Global existence and blow-up of weak solutions for a fourth-order parabolic equation with gradient nonlinearity

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Abstract. This article deals with the behaviors of solutions to the initial-boundary value problem for a fourth-order parabolic equation with gradient nonlinearity. More precisely, we first get a threshold result for the solutions to exist globally or to blow up in finite time when the initial energy is subcritical and critical, and give an upper bound estimate of the lifespan. Furthermore, we derive the sufficient conditions for the existence of global and blow-up solutions for supercritical initial energy. Finally, we also give a lower bound estimate of the lifespan and obtain some estimates for blow-up rate. These results extend and improve some recent results.

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1. Introduction

In this paper, we consider the following initial-boundary value problem for a fourth-order parabolic equation with gradient nonlinearity:

$$\begin{cases} u_t + (-\Delta)^2 u = -\nabla \cdot (|\nabla u|^{p-2}\nabla u), & (x,t) \in \Omega \times (0,T), \\ \partial_\nu u = \partial_\nu \Delta u = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geqslant 2)$ with smooth boundary, $u_0 \in L^2(\Omega)$, $T > 0$, and ∂_ν denotes the outer normal derivative to $\partial\Omega$. Moreover, the parameter p satisfies one of the following conditions:

$(H1)$ $2 < p < \infty$, if $N = 2$; $2 < p < \frac{2N}{N-2}$, if $N > 2$.

$(H2)$ $2 < p < 2 + \frac{4}{N+2}, N \geq 2$.

In recent years, the epitaxial growth of nanoscale thin films has attracted considerable attentions in materials science. To clarify such phenomena, we first sketch the lines along which the studied model is derived. Due to Zangwill in [14], for a spatial variable x in the domain $\Omega = [0,L]^2$, the continuum model for epitaxial thin-film growth reads

$$u_t = g - \nabla \cdot j + \eta, \tag{1.2}$$

where $u(x,t)$ denotes the height of a film in epitaxial growth with $g = g(x,t)$, $j = j(x,t)$ and $\eta = \eta(x,t)$ being the deposition flux, all processes of moving atoms along the surface and Gaussian noise, respectively. Purely phenomenologically, one can expand $j(x,t)$ in a power series involving the surface slope ∇u and various powers and derivatives thereof. This simple case (to keep only "sensible" terms (see [14] for details)) is

$$j = A_1 \nabla u + A_2 \nabla(\Delta u) + A_3 |\nabla u|^2 \nabla u. \tag{1.3}$$

The spatial derivatives in (1.3) have the following physical interpretations:

$$A_1 \Delta u : \text{diffusion due to evaporation} - \text{condensation} [2, 10];$$

$$A_2 \Delta^2 u : \text{capillarity driven surface diffusion} [5, 10];$$

$$A_3 \nabla \cdot (|\nabla u|^{p-2} \nabla u) : \text{(upward) hopping of atoms} [1].$$

Hence, if we drop Gaussian noise, then Eq. (1.2) becomes

$$u_t + A_1 \Delta u + A_2 \Delta^2 u + A_3 \nabla \cdot (|\nabla u|^{p-2} \nabla u) = g. \tag{1.4}$$

Since these models describe the complex process of making a thin-film layer on a substrate by chemical vapor deposition, one of some interesting problems is to analyze these processes quantitatively on the correct scale so that we can deeply understand and optimize the particular properties of the film. These equations such as (1.4) under different initial and boundary conditions have been investigated extensively during the past few years. King et al. [7] studied the existence of global-in-time solutions and large time behavior of solutions to (1.4) in an appropriate function space for the case $A_1 = A_3 = 1$ and $g = 0$. Later, Sandjo et al. [12] proved the local and global existence of solutions for similar problems. Recently, for the case $A_1 = 0$, $A_2 = 1$, $A_3 = 1$ and $g = 0$, Ishige et al. [6] first obtained the local existence and singularity behavior on the whole space. Later, Miyake and Okabe [9] combined the well-known potential well method with the Galerkin method to derive the precise asymptotic behavior of global-in-time solution to problem (1.1) when the initial energy is subcritical. However, some problems are unsolved in [9].

- Whether or not will the advection term $\nabla \cdot (|\nabla u|^{p-2} \nabla u)$ cause the finite-time blow-up?
- Whether or not can the supercritical initial energy also cause the finite-time blow-up?
- If the finite-time blow-up happens, can we give some estimates for blow-up rate?

In this paper, we give a positive answer to the problems above. To be precise, we combine some energy estimates from [9] and the modified potential well method, which was first proposed by Payne and Sattinger [11,13], with differential inequality arguments to prove that the solution globally exists when the initial energy starts from the stable set and fails to globally exist when the initial energy starts from the unstable set. For the high initial energy, to remedy the failure of potential well method, we borrow some ideas inspired by the study of dynamical system to prove that the functional $\int_\Omega |u|^2 dx$ is monotonically increasing with respect to time variable, which helps us establish a substitute for unstable sets. Meanwhile, we also obtain an upper bound of the lifespan. Finally, we obtain a lower bound estimate for the lifespan by constructing a first-order differential inequality. These results extended and improved some existing results [9].

This paper is organized as follows. In Sect. 2, we give some notations, definitions and lemmas concerning the basic properties of the related functionals and sets. Sections 3, 4 and 5 will be devoted to the cases $J(u_0) < d$, $J(u_0) = d$ and $J(u_0) > d$, respectively. In Sect. 6, we consider the lower bound estimate for the lifespan.

