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The essential spectrum of the volume integral operator in electromagnetic scattering by a homogeneous obstacle with Lipschitz boundary and regularization

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Abstract. Scattering of time-harmonic electromagnetic waves by penetrable obstacles admits an equivalent formulation in terms of a strongly singular volume integral equation (VIE). For the case of piecewise-constant physical parameters and Lipschitz interfaces, we give a characterization of the essential spectrum of the magnetic and the electromagnetic operators which describe the VIE, based on the spectral properties of the normal derivative of the Laplace single-layer potential $\frac{1}{2}\mathbb{I} + K_0'$. This extends the results obtained in a previous paper (Costabel et al., 2012) which were available only for smooth domains. The results on the spectrum will then be used to derive necessary and sufficient conditions to ensure that the diffraction problem is well-posed in the Fredholm sense. Also, we use the employed spectral methods to show that the construction of regularizers for the operators which appear in the VIE needs only the determination of a regularizer for the operator $\lambda\mathbb{I} + K_0'$.

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1. Introduction

We deal with the electromagnetic scattering of time-harmonic electromagnetic waves by a bounded penetrable medium of compact support. This scattering phenomenon is governed by Maxwell's system. As an efficient tool to treat this transmission problem in numerical modeling and simulation, one could convert it into an equivalent volume integral equation defined only on the support of the scattering body. This equation sometimes simply called "volume integral equation" (VIE) of electromagnetic scattering has been widely used by physicists for numerical calculations; see, for example [1,2,10,17]. However, to the author's best knowledge, very few publications can be found in the literature that address the mathematical properties, such as the unique solvability of the integral equations under consideration, see, for example, previous studies [5–7,9,12,13].

This VIE involves three strongly singular volume integral operators: the dielectric, the magnetic, and the electromagnetic operators. For scatterers with Lipschitz boundary and in the situation where the physical parameters are piecewise constant, the essential spectrum of the dielectric operator has been obtained in the previous study [6]. However, in this last paper, the essential spectrum of the magnetic and electromagnetic operators has been determined only for smooth domains. The first objective of our study is to extend those results to the Lipschitz case. We will show that this spectrum depends on the spectral properties of the normal derivative of the single-layer potential for the Laplace equation $\frac{1}{2}\mathbb{I} + K_0'$ on the zero-mean elements of $H^{-\frac{1}{2}}(\Gamma)$ (where K_0' is no longer compact when the boundary of the scatterer Γ is only Lipschitz).

The knowledge of such a spectrum plays an important role in the development of numerical methods for solving the integral equations (choice of an iterative algorithm, convergence rate, construction of a proper preconditioner, etc). This research is motivated by various applications in physics and engineering such as electromagnetic resonances, metamaterials (negative refraction index), etc.

In [17], it was shown by numerical computations that the eigenvalues of the coefficient matrix arising from a discretization of the volume integral equation of electromagnetic scattering for a compact spherical scatterer consist of a line segment plus some isolated point. In [10], a mathematical analysis of the essential spectrum (in the dielectric case) based on the Mikhlin's theory [16] of singular integral operators (construction of symbol of the volume integral operator) was given under the assumption of Hölder continuity of the physical parameters: the essential spectrum is given by the value 1 union an essential continuous part. However, for the case of discontinuous parameters and under numerical observations, one may affirm that the line segment connecting 1 and the relative permittivity parameter is contained in the essential spectrum which is not true as explained in the previous paper [6]. In fact, for a smooth boundary, the essential spectrum consists of the endpoints and the midpoint of that segment. Furthermore, it was demonstrated in [9] that the endpoints of the segment are isolated points of the essential spectrum for the case of a Lipschitz domain.

In [12,13], and under some sufficient hypotheses on the physical parameters (in particular the positivity or the negativity criterion), it has been demonstrated, using the Fredholm alternative, that the VIE is uniquely solvable. However, through the analysis of the essential spectrum of the integral operators, we will give in this paper the necessary and sufficient conditions on the physical coefficients to ensure the unique solvability of the problem, without any additional assumption on the coefficients' sign. These obtained conditions, if they are satisfied, guarantee also the existence of an operator, which applied to the VIE, transforms it into the form "identity plus compact". Such an operator will be called a "regularizer" for the one describing the VIE. Within this context, our second goal is then to find an explicit representation of that regularizer in terms of some integral operators. This is of great interest in the numerical techniques for solving the VIE as explained in [19]. In this paper, we will give a method to get a formula for this regularizer when the boundary of the domain is Lipschitz, by only knowing a regularizer for $\lambda\mathbb{I}+K_0'$. This approach shares some common points with the one used in the spectral analysis of the VIE. Moreover, this argument will also work in the case when the boundary is smooth because in this situation K_0' is compact. However, as far as we know, there are no papers where the construction of a regularizer for $\lambda\mathbb{I}+K_0'$ is discussed in the Lipschitz situation. This will be the subject of future work.

Before going on, we would like to point out that in the two-dimensional configuration, some results on the spectrum of the volume integral operator for the electromagnetic transmission problem have been established in a previous study [7].

The outline of this paper is as follows. First, we carefully present in Sect. 2 the electromagnetism transmission problem and give its volume integral formulation. Next, we collect in Sect. 3 some preliminaries and notations that will be used in the sequel. Then in Sect. 4, we treat the magnetic and the electromagnetic case where the spectrum of the volume integral operator is derived for a Lipschitz domain. The obtained results are then used to deduce necessary and sufficient conditions on the physical parameters for the well-posedness in the Fredholm sense of the general electromagnetic problem. Finally, the last section is devoted to the construction of regularizers in the dielectric, magnetic, and electromagnetic configuration.

2. Electromagnetic scattering by a penetrable, homogeneous, isotropic object and its volume integral equation formulation

Throughout this work, we denote by $\Omega \subset \mathbb{R}^3$ a bounded open set with a Lipschitz boundary Γ. Moreover, we use the standard notation for Sobolev spaces H^m, and we denote spaces of vector-valued functions by boldface letters.

We consider the diffraction of a time-harmonic electromagnetic wave $(\mathbf{E}^i, \mathbf{H}^i)$ by an obstacle Ω surrounded by a homogeneous medium with constant real positive electric permittivity ε_0, magnetic permeability μ_0, and vanishing electric conductivity σ_0. The incoming wave $(\mathbf{E}^i, \mathbf{H}^i)$ satisfies the free-space Maxwell's equations:

$$\mathrm{curl}\mathbf{E}^i - ik\mathbf{H}^i = 0, \mathrm{curl}\mathbf{H}^i + ik\mathbf{E}^i = \mathbf{J}, \text{ in } \mathbb{R}^3, \tag{1}$$

where $k = w\sqrt{\varepsilon_0\mu_0}$ is the wave number, $w > 0$ the frequency, and $\mathbf{J}$ is the current density assumed to be of compact support that is disjoint from Ω, and divergence-free.

In this paper, we consider the case where the dielectric permittivity ε and the magnetic permeability μ are piecewise-constant functions:

$$\varepsilon = \varepsilon_0,\ \mu = \mu_0 \text{ in } \mathbb{R}^3\backslash\Omega;\ \varepsilon = \varepsilon_r\varepsilon_0,\ \mu = \mu_r\mu_0 \text{ in } \Omega. \tag{2}$$

Here, the relative permittivity ε_r and permeability μ_r can be arbitrarily complex number.
We then define the total wave $(\mathbf{E}, \mathbf{H})$ by:

$$\mathbf{E} = \mathbf{E}^i + \mathbf{E}^s, \mathbf{H} = \mathbf{H}^i + \mathbf{H}^s, \tag{3}$$

where $(\mathbf{E}^s, \mathbf{H}^s)$ is the scattered field satisfying the Silver-Müller radiation condition:

$$\lim_{r\to 0}(\mathbf{H}^s \times x - r\mathbf{E}^s) = 0 \tag{4}$$

uniformly for all direction $\hat{x} = x/|x|$, where $r = |x|$.

We require that $(\mathbf{E}, \mathbf{H})$ satisfy the following Maxwell's equations:

$$\mathrm{curl}\mathbf{E} - ik\mu_r\mathbf{H} = 0,\ \mathrm{curl}\mathbf{H} + ik\varepsilon_r\mathbf{E} = \mathbf{J}, \text{ in } \mathbb{R}^3. \tag{5}$$

In addition, we need the following boundary transmission conditions:

$$[\mathbf{n}\times\mathbf{E}]_\Gamma = 0,\ [\mathbf{n}\cdot\mu_r\mathbf{H}]_\Gamma = 0,\ [\mathbf{n}\times\mathbf{H}]_\Gamma = 0,\ [\mathbf{n}\cdot\varepsilon_r\mathbf{E}]_\Gamma = 0, \tag{6}$$

where $\mathbf{n}$ is the unit outward normal to the boundary Γ and the brackets $[\cdot]_\Gamma$ denotes the jump across Γ.

