcal{R} : \mathcal{D}_N \to \mathcal{D}_N$ is completely continuous.*

Based on the above arguments, the Schauder's fixed point theorem expresses that $\mathcal{R}$ admits a fixed point $(u^*_{1N}, u^*_{2N}, v^*_{1N}, v^*_{2N}) \in \mathcal{D}_N$. In addition, $(u^*_{1N}(t+T,\cdot), u^*_{2N}(t+T,\cdot), v^*_{1N}(t+T,\cdot), v^*_{2N}(t+T,\cdot)) = (u^*_{1N}(t,\cdot), u^*_{2N}(t,\cdot),\ v^*_{1N}(t,\cdot), v^*_{2N}(t,\cdot))$ for all $t \in \mathbb{R}$. Note that $u^*_{iN}, v^*_{iN} \in C^{\frac{\theta}{2},\theta}(\mathbb{R}\times[-N,N])$ for some $\theta \in (0,1)$ and $i = 1,2$. By [24, Theorem 5.1.18 and 5.1.19], $u^*_{iN}, v^*_{iN} \in C^{1,2}(\mathbb{R}\times[-N,N])(i=1,2)$ satisfy

$$\begin{cases} \partial_t u^*_{1N} = D_1\partial_{zz}u^*_{1N} - c\partial_z u^*_{1N} - \big(\beta_1(t)v^*_{1N} + \beta_2(t)v^*_{2N}\big)u^*_{1N},\ \forall t\in\mathbb{R},\ z\in[-N,N],\\ \partial_t u^*_{2N} = D_2\partial_{zz}u^*_{2N} - c\partial_z u^*_{2N} - \beta_3(t)u^*_{2N}v^*_{1N},\ \forall t\in\mathbb{R},\ z\in[-N,N],\\ \partial_t v^*_{1N} = d_1\partial_{zz}v^*_{1N} - c\partial_z v^*_{1N} + \big(\beta_1(t)v^*_{1N} + \beta_2(t)v^*_{2N}\big)u^*_{1N} - r_1(t)v^*_{1N},\ \forall t\in\mathbb{R},\ z\in[-N,N],\\ \partial_t v^*_{2N} = d_2\partial_{zz}v^*_{2N} - c\partial_z v^*_{2N} + \beta_3(t)u^*_{2N}v^*_{1N} - r_2(t)v^*_{2N},\ \forall t\in\mathbb{R},\ z\in[-N,N],\\ u^*_{iN}(t,\pm N) = u^-_i(t,\pm N),\ v^*_i(t,\pm N) = v^-_{iN}(t,\pm N),\ \forall t\in\mathbb{R}, \end{cases} \tag{3.14}$$

where $i = 1,2$. Similar to [53, Theorem 3.6] and [48, Theorem 2.7], we have the following local uniform estimates on u^*_i and $v^*_i(i=1,2)$.

Lemma 3.7. *Let $p \geqslant 2$. For any given $L > 0$, there exists a constant $C := C(p,L) > 0$ such that for any $N > \max\{L, -\min\{z_2,z_3\}\}$ large enough, there hold*

$$\|u^*_{iN}\|_{W^{1,2}_p([0,T]\times[-L,L])}, \|v^*_{iN}\|_{W^{1,2}_p([0,T]\times[-L,L])} \leqslant C.$$

In addition, there exists a constant $\hat{C} := \hat{C}(L) > 0$ such that, for any $z_0 \in \mathbb{R}$,

$$\|u^*_{iN}\|_{C^{\frac{1+\theta}{2},1+\theta}([0,T]\times[z_0-L,z_0+L])}, \|v^*_{iN}\|_{C^{\frac{1+\theta}{2},1+\theta}([0,T]\times[z_0-L,z_0+L])} \leqslant \hat{C}$$

for any $N > \max\{L+|z_0|, -\min\{z_2,z_3\}\}$, $\theta\in(0,1)$ and $i=1,2$.

Now, we estimate the solution of system (3.14), denoted by $(u^*_{1N}, u^*_{2N}, v^*_{1N}, v^*_{2N})$.

Proposition 3.8. *Let N be large enough satisfying $N > -\min\{z_2,z_3\}$. There exists a constant C_0 independent upon N such that*

$$\frac{1}{T}\int_{-N}^{N}\int_0^T (\beta_1(t)v^*_{1N}(t,z) + \beta_2(t)v^*_{2N}(t,z))u^*_{1N}(t,z)\mathrm{d}t\mathrm{d}z < C_0,$$

$$\frac{1}{T}\int_{-N}^{N}\int_0^T \beta_3(t)v^*_{1N}(t,z)u^*_{2N}(t,z)\mathrm{d}t\mathrm{d}z < C_0,$$

$$\frac{1}{T}\int_{-N}^{N}\int_0^T v^*_{iN}(t,z)\mathrm{d}t\mathrm{d}z < C_0,\ \int_0^T \partial_z u^*_{iN}(t,z)\mathrm{d}t\mathrm{d}z \leqslant 0,\ i=1,2$$

for any $z\in[-N,N]$.

Proof. We firstly define

$$\tilde{u}^*_{iN}(z) = \frac{1}{T}\int_0^T u^*_{iN}(t,z)\mathrm{d}t,\ \tilde{v}^*_{iN}(z) = \frac{1}{T}\int_0^T v^*_{iN}(t,z)\mathrm{d}t,$$

$$\tilde{u}^\pm_i(z) = \frac{1}{T}\int_0^T u^\pm_i(t,z)\mathrm{d}t,\ \tilde{v}^\pm_i(z) = \frac{1}{T}\int_0^T v^\pm_i(t,z)\mathrm{d}t,\ \forall z\in[-N,N].$$

Obviously,

$$\tilde{u}_i^-(z) \leqslant \tilde{u}_{iN}^*(z) \leqslant \tilde{u}_i^+(z),\ \tilde{v}_i^-(z) \leqslant \tilde{v}_{iN}^*(z) \leqslant \tilde{v}_i^+(z),\ i=1,2,\ \forall z \in [-N,N].$$

According to (3.14), we have

$$c\tilde{u}_{1N,z}^*(z) = D_1\tilde{u}_{1N,zz}^*(z) - \frac{1}{T}\int_0^T (\beta_1(t)v_{1N}^*(t,z) + \beta_2(t)v_{2N}^*(t,z))u_{1N}^*(t,z)\mathrm{d}t,\ \forall z \in [-N,N], \quad (3.15)$$

where $\tilde{u}_{1N,z}^*(z) := \frac{\mathrm{d}\tilde{u}_{1N}^*(z)}{\mathrm{d}z}$ and $\tilde{u}_{1N,zz}^*(z) := \frac{\mathrm{d}^2\tilde{u}_{1N}^*(z)}{\mathrm{d}z^2}$. It follows from (3.15) that

$$\begin{aligned}\left(e^{-\frac{cz}{D_1}}\tilde{u}_{1N,z}^*\right)_z =& e^{-\frac{cz}{D_1}}\left(\tilde{u}_{1N,zz}^* - \frac{c}{D_1}\tilde{u}_{1N,z}^*\right)\\ =& \frac{e^{-\frac{cz}{D_1}}}{D_1 T}\int_0^T (\beta_1(t)v_{1N}^*(t,z) + \beta_2(t)v_{2N}^*(t,z))u_{1N}^*(t,z)\mathrm{d}t,\ \forall z \in [-N,N].\end{aligned}$$