2. Preliminaries

In this section, we first introduce some notations and definitions that will be used throughout the paper. In what follows, we denote by $\| \cdot \|_r (r \geq 1)$ the norm in $L^r(\Omega)$ and by $(\cdot, \cdot)$ the $L^2(\Omega)$-inner product. C denotes a generic positive constant, which may differ at each appearance. In addition, we set

$$\mathscr{V} := \left\{ \varphi \in H^1\left(0, T; L^2_{\mathscr{N}}(\Omega)\right) \cap L^2\left(0, T; H^2_{\mathscr{N}}(\Omega)\right) \mid \nabla\varphi \in (L^p\left(0, T; L^p(\Omega)\right))^N \right\},$$

where

$$L^2_{\mathcal{N}}(\Omega) := \left\{ v \in L^2(\Omega) \mid \int_{\Omega} v dx = 0 \right\} \subset L^2(\Omega),$$

$$H^2_{\mathcal{N}}(\Omega) := \left\{ v \in H^2(\Omega) \cap L^2_{\mathcal{N}}(\Omega) \mid \partial_{\nu} v = 0 \text{ on } \partial\Omega \right\} \subset H^2(\Omega).$$

We mention several remarks on $L^2_{\mathcal{N}}(\Omega)$ and $H^2_{\mathcal{N}}(\Omega)$. As stated in [7], the map $\Delta : H^2_{\mathcal{N}}(\Omega) \to L^2_{\mathcal{N}}(\Omega)$ is a homeomorphism and hence there exists a constant $c_1 = c_1(N) > 0$ such that

$$c_1^{-1} \sum_{k=0}^{2} \|\nabla^k v\|_2^2 \leq \|v\|_{H^2_{\mathcal{N}}}^2 := \|\Delta v\|_2^2, \quad v \in H^2_{\mathcal{N}}(\Omega).$$

Before stating our main results, we introduce the definition of weak solution to problem (1.1) in [9].

Definition 2.1. [9] Let $u_0 \in L^2_{\mathcal{N}}(\Omega)$ and $T > 0$. We say that a function

$$u \in C\left(0, T; L^2_{\mathcal{N}}(\Omega)\right) \cap L^2\left(0, T; H^2_{\mathcal{N}}(\Omega)\right) \quad \text{with} \quad \nabla u \in (L^p\left(0, T; L^p(\Omega)\right))^N$$

is a solution to problem (1.1) in $\Omega \times [0, T]$ if u satisfies

$$\int_{\Omega} [u(T)\varphi(T) - u_0\varphi(0)] \, dx$$

$$- \int_0^T \int_{\Omega} u \partial_t \varphi dx dt + \int_0^T \int_{\Omega} \left[\nabla u \nabla \varphi - |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \right] dx dt = 0, \tag{2.1}$$

for all $\varphi \in \mathcal{V}$. Moreover, we say that u is a global-in-time solution to problem (1.1) if u is a solution to problem (1.1) in $\Omega \times [0, T']$ for all $T' > 0$.

3. The case $J(u_0) < d$

In this section, on the basis of reference [9], we continue to study the blow-up properties of solutions to problem (1.1) under the condition that $J(u_0) < d$, and we will give the threshold result of solutions to exist globally or to blow up in finite time. First, we give the definition of the solution blow-up in finite time.

Definition 3.1. Let $u(t)$ be a weak solution to problem (1.1), define the maximal existence time of $u(t)$ by

$$T^* = \sup\{T > 0 : u = u(t) \text{ exists on } [0, T]\}.$$

We say that $u(t)$ blows up at a finite time $T^* < \infty$ provided that

$$\lim_{t \to T^*} \|u\|_2^2 = +\infty.$$

In order to investigate the blow-up properties of solutions to problem (1.1), we introduce the following functionals. For $u \in H^2_{\mathcal{N}}(\Omega)$, define the energy functional associated with problem (1.1)

$$J(u) = \frac{1}{2}\|\Delta u\|_2^2 - \frac{1}{p}\|\nabla u\|_p^p, \quad \forall\, t \geq 0. \tag{3.1}$$

By a simple calculation and density argument, it is not hard to verify that $J(u)$ is a nonincreasing function on $[0, +\infty)$ and satisfies

$$\frac{d}{dt}J(u) = -\|u_t\|_2^2 \leq 0, \quad \forall\, t \geq 0, \quad \text{i.e.,} \quad J(u_0) = J(u) + \int_0^t \|u_s\|_2^2 \, ds, \tag{3.2}$$

where

$$J(u_0) = \frac{1}{2}\|\Delta u_0\|_2^2 - \frac{1}{p}\|\nabla u_0\|_p^p.$$

Further, we also define Nehari functional

$$I(u) = <J'(u), u> = \|\Delta u\|_2^2 - \|\nabla u\|_p^p, \tag{3.3}$$

and the Nehari manifold

$$\mathcal{N} = \{u \in H_{\mathcal{N}}^2(\Omega)\backslash\{0\} \mid I(u) = 0\}.$$

Owing to $p \leq 2^*$, it is obvious to find that both $J(u)$ and $I(u)$ are well defined. So we define

$$W = \{u \in H_{\mathcal{N}}^2(\Omega) \mid J(u) < d, I(u) > 0\} \cup \{0\},$$
$$V = \{u \in H_{\mathcal{N}}^2(\Omega) \mid J(u) < d, I(u) < 0\},$$

where

$$d = \inf_{u \in H_{\mathcal{N}}^2(\Omega)\backslash\{0\}} \sup_{\lambda \geq 0} J(\lambda u) = \inf_{u \in \mathcal{N}} J(u)$$

is the depth of the potential well W.