Let

$$\mathbb{H}(\mathrm{curl}, \Omega) = \{\mathbf{u} \in \mathbb{L}^2(\Omega); \mathrm{curl}\mathbf{u} \in \mathbb{L}^2(\Omega)\}.$$

Following the idea developed in [12, Section 2], we will show that the scattering problem (1) to (6) is equivalent to the following volume integral equation (VIE):

$$\text{Find } \mathbf{u} \text{ in } \mathbb{H}(\mathrm{curl}, \Omega) \text{ such that } \mathbf{u}(x) - \eta A_k\mathbf{u}(x) - \nu B_k\mathbf{u}(x) = \mathbf{u}^i(x), \tag{7}$$

where $\mathbf{u}^i$ is a given data, $\eta = 1 - \varepsilon_r$ is the electric contrast, $\nu = 1 - \frac{1}{\mu_r}$ is the magnetic contrast, and where the integral operators A_k and B_k are given, for $x \in \Omega$ by:

$$A_k\mathbf{u}(x) = (-\nabla\mathrm{div} - k^2)\int_\Omega g_k(x-y)\,\mathbf{u}(y)\,\mathrm{d}y \text{ (the dielectric operator)}, \tag{8}$$

$$B_k\mathbf{u}(x) = \mathrm{curl}\int_\Omega g_k(x-y)\,\mathrm{curl}\mathbf{u}(y)\,\mathrm{d}y \text{ (the magnetic operator)}. \tag{9}$$

Here, g_k is the fundamental solution of the Helmholtz equation $\Delta u + k^2 u = 0$:

$$g_k(x-y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad x \neq y. \tag{10}$$

We start by taking the electric field $\mathbf{E}$ as the unknown. We note that the argument works also for the magnetic field $\mathbf{H}$ because $\mathbf{E}$ and $\mathbf{H}$ play symmetrical roles by interchanging $-\mu_r$ and ε_r in (5).

Eliminating $\mathbf{H}$ from the system (5) gives:

$$\mathrm{curl}\left(\frac{1}{\mu_r}\mathrm{curl}\mathbf{E}\right) - k^2\varepsilon_r\mathbf{E} = ik\mathbf{J} \text{ in } \mathbb{R}^3.$$

Doing the same operation to the field $\mathbf{E}^i$ with the system (1) leads to:

$$\text{curlcurl}\mathbf{E}^i - k^2\mathbf{E}^i = ik\mathbf{J} \text{ in } \mathbb{R}^3.$$

By taking the difference between the two above equations, we find that the scattered field $\mathbf{E}^s$ is a solution of the following equation:

$$\text{curl}\left(\frac{1}{\mu_r}\text{curl}\mathbf{E}^s\right) - k^2\varepsilon_r\mathbf{E}^s = \text{curl}\left(\left(-\frac{1}{\mu_r}+1\right)\text{curl}\mathbf{E}^i\right) + k^2(\varepsilon_r - 1)\mathbf{E}^i \text{ in } \mathbb{R}^3.$$

If we use the expressions $\eta = 1 - \varepsilon_r$ and $\nu = 1 - \frac{1}{\mu_r}$, the last formula becomes:

$$\text{curl}\left(\frac{1}{\mu_r}\text{curl}\mathbf{E}^s\right) - k^2\varepsilon_r\mathbf{E}^s = \text{curl}(\nu\text{curl}\mathbf{E}^i) - k^2\eta\mathbf{E}^i v \text{ in } \mathbb{R}^3. \tag{11}$$

Moreover, consider the following boundary transmission conditions and the Silver-Müller condition that are satisfied by $\mathbf{E}^s$:

$$[\mathbf{n}\cdot\mathbf{E}^s]_\Gamma = 0, [\mathbf{n}\times\mathbf{E}^s]_\Gamma = 0, [\mathbf{n}\cdot\text{curl}\mathbf{E}^s]_\Gamma = 0, \left[\mathbf{n}\times\frac{1}{\mu_r}\text{curl}\mathbf{E}^s\right]_\Gamma = 0 \tag{12}$$

$$\lim_{r\to 0} r(\text{curl}\mathbf{E}^s \times \hat{x} - ik\mathbf{E}^s) = 0$$

$$\text{uniformly for all direction } \hat{x} = \frac{x}{|x|}, \text{ where } r = |x|. \tag{13}$$

Then, we have the following lemma which states that the system (11) to (13) is equivalent to the following integral equation:

Find $\mathbf{E}^s$ in $\mathbb{H}(\text{curl},\Omega)$ such that

$$\mathbf{E}^s(x) = -\eta\,(k^2 + \nabla\text{div})\int_\Omega [\mathbf{E}^s(y) + \mathbf{E}^i(y)]g_k(x-y)\,\mathrm{d}y + \nu\,\text{curl}\int_\Omega [\text{curl}\mathbf{E}^s(y) + \text{curl}\mathbf{E}^i(y)]g_k(x-y)\,\mathrm{d}y. \tag{14}$$

Lemma 2.1. [12, Theorem 2.3] *Let $k \in \mathbb{C}\backslash\{0\}$ with $\operatorname{Re} k \geq 0$ and $\operatorname{Im} k \geq 0$. Define*

$$\mathbb{H}_{loc}(\text{curl},\mathbb{R}^3) = \{\mathbf{v} : \mathbb{R}^3 \to \mathbb{C}^3 \,|\, \mathbf{v}_{|B} \in \mathbb{H}(\text{curl}, B) \textit{ for all balls } B \subseteq \mathbb{R}^3\}.$$

1. *Let $\mathbf{v} \in \mathbb{H}_{loc}(\text{curl},\mathbb{R}^3)$ be a solution of the problem (11) to (13). Then, $\mathbf{v}_{|\Omega} \in \mathbb{H}(\text{curl},\Omega)$ solves (14).*
2. *Let $\mathbf{v} \in \mathbb{H}(\text{curl},\Omega)$ solve (14). Then, $\mathbf{v}$ can be extended by the right side of (14) to a solution of the system (11) to (13).*

With the help of (3), we obtain as a consequence the following:

Corollary 2.2. *Let $(\mathbf{E},\mathbf{H})$ be a solution to the problem (1) to (6) and consider $\mathbf{u} = \mathbf{E}_{|\Omega} \in \mathbb{H}(\text{curl},\Omega)$. Then, $\mathbf{u}$ solves (7).*

Conversely, let $\mathbf{u} \in \mathbb{H}(\text{curl},\Omega)$ be a solution of (7) with $\mathbf{u}^i = \mathbf{E}^i$. If we use the formula (7) to extend $\mathbf{u}$ to all of $\mathbb{R}^3$ and if we define $(\mathbf{E},\mathbf{H}) = (\mathbf{u}, \frac{1}{ik\mu_r}\text{curl}\mathbf{u})$, then $(\mathbf{E},\mathbf{H})$ is a solution to the diffraction problem (1) to (6).

Remark 1. The fact that the coefficients η and ν vanish outside of Ω permits to consider the volume integral equation (7) on any domain $\widehat{\Omega}$ with $\Omega \subset \widehat{\Omega} \subset \mathbb{R}^3$. If $\mathbf{u}$ solves (7) on $\widehat{\Omega}$, one can employ the same expression (7) to extend $\mathbf{u}$ outside of $\widehat{\Omega}$. Note that the resulting function $\mathbf{u}$ will not depend on $\widehat{\Omega}$ and will be a solution to the corresponding original scattering problem.

Recall that a bounded linear operator T mapping a Banach space X to itself is said to be a Weyl operator if T is a Fredholm operator of index zero. The Weyl essential spectrum of T, denoted by $\sigma_{\mathrm{ess}}(T)$, is then given by the set of all complex numbers λ for which the operator $\lambda\mathbb{I} - T$ is not Weyl.

Then, the first challenge that will be addressed in the present paper is to find conditions on the complex numbers η and ν for which the strongly singular volume integral operators $\mathbb{I} - \nu B_k$ and $\mathbb{I} - \eta A_k - \nu B_k$ are Fredholm of index zero in the space $\mathbb{H}(\mathrm{curl}, \Omega)$. In other words, when doesn't the number 1 belong to the Weyl essential spectrum of the operators νB_k and $\eta A_k + \nu B_k$. To find an answer to this question, we will study in Sect. 4 the sets $\sigma_{\mathrm{ess}}(B_k)$ and $\sigma_{\mathrm{ess}}(\eta A_k + \nu B_k)$ for a Lipschitz domain Ω. This extends the results obtained in [6] which are valid only for smooth Ω. In this context, we would like to point out that in the dielectric configuration, the essential spectrum of A_k is already known for a Lipschitz domain so conditions for the solvability of the dielectric volume integral equation can be deduced, see the previous work [6]. We also note that many of the operators studied in the next can be written as compact perturbations of self-adjoint operators in some Hilbert spaces, which implies that if they are Fredholm, they are of index zero.

If we define a regularizer for a bounded operator as:

Definition 2.3. Let X and Y be two vector spaces. Suppose $E \in \mathscr{L}(X,Y)$ and $F \in \mathscr{L}(Y,X)$. If $FE = \mathbb{I} + K_X$ where $K_X : X \to X$ is compact, then F is called a left regularizer for E. Likewise, if $EF = \mathbb{I} + K_Y$ where $K_Y : Y \to Y$ is compact, then F is called a right regularizer for E. If F is both a left and a right regularizer for E, then we say that F is a two-sided regularizer (or simply a regularizer) for E.