Then integrating two sides of the above equation from $z \in [-N,N)$ to N yields

$$\tilde{u}_{1N,z}^*(z) = e^{-\frac{c(N-z)}{D_1}}\tilde{u}_{1N,z}^*(N) - \frac{1}{D_1T}\int_z^N e^{-\frac{c(\xi-z)}{D_1}}\int_0^T (\beta_1(t)v_{1N}^*(t,\xi) + \beta_2(t)v_{2N}^*(t,\xi))u_{1N}^*(t,\xi)\mathrm{d}t\mathrm{d}\xi. \quad (3.16)$$

Due to $\tilde{u}_{1N}^*(z) \geqslant 0$ for $z \in [-N,N]$ and $\tilde{u}_{1N}^*(N) = \tilde{u}_1^-(N) = 0$, one has $\tilde{u}_{1N,z}^*(N) \leqslant 0$. According to (3.16), it has $\tilde{u}_{1N,z}^*(z) \leqslant 0$ and $\tilde{u}_{1N,z}^*(z) \not\equiv 0$ on $[-N,N]$. By using $\tilde{u}_{1N,z}^*(-N) \geqslant \tilde{u}_{1,z}^-(-N) = -S_H^0 M\epsilon_1 e^{-\epsilon_1 N} \geqslant -S_H^0$, integrating from $-N$ to N for equation (3.15) leads to

$$\begin{aligned}&\frac{1}{T}\int_{-N}^N\int_0^T (\beta_1(t)v_{1N}^*(t,z) + \beta_2(t)v_{2N}^*(t,z))u_{1N}^*(t,z)\mathrm{d}t\mathrm{d}z\\ &= c\big(\tilde{u}_{1N}^*(-N) - \tilde{u}_{1N}^*(N)\big) + D_1(\tilde{u}_{1N,z}^*(N) - \tilde{u}_{1N,z}^*(-N))\\ &\leqslant (c + D_1)S_H^0.\end{aligned}$$

In addition, $\frac{1}{T}\int_{-N}^N\int_0^T \beta_3(t)v_{1N}^*(t,z)u_{2N}^*(t,z)\mathrm{d}t\mathrm{d}z < C_0$ can be discussed similarly.

Let $\bar{r}_1 := \max_{t\in[0,T]} r_1(t)$. Then, $\tilde{v}_{1N}^*(z)$ satisfies

$$\begin{aligned}&-d_1\tilde{v}_{1N,zz}^*(z) + c\tilde{v}_{1N,z}^*(z) + \bar{r}_1\tilde{v}_{1N}^*(z)\\ &= \frac{1}{T}\int_0^T (\beta_1(t)v_{1N}^*(t,z) + \beta_2(t)v_{2N}^*(t,z))u_{1N}^*(t,z)\mathrm{d}t - \frac{1}{T}\int_0^T (r_1(t) - \bar{r}_1)v_{1N}^*(t,z)\mathrm{d}t.\end{aligned}$$

Similarly, one has $\tilde{v}_{1N,z}^*(N) \leqslant 0$, $\tilde{v}_{1N,z}^*(-N) \geqslant \tilde{v}_{1,z}^-(-N) \geqslant -\mathcal{K}\mu_{\epsilon_2}e^{-\epsilon_2 N}\tilde{\mathcal{P}}_1$, $\tilde{v}_{1N}^*(N) = 0$ and $\tilde{v}_{1N}^*(-N) = \tilde{v}_1^-(-N)$, where $\tilde{\mathcal{P}}_1 := \int_0^T \mathcal{P}_1(t)\mathrm{d}t$ and $\mathcal{P}_1(t)$ has been defined in Lemma 3.3. Then by integrating the two sides of the last equality on $[-N,N]$, one has

$$\int_{-N}^{N} \tilde{v}_{1N}^*(z)\mathrm{d}z \leqslant \frac{d_1}{\bar{r}_1}(\tilde{v}_{1N,z}^*(N) - \tilde{v}_{1N,z}^*(-N)) + \frac{c}{\bar{r}_1}(\tilde{v}_{1N}^*(-N) - \tilde{v}_{1N}^*(N))$$
$$+ \frac{1}{\bar{r}_1 T}\int_{-N}^{N}\int_{0}^{T}(\beta_1(t)v_{1N}^*(t,z) + \beta_2(t)v_{2N}^*(t,z))u_{1N}^*(t,z)\mathrm{d}t\mathrm{d}z$$
$$\leqslant \frac{1}{\bar{r}_1}\big(d_1\mathcal{K}\mu_{\epsilon_2}e^{-\epsilon_2 N}\tilde{\mathcal{P}}_1 + c\tilde{v}_{1N}^-(-N) + (c + D_1)S_H^0\big).$$

Furthermore, $\frac{1}{T}\int_0^T\int_{-N}^{N} v_{2N}^*(z)\mathrm{d}t\mathrm{d}z \leqslant C_0$ can be proved similarly. It completes the proof. □

Theorem 3.9. *Assume that $R_0 > 1$. For any $c > c^*$, system (3.2) admits a time-periodic solution $(u_1^*, u_2^*, v_1^*, v_2^*)$ satisfying (3.3). In addition, there hold $0 < \frac{1}{T}\int_0^T v_1^*(t,z)\mathrm{d}t \leqslant (S_H^0 - S_H^\infty)$ and $0 < \frac{1}{T}\int_0^T v_2^*(t,z)\mathrm{d}t \leqslant (S_V^0 - S_V^\infty)$ for any $z \in \mathbb{R}$, and*

$$\frac{1}{T}\int_{-\infty}^{+\infty}\int_0^T r_1(t)v_1^*(t,z)\mathrm{d}t\mathrm{d}z = \frac{1}{T}\int_{-\infty}^{+\infty}\int_0^T(\beta_1(t)v_1^*(t,z) + \beta_2(t)v_2^*(t,z))u_1^*(t,z)\mathrm{d}t\mathrm{d}z = c(S_H^0 - S_H^\infty),$$
$$\frac{1}{T}\int_{-\infty}^{+\infty}\int_0^T r_2(t)v_2^*(t,z)\mathrm{d}t\mathrm{d}z = \frac{1}{T}\int_{-\infty}^{+\infty}\int_0^T \beta_3(t)v_1^*(t,z)u_2^*(t,z)\mathrm{d}t\mathrm{d}z = c(S_V^0 - S_V^\infty).$$

Proof. The proof is divided into four steps.