Next, we state some existing results of the precise asymptotic behavior [9].

Theorem 3.1. [9] *Let $(H1)$ hold and $u_0 \in W$. Then problem (1.1) possesses the unique global-in-time solution u satisfying*

$$\|\Delta u\|_2 = O\left(e^{-\mu_1^2 t}\right) \quad as \quad t \to \infty.$$

Obviously, when $u_0 \in V$, there are seldom any results. Subsequently, we give the main result about the blow-up properties of solutions to problem (1.1).

Theorem 3.2. *Let $(H1)$ hold and u be a weak solution of problem (1.1) with $u_0 \in L_{\mathcal{N}}^2(\Omega)$. If $u_0 \in V$, then there exists a finite time T^* such that u blows up at T^*. Moreover, T^* can be estimated from above as follows*

$$T^* \leq \frac{pd}{(p-2)^2 B^p(d - J(u_0))\|u_0\|_2^{p-2}}.$$

Proofs of Theorem 3.2 begin with the following two important lemmas.

Lemma 3.1. *The potential depth d is positive.*

Proof. From the definition of d, we obtain

$$\begin{aligned} d &= \inf_{u \in H_{\mathcal{N}}^2(\Omega)\backslash\{0\}} \sup_{\lambda \geq 0} J(\lambda u) = \inf_{u \in H_{\mathcal{N}}^2(\Omega)\backslash\{0\}} \left\{ \frac{\lambda_0^2}{2}\|\Delta u\|_2^2 - \frac{\lambda_0^p}{p}\|\nabla u\|_p^p \right\} \\ &= \inf_{u \in H_{\mathcal{N}}^2(\Omega)\backslash\{0\}} \left\{ \frac{1}{2}\left(\frac{\|\Delta u\|_2^2}{\|\nabla u\|_p^p}\right)^{\frac{2}{p-2}}\|\Delta u\|_2^2 - \frac{1}{p}\left(\frac{\|\Delta u\|_2^2}{\|\nabla u\|_p^p}\right)^{\frac{p}{p-2}}\|\nabla u\|_p^p \right\} \\ &= \frac{p-2}{2p} \inf_{u \in H_{\mathcal{N}}^2(\Omega)\backslash\{0\}} \left(\frac{\|\Delta u\|_2^2}{\|\nabla u\|_p^2}\right)^{\frac{p}{p-2}} = \frac{p-2}{2p}S_p^{\frac{p}{p-2}} > 0, \end{aligned} \tag{3.4}$$

where

$$S_p = \inf_{u \in H^2_{\mathcal{N}}(\Omega)\setminus\{0\}} \frac{\|\Delta u\|_2^2}{\|\nabla u\|_p^2}, \quad \text{i.e.,} \quad S_p\|\nabla u\|_p^2 \le \|\Delta u\|_2^2.$$

$\square$

The following lemma shows that the set V is invariant under the semi-flow of problem (1.1). Meanwhile, as a by-product, we also establish the precise relation between $\min\left\{\|\nabla u\|_p^p, \|\Delta u\|_2^2\right\}$ and the depth d.

Lemma 3.2. *Assume that* $(H1)$ *holds and* $u_0 \in V$. *Then* $u(t) \in V$ *for all* $t \in [0, T^*)$ *and*

$$d < \frac{p-2}{2p} \min\left\{\|\nabla u\|_p^p, \|\Delta u\|_2^2\right\}. \tag{3.5}$$

Proof. We first prove $u(t) \in V$ for all $t \in [0, T^*)$ by arguing by contradiction. If there exists $t' > 0$ such that $u(t') \notin V$, then from $I(u_0) < 0$ and the continuity of $I(u(t))$ with respect to time variable t, we know that there exists a $t_0 \in [0, T^*)$ such that $I(u(t)) < 0$ for all $t \in [0, t_0)$ and $I(u(t_0)) = 0$, then $u(t_0) \in \mathcal{N}$.

On the one hand, the fact $u(t_0) \in \mathcal{N}$ and the definition of d show

$$J(u(t_0)) \ge d. \tag{3.6}$$

On the other hand, due to the monotonicity of $J(u)$, we get $J(u) \le J(u_0) < d$ for all $t \in [0, T^*)$, which contradicts with (3.6). Consequently, we prove that $u(t) \in V$ for all $t \in [0, T^*)$.

Next, we will prove (3.5). According to $u(t) \in V$ for all $t \in [0, T^*)$, we know $I(u) < 0$, which implies $\|\Delta u\|_2^2 < \|\nabla u\|_p^p$. Then, to utilize the embedding inequality $S_p\|\nabla u\|_p^2 \le \|\Delta u\|_2^2$ and the definition of d, we obtain (3.5). $\square$

Proof of Theorem 3.2.. Let

$$F_1(t) = \frac{1}{2}\int_\Omega |u|^2 dx.$$

Taking a derivative of $F_1(t)$, and combining with problem (1.1), Green formula, Lemma 3.2 and embedding inequality, we obtain

$$F_1'(t) = \int_\Omega uu_t dx = \int_\Omega u\left(-\Delta^2 u - \Delta_p u\right)dx = -\int_\Omega |\Delta u|^2 dx + \int_\Omega |\nabla u|^p dx$$

$$= \frac{p-2}{p}\int_\Omega |\nabla u|^p dx - 2J(u) > \left(1 - \frac{J(u_0)}{d}\right)\frac{p-2}{p}\int_\Omega |\nabla u|^p dx$$

$$\ge B^p\left(1 - \frac{J(u_0)}{d}\right)\frac{p-2}{p}\left(\int_\Omega |u|^2 dx\right)^{\frac{p}{2}}$$

$$= \left(\sqrt{2}B\right)^p\left(1 - \frac{J(u_0)}{d}\right)\frac{p-2}{p}F_1^{\frac{p}{2}}(t) := CF_1^{\frac{p}{2}}(t), \quad \forall\, t \in [0, T^*), \tag{3.7}$$

where $B\|u\|_2 \le \|\nabla u\|_p$ and $C = \left(\sqrt{2}B\right)^p\left(1 - \frac{J(u_0)}{d}\right)\frac{p-2}{p}$.