Then, the second goal of this work is to find, on the space $\mathbb{H}(\mathrm{curl}, \Omega)$, an expression of a regularizer for the operator $\mathbb{I} - \eta A_k - \nu B_k$. This topic will be discussed in Sect. 5. We note that the proofs of the results obtained in this section share some common points with those of the preceding one, like the spectral analysis techniques.

To address all the above-mentioned questions, we begin by collecting some definitions and tools, which will be the subject of the next section.

3. Preliminary results

3.1. Function spaces, integral operators

We use the function spaces:

$$\mathbb{H}(\mathrm{div}, \Omega) = \{\mathbf{u} \in \mathbb{L}^2(\Omega); \mathrm{div}\mathbf{u} \in L^2(\Omega)\}, \tag{15}$$

$$\mathbb{H}(\mathrm{div}0, \Omega) = \{\mathbf{u} \in \mathbb{H}(\mathrm{div}, \Omega); \mathrm{div}\mathbf{u} = 0\}, \tag{16}$$

$$\mathbb{H}_0(\mathrm{div}0, \Omega) = \{\mathbf{u} \in \mathbb{H}(\mathrm{div}0, \Omega); \mathbf{n} \cdot \mathbf{u} = 0\}. \tag{17}$$

As abbreviations for the restrictions onto the boundary Γ, we write for the trace and the normal derivative of a scalar function u:

$$\gamma_0 u = u_{|\Gamma} \quad \text{and} \quad \partial_{\mathbf{n}} u = \mathbf{n} \cdot \nabla u_{|\Gamma},$$

and for the normal trace of a vector function $\mathbf{u}$:

$$\gamma_{\mathbf{n}} \mathbf{u} = \mathbf{n} \cdot \mathbf{u}_{|\Gamma}.$$

We introduce then the Newton potential:

$$\mathcal{N}_k \mathbf{u}(x) := \int_{\mathbb{R}^3} g_k(x-y)\, \mathbf{u}(y)\, \mathrm{d}y, \tag{18}$$

and the single-layer potential:

$$\mathcal{S}_k v(x) := \int_\Gamma g_k(x-y)\, v(y)\, \mathrm{d}s(y), \tag{19}$$

where g_k is defined by formula (10).

We note that the operator $\mathcal{N}_k$ is bounded from $\mathbb{H}^s_{\text{comp}}(\mathbb{R}^3)$ into $\mathbb{H}^{s+2}_{\text{loc}}(\mathbb{R}^3)$, $\forall s \in \mathbb{R}$ and $\mathcal{S}_k$ is bounded from $H^{-\frac{1}{2}+s}(\Gamma)$ into $H^{1+s}_{loc}(\mathbb{R}^3)$ for $|s| < \frac{1}{2}$, see for example [18, Chapter 3].

Moreover, given a potential Φ, for $x \in \Gamma$ we set:

$$\Phi^{\pm}(x) := \lim_{h\to 0^+} \Phi(x \pm h\, \mathbf{n}(x)),$$
$$\partial^{\pm}_{\mathbf{n}(x)}\Phi(x) := \lim_{h\to 0^+} \nabla\Phi(x \pm h\, \mathbf{n}(x)) \cdot \mathbf{n}(x).$$

In the paper, we will use the following jump relation [3, Chapter 3]:

Lemma 3.1. *For $u \in H^{-\frac{1}{2}}(\Gamma)$ and $x \in \Gamma$:*

$$\partial^{\mp}_{\mathbf{n}(x)}(\mathcal{S}_k u)(x) = K'_k u(x) \pm \frac{1}{2} u(x),$$

where

$$K'_k u(x) = \int_\Gamma u(y)\, \partial_{\mathbf{n}(x)} g_k(x-y)\, \mathrm{d}s(y).$$

As an important tool, we need the essential spectrum of the operator: $\partial_{\mathbf{n}}\mathcal{S}_0 = \frac{1}{2}\mathbb{I} + K'_0$ in the space $H^{-\frac{1}{2}}(\Gamma)$. Let us define:

$$\Sigma = \sigma_{\text{ess}}\left(\tfrac{1}{2}\mathbb{I} + K'_0\right). \tag{20}$$

The following result is known [4, Theorem 1]:

Lemma 3.2. *Let Γ be the boundary of a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$. If we define $H_*^{-\frac{1}{2}}(\Gamma) = \left\{ v \in H^{-\frac{1}{2}}(\Gamma); \int_\Gamma v\, \mathrm{d}s = 0 \right\}$, then we have:*

$$\frac{1}{2}\mathbb{I} + K'_0 : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$$

is a self-adjoint contraction with respect to a certain scalar product in $H_*^{-\frac{1}{2}}(\Gamma)$. Its essential spectrum Σ is a compact subset of the open interval $(0,1)$.

If Γ is smooth (of class $C^{1+\alpha}$, $\alpha > 0$), then K'_0 is compact, hence $\Sigma = \{\frac{1}{2}\}$.

3.2. Some spectral analysis tools

We start this paragraph with one of the most used tools in our paper, which is the following (a proof can be found in [11, page 38]):

Lemma 3.3. (Recombination Lemma) *Let X and Y be two vector spaces, $S : Y \to X$ and $T : X \to Y$ two linear operators. Then we have:*

$$\sigma_{\text{ess}}(ST)\backslash\{0\} = \sigma_{\text{ess}}(TS)\backslash\{0\}.$$

Remark 2. There is another version of the last lemma whose statement is as follows:

Let X and Y be vector spaces and $S : Y \to X$ and $T : X \to Y$ linear operators. Then for $\lambda \neq 0$, T induces isomorphisms from $\ker(\lambda\mathbb{I} - ST)$ to $\ker(\lambda\mathbb{I} - TS)$ and from $X/(\lambda\mathbb{I} - ST)X$ to $Y/(\lambda\mathbb{I} - TS)Y$. In particular, $\lambda\mathbb{I} - ST$ is Fredholm of index zero in X if and only if $\lambda\mathbb{I} - TS$ is Fredholm of index zero in Y.

In addition, we will need to reinterpret the "Recombination Lemma" in terms of regularization operators, which is given by the following lemma:

Lemma 3.4. *Let X and Y be two vector spaces, $S : Y \to X$ and $T : X \to Y$ two linear operators and $\lambda \neq 0$.*

If $(\lambda\mathbb{I} - ST)'$ is a regularizer for $\lambda\mathbb{I} - ST$ on X, then:

$$(\lambda\mathbb{I} - TS)' = \frac{1}{\lambda}T(\lambda\mathbb{I} - ST)'S + \frac{1}{\lambda}\mathbb{I}$$

is a regularizer for $\lambda\mathbb{I} - TS$ on Y.

Proof. For $\lambda \neq 0$, let $(\lambda\mathbb{I} - ST)'$ denote a regularizer for $\lambda\mathbb{I} - ST$ on X. This implies the existence of two compact operators K_l and K_r on X such that:

$$\begin{cases} (\lambda\mathbb{I} - ST)'(\lambda\mathbb{I} - ST) = \mathbb{I} + K_l, \\ (\lambda\mathbb{I} - ST)(\lambda\mathbb{I} - ST)' = \mathbb{I} + K_r. \end{cases}$$

We then have:

$$\begin{aligned} \left(\frac{1}{\lambda}T(\lambda\mathbb{I} - ST)'S + \frac{1}{\lambda}\mathbb{I}\right)(\lambda\mathbb{I} - TS) &= \frac{1}{\lambda}T(\lambda\mathbb{I} - ST)'S(\lambda\mathbb{I} - TS) + \frac{1}{\lambda}(\lambda\mathbb{I} - TS) \\ &= \frac{1}{\lambda}T(\lambda\mathbb{I} - ST)'(\lambda\mathbb{I} - ST)S + \mathbb{I} - \frac{1}{\lambda}TS \\ &= \frac{1}{\lambda}T(\mathbb{I} + K_l)S + \mathbb{I} - \frac{1}{\lambda}TS = \mathbb{I} + \frac{1}{\lambda}TK_lS. \end{aligned}$$

The operator $TK_lS : Y \to Y$ is clearly compact. From this, we deduce that $\frac{1}{\lambda}T(\lambda\mathbb{I} - ST)'S + \frac{1}{\lambda}\mathbb{I}$ is a left regularizer on Y for $\lambda\mathbb{I} - TS$.

In the same manner, we check that $\frac{1}{\lambda}T(\lambda\mathbb{I} - ST)'S + \frac{1}{\lambda}\mathbb{I}$ is also a right regularizer on Y for $\lambda\mathbb{I} - TS$. □

Note that the above lemma gives another proof of the recombination lemma (Lemma 3.3) because it immediately implies, for $\lambda \neq 0$, $\lambda\mathbb{I} - ST$ and $\lambda\mathbb{I} - TS$ are simultaneously Fredholm or are simultaneously not Fredholm.