Firstly, we show existence of a periodic solution for system (3.2). Assume that $\{n_m\}_{m\geqslant 1}$ is an increasing sequence such that $n_m \geqslant -\min\{z_2, z_3\}$ for $m \in \mathbb{N}^+$ and $\lim_{m\to\infty} n_m = \infty$. It then follows that the solution sequence $(u_{1,n_m}, u_{2,n_m}, v_{1,n_m}, v_{2,n_m}) \in \mathcal{D}_{n_m}$ satisfies Lemma 3.7 and (3.14). By virtue of the periodicity of the solution sequence $(u_{1,n_m}, u_{2,n_m}, v_{1,n_m}, v_{2,n_m})$ with $t \in \mathbb{R}$, we can extract a subsequence of it, still denoted by $(u_{1,n_m}, u_{2,n_m}, v_{1,n_m}, v_{2,n_m})$, converging to a function $(u_1^*, u_2^*, v_1^*, v_2^*) \in C_{loc}(\mathbb{R}^4)$ in the following topologies

$$\begin{aligned}(u_{1,n_m}, u_{2,n_m}, v_{1,n_m}, v_{2,n_m}) \to (u_1^*, u_2^*, v_1^*, v_2^*) \text{ in } C_{loc}^{\frac{1+\beta}{2},1+\beta}(\mathbb{R}^4), \text{ in } H_{loc}^1(\mathbb{R}^4)\\ \text{and in } L_{loc}^2(\mathbb{R}, H_{loc}^2(\mathbb{R}^4)) \text{ weakly,}\end{aligned} \tag{3.17}$$

where $\beta \in (0,\theta)$ and $\theta \in (0,1)$. Clearly,

$$(u_1^*, u_2^*, v_1^*, v_2^*) \in C_{loc}^{\frac{1+\beta}{2},1+\beta}(\mathbb{R}^4) \cap H_{loc}^1(\mathbb{R}^4) \cap L_{loc}^2(\mathbb{R}, H_{loc}^2(\mathbb{R}^4)).$$

It follows from Lemma 3.7 that for any $N > 0$, there exists a constant C_3 such that

$$\|u_i^*\|_{C^{\frac{1+\theta}{2},1+\theta}([0,T]\times[-N,N])}, \|v_i^*\|_{C^{\frac{1+\theta}{2},1+\theta}([0,T]\times[-N,N])} \leqslant C_3. \tag{3.18}$$

Then using the similar arguments to those in [48, Theorem 2.9], $(u_1^*, u_2^*, v_1^*, v_2^*)$ satisfies

$$\begin{cases}\partial_t u_1^*(t,z) = D_1\partial_{zz}u_1^*(t,z) - c\partial_z u_1^*(t,z) - u_1^*(t,z)(\beta_1(t)v_1^*(t,z) + \beta_2(t)v_2^*(t,z)),\\ \partial_t v_1^*(t,z) = d_1\partial_{zz}v_1^*(t,z) - c\partial_z v_1^*(t,z) + u_1^*(t,z)(\beta_1(t)v_1^*(t,z) + \beta_2(t)v_2^*(t,z)) - r_1(t)v_1^*(t,z),\\ \partial_t u_2^*(t,z) = D_2\partial_{zz}u_2^*(t,z) - c\partial_z u_2^*(t,z) - \beta_3(t)u_2^*(t,z)v_1^*(t,z),\\ \partial_t v_2^*(t,z) = d_2\partial_{zz}v_2^*(t,z) - c\partial_z v_2^*(t,z) + \beta_3(t)u_2^*(t,z)v_1^*(t,z) - r_2(t)v_2^*(t,z),\end{cases}$$

where $(t,z)\in\mathbb{R}^2$. It further follows from Proposition 3.8 that there exists a constant $C_0>0$ such that

$$\begin{aligned}&\frac{1}{T}\int_{-\infty}^{+\infty}\int_0^T(\beta_1(t)v_1^*(t,z)+\beta_2(t)v_2^*(t,z))u_1^*(t,z)\mathrm{d}t\mathrm{d}z<C_0,\ \frac{1}{T}\int_{-\infty}^{+\infty}\int_0^T\beta_3(t)v_1^*(t,z)u_2^*(t,z)\mathrm{d}t\mathrm{d}z<C_0,\\&\frac{1}{T}\int_{-\infty}^{+\infty}\int_0^T v_i^*(t,z)\mathrm{d}t\mathrm{d}z<C_0,\ \int_0^T\partial_z u_i^*(t,z)\mathrm{d}t\mathrm{d}z\leqslant 0,\ i=1,2.\end{aligned}\tag{3.19}$$

Note that $(u_1^*,u_2^*,v_1^*,v_2^*)$ satisfies that

$$u_i^-(t,z)\leqslant u_i^*(t,z)\leqslant S_H^0(S_V^0),\ v_i^-(t,z)\leqslant v_i^*(t,z)\leqslant v_i^+(t,z),\ i=1,2,\ \forall(t,z)\in\mathbb{R}^2.$$

As a consequence, there holds $u_i^*(t,z)\to S_H^0(S_V^0)$ and $v_i^*(t,z)\to 0$ uniformly for $t\in\mathbb{R}$ and $i=1,2$, as $z\to-\infty$.

Secondly, we prove the asymptotic behavior of v_i^* as $z\to+\infty$. Define $\hat{v}_1(z)=\frac{1}{T}\int_0^T v_1^*(t,z)\mathrm{d}t$. Then $\hat{v}_1(t)$ satisfies

$$\begin{aligned}&-d_1\hat{v}_{1,zz}(z)+c\hat{v}_{1,z}(z)+\bar{r}_1\hat{v}_1(z)\\&=\frac{1}{T}\int_0^T(\beta_1(t)v_1^*(t,z)+\beta_2(t)v_2^*(t,z))u_1^*(t,z)\mathrm{d}t-\frac{1}{T}\int_0^T(r_1(t)-\bar{r}_1)v_1^*(t,z)\mathrm{d}t,\end{aligned}\tag{3.20}$$

where $\bar{r}_1:=\max_{t\in[0,T]}r_1(t)$. Denote the two roots of the characteristic equation

$$-d_1\eta^2+c\eta+\bar{r}_1=0$$

by

$$\eta^\pm:=\frac{c\pm\sqrt{c^2+4d_1\bar{r}_1}}{2d_1}.$$

Furthermore, let $\rho:=d_1(\eta^+-\eta^-)=\sqrt{c^2+4d_1\bar{r}_1}$. Then it is easy to see that $\eta^-<0<\eta^+$. It follows from (3.20) that

$$\begin{aligned}\hat{v}_1(z)=&\frac{1}{\rho T}\int_{-\infty}^{z}e^{\eta^-(z-y)}\left[\int_0^T(\beta_1(t)v_1^*(t,y)+\beta_2(t)v_2^*(t,y))u_1^*(t,y)-\frac{1}{T}\int_0^T(r_1(t)-\bar{r}_1)v_1^*(t,y)\right]\mathrm{d}t\mathrm{d}y\\&+\frac{1}{\rho T}\int_z^{+\infty}e^{\eta^+(z-y)}\left[\int_0^T(\beta_1(t)v_1^*(t,y)+\beta_2(t)v_2^*(t,y))u_1^*(t,y)-\frac{1}{T}\int_0^T(r_1(t)-\bar{r}_1)v_1^*(t,y)\right]\mathrm{d}t\mathrm{d}y\end{aligned}$$