A simple integration of (3.7) over $(0, t)$ is easy to calculate that

$$\|u\|_2^2 \ge \frac{2}{\left[\left(\frac{1}{2}\|u_0\|_2^2\right)^{1-\frac{p}{2}} - \frac{p-2}{2}Ct\right]^{\frac{2}{p-2}}}. \tag{3.8}$$

Therefore,

$$\lim_{t \to \frac{2}{C(p-2)} \left(\frac{1}{2} \|u_0\|_2^2 \right)^{1-\frac{p}{2}}} \|u\|_2^2 = +\infty. \tag{3.9}$$

At the same time, it is easy to see that the upper bound for the blow-up time T^* satisfies

$$T^* \leq \frac{pd}{(p-2)^2 B^p (d - J(u_0)) \|u_0\|_2^{p-2}}. \tag{3.10}$$

The proof of Theorem 3.2 is complete. $\square$

To sum up, we may obtain the following sharp results for the subcritical case.

Remark 3.1. (*Sharp condition for $J(u_0) < d$.*) Let (H1) hold and $u_0 \in H^2_{\mathcal{N}}(\Omega)$. Assume that $J(u_0) < d$. If $I(u_0) > 0$, then problem (1.1) admits a global weak solution; if $I(u_0) < 0$, then the solution to problem (1.1) blows up in finite time.

4. The case $J(u_0) = d$

For the critical case $J(u_0) = d$, the invariance of W cannot be true in general. To overcome this difficulty, we borrow some ideas from [8]; we find out a substitute for unstable sets to obtain similar results as the subcritical case. First, we give a key lemma.

Lemma 4.1. *Let (H1) hold. Then for any $u \in H^2_{\mathcal{N}}(\Omega) \backslash \{0\}$, we have*
(i) $\lim_{\lambda \to 0^+} J(\lambda u) = 0$, $\lim_{\lambda \to +\infty} J(\lambda u) = -\infty$.
(ii) *There exists a unique $\lambda^* = \lambda^*(u) > 0$ such that $\frac{d}{d\lambda} J(\lambda u)|_{\lambda = \lambda^*} = 0$. $J(\lambda u)$ is increasing on $0 < \lambda \leq \lambda^*$, decreasing on $\lambda^* \leq \lambda < +\infty$ and takes its maximum at $\lambda = \lambda^*$.*
(iii) $I(\lambda u) > 0$ *on* $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ *on* $\lambda^* < \lambda < +\infty$ *and* $I(\lambda^* u) = 0$.

Since the proof is standard, we will omit more details here. The interested readers may refer to [3,4,8,15]. The forthcoming theorem deals with the critical case.

Theorem 4.1. (Global Existence for $J(u_0) = d$.) *Let (H1) hold and $u_0 \in H^2_{\mathcal{N}}(\Omega)$. If $J(u_0) = d$ and $I(u_0) \geq 0$, then problem (1.1) admits a global weak solution.*

Proof. Let $\lambda_k = 1 - \frac{1}{k}, k = 1, 2, \ldots$. Consider the following initial boundary value problem

$$\begin{cases} u_t + (-\Delta)^2 u = -\nabla \cdot (|\nabla u|^{p-2} \nabla u), & (x, t) \in \Omega \times (0, T), \\ \partial_\nu u = \partial_\nu \Delta u = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = \lambda_k u_0(x) \triangleq u_0^k, & x \in \Omega. \end{cases} \tag{4.1}$$

Noticing that $I(u_0) \geq 0$, by Lemma 4.1(iii) we can deduce that there exists a unique $\lambda^* = \lambda^*(u_0) \geq 1$ such that $I(\lambda^* u_0) = 0$. Then from $0 < \lambda_k < 1 \leq \lambda^*$ and Lemma 4.1(ii), we get $I(u_0^k) = I(\lambda_k u_0) > 0$ and $J(u_0^k) = J(\lambda_k u_0) < J(u_0) = d$. In view of Theorem 3.1 in [9], it follows that for each k problem (4.1) admits a global weak solution $u^k \in C\left(0, T; L^2_{\mathcal{N}}(\Omega)\right) \cap L^2\left(0, T; H^2_{\mathcal{N}}(\Omega)\right)$ with $\nabla u^k \in (L^p\left(0, T; L^p(\Omega)\right))^N$ and $u^k \in W$ for $0 \leq t < \infty$ satisfying

$$\int_0^t \|u_s^k\|_2^2 \, ds + J(u^k) = J(u_0^k) < d.$$

Applying the arguments similar to those in Theorem 3.1, we see that there exist a subsequence of $\{u^k\}$ and a function u such that u is a weak solution of problem (1.1) with $I(u) \geq 0$ and $J(u) \leq d$ for $0 \leq t < \infty$. $\square$