As another important tool, we will employ the following results on the essential spectrum of upper triangular operator matrix:

Lemma 3.5. *Let X and Y be two Hilbert spaces. Let also $C : X \to X$, $E : Y \to Y$ and $D : Y \to X$ be bounded linear operators. Define the operator matrix $M_D = \left(\begin{smallmatrix} C & D \\ 0 & E \end{smallmatrix}\right)$. Then, we have:*

1. $$\begin{pmatrix} C & 0 \\ 0 & E \end{pmatrix} \text{ is Fredholm of index zero } \implies M_D \text{ is Fredholm of index zero for all } D \text{ in } \mathscr{L}(Y, X).$$

   In particular, we have:

   $$\sigma_{\mathrm{ess}}(M_D) \subseteq \sigma_{\mathrm{ess}}\begin{pmatrix} C & 0 \\ 0 & E \end{pmatrix} \subseteq \sigma_{\mathrm{ess}}(C) \cup \sigma_{\mathrm{ess}}(E). \tag{21}$$

   Moreover, (21) holds as equality if C and E are self-adjoint.

2. *If* $\begin{pmatrix} \tilde{C} & \tilde{D} \\ 0 & \tilde{E} \end{pmatrix}$ *is a regularizer for* $\begin{pmatrix} C & D \\ 0 & E \end{pmatrix}$ *on* $X \times Y$*, then* $\tilde{C} = C'$ *and* $\tilde{E} = E'$ *are regularizers for* C *and* E *on* X *and* Y*, respectively.*
Conversely, if C' *and* E' *are regularizers for* C *and* E *on* X *and* Y*, respectively, then:*
$$\begin{pmatrix} C' & -C'DE' \\ 0 & E' \end{pmatrix} \text{ is a regularizer for } \begin{pmatrix} C & D \\ 0 & E \end{pmatrix} \text{ on } X \times Y.$$

Proof. For the first affirmation, we refer the reader to the paper [14]. Let us prove now the second statement.

Assume that $\begin{pmatrix} \tilde{C} & \tilde{D} \\ 0 & \tilde{E} \end{pmatrix}$ is a regularizer for $\begin{pmatrix} C & D \\ 0 & E \end{pmatrix}$ on $X \times Y$. Then, we have:

$$\begin{cases} \begin{pmatrix} \tilde{C} & \tilde{D} \\ 0 & \tilde{E} \end{pmatrix} \begin{pmatrix} C & D \\ 0 & E \end{pmatrix} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} + \begin{pmatrix} K_{1,l} & K_{2,l} \\ K_{3,l} & K_{4,l} \end{pmatrix} = \begin{pmatrix} \mathbb{I} + K_{1,l} & K_{2,l} \\ K_{3,l} & \mathbb{I} + K_{4,l} \end{pmatrix}, \\ \begin{pmatrix} C & D \\ 0 & E \end{pmatrix} \begin{pmatrix} \tilde{C} & \tilde{D} \\ 0 & \tilde{E} \end{pmatrix} = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} + \begin{pmatrix} K_{1,r} & K_{2,r} \\ K_{3,r} & K_{4,r} \end{pmatrix} = \begin{pmatrix} \mathbb{I} + K_{1,r} & K_{2,r} \\ K_{3,r} & \mathbb{I} + K_{4,r} \end{pmatrix}, \end{cases}$$

where, for $i = 1, \ldots, 4$, $K_{i,l}$ and $K_{i,r}$ are some compact operators.
But, we also have:

$$\begin{cases} \begin{pmatrix} \tilde{C} & \tilde{D} \\ 0 & \tilde{E} \end{pmatrix} \begin{pmatrix} C & D \\ 0 & E \end{pmatrix} = \begin{pmatrix} \tilde{C}C & \tilde{C}D + \tilde{D}E \\ 0 & \tilde{E}E \end{pmatrix}, \\ \begin{pmatrix} C & D \\ 0 & E \end{pmatrix} \begin{pmatrix} \tilde{C} & \tilde{D} \\ 0 & \tilde{E} \end{pmatrix} = \begin{pmatrix} C\tilde{C} & C\tilde{D} + D\tilde{E} \\ 0 & E\tilde{E} \end{pmatrix}. \end{cases}$$

By identification, we find the relations:

$$\begin{cases} \tilde{C}C = \mathbb{I} + K_{1,l} \\ C\tilde{C} = \mathbb{I} + K_{1,r} \end{cases}, \begin{cases} \tilde{E}E = \mathbb{I} + K_{4,l} \\ E\tilde{E} = \mathbb{I} + K_{4,r} \end{cases},$$

which mean that $\tilde{C}$ and $\tilde{E}$ are regularizers for C and E, respectively.

Reciprocally, suppose that C' and E' are regularizers for C and E on X and Y, respectively, i.e., there exist some compact operators $K_{C,l}$, $K_{C,r}$, $K_{E,l}$ and $K_{E,r}$ such that we have:

$$\begin{cases} C'C = \mathbb{I} + K_{C,l} \\ CC' = \mathbb{I} + K_{C,r} \end{cases}, \begin{cases} E'E = \mathbb{I} + K_{E,l} \\ EE' = \mathbb{I} + K_{E,r} \end{cases}.$$

We now write:

$$\begin{aligned} \begin{pmatrix} C' & -C'DE' \\ 0 & E' \end{pmatrix} \begin{pmatrix} C & D \\ 0 & E \end{pmatrix} &= \begin{pmatrix} C'C & C'D - C'DE'E \\ 0 & E'E \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{I} + K_{C,l} & C'D(\mathbb{I} - E'E) \\ 0 & \mathbb{I} + K_{E,l} \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{I} + K_{C,l} & C'D(-K_{E,l}) \\ 0 & \mathbb{I} + K_{E,l} \end{pmatrix} \\ &= \begin{pmatrix} \mathbb{I} & 0 \\ 0 & \mathbb{I} \end{pmatrix} + \begin{pmatrix} K_{C,l} & -C'DK_{E,l} \\ 0 & K_{E,l} \end{pmatrix}. \end{aligned}$$

The matrix operator $\begin{pmatrix} K_{C,l} & -C'DK_{E,l} \\ 0 & K_{E,l} \end{pmatrix}$ is compact, so $\begin{pmatrix} C' & -C'DE' \\ 0 & E' \end{pmatrix}$ is a left regularizer for $\begin{pmatrix} C & D \\ 0 & E \end{pmatrix}$.

Following the same steps, we show that $\begin{pmatrix} C' & -C'DE' \\ 0 & E' \end{pmatrix}$ is also a right regularizer.

This ends the proof of the lemma. □

We finish this paragraph with a characterization of Fredholm operators which is known as the Atkinson theorem:

Lemma 3.6. [15, theorem 2.24] *Let X and Y be two vector spaces. For $E \in \mathscr{L}(X, Y)$, the following statements are equivalent:*

- *E is Fredholm,*
- *E has a left regularizer and a right regularizer,*
- *E has a regularizer.*

4. Results on the essential spectrum and the well-posedness

4.1. The essential spectrum of the magnetic volume integral operator

We first consider the magnetic scattering problem. When the dielectric contrast η is null, the volume integral equation involves only the operator B_k given by formula (9) and writes:

$$\text{Find } \mathbf{u} \text{ in } \mathbb{H}(\mathrm{curl}, \Omega) \text{ such that } \mathbf{u}(x) - \nu B_k \mathbf{u}(x) = \mathbf{u}^i(x). \tag{22}$$

We start by recalling some facts related to the operator B_k that have been established in [6, Section 4]:

Lemma 4.1. *The operator B_k is bounded from $\mathbb{H}(\mathrm{curl}, \Omega)$ to itself, it cannot be extended to $\mathbb{L}^2(\Omega)$ as a bounded operator and it has the same essential spectrum as the operator B_0.*

Thanks to the above result, we need to only focus on the essential spectrum of B_0 on $\mathbb{H}(\mathrm{curl}, \Omega)$. We use the same decomposition of the space $\mathbb{H}(\mathrm{curl}, \Omega)$ as the one employed in the previous work, namely the following decomposition which is satisfied both for the $\mathbb{L}^2(\Omega)$ and $\mathbb{H}(\mathrm{curl}, \Omega)$ norm:

$$\mathbb{H}(\mathrm{curl}, \Omega) = \nabla H_0^1(\Omega) \oplus \mathbb{X},$$

with $\mathbb{X} = \mathbb{H}(\mathrm{curl}, \Omega) \cap \mathbb{H}(\mathrm{div}0, \Omega)$ and where $\mathbb{H}(\mathrm{div}0, \Omega)$ is defined by (16).

On the space $\nabla H_0^1(\Omega)$, B_0 is the null operator because $\mathrm{curl}\nabla = 0$. We then consider the space $\mathbb{X}$. We will give another method different from the one used in the previous paper [6] which allows us to deduce the essential spectrum of the restriction of B_0 to the subspace $\mathbb{X}$, even for Lipschitz domains. This approach is based on the application of Lemma 3.3 twice. We then have the following:

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. The essential spectrum of the operator B_k in $\mathbb{H}(\mathrm{curl}, \Omega)$ is given by:*

$$\sigma_{\mathrm{ess}}(B_k) = \{0\} \cup \{1 - \Sigma\} \cup \{1\},$$

where Σ is defined in (20).