and

$$\begin{aligned}\hat{v}_{1,z}(z) =& \frac{\eta^-}{\rho T}\int_{-\infty}^{z} e^{\eta^-(z-y)}\left[\int_0^T(\beta_1(t)v_1^*(t,y)+\beta_2(t)v_2^*(t,y))u_1^*(t,y)-\frac{1}{T}\int_0^T(r_1(t)-\bar{r}_1)v_1^*(t,y)\right]\mathrm{d}t\mathrm{d}y\\ &+\frac{\eta^+}{\rho T}\int_{z}^{+\infty} e^{\eta^+(z-y)}\left[\int_0^T(\beta_1(t)v_1^*(t,y)+\beta_2(t)v_2^*(t,y))u_1^*(t,y)-\frac{1}{T}\int_0^T(r_1(t)-\bar{r}_1)v_1^*(t,y)\right]\mathrm{d}t\mathrm{d}y\\ \leqslant& \frac{\eta^-}{\rho T}\int_{-\infty}^{z} e^{\eta^-(z-y)}\int_0^T(\beta_1(t)v_1^*(t,y)+\beta_2(t)v_2^*(t,y))u_1^*(t,y)\mathrm{d}t\mathrm{d}y\\ &+\frac{\eta^+}{\rho T}\int_{z}^{+\infty} e^{\eta^+(z-y)}\int_0^T(\beta_1(t)v_1^*(t,y)+\beta_2(t)v_2^*(t,y))u_1^*(t,y)\mathrm{d}t\mathrm{d}y\\ =& \frac{\eta^-}{\rho T}\int_{0}^{+\infty} e^{\eta^- y}\int_0^T(\beta_1(t)v_1^*(t,z-y)+\beta_2(t)v_2^*(t,z-y))u_1^*(t,z-y)\mathrm{d}t\mathrm{d}y\\ &+\frac{\eta^+}{\rho T}\int_{-\infty}^{0} e^{\eta^+ y}\int_0^T(\beta_1(t)v_1^*(t,z-y)+\beta_2(t)v_2^*(t,z-y))u_1^*(t,z-y)\mathrm{d}t\mathrm{d}y.\end{aligned}$$

According to $\rho := d_1(\eta^+ - \eta^-)$ and $\eta^- < 0 < \eta^+$, it has

$$\|\hat{v}_{1,z}\| \leqslant \frac{1}{d_1 T}\int_{-\infty}^{+\infty}\int_0^T(\beta_1(t)v_1^*(t,z)+\beta_2(t)v_2^*(t,z))u_1^*(t,z)\mathrm{d}t\mathrm{d}z,$$

which implies that $\hat{v}_{1,z}(z)$ is uniformly bounded. Consequently, following $\int_{-\infty}^{+\infty}\hat{v}_1(z)\mathrm{d}z < C_0$, we must have $\hat{v}_1(z) \to 0$ as $z \to +\infty$. Using the similar arguments to those in [48, Theorem 2.9], $v_1^*(t,z) \to 0$ as $z \to +\infty$ uniformly for each $t \in \mathbb{R}$. As a consequence, $v_1^*(t,z) \leqslant C_0$ holds for any $(t,z) \in \mathbb{R}^2$. On the same way, $v_2^*(t,z) \to 0$ as $z \to +\infty$ uniformly for every $t \in \mathbb{R}$.

Thirdly, the asymptotic behavior of $u_i^*(i=1,2)$ is shown. By using the estimate of (3.18) and Laudau-type inequality (see, e.g., [3,20]), one has

$$\|\partial_z u_1^*\|_{L^\infty([0,T]\times(-\infty,M))} \leqslant 2\|u_1^* - S_H^0\|_{L^\infty([0,T]\times(-\infty,M))}\|\partial_{zz}u_1^*\|_{L^\infty([0,T]\times(-\infty,M))}.$$

As a consequence,

$$\lim_{z\to-\infty}\partial_z u_1^*(t,z) = 0 \text{ uniformly for } t \in \mathbb{R}.$$

Define $\hat{u}_1^*(z) = \frac{1}{T}\int_0^T u_1^*(t,z)\mathrm{d}t$. It is easy to see that $\hat{u}_{1,z}^*(z) \to 0$ as $z \to -\infty$. In addition, $\hat{u}_1^*(z)$ satisfies

$$c\hat{u}_{1,z}^*(z) = d_1\hat{u}_{1,zz}^*(z) - \frac{1}{T}\int_0^T(\beta_1(t)v_1^*(t,z)+\beta_2(t)v_2^*(t,z))u_1^*(t,z)\mathrm{d}t, \tag{3.21}$$

which implies that

$$(e^{-\frac{cz}{d_1}}\hat{u}_{1,z}^*(z))_z = e^{-\frac{cz}{d_1}}(\hat{u}_{1,zz}^*(z) - \frac{c}{d_1}\hat{u}_{1,z}^*(z)) = \frac{e^{-\frac{cz}{d_1}}}{d_1 T}\int_0^T(\beta_1(t)v_1^*(t,z)+\beta_2(t)v_2^*(t,z))u_1^*(t,z)\mathrm{d}t.$$

Then, an integration from z to ∞ for the above equality yields

$$e^{-\frac{cz}{d_1}}\hat{u}^*_{1,z}(z) = -\int_z^{\infty} \frac{e^{-\frac{cy}{d_1}}}{d_1 T}\int_0^T (\beta_1(t)v_1^*(t,y)+\beta_2(t)v_2^*(t,y))u_1^*(t,y)\mathrm{d}t\mathrm{d}y,$$

indicating that $\hat{u}^*_{1,z}(z)<0$ for $z\in\mathbb{R}$. Furthermore, $\hat{u}_1^*(\infty)$ exists and $\hat{u}_1^*(\infty)<\hat{u}_1^*(-\infty)=S_H^0$. barbălat's lemma implies that $\hat{u}^*_{1,z}(z)\to 0$ as $z\to\infty$. Integrating two sides of (3.21) from $-\infty$ to ∞ on z leads to

$$\frac{1}{T}\int_{-\infty}^{\infty}\int_0^T(\beta_1(t)v_1^*(t,z)+\beta_2(t)v_2^*(t,z))u_1^*(t,z)\mathrm{d}t\mathrm{d}z = c(S_H^0-\hat{u}_1(\infty)) = c(S_H^0-S_H^\infty),$$

where $S_H^\infty := \hat{u}_1(\infty) < S_H^0$. Using the similar arguments to those in [40, Theorem 2.10] and [48, Theorem 2.9], we get $u_1^*(t,z)\to S_H^\infty$ uniformly for $t\in\mathbb{R}$, as $z\to+\infty$. In addition, $u_2^*(t,z)$ can be discussed similarly.