Theorem 4.2. (*Blow-up for $J(u_0) = d$.*) *Let (H1) hold and u be a weak solution of problem* (1.1) *with $u_0 \in H^2_{\mathcal{N}}(\Omega)$. If $J(u_0) = d$ and $I(u_0) < 0$, then there exists a finite time T^* such that u blows up at T^*.*

Proof. Similarly to the proof of Theorem 3.2, we can get

$$F_1'(t) \geq \frac{p-2}{p} \int\limits_{\Omega} |\nabla u|^p dx - 2J(u).$$

Since $J(u_0) = d$, $I(u_0) < 0$, by the continuity of $J(u)$ and $I(u)$ with respect to t, there exists a $t_0 > 0$ such that $J(u(x,t)) > 0$ and $I(u(x,t)) < 0$ for $0 < t \leq t_0$. From $(u_t, u) = -I(u)$, we have $u_t \not\equiv 0$ for $0 < t \leq t_0$. Furthermore, we have

$$J(u(t_0)) \leq d - \int\limits_{0}^{t_0} \|u_s\|_2^2 \, ds < d.$$

Taking $t = t_0$ as the initial time and by Lemma 3.2, we know that $u(x,t) \in V$ for $t > t_0$. The reminder of the proof is almost the same as that of Theorem 3.2 and hence is omitted. $\qquad\square$

In short, we also have the following conclusions.

Remark 4.1. (*Sharp condition for $J(u_0) = d$.*) Let (H1) hold and $u_0 \in H^2_{\mathcal{N}}(\Omega)$. Assume that $J(u_0) = d$. If $I(u_0) \geq 0$, then problem (1.1) admits a global weak solution; if $I(u_0) < 0$, then problem (1.1) admits no global weak solution.

5. The case $J(u_0) > d$

In this section, inspired by some ideas from [4], we will investigate the conditions to ensure the existence of global solutions or blow-up solutions to problem (1.1) with $J(u_0) > d$. Before moving on to our result, let us pause to give some pivotal sets and functionals.

$$\mathcal{N}_+ = \{u \in H^2_{\mathcal{N}}(\Omega)\backslash\{0\} \mid I(u) > 0\}, \ \mathcal{N}_- = \{u \in H^2_{\mathcal{N}}(\Omega)\backslash\{0\} \mid I(u) < 0\},$$
$$J^\alpha = \left\{u \in H^2_{\mathcal{N}}(\Omega)\backslash\{0\} \mid J(u) < \alpha\right\},$$
$$\mathcal{N}^\alpha = \mathcal{N} \cap J^\alpha = \{u \in \mathcal{N} \mid J(u) < \alpha\}, \quad \forall\, \alpha > d.$$

Also define

$$\lambda_\alpha = \inf\left\{\|u\|_2^2 \mid u \in \mathcal{N}^\alpha\right\}, \quad \Lambda_\alpha = \sup\left\{\|u\|_2^2 \mid u \in \mathcal{N}^\alpha\right\}. \tag{5.1}$$

It is clear that λ_α (Λ_α) is nonincreasing (nondecreasing) with respect to α. In addition, we introduce the following three sets:

$$\mathcal{B} = \left\{u_0 \in H^2_{\mathcal{N}}(\Omega) \mid \text{the solution } u = u(t) \text{ of (1.1) blows up in finite time }\right\},$$
$$\mathcal{G} = \left\{u_0 \in H^2_{\mathcal{N}}(\Omega) \mid \text{the solution } u = u(t) \text{ of (1.1) exists for all } t > 0\right\},$$
$$\mathcal{G}_0 = \left\{u_0 \in H^2_{\mathcal{N}}(\Omega) \mid u(t) \to 0 \text{ in } H^2_{\mathcal{N}}(\Omega) \text{ as } t \to \infty\right\}.$$

To better analyze the behavior of the solutions to problem (1.1) with high energy level, we first present some useful lemmas.

Lemma 5.1. *Let (H1) hold. Then*
(i) *0 is away from both $\mathcal{N}$ and $\mathcal{N}_-$, i.e., $dist(0, \mathcal{N}) > 0$, $dist(0, \mathcal{N}_-) > 0$.*
(ii) *For any $\alpha > 0$, the set $J^\alpha \cap \mathcal{N}_+$ is bounded in $H^2_{\mathcal{N}}(\Omega)$.*

Proof. (i) For any $u \in \mathcal{N}$, according to the definition of d and $I(u)$, we have

$$d \le J(u) = \frac{1}{2}\|\Delta u\|_2^2 - \frac{1}{p}\|\nabla u\|_p^p = \frac{p-2}{2p}\|\Delta u\|_2^2, \tag{5.2}$$

which implies

$$\|\Delta u\|_2^2 \ge \frac{2pd}{p-2}.$$

Therefore, we know that $dist(0, \mathcal{N}) = \inf\limits_{u \in \mathcal{N}} \|u\|_{H_{\mathcal{N}}^2} = \inf\limits_{u \in \mathcal{N}} \|\Delta u\|_2 > 0$.

For any $u \in \mathcal{N}_-$, from the embedding inequality, we have

$$\|\Delta u\|_2^2 < \|\nabla u\|_p^p \le S_p^{-\frac{p}{2}}\|\Delta u\|_2^p, \tag{5.3}$$

which implies

$$\|\Delta u\|_2^2 \ge S_p^{\frac{p}{p-2}},$$

where $S_p > 0$ is given in Lemma 3.1. Thus, $dist(0, \mathcal{N}_-) = \inf\limits_{u \in \mathcal{N}_-} \|u\|_{H_{\mathcal{N}}^2} = \inf\limits_{u \in \mathcal{N}_-} \|\Delta u\|_2 > 0$.