Proof. First, let us remember the expression of the operator B_0 in $\mathbb{H}(\mathrm{curl}, \Omega)$:

$$B_0 \mathbf{u}(x) = \mathrm{curl} \int_{\Omega} g_0(x-y)\, \mathrm{curl}\mathbf{u}(y)\, \mathrm{d}y = \mathrm{curl}\, \mathcal{N}_0\, \mathrm{curl}\mathbf{u}(x)$$

with $\mathcal{N}_0$ the Newton potential defined by formula (18) for $k = 0$.

We will use twice Lemma 3.3:

First, we apply the lemma to these choices of maps:

$$X = \mathbb{X}, \quad Y = \mathbb{H}(\mathrm{div}0, \Omega)$$

$$\begin{aligned} T : X &\to Y & \qquad S : Y &\to X \\ \mathbf{u} &\mapsto \mathrm{curl}\mathbf{u} & \mathbf{w} &\mapsto \mathrm{curl}\, \mathcal{N}_0 \mathbf{w}. \end{aligned}$$

We found then that $\sigma_{\rm ess}(B_0)\backslash\{0\} = \sigma_{\rm ess}(\mathrm{curlcurl}\,\mathcal{N}_0)\backslash\{0\}$.

Now, let us simplify the expression of the operator $\mathrm{curlcurl}\,\mathcal{N}_0$ on the subspace $\mathbb{X}$. We have, for $\mathbf{u} \in \mathbb{X}$:

$$\begin{aligned}\mathrm{curlcurl}\,\mathcal{N}_0\mathbf{u} &= (\nabla\mathrm{div} - \Delta)\mathcal{N}_0\mathbf{u}\\ &= -A_0\mathbf{u} - \Delta\mathcal{N}_0\mathbf{u},\end{aligned}$$

where A_0 is the volume integral operator defined in formula (8).

Using the fact that the restriction of A_0 on the set $\mathbb{H}(\mathrm{div}0,\Omega)$ is the operator $\nabla\mathcal{S}_0\gamma_{\mathbf{n}}$ where $\mathcal{S}_0$ is the single layer associated to the Laplace operator defined by (19) (see [6, Section 3]) and that the Newton potential $\mathcal{N}_0$ satisfies $-\Delta\mathcal{N}_0\mathbf{u} = \mathbf{u}$ in the distributional sense [18, Theorem 3.14], we deduce:

$$\mathrm{curlcurl}\,\mathcal{N}_0\mathbf{u} = (\mathbb{I} - \nabla\mathcal{S}_0\gamma_{\mathbf{n}})\mathbf{u}. \tag{23}$$

This leads to the first relation:

$$\sigma_{\rm ess}(B_0)\backslash\{0\} = \sigma_{\rm ess}(\mathbb{I} - \nabla\mathcal{S}_0\gamma_{\mathbf{n}})\backslash\{0\}.$$

Second, we apply the same lemma to:

$$\begin{aligned} X &= \mathbb{H}(\mathrm{div}0,\Omega), & Y &= H_*^{-\frac{1}{2}}(\Gamma),\\ T &: X \to Y & S &: Y \to X\\ &\mathbf{w} \mapsto \gamma_{\mathbf{n}}\mathbf{w} & &\varphi \mapsto \nabla\mathcal{S}_0\varphi,\end{aligned}$$

where $H_*^{-\frac{1}{2}}(\Gamma)$ is the space of zero-mean elements of $H^{-\frac{1}{2}}(\Gamma)$ (see Lemma 3.2).

This gives rise, with the help of Lemma 3.1, to the following second identity:

$$\sigma_{\rm ess}(\mathbb{I} - \nabla\mathcal{S}_0\gamma_{\mathbf{n}})\backslash\{1\} = \sigma_{\rm ess}\left(\frac{1}{2}\mathbb{I} - K_0'\right)\backslash\{1\}.$$

To end the proof, we will show that the value 1 belongs to the essential spectrum of B_0 using the fact that its restriction on the subspace $\mathbb{Y} := \mathbb{H}_0(\mathrm{div}0,\Omega) \cap \{\mathbf{u} \in \mathbb{H}(\mathrm{curl},\Omega); \mathbf{n}\times\mathbf{u} = 0\}$ is the identity operator.

Indeed, integration by parts and the identity (23) give, for $\mathbf{u} \in \mathbb{Y}$:

$$\begin{aligned} B_0\mathbf{u} &= \mathrm{curl}\left(\int_\Omega \mathrm{curl}_y(g_0(x-y)\mathbf{u}(y))\,\mathrm{d}y + \mathrm{curl}\int_\Omega g_0(x-y)\mathbf{u}(y)\,\mathrm{d}y\right)\\ &= \mathrm{curl}\int_\Gamma g_0(x-y)\mathbf{n}\times\mathbf{u}(y)\,\mathrm{d}s(y) + \mathrm{curlcurl}\int_\Omega g(x-y)\mathbf{u}(y)\,\mathrm{d}y\\ &= (\mathbb{I} - \nabla\mathcal{S}_0\gamma_{\mathbf{n}})\,\mathbf{u}\\ &= \mathbf{u}.\end{aligned}$$

□

Remark 3. The identities obtained in the above proof enable one to have information for the surface-operators system $\hat{B}_0$ introduced in [6, Formula (7)], namely the operator:

$$\hat{B}_0 = \begin{pmatrix} \frac{1}{2}\mathbb{I} - K_0' & \mathbf{n}\cdot\mathrm{curl}\mathcal{S}_0\\ -\mathbf{n}\times\nabla\mathcal{S}_0 & M_0 + \frac{1}{2}\mathbb{I}\end{pmatrix}$$

which is defined on the space $H^{-\frac{1}{2}}(\Gamma) \times H_\times^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$.

In fact, we deduce that $\sigma_{\rm ess}(\hat{B}_0) = \{1 - \Sigma\}$ for a Lipschitz boundary Γ.

As a straightforward corollary of the previous theorem, we obtain the following sufficient and necessary conditions for the solvability of the magnetic volume integral equation (22):

Corollary 4.3. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let the coefficient μ_r be a complex constant. Then the integral operator of Eq. (22), namely $\mathbb{I} - \nu B_k$, is Fredholm in $\mathbb{H}(\mathrm{curl}, \Omega)$ if and only if:*

$$\mu_r \neq 0 \quad \text{and} \quad \mu_r \neq 1 - \frac{1}{\sigma} \text{ for all } \sigma \in \Sigma.$$

4.2. The essential spectrum of the complete volume integral operator

We first remind the reader that the general configuration is modeled by the volume integral equation (7) defined in $\mathbb{H}(\mathrm{curl}, \Omega)$ and involving the operator $\eta A_k + \nu B_k$, where A_k and B_k are given by (8) and (9).

In this section, we will determine the essential spectrum of the operator $\eta A_k + \nu B_k$ in $\mathbb{H}(\mathrm{curl}, \Omega)$ when the domain Ω is Lipschitz. We obtain the following:

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. For any $\eta, \nu \in \mathbb{C}$, the essential spectrum of the operator $\eta A_k + \nu B_k$ in the space $\mathbb{H}(\mathrm{curl}, \Omega)$ is given by:*

$$\sigma_{\mathrm{ess}}(\eta A_k + \nu B_k) = \{0, \eta, \nu\} \cup \eta\Sigma \cup \nu(1 - \Sigma),$$

where Σ is defined in (20).

Proof. We begin by recalling that, in $\mathbb{H}(\mathrm{curl}, \Omega)$, we have the following property [6, Sections 3 and 4]:

$$\eta A_k + \nu B_k \text{ has the same essential spectrum as } \eta A_0 + \nu B_0. \tag{24}$$

Thanks to this result, it suffices to consider the operator $\eta A_0 + \nu B_0$. Once again, we will use the following orthogonal decomposition:

$$\mathbb{H}(\mathrm{curl}, \Omega) = \nabla H_0^1(\Omega) \oplus \mathbb{X}$$

with $\mathbb{X} = \mathbb{H}(\mathrm{curl}, \Omega) \cap \mathbb{H}(\mathrm{div}0, \Omega)$.

For the space $\nabla H_0^1(\Omega)$, we obtain the essential spectrum of ηA_0 which is the set $\{\eta\}$ because the restriction of A_0 to that subspace is the identity operator [6, Theorem 3.2].

Let us focus now on the second space $\mathbb{X} = \mathbb{H}(\mathrm{div}0, \Omega) \cap \mathbb{H}(\mathrm{curl}, \Omega)$. For a Lipschitz boundary, we will proceed as follows.