Finally, we discuss the properties of v_1^*. Since $\hat{v}_1$ satisfies

$$-d_1\hat{v}_{1,zz}(z)+c\hat{v}_{1,z}(z) = \frac{1}{T}\int_0^T(\beta_1(t)v_1^*(t,z)+\beta_2(t)v_2^*(t,z))u_1^*(t,z)\mathrm{d}t - \frac{1}{T}\int_0^T r_1(t)v_1^*(t,z)\mathrm{d}t. \quad (3.22)$$

An integrating of (3.22) on $\mathbb{R}$ leads to

$$\begin{aligned}&\frac{1}{T}\int_0^T\int_{-\infty}^{\infty} r_1(t)v_1^*(t,z)\mathrm{d}t\mathrm{d}z\\ &\quad=\frac{1}{T}\int_0^T\int_{-\infty}^{\infty}(\beta_1(t)v_1^*(t,z)+\beta_2(t)v_2^*(t,z))u_1^*(t,z)\mathrm{d}t\mathrm{d}z = c(S_H^0-S_H^\infty).\end{aligned}$$

By using the above arguments on the asymptotic behavior of $v_1^*(t,z)$ as $z\to-\infty$, it is obvious that

$$\lim_{z\to\pm\infty}\partial_z v_1^*(t,z)=0 \text{ uniformly for } t\in\mathbb{R}.$$

For any $t\in\mathbb{R}$, consider the following equation

$$c\bar{v}_{1,z}(z) = d_1\bar{v}_{1,zz}(z)+\frac{1}{T}\int_0^T r_1(t)v_1^*(t,z)\mathrm{d}t,\ \forall z\in\mathbb{R}. \quad (3.23)$$

Then the solution of (3.23) satisfies

$$\begin{aligned}\bar{v}_1(z) =& \frac{1}{cT}\int_{-\infty}^{z}\int_0^T r_1(t)v_1^*(t,y)\mathrm{d}t\mathrm{d}y\\ &+\frac{1}{cT}\int_z^{+\infty} e^{\frac{c(z-y)}{d_1}}\int_0^T r_1(t)v_1^*(t,y)\mathrm{d}t\mathrm{d}y.\end{aligned}$$

Based on (3.22) and L'Hôpital's rule, it follows that

$$\lim_{z\to-\infty}\bar{v}_1(z)=0,\ \lim_{z\to+\infty}\bar{v}_1(z)=\frac{1}{T}\int_0^T\int_{-\infty}^{\infty}(\beta_1(t)v_1^*(t,z)+\beta_2(t)v_2^*(t,z))u_1^*(t,z)\mathrm{d}t\mathrm{d}z = c(S_H^0-S_H^\infty)$$

and

$$\lim_{z\to\pm\infty}\bar{v}_{1,z}(z)=0.$$

Define a new function

$$\check{v}_1(z):=\hat{v}_1(z)+\bar{v}_1(z),\ \forall z\in\mathbb{R},$$

where $\hat{v}_1(z)=\frac{1}{T}\int\limits_0^T v_1^*(t,z)\mathrm{d}t$. On the basis of (3.22) and (3.23), $\check{v}_1(z)$ satisfies

$$-d_1\check{v}_{1,zz}(z)+c\check{v}_{1,z}(z)=\frac{1}{T}\int\limits_0^T(\beta_1(t)v_1^*(t,z)+\beta_2(t)v_2^*(t,z))u_1^*(t,z)\mathrm{d}t.$$

Multiplying two sides of the above equation by $e^{-\frac{c}{d_1 z}}$ and integrating from z to ∞, one has

$$\check{v}_{1,z}(z)=\frac{1}{d_2T}\int\limits_z^\infty e^{-\frac{c(z-y)}{d_2}}\int\limits_0^T(\beta_1(t)v_1^*(t,y)+\beta_2(t)v_2^*(t,y))u_1^*(t,y)\mathrm{d}t\mathrm{d}y.$$

Then, it is easy to see that $\check{v}_1(z)$ is non-decreasing in $\mathbb{R}$ and $\lim_{z\to\infty}\check{v}_1(z)=S_H^0-S_H^\infty$, indicating that $\check{v}_1(z)\leqslant S_H^0-S_H^\infty$ for all $z\in\mathbb{R}$. In light of the definition of $\check{v}_1(z)$ and $\bar{v}_1(z)$, we conclude that $\hat{v}_1(z)\leqslant\check{v}_1(z)\leqslant S_H^0-S_H^\infty$ on $\mathbb{R}$. That is, $0\leqslant\frac{1}{T}\int\limits_0^T v_1^*(t,z)\mathrm{d}t\leqslant S_H^0-S_H^\infty$ for any $z\in\mathbb{R}$. In addition, $v_2^*(t,z)$ has the similar conclusion as $v_1^*(t,z)$. The proof is completed. □

Remark 3.10. The existence of critical periodic traveling waves is complex, which will be investigated in our future work.

4. Non-existence of periodic traveling wave solutions

In the section, we establish the non-existence of the time-periodic traveling wave solutions of model (1.1) for these cases as below: $R_0\leqslant 1$ or $R_0>1$ and $0<c<c^*$.

4.1. Case 1: $R_0>1$ and $0<c<c^*$

With the aim of it, we need to study the following lemma. Firstly, for some $c\in(0,c^*)$, fix $c_0\in(c,c^*)$. Let $\upsilon_{c_0}=\frac{c_0}{2}$, $d_1=d_2=1$ and ϵ be small enough, consider the following system

$$\begin{cases}\frac{\mathrm{d}\tilde{\eta}_1}{\mathrm{d}t}(t)=\upsilon_{c_0}^2\tilde{\eta}_1(t)+(\beta_1(t)\tilde{\eta}_1(t)+\beta_2(t)\tilde{\eta}_2(t))S_H^0(1-\epsilon)-r_1(t)\tilde{\eta}_1(t),\\ \frac{\mathrm{d}\tilde{\eta}_2}{\mathrm{d}t}(t)=\upsilon_{c_0}^2\tilde{\eta}_2(t)+\beta_3(t)\tilde{\eta}_1(t)S_V^0(1-\epsilon)-r_2(t)\tilde{\eta}_2(t).\end{cases}\tag{4.1}$$

Denote the solution map of system (4.1) by $(\eta_1^\epsilon,\eta_2^\epsilon)_t(\tilde{\eta}_{10},\tilde{\eta}_{20}):=(\eta_1^\epsilon,\eta_2^\epsilon)(t;\tilde{\eta}_{10},\tilde{\eta}_{20})$, where $(\eta_1^\epsilon,\eta_2^\epsilon)(t;\tilde{\eta}_{10},\tilde{\eta}_{20})$ is the solution of system (4.1) with initial value $(\tilde{\eta}_{10},\tilde{\eta}_{20})\in\mathbb{R}_+^2$. In addition, let $\lambda_{c_0,\epsilon}=\frac{\ln\rho^\epsilon(\upsilon_{c_0})}{T}$, where $\rho^\epsilon(\upsilon_{c_0})$ is the spectral radius of the Poincaré map $B_{c_0,\epsilon}:=(\eta_1^\epsilon,\eta_2^\epsilon)_T$ of system (4.1). By using the similar arguments as those in [45], (η_1^*,η_2^*) is a eigenvalue vector of $B_{c_0,\epsilon}$ associated with the corresponding principal eigenvalue $\rho^\epsilon(\upsilon_{c_0})$.

Based on the above arguments, we can obtain the following conclusion.

Lemma 4.1. *Suppose that $\upsilon_{c_0} = \frac{c_0}{2}$, $L > 0$ is large enough and $\epsilon > 0$ is small enough, consider the principal eigenvalue problem of the cooperative elliptic system as below*

$$\begin{cases} \frac{\mathrm{d}\bar{\eta}_1}{\mathrm{d}t}(t) - \upsilon_{c_0}^2\bar{\eta}_1(t) - (\beta_1(t)\bar{\eta}_1(t) + \beta_2(t)\bar{\eta}_2(t))S_H^0(1-\epsilon) + r_1(t)\bar{\eta}_1(t) = -\lambda_{c_0,\epsilon}\bar{\eta}_1, \\ \frac{\mathrm{d}\bar{\eta}_2}{\mathrm{d}t}(t) - \upsilon_{c_0}^2\bar{\eta}_2(t) - \beta_3(t)S_V^0\bar{\eta}_1(t)(1-\epsilon) + r_2(t)\bar{\eta}_2(t) = -\lambda_{c_0,\epsilon}\bar{\eta}_2, \end{cases} \tag{4.2}$$

Then system (4.2) generates a positive time-periodic solution with the period $T > 0$.