(ii) For any $u \in J^\alpha \cap \mathcal{N}_+$, we obtain

$$\alpha \ge J(u) = \frac{1}{p}I(u) + \frac{p-2}{2p}\|\Delta u\|_2^2 > \frac{p-2}{2p}\|\Delta u\|_2^2, \tag{5.4}$$

which yields

$$\|u\|_{H_{\mathcal{N}}^2}^2 = \|\Delta u\|_2^2 < \frac{2p\alpha}{p-2}.$$

Therefore, the set $J^\alpha \cap \mathcal{N}_+$ is bounded in $H_{\mathcal{N}}^2(\Omega)$. $\qquad\square$

Next, we discuss the properties of λ_α and Λ_α.

Lemma 5.2. *Let $(H1)$ hold. Then for any $\alpha > d$, λ_α and Λ_α defined in (5.1) satisfy $0 < \lambda_\alpha \le \Lambda_\alpha \le M_\alpha < +\infty$.*

Proof. First, we prove $\Lambda_\alpha \le M_\alpha < +\infty$, where $\Lambda_\alpha = \sup\left\{\|u\|_2^2 \mid u \in \mathcal{N}^\alpha\right\}$. For any $u \in \mathcal{N}^\alpha$,

$$J(u) = \frac{1}{2}I(u) + \frac{p-2}{2p}\|\nabla u\|_p^p = \frac{p-2}{2p}\|\nabla u\|_p^p \le \alpha, \tag{5.5}$$

which implies

$$\|\nabla u\|_p^p \le \frac{2p\alpha}{p-2}.$$

Then from embedding inequality, we have

$$\|u\|_2^2 \le B^{-2}\|\nabla u\|_p^2 \le B^{-2}\left(\frac{2p\alpha}{p-2}\right)^{\frac{2}{p}} := M_\alpha.$$

Therefore, we get $\Lambda_\alpha \le M_\alpha < +\infty$.

Next, we prove $\lambda_\alpha > 0$, where $\lambda_\alpha = \inf\left\{\|u\|_2^2 \mid u \in \mathcal{N}^\alpha\right\}$. The Gagliardo–Nirenberg inequality indicates that there exists a positive constant B_2 such that

$$\|\nabla u\|_p \le B_2\|\Delta u\|_2^\theta\|u\|_2^{1-\theta},$$

where $\theta = \frac{1}{2} + \frac{N(p-2)}{4p} \in \left(\frac{1}{2}, 1\right)$. Moreover, noticing that $u \in \mathcal{N} \Rightarrow \|\Delta u\|_2^{\frac{2}{p}} = \|\nabla u\|_p$, we have

$$\|u\|_2^2 \ge B_2^{-\frac{2}{1-\theta}}\left(\|\Delta u\|_2^2\right)^{\frac{1}{1-\theta}\left(\frac{2}{p}-\theta\right)}.$$

Obviously, by Lemma 5.1(i) and the definition of $\mathcal{N}^\alpha$, the right-hand side of the above inequality remains bounded away from 0 no matter what the sign of $\frac{2}{p} - \theta$ is. Therefore, $\lambda_\alpha > 0$. $\qquad\square$

In the following, we give a criterion for the existence of global solutions that tend to 0 as t tends to ∞ or finite-time blow-up solutions in terms of λ_α and Λ_α for supercritical initial energy, i.e., $J(u_0) > d$. Noticing that $\lambda_{J(u_0)} > 0$, Theorem 5.1(i) is nontrivial. Our main results are as follows:

Theorem 5.1. *Let* $(H1)$ *hold and* $J(u_0) > d$. *Then we have*

(1) *If* $u_0 \in \mathcal{N}_+$ *and* $\|u_0\|_2^2 \le \lambda_{J(u_0)}$, *then* $u_0 \in \mathcal{G}$. *That is, the weak solution* u *of problem* (1.1) *exists globally and* $u(t) \to 0$ *as* $t \to +\infty$.

(2) *If* $u_0 \in \mathcal{N}_-$ *and* $\|u_0\|_2^2 \ge \Lambda_{J(u_0)}$, *then* $u_0 \in \mathcal{B}$. *That is, the weak solution* u *of problem* (1.1) *blows up in finite time.*

Proof. We denote by $\omega(u_0) = \cap_{t \ge 0} \overline{\{u(s) : s \ge t\}}$ the $\omega-$limit of u_0. And as shown by Definition 3.1, T^* represents the maximum existence time of the solution.

(1) If $u_0 \in \mathcal{N}_+$ and $\|u_0\|_2^2 \le \lambda_{J(u_0)}$.

First, we claim that $u \in \mathcal{N}_+$, for all $t \in [0, T^*)$. By contradiction, there exists a $t_0 \in (0, T^*)$ such that $u \in \mathcal{N}_+$ for $t \in [0, t_0)$ and $u(t_0) \in \mathcal{N}$. Taking $\varphi = u$ in the definition of weak solution (2.1), we obtain

$$\frac{d}{dt}\|u\|_2^2 = 2(u, u_t) = -2I(u) < 0, \quad \forall\, t \in [0, t_0). \tag{5.6}$$

Then, we have

$$\|u(t_0)\|_2^2 < \|u_0\|_2^2 \le \lambda_{J(u_0)}. \tag{5.7}$$

On the other hand, (5.6) implies that $u_t \not\equiv 0$ for $(x, t) \in \Omega \times (0, t_0)$. It follows from (3.2) that $J(u(t_0)) < J(u_0)$, which yields $u(t_0) \in J^{J(u_0)}$. Therefore, $u(t_0) \in \mathcal{N}^{J(u_0)}$. By the definition of $\lambda_{J(u_0)}$, we obtain

$$\|u(t_0)\|_2^2 \ge \lambda_{J(u_0)}, \tag{5.8}$$

which contradicts (5.7). Therefore, $u \in \mathcal{N}_+$, for all $t \in [0, T^*)$. Further, we know that $u \in \mathcal{N}_+ \cap J^{J(u_0)}$ for all $t \in [0, T^*)$.