First, we remember that [6, Section 3]:

$$A_0\mathbf{u} = \nabla \mathcal{S}_0 \gamma_{\mathbf{n}} \mathbf{u}, \quad \text{for} \quad \mathbf{u} \in \mathbb{H}(\mathrm{div}0, \Omega).$$

We then apply Lemma 3.3 to (we may also take $H_*^{-\frac{1}{2}}(\Gamma)$ instead of $H^{-\frac{1}{2}}(\Gamma)$):

$$\begin{aligned} &X = \mathbb{X},\, Y = H^{-\frac{1}{2}}(\Gamma) \times \mathbb{H}(\mathrm{div}0, \Omega), \\ &T : X \to Y, \mathbf{u} \mapsto (\gamma_{\mathbf{n}}\mathbf{u}, \mathrm{curl}\mathbf{u}), \\ &S : Y \to X, (\phi, \mathbf{w}) \mapsto \eta \nabla \mathcal{S}_0 \phi + \nu\, \mathrm{curl}\, \mathcal{N}_0 \mathbf{w}. \end{aligned} \tag{25}$$

This gives:

$$\sigma_{\mathrm{ess}}(\eta A_0 + \nu B_0) \backslash \{0\} = \sigma_{\mathrm{ess}}(TS) \backslash \{0\}.$$

Moreover, due to Lemma 3.1 and formula (23), we get:

$$TS = \begin{pmatrix} \eta\, \partial_{\mathbf{n}} \mathcal{S}_0 & \nu\, \gamma_{\mathbf{n}} \mathrm{curl}\, \mathcal{N}_0 \\ 0 & \nu\, \mathrm{curl}\, \mathrm{curl}\, \mathcal{N}_0 \end{pmatrix} = \begin{pmatrix} \eta \left(\dfrac{1}{2}\mathbb{I} + K_0' \right) & \nu\, \gamma_{\mathbf{n}} \mathrm{curl}\, \mathcal{N}_0 \\ 0 & \nu(\mathbb{I} - \nabla \mathcal{S}_0 \gamma_{\mathbf{n}}) \end{pmatrix}.$$

The next step is to employ the first statement of Lemma 3.5 with $C = \eta \left(\frac{1}{2}\mathbb{I} + K_0'\right)$ and $E = \nu(\mathbb{I} - \nabla \mathcal{S}_0 \gamma_{\mathbf{n}})$ which are self-adjoint [6, Section 3]. We already know the essential spectrum of the operator $\mathbb{I} - \nabla \mathcal{S}_0 \gamma_{\mathbf{n}}$ in the space $\mathbb{H}(\mathrm{div}0, \Omega)$ which is the set $\{1\} \cup (1 - \Sigma)$ (see the proof of Theorem 4.2). The result follows. □

From the precedent theorem, we could deduce the following:

Corollary 4.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let the coefficients ε_r and μ_r be complex constants.*

Then the integral operator of Eq. (7), namely $\mathbb{I} - \eta A_k - \nu B_k$, is Fredholm in $\mathbb{H}(\mathrm{curl}, \Omega)$ if and only if:

$$\varepsilon_r \neq 0,\ \mu_r \neq 0 \text{ and } \varepsilon_r \neq 1 - \frac{1}{\sigma},\ \mu_r \neq 1 - \frac{1}{\sigma} \text{ for all } \sigma \in \Sigma. \tag{26}$$

According to the approach given above, with the help of Remark 2, we could deduce the following theorem which states that the electromagnetic volume integral equation (7) (and so the Maxwell transmission problem) is well-posed in the Fredholm sense if and only if two scalar integral equations, one involving the electric permittivity and the other one the magnetic permeability, are well-posed:

Theorem 4.6. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let the coefficients ε_r and μ_r be complex constants.*

Then, the electromagnetic volume integral equation $(\mathbb{I} - \eta A_k + \nu B_k)\mathbf{u} = \mathbf{f}$ is well-posed in the Fredholm sense in $\mathbb{H}(\mathrm{curl}, \Omega)$ if and only if the following two independent scalar problems are well-posed:

$$\begin{cases} \textit{Problem 1: Find } u_1 \in H^{-\frac{1}{2}}(\Gamma) \textit{ such that} \\ \left(\mathbb{I} - \eta\left(\frac{1}{2}\mathbb{I} + K_0'\right)\right) u_1 = f_1,\ f_1 \in H^{-\frac{1}{2}}(\Gamma) \textit{ is a data,} \\ \textit{Problem 2: Find } u_2 \in H^{-\frac{1}{2}}(\Gamma) \textit{ such that} \\ \left((1-\nu)\mathbb{I} + \nu\left(\frac{1}{2}\mathbb{I} + K_0'\right)\right) u_2 = f_2,\ f_2 \in H^{-\frac{1}{2}}(\Gamma) \textit{ is a data,} \end{cases}$$

with $\frac{1}{2}\mathbb{I} + K_0'$ the adjoint Neumann–Poincaré operator defined in Sect. 3.1.

Before closing this section, we would like to mention that, in our future research, we intend to concentrate on the identification of the two scalar transmission problems which are equivalent to the two integral equations described in the above theorem and to compare the result with that obtained, in a slightly different functional framework, in [8] for the case of Maxwell's problem modeling the propagation of waves in composite objects mixing positive and negative materials.

5. Construction of regularizers for the volume integral operators

If the complex constant ε_r and μ_r are chosen such that the operator $\mathbb{I} - \eta A_k + \nu B_k$ is Fredholm on $\mathbb{H}(\mathrm{curl}, \Omega)$, i.e., they satisfy the condition (26) in Corollary 4.5, then the goal of the present section is to give an explicit construction of an operator which, applied to the volume integral equation (7), transforms it to the form "identity plus compact". In other words, we wish to find a regularizer for the operator $\mathbb{I} - \eta A_k + \nu B_k$ (existence of such an operator is assured by Lemma 3.6).

5.1. Results

In this paragraph, we will give explicit formulas for the regularizers for A_k, B_k, and their linear combinations with the identity operator, by only knowing a regularizer for $\lambda\mathbb{I} + K_0'$. This regularizer is trivial, of course, if the boundary is smooth, since in this case K_0' is compact (see Lemma 3.2) but the argument works for the Lipschitz case.

Recall that $\mathcal{N}_0$ and $\mathcal{S}_0$ denote, respectively, the single-layer potential and the Newton potential, $\gamma_{\mathbf{n}}$ is the normal trace operator, and that Σ denotes the essential spectrum of $\frac{1}{2}\mathbb{I} + K_0'$ (see Sect. 3.1).

We begin with the dielectric case ($\nu = 0$) and its corresponding integral equation which is the following:

$$\text{Find } \mathbf{u} \in \mathbb{H}(\mathrm{curl}, \Omega) \text{ such that } \mathbf{u}(x) - \eta A_k \mathbf{u}(x) = \mathbf{u}^i(x). \tag{27}$$

We first recall the following results for the dielectric operator which will be employed in the next [6, Theorem 3.1]:

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then we have:*

- *The operator A_k can be extended to $\mathbb{L}^2(\Omega)$ as a bounded operator. It has $\mathbb{H}(\mathrm{curl}, \Omega)$ and $\mathbb{H}(\mathrm{div}, \Omega)$ as invariant subspaces.*
- *For $\mathbf{u}^i$ in $\mathbb{H}(\mathrm{curl}, \Omega) \cap \mathbb{H}(\mathrm{div}, \Omega)$, the integral equation (27) in $\mathbb{L}^2(\Omega)$ has the same solutions as in $\mathbb{H}(\mathrm{curl}, \Omega)$ or in $\mathbb{H}(\mathrm{div}, \Omega)$.*
- *$A_k - A_0$ defines a compact operator on $\mathbb{L}^2(\Omega)$.*

Thanks to this result, one can understand that the construction of a regularizer can be investigated in $\mathbb{L}^2(\Omega)$. We then have:

Theorem 5.2. (The dielectric case) *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let the coefficient ε_r be a complex constant verifying:*

$$\varepsilon_r \neq 0, \text{ and } \varepsilon_r \neq 1 - \frac{1}{\sigma} \text{ for all } \sigma \in \Sigma.$$

If we denote by R_η a regularizer for $\frac{2-\eta}{2}\mathbb{I} - \eta K_0'$ on $H^{-\frac{1}{2}}(\Gamma)$, then the operator:

$$\frac{\eta}{1-\eta}\mathbf{P}_{\nabla H_0^1} + \eta \nabla \mathcal{S}_0 R_\eta \gamma_{\mathbf{n}} \mathbf{P}_{\mathbb{W}} + \mathbb{I} \tag{28}$$

is a regularizer for $\mathbb{I} - \eta A_k$ on $\mathbb{L}^2(\Omega)$, where $\mathbf{P}_{\nabla H_0^1}$ and $\mathbf{P}_{\mathbb{W}}$ are the orthogonal projections onto ∇H_0^1 and $\mathbb{W} = \nabla H^1(\Omega) \cap \mathbb{H}(\mathrm{div}0, \Omega)$, respectively.