Proof. Consider the following system

$$\begin{cases} \frac{\mathrm{d}\tilde{\eta}_1}{\mathrm{d}t}(t) = (\upsilon_{c_0}^2 - \lambda)\tilde{\eta}_1(t) + (\beta_1(t)\tilde{\eta}_1(t) + \beta_2(t)\tilde{\eta}_2(t))S_H^0(1-\epsilon) - r_1(t)\tilde{\eta}_1(t), \\ \frac{\mathrm{d}\tilde{\eta}_2}{\mathrm{d}t}(t) = (\upsilon_{c_0}^2 - \lambda)\tilde{\eta}_2(t) + \beta_3(t)\tilde{\eta}_1(t)S_V^0(1-\epsilon) - r_2(t)\tilde{\eta}_2(t). \end{cases} \tag{4.3}$$

Define the semiflow of system (4.3) by $(\tilde{\eta}_1, \tilde{\eta}_2)_t(\tilde{\eta}_{10}, \tilde{\eta}_{20}) := (\tilde{\eta}_1, \tilde{\eta}_2)(t; \tilde{\eta}_{10}, \tilde{\eta}_{20})$, where $(\tilde{\eta}_1, \tilde{\eta}_2)(t; \tilde{\eta}_{10}, \tilde{\eta}_{20})$ is the solution of system (4.3) with initial value $(\tilde{\eta}_{10}, \tilde{\eta}_{20}) \in \mathbb{R}_+^2$. In addition, denote the Poincaré map of system (4.3) by $\mathcal{P}_{c_0,\epsilon} := (\tilde{\eta}_1, \tilde{\eta}_2)_T$. It further follows that

$$\mathcal{P}_{c_0,\epsilon}(\kappa_1, \kappa_2) = (\tilde{\eta}_1, \tilde{\eta}_2)_T(\kappa_1, \kappa_2) = (\tilde{\eta}_1, \tilde{\eta}_2)(T; \kappa_1, \kappa_2) = e^{-\lambda T}(\eta_1^\epsilon, \eta_2^\epsilon)(T; \kappa_1, \kappa_2),$$

where (κ_1, κ_2) is the initial value of system (4.3) and $(\eta_1^\epsilon, \eta_2^\epsilon)$ is the solution of system (4.1). Consequently, one has

$$\mathcal{P}_{c_0,\epsilon}(\eta_1^*, \eta_2^*) = e^{-\lambda T}(B_{c_0,\epsilon}(\eta_1^*, \eta_2^*)) = e^{-\lambda T}\rho^\epsilon(\nu_{c_0})(\eta_1^*, \eta_2^*),$$

where (η_1^*, η_2^*) has been defined in (4.1). If $\lambda = \lambda_{c_0,\epsilon} = \frac{\ln \rho^\epsilon(\upsilon_{c_0})}{T}$, then (η_1^*, η_2^*) is a fixed point of the Poincaré map $\mathcal{P}_{c_0,\epsilon}$. As a consequence, $(\tilde{\eta}_1, \tilde{\eta}_2)_t := (\tilde{\eta}_1, \tilde{\eta}_2)(t; \eta_1^*, \eta_2^*)$ is a positive time-periodic solution of system (4.3) with $\lambda = \lambda_{c_0,\epsilon}$. This completes the proof. □

Theorem 4.2. *Assume that $R_0 > 1$, $0 < c < c^*$ and $d_1 = d_2 = 1$. Then system (1.1) admits no nontrivial T-periodic traveling waves (u_1, u_2, v_1, v_2) satisfying (3.2) and (3.3).*

Proof. Suppose, by a contradiction way, that there exists such a solution (u_1, u_2, v_1, v_2) satisfying (3.2) and (3.3) for some $c < c^*$. Firstly, according to $\lim_{t\to-\infty} u_1(t, z) = S_H^0$, $\forall t \in \mathbb{R}$, we can choose a $M_\epsilon > 0$ large enough and a $\epsilon > 0$ sufficiently small such that

$$S_H^0 - \epsilon \leqslant u_1(t, z) \leqslant S_H^0 + \epsilon, \ \forall z < -M_\epsilon \tag{4.4}$$

uniformly for $t \in \mathbb{R}$. Let $y_1, y_2 < -M_\epsilon$, we take into account the following system

$$\begin{cases} (\partial_t + c_0\partial_z - \Delta + r_1(t))\, w_1(t, z) = S_H^0(1-\epsilon)(\beta_1(t)w_1(t, z) + \beta_2(t)w_2(t, z)), \\ (\partial_t + c_0\partial_z - \Delta + r_2(t))\, w_2(t, z) = S_V^0(1-\epsilon)\beta_3(t)w_1(t, z), \ t \geqslant 0, \ z \in (y_1, y_2), \\ w_1(t, y_1) = w_1(t, y_2) = 0, \ w_2(t, y_1) = w_2(t, y_2) = 0, \ t \geqslant 0. \end{cases} \tag{4.5}$$

Furthermore, one has

$$c < c_\epsilon^* := \inf_{\mu > 0} \frac{\ln r^\epsilon(\mu)}{T\mu} \leqslant \frac{\ln r^\epsilon(\upsilon_{c_0})}{T\upsilon_{c_0}} = \frac{\lambda_{c_0,\epsilon}}{\upsilon_{c_0}},$$

expressing that $c\upsilon_{c_0} < \lambda_{c_0,\epsilon}$, where $\lambda_{c_0,\epsilon}$ has been defined in Lemma 4.1, $r^\epsilon(\mu)$ and c_ϵ^* have been defined in (2.5) and $\upsilon_{c_0} = \frac{c_0}{2}$.

Secondly, denote $\binom{\bar{w}_1}{\bar{w}_2}(t, z) := e^{\lambda^* t}e^{\upsilon_{c_0} z}p(z)\binom{k_1(t)}{k_2(t)}$, where $\lambda^* \in (0, \lambda_{c_0,\epsilon} - c_0\upsilon_{c_0})$ is a constant, $(k_1(t), k_2(t))$ is a solution of system (4.2) and $p(z)$ is the eigenfunction of the principal eigenvalue problem as below

$$\begin{cases} -\partial_{zz}p(z) = \rho_L p(z), \ z \in (y_1, y_2), \\ p(z) > 0, \ z \in (y_1, y_2), \\ p(y_1) = p(y_2) = 0, \end{cases}$$

where $L := |y_1 - y_2|$. Furthermore, one has $\lim_{L\to\infty} \rho_L = 0$, indicating that $\lambda^* + c_0 \upsilon_{c_0} - \lambda_{c_0,\epsilon} + \rho_L \leqslant 0$. According to Lemma 4.1 and the above arguments, plugging $\bar{w}_1(t,z)$ into the first equation of system (4.5) becomes to