Next, Lemma 5.1(ii) shows that the orbit $u(t)$ remains bounded in $H^2_{\mathcal{N}}(\Omega)$ for $t \in [0, T^*)$ so that $T^* = \infty$.

Finally, we prove $\omega(u_0) = 0$, i.e., $u(t) \to 0$ when $t \to +\infty$. For any $\omega \in \omega(u_0)$, from (5.6) and the hypothesis, we can infer that

$$\|\omega\|_2^2 < \|u_0\|_2^2 \le \lambda_{J(u_0)}. \tag{5.9}$$

In addition, according to (3.2), we can deduce $\omega \in J^{J(u_0)}$ from $J(\omega) < J(u_0)$. Notice that (5.9) and $\lambda_{J(u_0)} = \inf \left\{ \|u\|_2^2 \,\middle|\, u \in \mathcal{N}^{J(u_0)} \right\}$, we obtain $\omega \notin \mathcal{N}^{J(u_0)}$, further $\omega \notin \mathcal{N}$. Thus, $\omega(u_0) \cap \mathcal{N} = \emptyset$, which indicates $\omega(u_0) = \{0\}$. Therefore, the weak solution u of problem (1.1) exists globally and $u(t) \to 0$ as $t \to +\infty$.

(2) If $u_0 \in \mathcal{N}_-$ and $\|u_0\|_2^2 \ge \Lambda_{J(u_0)}$.

We first claim that $u \in \mathcal{N}_-$, for all $t \in [0, T^*)$. By contradiction, there exists a $t_0 \in (0, T^*)$ such that $u \in \mathcal{N}_-$ for $t \in [0, t_1)$ and $u(t_1) \in \mathcal{N}$. Similar to case (1), we get $u(t_1) \in \mathcal{N}^{J(u_0)}$. Review the definition of $\Lambda_{J(u_0)}$, we get

$$\|u(t_1)\|_2^2 \le \Lambda_{J(u_0)}. \tag{5.10}$$

Moreover, noticing that $I(u(t)) < 0$ for $t \in [0, t_1)$, it follows from (5.6) that

$$\frac{d}{dt}\|u\|_2^2 = 2(u, u_t) = -2I(u) > 0, \quad \forall\, t \in [0, t_1). \tag{5.11}$$

Then, we have

$$\|u(t_1)\|_2^2 > \|u_0\|_2^2 \ge \Lambda_{J(u_0)}, \tag{5.12}$$

which contradicts (5.10). Therefore, $u \in \mathcal{N}_-$ for all $t \in [0, T^*)$. Further, we obtain $u \in \mathcal{N}_- \cap J^{J(u_0)}$ for all $t \in [0, T^*)$.

Next, we prove that $\omega(u_0) \cap \mathcal{N} = \emptyset$, $T^* < +\infty$. If not, suppose $T^* = +\infty$, then for any $\omega \in \omega(u_0)$, we have

$$\|\omega\|_2^2 > \|u_0\|_2^2 \geq \Lambda_{J(u_0)}, \tag{5.13}$$

and $J(\omega) < J(u_0)$ from (5.11) and (3.2), respectively. The second inequality implies $\omega \in J^{J(u_0)}$. Noting that the definition of $\Lambda_{J(u_0)}$ and (5.13), we derive $\omega \notin \mathcal{N}^{J(u_0)}$, further $\omega \notin \mathcal{N}$. Thus, $\omega(u_0) \cap \mathcal{N} = \emptyset$, which indicates $\omega(u_0) = \{0\}$, which is contradictive with Lemma 5.1(i). Hence, $\omega(u_0) = \emptyset$, $T^* < +\infty$. That is, the weak solution u of problem (1.1) blows up in finite time.

Based on the above discussion, we have completed the proof of Theorem 5.1. $\qquad\square$

Proposition 5.1. *If $J(u_0)$ satisfies $d < J(u_0) < \frac{p-2}{2p}B^p\|u_0\|_2^p$, then $u_0 \in \mathcal{N}_- \cap \mathcal{B}$. That is, the weak solution u of problem (1.1) blows up in finite time.*

Proof. From (3.3), (3.1) and embedding inequality, we obtain

$$J(u_0) = \frac{1}{2}I(u_0) + \frac{p-2}{2p}\|\nabla u_0\|_p^p \geq \frac{1}{2}I(u_0) + \frac{p-2}{2p}B^p\|u_0\|_2^p.$$

Combining the above inequality with the given assumption, we can obtain $I(u_0) < 0$, i.e., $u_0 \in \mathcal{N}_-$. In addition, according to the assumption and Lemma 5.2, we know that

$$\|u_0\|_2^2 > B^{-2}\left(\frac{2pJ(u_0)}{p-2}\right)^{\frac{2}{p}} = M_{J(u_0)} \geq \Lambda_{J(u_0)}.$$

Finally, by utilizing Theorem 5.1, it is easy to derive the conclusion of Proposition 5.1. $\qquad\square$

Theorem 5.2. *For any $M > d$, there exists initial value $u_M \in \mathcal{N}_-$ such that $J(u_M) \geq M$ and $u_M \in \mathcal{B}$.*