In particular, if Γ is smooth then (28) becomes:

$$\frac{\eta}{1-\eta}\mathbf{P}_{\nabla H_0^1} + \frac{2\eta}{2-\eta} \nabla \mathcal{S}_0 \gamma_{\mathbf{n}} \mathbf{P}_{\mathbb{W}} + \mathbb{I}.$$

Next, we consider the volume integral equation formulation for the magnetic case ($\eta = 0$):

$$\text{Find } \mathbf{u} \text{ in } \mathbb{H}(\mathrm{curl}, \Omega) \text{ such that } \mathbf{u}(x) - \nu B_k \mathbf{u}(x) = \mathbf{u}^i(x).$$

In this situation, we have the following:

Theorem 5.3. (The magnetic case) *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let the coefficient μ_r be a complex constant such that:*

$$\mu_r \neq 0, \quad \text{and} \quad \mu_r \neq 1 - \frac{1}{\sigma} \text{ for all } \sigma \in \Sigma.$$

If we denote by R_ν a regularizer for $\frac{2-\nu}{2}\mathbb{I} + \nu K_0'$ on $H^{-\frac{1}{2}}(\Gamma)$, and by $\mathbf{P}_{\mathbb{X}}$ the orthogonal projection onto the subspace $\mathbb{X} = \mathbb{H}(\mathrm{curl}, \Omega) \cap \mathbb{H}(\mathrm{div}0, \Omega)$, then the operator:

$$\left(\frac{\nu}{1-\nu}\mathrm{curl}\mathcal{N}_0\mathrm{curl} - \frac{\nu^2}{1-\nu}\mathrm{curl}\mathcal{N}_0 \nabla \mathcal{S}_0 R_\nu \gamma_{\mathbf{n}} \mathrm{curl}\right)\mathbf{P}_{\mathbb{X}} + \mathbb{I}$$

is a regularizer for $\mathbb{I} - \nu B_k$ on $\mathbb{H}(\mathrm{curl}, \Omega)$.

In particular, if Γ is smooth then the last expression becomes:

$$\left(\frac{\nu}{1-\nu}\mathrm{curl}\mathcal{N}_0\mathrm{curl} - \frac{2\nu^2}{(1-\nu)(2-\nu)}\mathrm{curl}\mathcal{N}_0 \nabla \mathcal{S}_0 \gamma_{\mathbf{n}} \mathrm{curl}\right)\mathbf{P}_{\mathbb{X}} + \mathbb{I}.$$

We finish this paragraph with a result for the general configuration:

Theorem 5.4. (The electromagnetic case) *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Let the coefficients ε_r and μ_r be complex constants satisfying:*

$$\varepsilon_r \neq 0,\ \mu_r \neq 0 \text{ and } \varepsilon_r \neq 1 - \frac{1}{\sigma},\ \mu_r \neq 1 - \frac{1}{\sigma} \text{ for all } \sigma \in \Sigma.$$

If we denote by R_η (resp. R_ν) a regularizer for $\frac{2-\eta}{2}\mathbb{I} - \eta K_0'$ (resp. $\frac{2-\nu}{2}\mathbb{I} + \nu K_0'$) on $H^{-\frac{1}{2}}(\Gamma)$, and by $\mathbf{P}_{\mathbb{X}}$ the orthogonal projection onto the subspace $\mathbb{X} = \mathbb{H}(\mathrm{curl}, \Omega) \cap \mathbb{H}(\mathrm{div}0, \Omega)$, then the operator:

$$\frac{1}{1-\eta}\mathbb{I} + \left(LMN - \frac{\eta}{1-\eta}\mathbb{I}\right)\mathbf{P}_{\mathbb{X}},$$

where

$$L : (u, \mathbf{v}) \in H^{-\frac{1}{2}}(\Gamma) \times \mathbb{H}(\mathrm{div}0, \Omega) \mapsto \eta\nabla\mathcal{S}_0 u + \nu\mathrm{curl}\mathcal{N}_0\mathbf{v} \in \mathbb{X},$$
$$N : \mathbf{a} \in \mathbb{X} \mapsto (\gamma_{\mathbf{n}}\mathbf{a}, \mathrm{curl}\mathbf{a}) \in H^{-\frac{1}{2}} \times \mathbb{H}(\mathrm{div}0, \Omega),$$

and

$$M : H^{-\frac{1}{2}} \times \mathbb{H}(\mathrm{div}0, \Omega) \to H^{-\frac{1}{2}} \times \mathbb{H}(\mathrm{div}0, \Omega)$$
$$(v, \mathbf{w}) \mapsto \begin{pmatrix} R_\eta & \frac{\nu}{1-\nu} R_\eta\gamma_{\mathbf{n}}\mathrm{curl}\mathcal{N}_0\left(\mathbb{I} - \nu\nabla\mathcal{S}_0 R_\nu\gamma_{\mathbf{n}}\right) \\ 0 & \frac{1}{1-\nu}\mathbb{I} - \frac{\nu}{1-\nu}\nabla\mathcal{S}_0 R_\nu\gamma_{\mathbf{n}} \end{pmatrix} \begin{pmatrix} v \\ \mathbf{w} \end{pmatrix},$$

is a regularizer for $\mathbb{I} - \eta A_k - \nu B_k$ on $\mathbb{H}(\mathrm{curl}, \Omega)$.

In particular, if Γ is smooth then the operator matrix M becomes:

$$\begin{pmatrix} \frac{2}{2-\eta}\mathbb{I} & \frac{2\nu}{(1-\nu)(2-\eta)}\gamma_{\mathbf{n}}\mathrm{curl}\mathcal{N}_0\left(\mathbb{I} - \frac{2\nu}{2-\nu}\nabla\mathcal{S}_0\gamma_{\mathbf{n}}\right) \\ 0 & \frac{1}{1-\nu}\mathbb{I} - \frac{2\nu}{(1-\nu)(2-\nu)}\nabla\mathcal{S}_0\gamma_{\mathbf{n}} \end{pmatrix}.$$

5.2. Proof

The steps to proving the last three theorems are based on the ideas described in the previous work [6] and the previous section on the essential spectrum, in addition to the two Lemmas 3.4 and 3.5.

Let us begin with the proof of the first theorem:

Proof of Theorem 5.2. First, note that the given assumption on ε_r guarantees that the operator $\mathbb{I} - \eta A$ is Fredholm on $\mathbb{L}^2(\Omega)$ [6, Corollary 3.3], and hence the existence of a regularizer for that operator is assured (Lemma 3.6).

We already know that $A_k - A_0$ defines a compact operator (Lemma 5.1). Thus, it suffices to search a regularizer for $\mathbb{I} - \eta A_0$ on $\mathbb{L}^2(\Omega)$.

We use the following orthogonal decomposition:

$$\mathbb{L}^2(\Omega) = \nabla H_0^1(\Omega) \oplus \mathbb{H}_0(\mathrm{div}0, \Omega) \oplus \mathbb{W},$$

where

$$\mathbb{W} = \nabla H^1 \cap \mathbb{H}(\mathrm{div}0, \Omega).$$

We construct a regularizer for $\mathbb{I} - \eta A_0$ on each subspace mentioned in the above decomposition then we end up using the properties of the associated orthogonal projections.

Using the fact that [6, Section 3]

$$A_0 = \mathbb{I} \text{ on } \nabla H_0^1(\Omega),\ A_0 = 0 \text{ on } \mathbb{H}_0(\mathrm{div}0, \Omega),\ \text{and } A_0 = \nabla\mathcal{S}_0\gamma_{\mathbf{n}} \text{ on } \mathbb{W}, \tag{29}$$

we get:

$$(\mathbb{I}-\eta A_0)|_{\nabla H_0^1(\Omega)} = (1-\eta)\mathbb{I},\ (\mathbb{I}-\eta A_0)|_{H_0(\mathrm{div}0,\Omega)} = \mathbb{I},$$
$$(\mathbb{I}-\eta A_0)|_{\mathbb{W}} = \mathbb{I}-\eta\nabla\mathcal{S}_0\gamma_{\mathbf{n}}$$

So, we could take $\frac{1}{1-\eta}\mathbb{I}$ and $\mathbb{I}$ as regularizers for $\mathbb{I}-\eta A_0$ on the first two subspaces, in the order. For the third subspace, we apply Lemma 3.4 to these choices of maps for $\lambda=1$:

$$Y=\mathbb{W},\ X=H^{-\frac{1}{2}}(\Gamma),\ T=\eta\nabla\mathcal{S}_0,\ S=\gamma_{\mathbf{n}}.$$

This gives:

$$\begin{aligned}((\mathbb{I}-\eta A_0)|_{\mathbb{W}})' &= (\mathbb{I}-\eta\nabla\mathcal{S}_0\gamma_{\mathbf{n}})' \\ &= \eta\nabla\mathcal{S}_0\left(\mathbb{I}-\eta\gamma_{\mathbf{n}}\nabla\mathcal{S}_0\right)'\gamma_{\mathbf{n}}+\mathbb{I} \\ &= \eta\nabla\mathcal{S}_0\left(\mathbb{I}-\eta\left(\frac{1}{2}\mathbb{I}+K_0'\right)\right)'\gamma_{\mathbf{n}}+\mathbb{I} \\ &= \eta\nabla\mathcal{S}_0\left(\frac{2-\eta}{2}\mathbb{I}-\eta K_0'\right)'\gamma_{\mathbf{n}}+\mathbb{I}.\end{aligned}$$

and the desired statement follows. □

We now pass on to the proof of the second theorem:

Proof of Theorem 5.3. We start in the same manner as was done at the beginning of the precedent proof.