$$\begin{aligned}&(\partial_t + c_0\partial_z - \Delta + r_1(t))\,\bar{w}_1(t,z) - S_H^0(1-\epsilon_0)\big(\beta_1(t)\bar{w}_1(t,z) + \beta_2(t)\bar{w}_2(t,z)\big)\\ &\quad= \lambda^*\bar{w}_1(t,z) + e^{\lambda^* t}e^{\upsilon_{c_0}x}p(z)k_1'(t) + c_0\big(\upsilon_{c_0}e^{\upsilon_{c_0}z}p(z) + e^{\upsilon_{c_0}z}p'(z)\big)e^{\lambda^* t}k_1(t) - \big(\upsilon_{c_0}^2 e^{\upsilon_{c_0}z}p(z)\\ &\qquad + 2\upsilon_{c_0}e^{\upsilon_{c_0}z}p'(z) + e^{\upsilon_{c_0}z}p''(z)\big)e^{\lambda^* t}k_1(t) - S_H^0(1-\epsilon_0)\big(\beta_1(t)\bar{w}_1(t,z) + \beta_2(t)\bar{w}_2(t,z)\big) + r_1(t)\bar{w}_1(t,z)\\ &\quad= \lambda^*\bar{w}_1(t,z) + c_0\upsilon_{c_0}\bar{w}_1(t,z) + (c_0 - 2\upsilon_{c_0})p'(z)e^{\upsilon_{c_0}z}k_1(t)e^{\lambda^* t} - p''(z)e^{\upsilon_{c_0}z}k_1(t)e^{\lambda^* t} + p(z)e^{\lambda^* t}e^{\upsilon_{c_0}z}\\ &\quad\big(k_1'(t) - \upsilon_{c_0}^2 k_1(t) + r_1(t)k_1(t) - S_H^0(1-\epsilon_0)\big(\beta_1(t)k_1(t) + \beta_2(t)k_2(t)\big)\big)\\ &\quad= \big(\lambda^* + c_0\upsilon_{c_0} - \lambda_{c_0,\epsilon} + \rho_L\big)\bar{w}_1(t,z) \leqslant 0.\end{aligned} \tag{4.6}$$

Thirdly, let $\delta > 0$ be small enough such that $v_1(0,z) \geqslant \delta\bar{w}_1(0,z),\ \forall z \in (y_1,y_2)$. Consider functions $u_i(t, z + (c-c_0)t)$ and $v_i(t, z+(c-c_0)t)$ for any $t \in \mathbb{R}$ and $z \in (y_1,y_2)$. Denote $\hat{v}_i(t,z) := v_i(t, z + (c - c_0)t)(i = 1,2)$, which satisfies

$$\partial_t\hat{v}_1(t,z) = \Delta\hat{v}_1(t,z) - c_0\partial_z\hat{v}_1(t,z) + u_1(t, z+(c-c_0)t)\big(\beta_1(t)\hat{v}_1(t,z) + \beta_2(t)\hat{v}_2(t,z)\big) - r_1(t)\hat{v}_1(t,z).$$

In view of $c - c_0 < 0$, $z \in (y_1,y_2)$ and $y_1 < y_2 < -M_\epsilon$, one has $z + (c-c_0)t < -M_\epsilon,\ \forall t \geqslant 0,\ z \in [y_1,y_2]$. Due to (4.4), $\hat{v}_1(t,z)$ satisfies

$$\partial_t\hat{v}_1(t,z) \geqslant \Delta\hat{v}_1(t,z) - c_0\partial_z\hat{v}_1(t,z) + S_H^0(1-\epsilon)\big(\beta_1(t)\hat{v}_1(t,z) + \beta_2(t)\hat{v}_2(t,z)\big) - r_1(t)\hat{v}_1(t,z)$$

for any $t \geqslant 0$ and $z \in [y_1,y_2]$. Since there are

$$\begin{aligned}&\delta\bar{w}_1(0,z) \leqslant \hat{v}_1(0,z) \text{ for } z \in (y_1,y_2) \text{ and}\\ &\bar{w}_1(t,z) = 0 \leqslant \hat{v}_1(t,z) \text{ for } t \geqslant 0 \text{ and } z = y_1 \text{ or } y_2,\end{aligned}$$

we infer from the parabolic maximum principle that

$$\bar{w}_1(t,z) = e^{\lambda^* t}e^{\upsilon_{c_0}z}p(z)k_1(t) \leqslant v_1(t, z+(c-c_0)t),\ \forall t \geqslant 0,\ z \in (y_1,y_2).$$

Due to $\lambda^* > 0$, we obtain $v_1(t, z+(c-c_0)t) \to \infty$ as $t \to \infty$, which leads to a contradiction. On the same way, v_2 is proved similarly and thus we omit it. The proof is completed. □

4.2. Case 2: $R_0 < 1$

Theorem 4.3. *Assume that $R_0 < 1$. Then for any $c \geqslant 0$, system (3.2) admits no nontrivial T-periodic solution (u_1,u_2,v_1,v_2) satisfying (3.3).*

Proof. Assume that there exists a nontrivial T-periodic solution (u_1,u_2,v_1,v_2) of system (3.2)–(3.3) by a contradiction way. Let $\bar{v}_i(t) := \int_{-\infty}^{+\infty} v_i(t,z)\mathrm{d}z$ on $\mathbb{R}$ for $i = 1,2$. Obviously, $\bar{v}_i(t) = \bar{v}_i(t+T),\ \forall t \in \mathbb{R}$ for $i = 1,2$. In light of inequality (3.19), one gets that $\bar{v}_i(t)$ is bounded on $[0,T)$. In addition, for any given $t \in [0,T)$, there exists a $\epsilon_0(t)$ depending upon t such that

$$\bar{v}_i(t) > \epsilon_0(t). \tag{4.7}$$

Furthermore, it follows from $u_i(t,z) \leqslant S_H^0(S_V^0)(i = 1,2)$ that

$$\begin{cases}\partial_t v_1(t,z) \leqslant d_1\partial_{zz}v_1(t,z) - c\partial_z v_1(t,z) + \big(\beta_1(t)S_H^0 - r_1(t)\big)\,v_1(t,z) + \beta_2(t)S_H^0 v_2^*(t,z),\\ \partial_t v_2(t,z) \leqslant d_2\partial_{zz}v_2(t,z) - c\partial_z v_2(t,z) + \beta_3(t)S_V^0 v_1(t,z) - r_2(t)v_2(t,z).\end{cases}$$

Integrating both two side of the above equations from $-\infty$ to ∞, we obtain

$$\begin{cases} \frac{\mathrm{d}\bar{v}_1}{\mathrm{d}t} \leqslant \left(\beta_1(t)S_H^0 - r_1(t)\right)\bar{v}_1(t) + \beta_2(t)S_H^0\bar{v}_2(t), \\ \frac{\mathrm{d}\bar{v}_2}{\mathrm{d}t} \leqslant \beta_3(t)S_V^0\bar{v}_1(t) - r_2(t)\bar{v}_2. \end{cases}$$