Proof. Assume that $M > d$ and Ω_1, Ω_2 are two arbitrary disjoint open subdomains of Ω. Furthermore, we assume that $\nu \in H_{\mathcal{N}}^2(\Omega_1)$ is an arbitrary nonzero function. Then, we take $\xi > 0$ large enough such that

$$J(\xi u) = \frac{\xi^2}{2}\|\Delta u\|_2^2 - \frac{\xi^p}{p}\|\nabla u\|_p^p \leq 0, \quad M < \frac{p-2}{2p}B^p\|\xi\nu\|_2^p.$$

We fix such a number $\xi > 0$ and choose a function $\mu \in H_{\mathcal{N}}^2(\Omega_2)$ satisfying $M = J(\mu) + J(\xi\nu)$. Extend ν and μ to be 0 in $\Omega\backslash\Omega_1$ and $\Omega\backslash\Omega_2$, respectively, and set $u_M = \mu + \xi\nu$, then $J(u_M) = J(\mu) + J(\xi\nu) = M$, and it follows that

$$\frac{p-2}{2p}B^p\|u_M\|_2^p \geq \frac{p-2}{2p}B^p\|\xi\nu\|_2^p > M = J(u_M) > d.$$

By Proposition 5.1 it is seen that $u_M \in \mathcal{N}_- \cap \mathcal{B}$. $\qquad\square$

6. Lower bound for the lifespan

We all know that the upper bound guarantees blowing up of the solution and the importance of the lower bound is that it may provide us a safe time interval for operation if we use problem (1.1) to model a physical process. In this section, we mainly give the lower bound estimate for the lifespan.

Theorem 6.1. *If T^* is blow-up time, then T^* satisfies the following estimate*

$$T^* \geq \frac{\|u_0\|_2^{-2\kappa}}{B_2^{\frac{2p\kappa}{p-2}}\kappa}, \quad 2 < p \leq p_*, \ N \geq 2,$$

where $\kappa = \frac{4(p-2)}{8-2p-N(p-2)}$ and $p_* = 2 + \frac{4}{N+2}$.

Proof. Set

$$F_2(t) = \int\limits_\Omega |u|^2 dx.$$

Taking the first derivative of $F_2(t)$, then using Gagliardo–Nirenberg inequality, we have

$$
\begin{aligned}
F_2'(t) = \int\limits_\Omega u u_t dx &= \int\limits_\Omega u \left(-\Delta^2 u - \Delta_p u\right) dx \\
&= -\|\Delta u\|_2^2 + \|\nabla u\|_p^p \\
&\leq -\|\Delta u\|_2^2 + B_2^p \|\Delta u\|_2^{\theta p} \|u\|_2^{(1-\theta)p},
\end{aligned}
\tag{6.1}
$$

where $\theta p = \frac{N(p-2)}{4} + \frac{p}{2}$. Since $2 < p \leq p_*$, this shows $\theta p > 2$. Then, together with the ε-Young's inequality, we have

$$F_2'(t) \leq -\|\Delta u\|_2^2 + B_2^p \left(\varepsilon^\beta \|\Delta u\|_2^2 + \varepsilon^{-\beta'} \|u\|_2^{(1-\theta)p\beta'}\right), \tag{6.2}$$

where $\beta = \frac{2}{\theta p}$ and $\beta' = \frac{2}{2-\theta p}$. Next, we choose ε, such that $B_2^p \varepsilon^\beta = 1$, which implies $B_2^p \varepsilon^{-\beta'} = B_2^{p\beta'}$ by virtue of $\frac{1}{\beta} + \frac{1}{\beta'} = 1$. Then (6.2) can be reduced to

$$F_2'(t) \leq B_2^{p\beta'} F_2^{\frac{(1-\theta)p\beta'}{2}}(t). \tag{6.3}$$

A simple integration of (6.3) over $(0, t)$ is easy to calculate that

$$\|u\|_2^2 \leq \left[\frac{1}{\|u_0\|_2^{-2\kappa} - B_2^{p\beta'}\kappa t}\right]^{\frac{1}{\kappa}}. \tag{6.4}$$

Therefore,

$$\lim_{t \to \kappa^{-1} B_2^{-p\beta'} \|u_0\|_2^{-2\kappa}} \|u\|_2^2 = +\infty. \tag{6.5}$$

In other words, we also obtain some estimate for the upper bound of the blow-up time T^*

$$T^* \geq \kappa^{-1} B_2^{\frac{2p\kappa}{2-p}} \|u_0\|_2^{-2\kappa}, \quad 2 < p \leq p_*, \ N \geq 2, \tag{6.6}$$

where $\kappa = \frac{4(p-2)}{8-2p-N(p-2)}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 6.1. From the above analysis, it is not difficult to find, if all the assumptions in Theorems 3.2 and 6.1 hold, then the following blow-up rate holds

$$C_1 \left(T_1^* - t\right)^{-\frac{2}{p-2}} \leq \|u\|_2^2 \leq C_2 \left(T_2^* - t\right)^{-\frac{2}{p-2}}, \quad 2 < p \leq 2 + \frac{4}{N+2}, N \geq 2,$$

where $T_1^* = \frac{pd}{(p-2)^2 B^p (d - J(u_0)) \|u_0\|_2^{p-2}}$, $T_2^* = \kappa^{-1} B_2^{\frac{2p\kappa}{2-p}} \|u_0\|_2^{-2\kappa}$, $\kappa = \frac{4(p-2)}{8-2p-N(p-2)}$.

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