Due to the hypothesis on μ_r, with the help of Lemma 3.6 and Corollary 4.3, the operator $\mathbb{I}-\nu B_k$ has a regularizer on $\mathbb{H}(\mathrm{curl},\Omega)$.

Because the difference B_k-B_0 is compact on $\mathbb{H}(\mathrm{curl},\Omega)$, we could consider the operator $\mathbb{I}-\nu B_0$.

We will use once again the following orthogonal decomposition:

$$\mathbb{H}(\mathrm{curl},\Omega) = \nabla H_0^1(\Omega)\oplus\mathbb{X} \tag{30}$$

with $\mathbb{X}=\mathbb{H}(\mathrm{curl},\Omega)\cap\mathbb{H}(\mathrm{div}0,\Omega)$.

As a regularizer on $\nabla H_0^1(\Omega)$ for $(\mathbb{I}-\nu B_0)|_{\mathbb{X}}=\mathbb{I}$ is trivial, let us focus on the second subspace $\mathbb{X}$.

Remember that $(\mathbb{I}-\nu B_0)\mathbf{u}=(\mathbb{I}-\nu\mathrm{curl}\mathcal{N}_0\mathrm{curl})\mathbf{u}$ for $\mathbf{u}\in\mathbb{H}(\mathrm{curl},\Omega)$, and so also on $\mathbb{X}$. For $\lambda=1$, we apply then Lemma 3.4 in the case where

$$\begin{aligned}&Y=\mathbb{X},\ X=\mathbb{H}(\mathrm{div}0,\Omega),\\ &S: Y\to X,\ \mathbf{u}\mapsto\mathrm{curl}\mathbf{u},\\ &T: X\to Y,\ \mathbf{w}\mapsto\nu\mathrm{curl}\mathcal{N}_0\mathbf{w}.\end{aligned}$$

This gives, for $\mathbf{u}\in\mathbb{X}$ and using formula (23):

$$\begin{aligned}(\mathbb{I}-\nu\mathrm{curl}\mathcal{N}_0\mathrm{curl})'\mathbf{u} &= (\nu\mathrm{curl}\mathcal{N}_0(\mathbb{I}-\nu\mathrm{curl}\mathrm{curl}\mathcal{N}_0)'\mathrm{curl}+\mathbb{I})\mathbf{u}\\ &= (\nu\mathrm{curl}\mathcal{N}_0(\mathbb{I}-\nu(\mathbb{I}-\nabla\mathcal{S}_0\gamma_{\mathbf{n}}))'\mathrm{curl}+\mathbb{I})\mathbf{u}\\ &= (\nu\mathrm{curl}\mathcal{N}_0((1-\nu)\mathbb{I}+\nu\nabla\mathcal{S}_0\gamma_{\mathbf{n}})'\mathrm{curl}+\mathbb{I})\mathbf{u}.\end{aligned} \tag{31}$$

To obtain a regularizer for $(1-\nu)\mathbb{I}+\nu\nabla\mathcal{S}_0\gamma_{\mathbf{n}}$, we use again Lemma 3.4 for $\lambda=1-\nu\neq 0$ with:

$$\begin{aligned}&X=H^{-\frac{1}{2}}(\Gamma),\ Y=\mathbb{H}(\mathrm{div}0,\Omega),\\ &S: Y\to X,\ \mathbf{u}\mapsto\gamma_{\mathbf{n}}\mathbf{u},\\ &T: X\to Y,\ v\mapsto -\nu\nabla\mathcal{S}_0 v.\end{aligned}$$

This leads to:

$$((1-\nu)\mathbb{I}+\nu\nabla\mathcal{S}_0\gamma_{\mathbf{n}})' = \frac{-\nu}{1-\nu}\nabla\mathcal{S}_0((1-\nu)\mathbb{I}+\nu\gamma_{\mathbf{n}}\nabla\mathcal{S}_0)'\gamma_{\mathbf{n}} + \frac{1}{1-\nu}\mathbb{I}.$$
$$= \frac{1}{1-\nu}\mathbb{I} - \frac{\nu}{1-\nu}\nabla\mathcal{S}_0\left(\frac{2-\nu}{2}\mathbb{I}+\nu K_0'\right)'\gamma_{\mathbf{n}}. \quad (32)$$

If we use this last identity in expression (31), we find on $\mathbb{X}$:

$$(\mathbb{I}-\nu\mathrm{curl}\mathcal{N}_0\mathrm{curl})' = \frac{\nu}{1-\nu}\mathrm{curl}\mathcal{N}_0\mathrm{curl} - \frac{\nu^2}{1-\nu}\mathrm{curl}\mathcal{N}_0\nabla\mathcal{S}_0R_\nu\gamma_{\mathbf{n}}\mathrm{curl}+\mathbb{I},$$

where $R_\nu = \left(\frac{2-\nu}{2}\mathbb{I}+\nu K_0'\right)'$ denotes a regularizer for $\frac{2-\nu}{2}\mathbb{I}+\nu K_0'$ on $H^{-\frac{1}{2}}(\Gamma)$.

This completes the proof of the theorem. □

We end this paper with a demonstration of the third and last theorem:

Proof of Theorem 5.4. In a similar way to the previous proof, the hypothesis on ε_r and μ_r guarantee the existence of a regularizer for $\mathbb{I}-\eta A_k+\nu B_k$ on $\mathbb{H}(\mathrm{curl},\Omega)$.

Using property (24), we need only consider the operator $\mathbb{I}-\eta A_0-\nu B_0$.

We will use the same orthogonal decomposition as in the precedent proof, i.e., the decomposition (30).

With the help of (29), we obtain:

- For $\mathbf{u}\in\nabla H_0^1(\Omega)$, $(\mathbb{I}-\eta A_0+\nu B_0)\mathbf{u} = (\mathbb{I}-\eta A_0)\mathbf{u} = (1-\eta)\mathbf{u}$.
- On the second subspace $\mathbb{X}$:

$$(\mathbb{I}-\eta A_0-\nu B_0)\mathbf{u} = (\mathbb{I}-\eta\nabla\mathcal{S}_0\gamma_{\mathbf{n}}-\nu\mathrm{curl}\mathcal{N}_0\mathrm{curl})\mathbf{u}.$$

To find a regularizer for $\mathbb{I}-\eta A_0-\nu B_0$ on $\mathbb{X}$, we apply the second statement of Lemma 3.4 to the operators S and T defined by (25) for $\lambda=1$. This gives:

$$(\mathbb{I}-\eta A_0-\nu B_0)' = S(\mathbb{I}-TS)'T+\mathbb{I},$$

where $(\mathbb{I}-TS)'$ is a regularizer on $H^{-\frac{1}{2}}(\Gamma)\times\mathbb{H}(\mathrm{div}0,\Omega)$ for

$$\mathbb{I}-TS = \begin{pmatrix} \frac{2-\eta}{2}\mathbb{I}-\eta K_0' & -\nu\gamma_{\mathbf{n}}\mathrm{curl}\mathcal{N}_0 \\ 0 & \mathbb{I}-\nu\mathrm{curl}\mathrm{curl}\mathcal{N}_0 \end{pmatrix}.$$

To find an expression of $(\mathbb{I}-TS)'$, we write the last operator matrix in the form $\begin{pmatrix} C & D \\ 0 & E\end{pmatrix}$ where

$$C : v\in H^{-\frac{1}{2}}(\Gamma)\mapsto \frac{2-\eta}{2}v-\eta K_0'v\in H^{-\frac{1}{2}}(\Gamma),$$
$$D : \mathbf{u}\in\mathbb{H}(\mathrm{div}0,\Omega)\mapsto -\nu\gamma_{\mathbf{n}}\mathrm{curl}\mathcal{N}_0\mathbf{u}\in H^{-\frac{1}{2}}(\Gamma),$$
$$E : \mathbf{u}\in\mathbb{H}(\mathrm{div}0,\Omega)\mapsto \mathbf{u}-\nu\mathrm{curl}\mathrm{curl}\mathcal{N}_0\mathbf{u}\in\mathbb{H}(\mathrm{div}0,\Omega),$$

and then apply the second statement of Lemma 3.5. This gives a regularizer of the form:

$$(\mathbb{I}-TS)' = \begin{pmatrix} C' & -C'DE' \\ 0 & E'\end{pmatrix},$$

where C' and E' are regularizers for C and E on $H^{-\frac{1}{2}}(\Gamma)$ and $\mathbb{H}(\mathrm{div}0,\Omega)$, respectively.

We already know from the precedent proof that:

$$E = \mathbb{I}-\nu\mathrm{curl}\mathrm{curl}\mathcal{N}_0 = (1-\nu)\mathbb{I}+\nu\nabla\mathcal{S}_0\gamma_{\mathbf{n}}.$$

Also, we have obtained from that proof an expression of $((1-\nu)\mathbb{I}+\nu\nabla\mathcal{S}_0\gamma_{\mathbf{n}})'$, see formula (32), and the statement of the theorem follows. □

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