Then by using the parabolic maximum principle, one has

$$(\bar{v}_1(t), \bar{v}_2(t)) \leqslant (\tilde{v}_1(t), \tilde{v}_2(t)),\ t \geqslant 0,$$

where $(\tilde{v}_1(t), \tilde{v}_2(t))$ is the solution of the system as below

$$\begin{cases} \frac{\mathrm{d}\tilde{v}_1}{\mathrm{d}t} = \left(\beta_1(t)S_H^0 - r_1(t)\right)\tilde{v}_1(t) + \beta_2(t)S_H^0\tilde{v}_2(t), \\ \frac{\mathrm{d}\tilde{v}_2}{\mathrm{d}t} = \beta_3(t)S_V^0\tilde{v}_1(t) - r_2(t)\tilde{v}_2(t), \\ \tilde{v}_1(0) = \bar{v}_1(0),\ \tilde{v}_2(0) = \bar{v}_2(0). \end{cases}$$

Due to [50, Theorem 2.1] associated with $R_0 < 1$, one has $\lim_{t\to+\infty}\tilde{v}_i(t) = 0 (i = 1, 2)$, implying that

$$\lim_{t\to+\infty}\bar{v}_i(t) = 0,\ i = 1, 2,$$

which leads to a contradiction with (4.7). This completes the proof. □

4.3. Case 3: $R_0 = 1$

Theorem 4.4. *Assume that $R_0 = 1$. Then for any $c \geqslant 0$, system (3.2) admits no nontrivial T-periodic solution (u_1, u_2, v_1, v_2) satisfying (3.3).*

Proof. Assume that there exists a nontrivial T-periodic solution (u_1, u_2, v_1, v_2) of system (3.2)–(3.3) by a contradiction way. Let $\bar{v}_i(t) := \int_{-\infty}^{+\infty} v_i(t, z)\mathrm{d}z$ on $\mathbb{R}$ for $i = 1, 2$. Due to (3.19), we can get that $\bar{v}_i(t)$ bounded on $[0, T]$. In addition, $\bar{v}_i(t)$ satisfies

$$\begin{aligned} \frac{\mathrm{d}\bar{v}_1}{\mathrm{d}t} &= S_H^0(\beta_1(t)\bar{v}_1(t) + \beta_2(t)\bar{v}_2(t)) - r_1(t)\bar{v}_1(t) + f_1(t), \\ \frac{\mathrm{d}\bar{v}_2}{\mathrm{d}t} &= S_V^0\beta_3(t)\bar{v}_1(t) - r_2(t)\bar{v}_2 + f_2(t), \end{aligned} \tag{4.8}$$

where $f_1(t) = \beta_1(t)\int_{-\infty}^{+\infty}(u_1(t, z) - S_H^0)v_1(t, z)\mathrm{d}z + \beta_2(t)\int_{-\infty}^{+\infty}(u_1(t, z) - S_H^0)v_2(t, z)\mathrm{d}z$ and $f_2(t) = \beta_3(t) \int_{-\infty}^{+\infty}(u_2(t, z) - S_V^0)v_1(t, z)\mathrm{d}z$ and $f(t) = (f_1(t), f_2(t))^T$. System (4.8) owns a positive T-periodic solution $\bar{v}(t) := (\bar{v}_1(t), \bar{v}_2(t))^T$. Thus, we get

$$\bar{v}(t) = U(t, 0)\bar{v}(0) + \int_0^t U(t, t-s)\big(\mathcal{F}(t-s)\bar{v}(t-s) + f(t)\big)\mathrm{d}s,\ \forall t \geqslant 0, \tag{4.9}$$

where $U(t, s)$ and $\mathcal{F}(t)$ have been defined in Sect. 2. In addition, it is not difficult to show that $u_1(t, z) \leqslant S_H^0$ for $(t, z) \in \mathbb{R}^2$. In fact, suppose that there exists (t_0, z_0) such that $\max_{(t,z)\in\mathbb{R}^2} u_1(t, z) = u_1(t_0, z_0) > S_H^0$. Thus,

$$\begin{aligned} 0 =&\partial_t u_1(t, z)\mid_{(t_0,z_0)} \\ &= d_1\partial_{zz}u_1(t, z)\mid_{(t_0,z_0)} - c\partial_z u_1(t, z)\mid_{(t_0,z_0)} - u_1(t_0, z_0)(\beta_1(t_0)v_1(t_0, z_0) + \beta_2(t_0)v_2(t_0, z_0)) < 0, \end{aligned}$$

which is a contradiction. Furthermore, $u_2(t, z)$ can be proved similarly. As a consequence, it has

$$f_i(t) \leqslant 0,\ \forall t \in [0, T]. \tag{4.10}$$

Consider the following problem:

$$\begin{cases} \frac{\mathrm{d}\tilde{v}_1}{\mathrm{d}t} = \left(S_H^0\beta_1(t) - r_1(t)\right)\tilde{v}_1(t) + \beta_2(t)S_H^0\tilde{v}_2(t), \\ \frac{\mathrm{d}\tilde{v}_2}{\mathrm{d}t} = S_V^0\beta_3(t)\tilde{v}_1(t) - r_2(t)\tilde{v}_2(t), \\ \tilde{v}_1(0) = \bar{v}_1(0),\ \tilde{v}_2(0) = \bar{v}_2(0). \end{cases}$$

Due to [50, Theorem 2.1] associated with $R_0 = 1$, there exists a positive T-periodic solution $\tilde{v}(t) := (\tilde{v}_1(t), \tilde{v}_2(t))^T$ satisfying the above problem. A straightforward computation leads to

$$\tilde{v}(t) = U(t,0)\tilde{v}(0) + \int_0^t U(t,t-s)\mathcal{F}(t-s)\tilde{v}(t-s)\mathrm{d}s,\ \forall t \geqslant 0. \tag{4.11}$$

It further follows from the parabolic maximum principle together with (4.10) that

$$\tilde{v}(t) \geqslant \bar{v}(t), \forall t \in [0, +\infty). \tag{4.12}$$

However, due to the periodicity of $\bar{v}(t)$ and $\tilde{v}(t)$, one has $\tilde{v}(T) = \tilde{v}(0) = \bar{v}(0) = \bar{v}(T)$, that is,

$$\begin{aligned} &U(T,0)\bar{v}(0) + \int_0^T U(T,T-s)\big(\mathcal{F}(T-s)\bar{v}(T-s) + f(T-s)\big)\mathrm{d}s \\ &= U(T,0)\tilde{v}(0) + \int_0^T U(T,T-s)\mathcal{F}(T-s)\tilde{v}(T-s)\mathrm{d}s. \end{aligned}$$

In view of (4.10), one has

$$0 > \int_0^T U(T,T-s)f(T-s)\mathrm{d}s = \int_0^T U(T,T-s)\mathcal{F}(T-s)\big(\tilde{v}(T-s) - \bar{v}(T-s)\big)\mathrm{d}s,$$

implying that there exists a $t_0 \in [0, T)$ satisfying

$$\tilde{v}(t_0) < \bar{v}(t_0).$$

As a consequence, it contradicts with (4.12). It completes the proof. □

Author contributions This paper was written by myself.

Funding Researcher was supported by National Natural Science Foundation of China (12161052) and Natural Science Foundation of Gansu, China (21JR7RA240).

Data Availability Statement The authors declare that the data are available on request from the authors.

Declarations

Conflict of interest No potential conflict of interest was reported by the authors.

Ethical statement This paper does not contain any studies with human participants or animals performed by any of the authors. I also certify that this paper is original and has not been published and will not be submitted elsewhere for publication.

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(Received: July 9, 2023; revised: October 29, 2023; accepted: December 20, 2023)